Branched covers in low dimensions

<u>exam</u> : homework (weekly/fortnightly)

Introduction

We're gonna talk about smooth mflds of dimension (1), 2, 3, and 4.

Usually : compact,

often : orientable, without boundary

Why ≤ 4 ?

Differential topology
- $\boxed{\text{low-dimensional}}$
- $\sim$ threshold at 4.
- high-dimensional

different tools : h-cobordism thm, S-cobordism thm, surgery theory

algebraic topology "dominates" topology

$\rightsquigarrow$ different tool, more ad hoc

Main problem in low-dim is the failure of the Whitney trick

$$\boxed{\begin{array}{ll} D^2 \looparrowright X & \text{immersed} \\ \quad \rightsquigarrow & \text{perturbation} \\ D^2 \hookrightarrow X & \text{embedded} \end{array}} \quad \text{if } \dim \geq 5$$

↳ most of cxity of a manifold is in π_1.

In dim ≤ 3 : "geometry" dominates (in particular, "hyperbolic geometry")

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dim $= 4$ is wild : all sorts of weird things happen.

- How do we study manifolds?

$\longrightarrow$ look at maps between them:

If we have X a fixed mfld, we can look at maps:

- out of X: $\boxed{X \longrightarrow \mathbb{R}}$ Morse theory

  $\boxed{X \longrightarrow S^N}$ $\to$ cohom-theory

- into X: $\boxed{S^N \longrightarrow X}$ homotopy theory: π_1

  $\boxed{Y \subset\!\!\longrightarrow X}$

  $\hookrightarrow$ in some sense, this is homology.

- onto X: $\tilde{X} \longrightarrow \boxed{X}$ $\leftarrow$

  that are onto + some special properties

  Branched covers.

recep. $\leftarrow$ $\boxed{\rightsquigarrow \text{gen. of covers.}}$

Plan of the course:

today

$\boxed{2} \longrightarrow 3 \longrightarrow 4$

2:
b.c. of surfaces
Riemann–Hurwitz
degree-genus

3:
knots & links
examples
$\Omega_3 = 0$

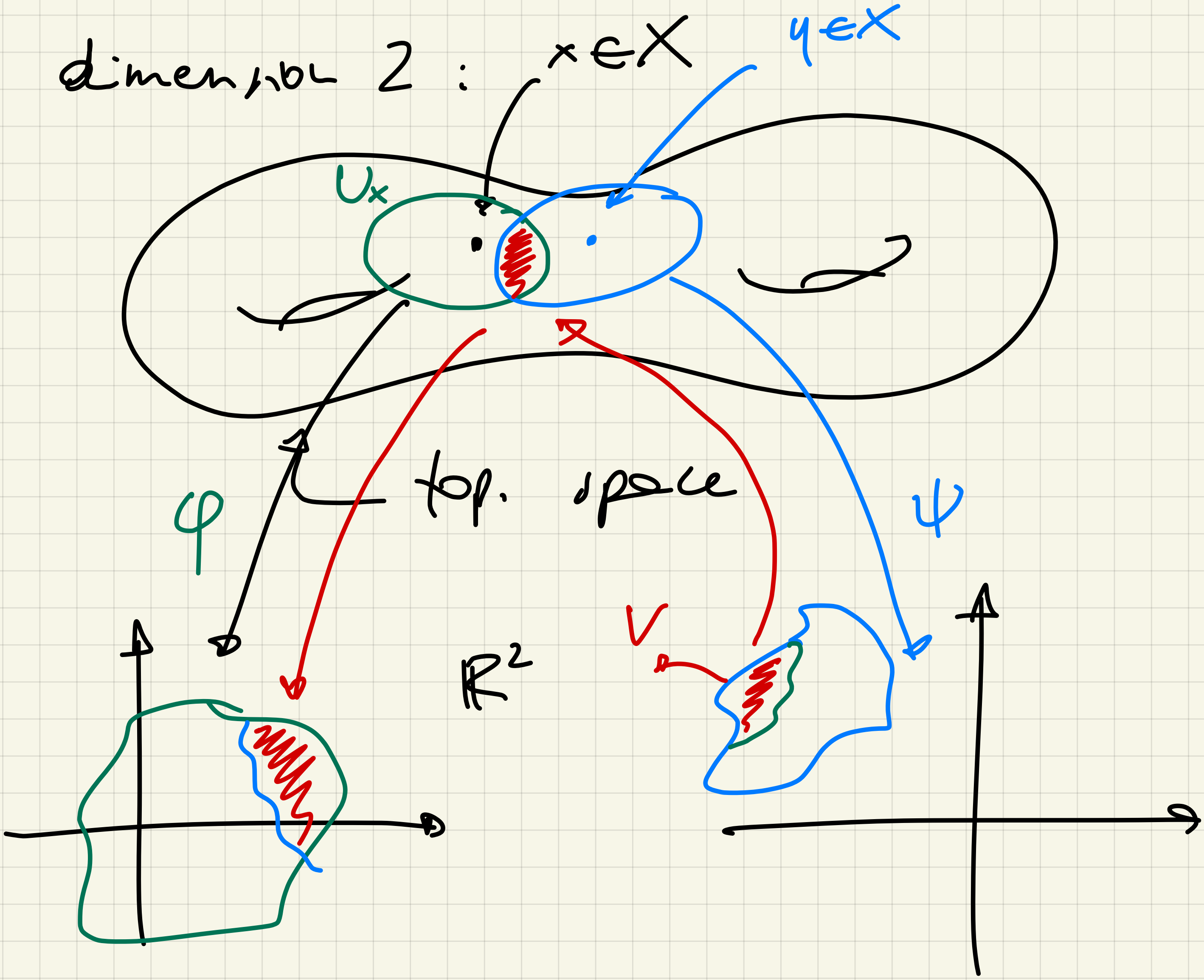
4:
zoo, knotted surfaces in 4D
g-signature theorem
applications.

interludes:
- recap (surfaces, covering spaces, algeb. topology)
- group actions (Riemannian metrics)
- vector bundles.

Recap

- Surfaces

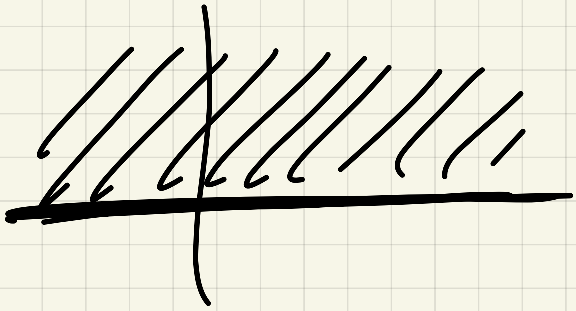
def A surface is a smooth mfld X of dimension 2:



$\varphi \circ \psi^{-1} : V \longrightarrow \mathbb{R}^2$ is a diffeo

↳ orientability: check the $\varphi : U \to \mathbb{R}^2$ to be consistently oriented → $\mathrm{Jac}(\varphi \circ \psi^{-1})$ has constant sign.

boundary: ∂



$(\mathbb{R}^2, y \geq 0)$

<u>ex</u> Take a surface and remove an open ball $\rightsquigarrow$ creates ∂.

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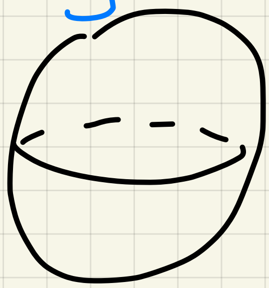
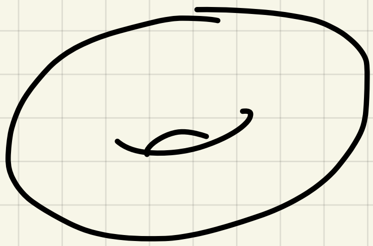
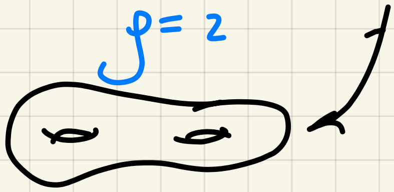
known things: $\circledast$ all surfaces admit a triangulation (i.e. are homeo to a simplicial cx), and all triangulated surfaces can be smoothed (uniquely): <u>moral</u>: surfaces can be treated combinatorially.

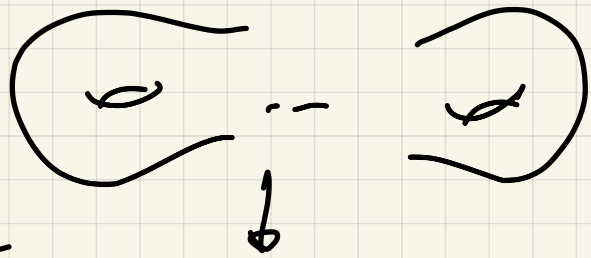
$\circledast$ <u>Classify surfaces</u>:

→ orientable surfaces

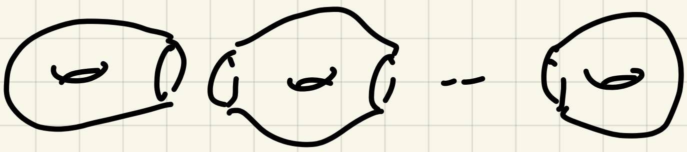
$g=0$ 
↳ sphere S^2

$g=1$ 
↳ torus T^2

$g=2$ 



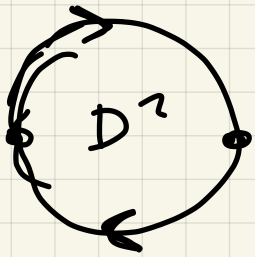
conn. sums:



number of tori in the decomposition is called the **genus** of the surface.

→ non orientable surfaces

$\mathbb{RP}^2$ real projective plane:

$S^2/_{x\sim -x}$ or 

$= D^2/_{x\sim -x}$ on ∂D^2.

every non-or. surface

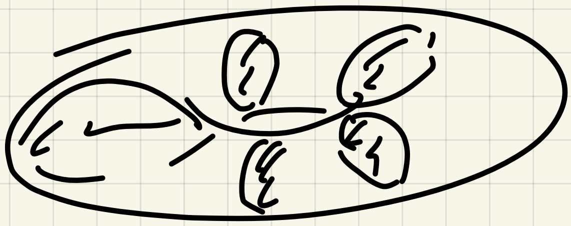
$\underbrace{\quad}_{\mathbb{RP}^2}$ ... $\underbrace{\quad}_{\mathbb{RP}^2}$ $\underbrace{\quad}_{\mathbb{RP}^2}$

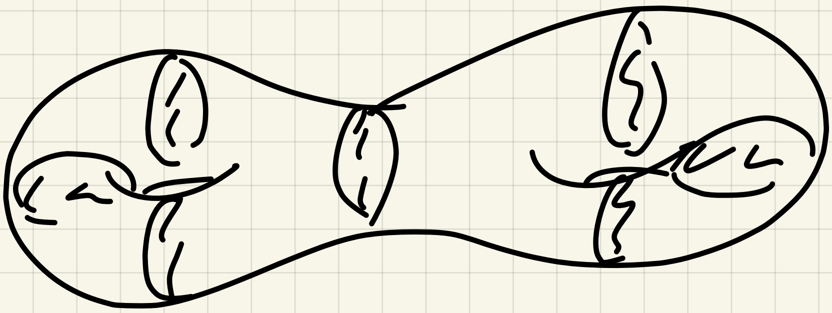
note Every orientable surface bounds a 3-manifold:

$$S^2 = \partial B^3$$

solid torus

$$T^2 = \partial \text{ "donut" } = \partial(\boxed{S^1 \times D^2})$$





When do we find surfaces?

Algebraic geometry (cx geometry)

$$\mathbb{C}^2 \supset \{ f(x,y) = 0 \} =: C$$

f is a polynomial in two variables w/ $\mathbb{C}$-coefficients

If not unlucky (if f is generic)

C is a surface in $\mathbb{C}^2$.

however: C non compact.

Look at $\mathbb{CP}^2$ cx proj. space:

$$\mathbb{CP}^2 = \mathbb{C}^3 \setminus \{(0,0,0)\} \big/ \sim$$

$$\left[\begin{array}{l} (x,y,z) \sim (\lambda x, \lambda y, \lambda z) \\ \lambda \in \mathbb{C}^* \\ [(x,y,z)] = (x:y:z). \end{array}\right.$$

Now, if $F(x,y,z) \in \mathbb{C}[x,y,z]$ & F is homogeneous of degree d, then

$C = \{F = 0\} \subset \mathbb{CP}^2$

is a surface in $\mathbb{CP}^2$ (if F is generic)

generic here means that the curve

C is non-singular, i.e.

$$\nabla F \neq 0 \quad \text{on} \quad C.$$

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exemple / exercise $\qquad F(x,y,z) = x^d + y^d + z^d$

$\rightsquigarrow$ the curve C is non-singular.

In $\mathbb{C}P^2 \supset \mathbb{R}P^2$

$\mathbb{C}P^2 = (x:y:z)$ all cx

$\mathbb{R}P^2 = (x:y:z)$ $\hookrightarrow$ all reals

$\pi_1(\text{surface}) = \longrightarrow$

$\left.\begin{array}{l}\text{orientable, w/o } \partial \\ \text{genus} = g\end{array}\right\} \longrightarrow \pi_1 = \left\langle \begin{array}{c} x_1 \dots x_g \\ y_1 \dots y_g \end{array} \middle| \text{rel.} \right\rangle$

$$\text{rel} = [x_1 y_1] \cdot \ldots \cdot [x_g y_g]$$

$$x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1} \ldots x_g y_g x_g^{-1} y_g^{-1} = 1$$

e.g. $\pi_1(T^2) = \langle x, y \mid [x, y] \rangle =$

$$= \mathbb{Z}^2.$$

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if surface is orientable

genus $= g$, has $r+1$ $\rightsquigarrow$ $\pi_1 =$ free on

boundary cpts $(r \geq 0)$ $2g + r$

generators.

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if surface is non-orientable

$\partial = \emptyset$, conn. of h

copies of $\mathbb{RP}^2$,

$$\pi_1 = \langle x_1 \cdots x_h \mid x_1^2 \cdots x_h^2 \rangle$$

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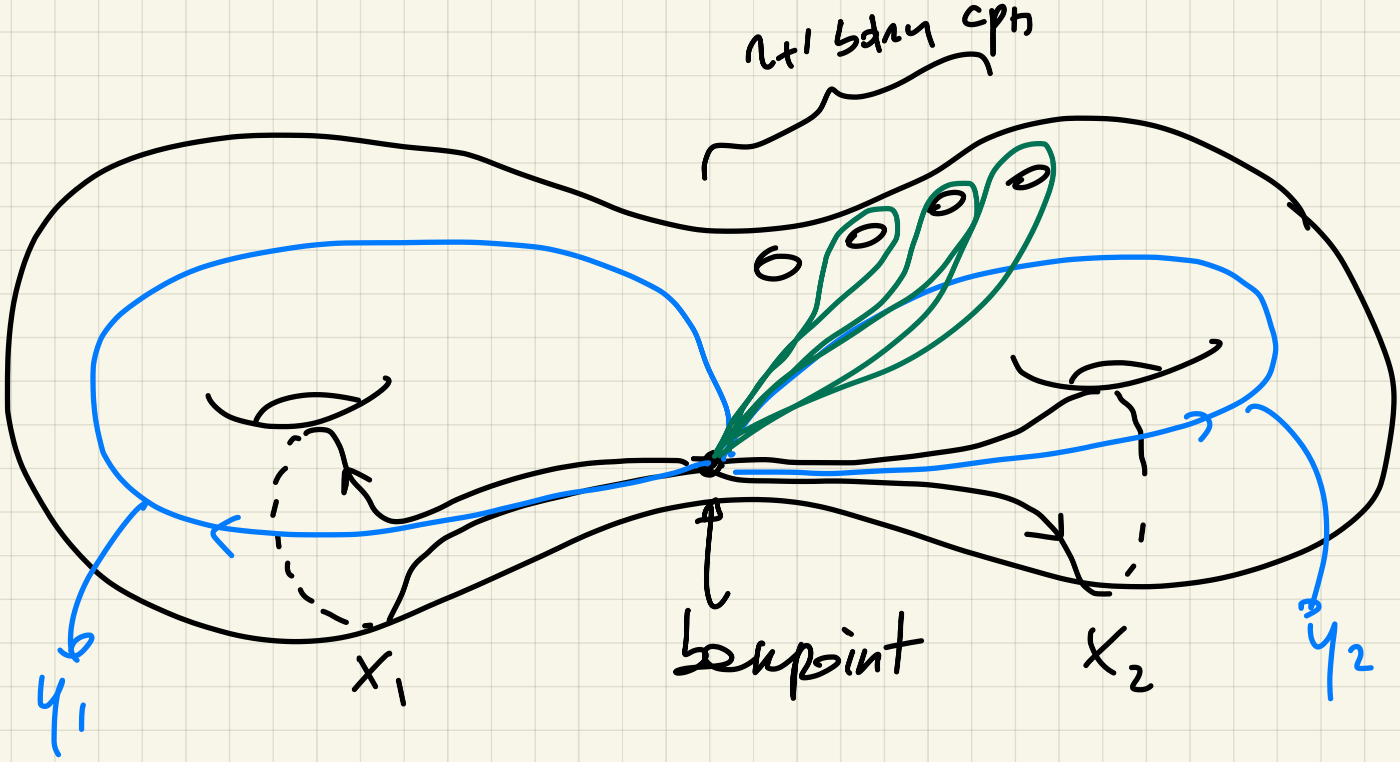
if non-or., non-empty ∂,

π_1 is free on $h + r$ of generators.

h copies of $\mathbb{RP}^2$, $r+1$ boundary cpts



- <u>Covering spaces</u>

A covering (space, map)

$$p : \tilde{X} \longrightarrow X$$

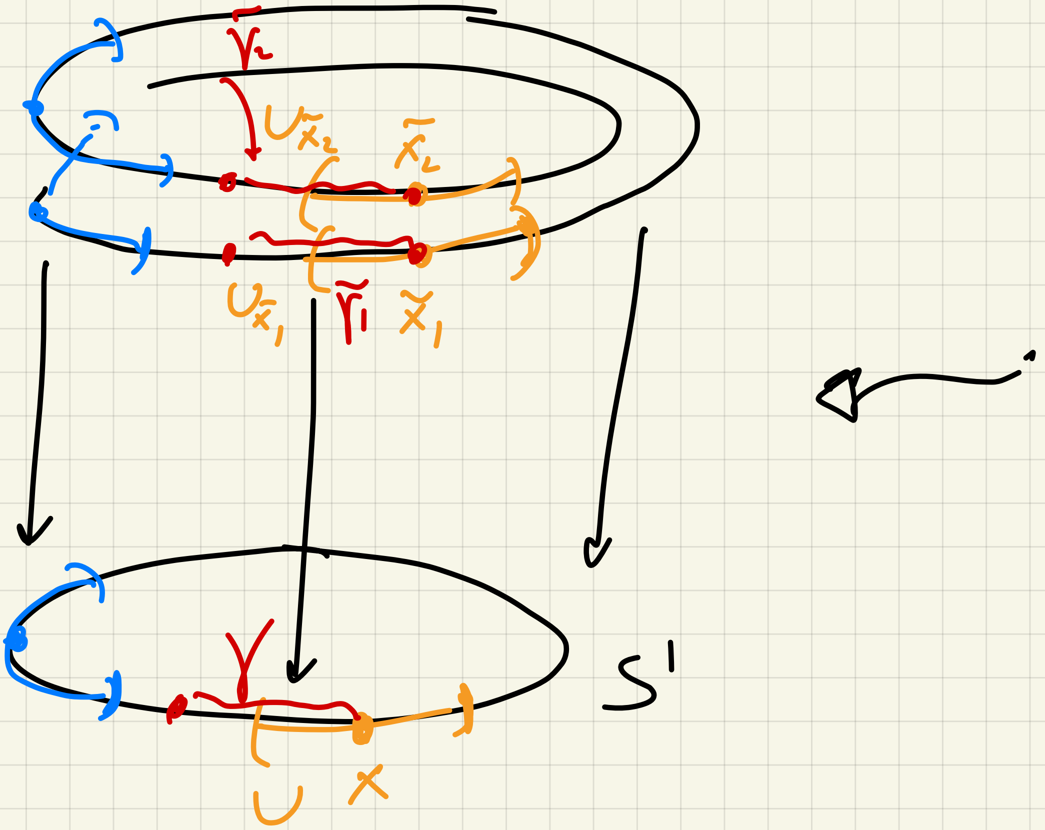
Cont. between top. spaces &

such that $\forall\, x \in X$ $\exists\, U \ni x$ open set

such that $p^{-1}(U) = \coprod_{p(\tilde{x}) = x} U_{\tilde{x}}$

where $p|_{U_{\tilde{x}}} : U_{\tilde{x}} \longrightarrow U$ is

a homeo $\forall\, \tilde{x} \in p^{-1}(x)$



<u>properties</u> : • any path γ in X lifts to a path in $\tilde{X}$ for each preimage of the st. point of γ

• $p_* : \pi_1(\tilde{X}, \tilde{x}) \to \pi_1(X, p(\tilde{x}))$

i) <u>injective</u>. $\pi_1(\tilde{X}) < \pi_1(X)$

& every subgroup of $\pi_1(X)$ corr. to a connected covering space (if X is not pathological)

- we can take $\{1\} < \pi_1(X)$

  $\downarrow$

  this gives a covering space that is both larger & simpler (π_1 is simpler) than all others.

$\rightsquigarrow$ the <u>universal cover</u> of X.

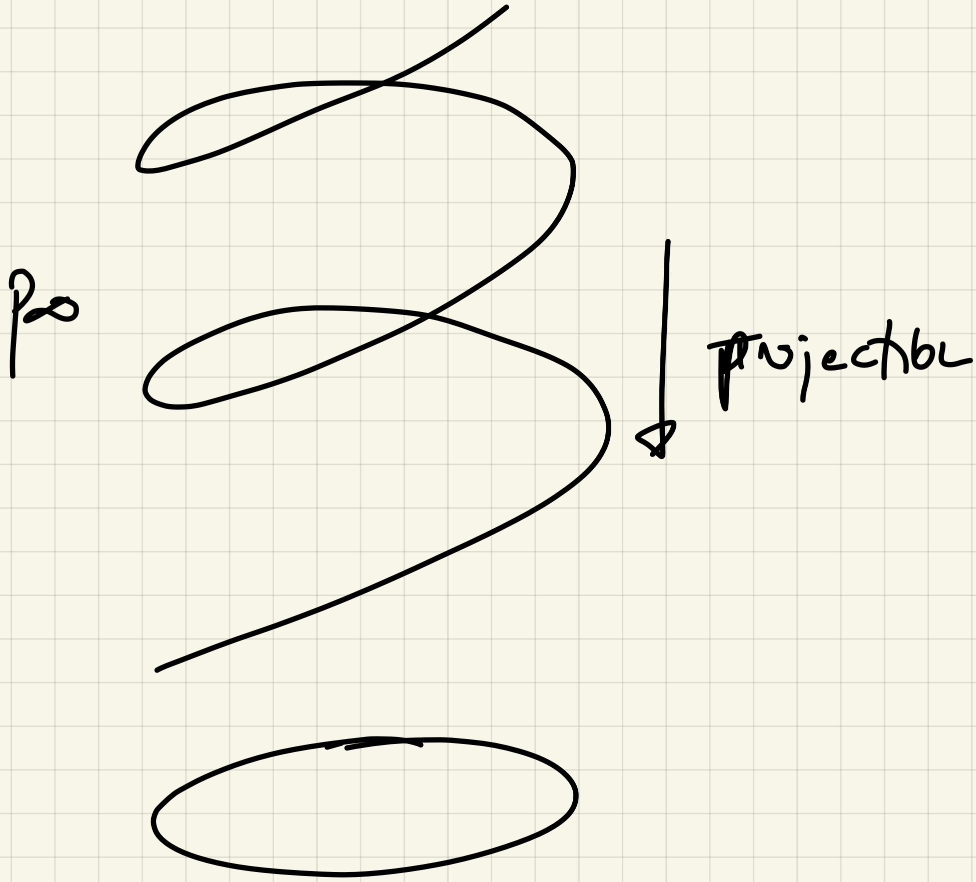
<u>ex</u> $S^1 \subset \mathbb{C}$ unit circle in $\mathbb{C}$

$$\begin{aligned} p_k : S^1 &\longrightarrow S^1 \\ z &\longmapsto z^k \end{aligned}$$

$\rightarrow$ is a covering map. if $k \neq 0$.

$$\begin{aligned} p_k \times p_\ell : S^1 \times S^1 &\longrightarrow S^1 \times S^1 \\ (z, w) &\longmapsto (z^k, w^\ell) \end{aligned}$$

$$\begin{aligned} p_\infty : \mathbb{R} &\longrightarrow S^1 \\ t &\longmapsto e^{2\pi i t} \end{aligned}$$



<u>exercise/fact</u>: • p_k corresponds to the subgroup $k \cdot \mathbb{Z} \subset \mathbb{Z} = \pi_1(S^1)$

• p_∞ corresponds $\{0\} \subset \mathbb{Z} = \pi_1(S^1)$

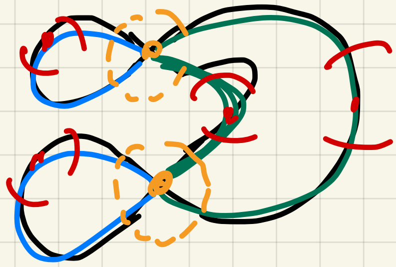
$$\pi : S^2 \longrightarrow \mathbb{RP}^2 = S^2/\sim$$

<u>fact</u>: this is a covering map.

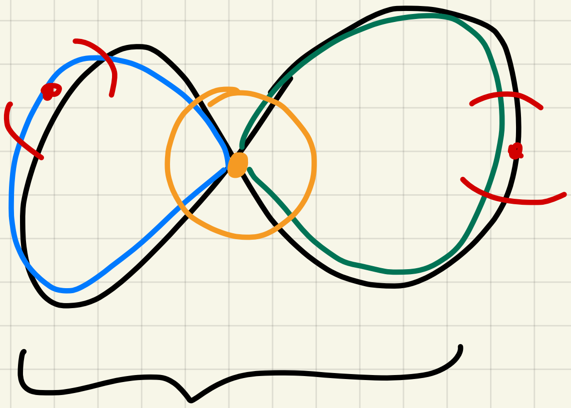
π $\downarrow$ associated to $\{0\} \subset \mathbb{Z}/2\mathbb{Z} \cong \pi_1(\mathbb{RP}^1)$.

$$\pi_1(\mathbb{RP}^2) = \langle x \mid x^2 \rangle = \mathbb{Z}/2\mathbb{Z}$$

ex



$\rightsquigarrow$ this is a covering map.



wedge of two circles

Branched covers of surfaces

non-branched: S^2 does not have any cover; T^2 is only covered by itself, $\mathbb{R} \times S^1$, $\mathbb{R} \times \mathbb{R}$.

add more flexibility, by softening the condition of covering spaces.

Idea comes from complex geometry / complex analysis : gives us a local model :

set of points

<u>def</u> A <u>branched cover</u> between two surfaces X, Y is a map $p: X \to Y$ such that $\exists B \subset Y$ finite &:

$$\boxed{p|_{p^{-1}(Y \setminus B)} : p^{-1}(Y \setminus B) \longrightarrow Y \setminus B}$$

branch locus

principal part

is a cover & around each point $b \in B$ & $x \in p^{-1}(b)$ there are coordinates such that

$$p : z \longmapsto z^k \text{ for some } k.$$

local model of a hol^c map.

If $|p^{-1}(y)| < \infty$ $(y \in Y \setminus B)$,

then its cardinality is called the <u>degree</u> of p.

<u>ex</u> • $\bar{p}_k : \mathbb{C} \longrightarrow \mathbb{C}$ is a branched cover
$\qquad z \longmapsto z^k$

br. loc. is $\{0\}$.

• $p_k : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$
$\qquad (z:w) \longmapsto (z^k : w^k)$

branch loc is $\{(0:1), (1:0)\}$

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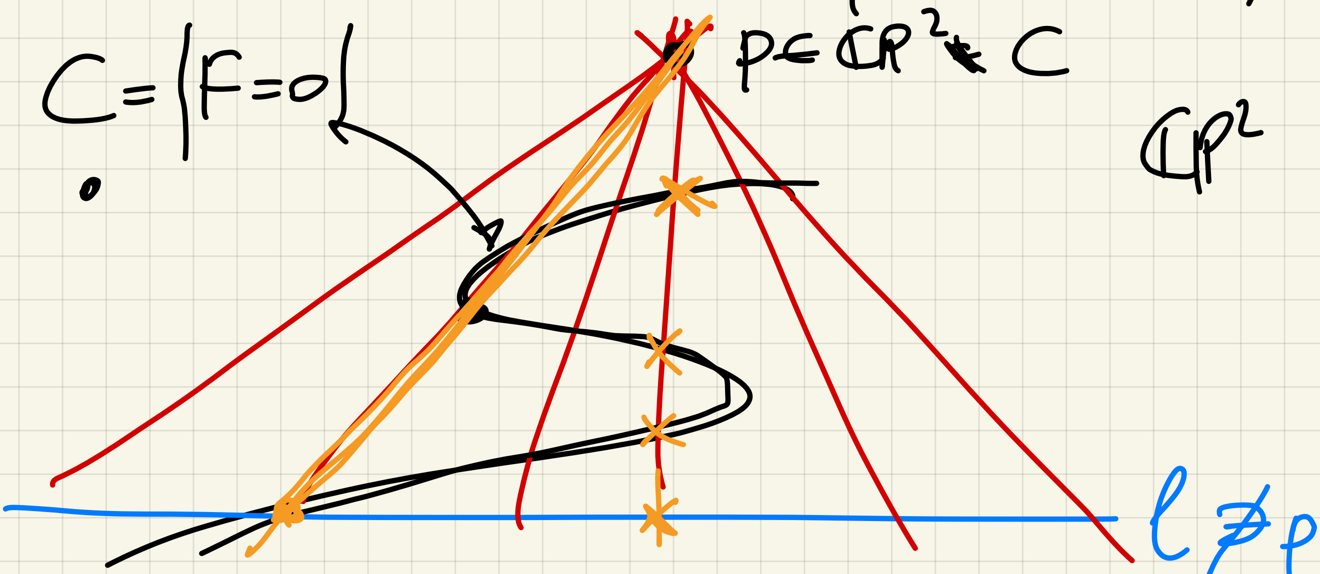
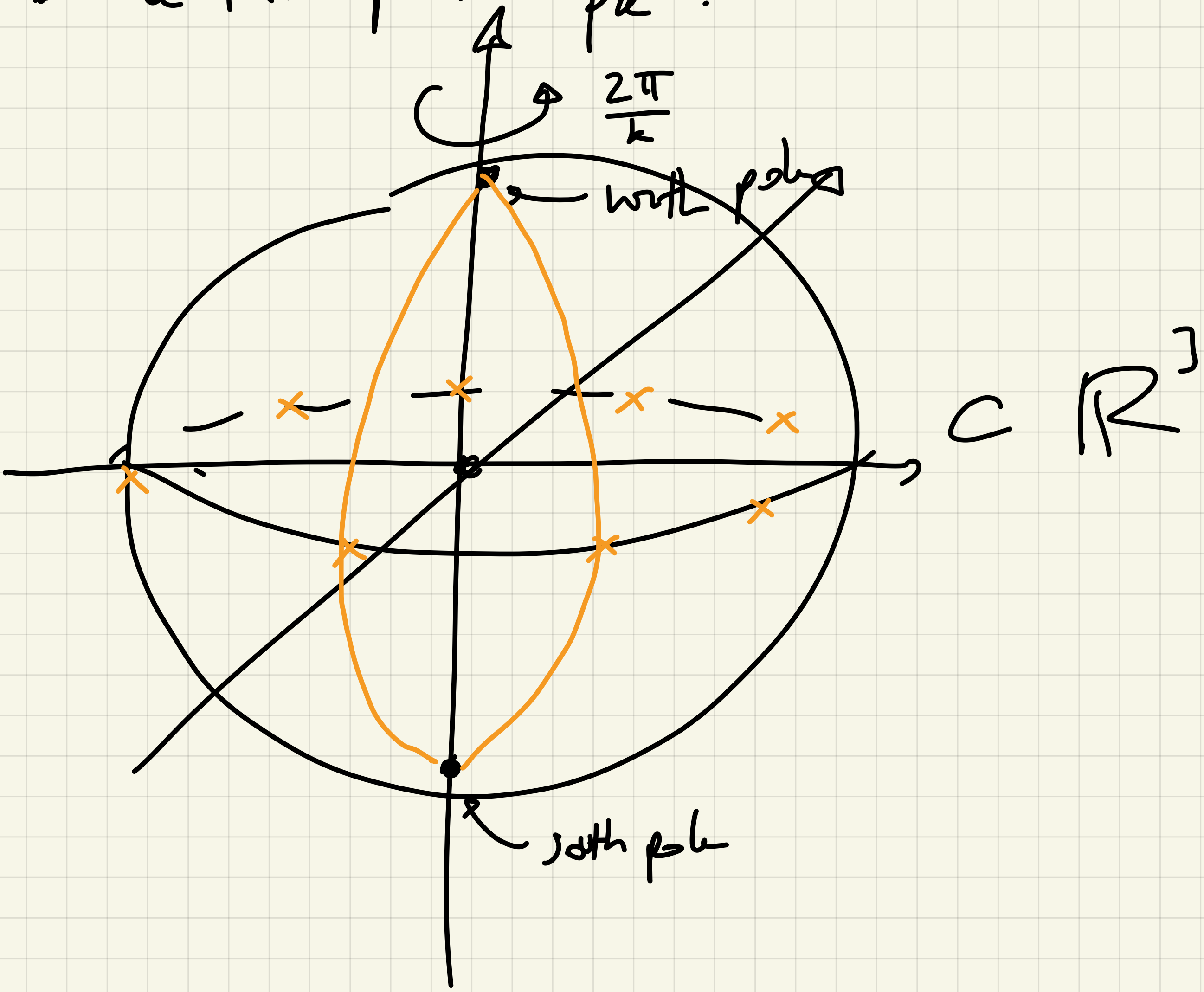


$$C \to \ell \simeq \mathbb{CP}^1 \simeq S^2$$

this is a b.c. because it's a holomorphic map.

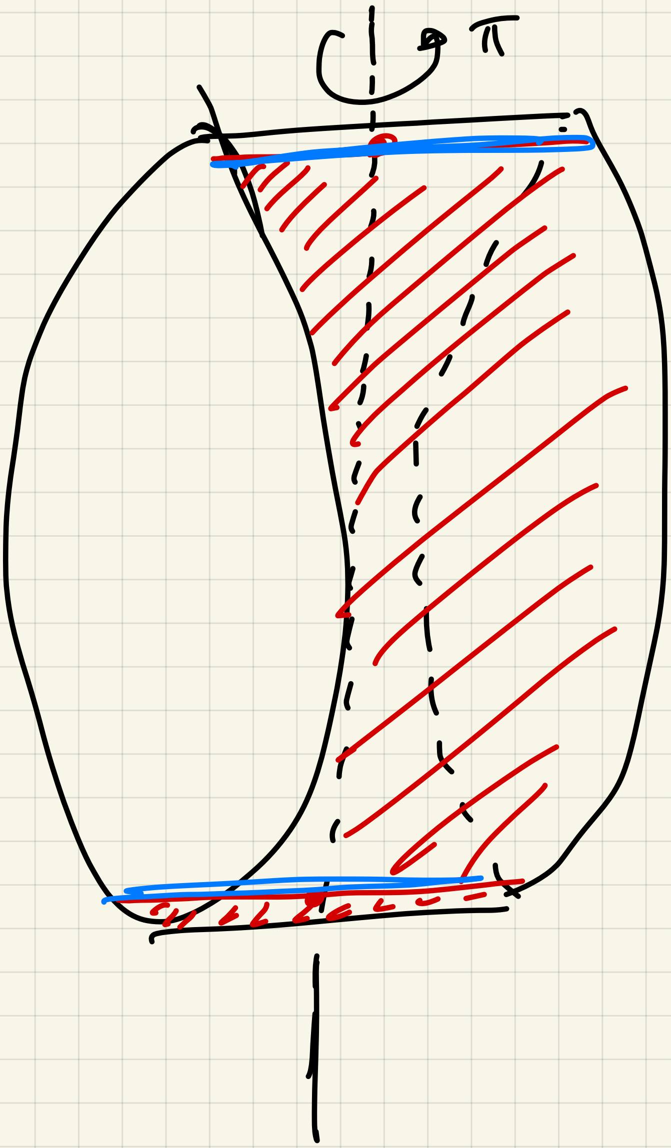
- re-interpret p_k :



$\sim_k$ on S^2 : $p \sim rot(\varphi, \frac{2\pi j}{k})$

p_k is conj. to the quotient map

$S^2 \longrightarrow S^2/\sim_k$ "is" p_k.



annulus : $S^1 \times I$

A

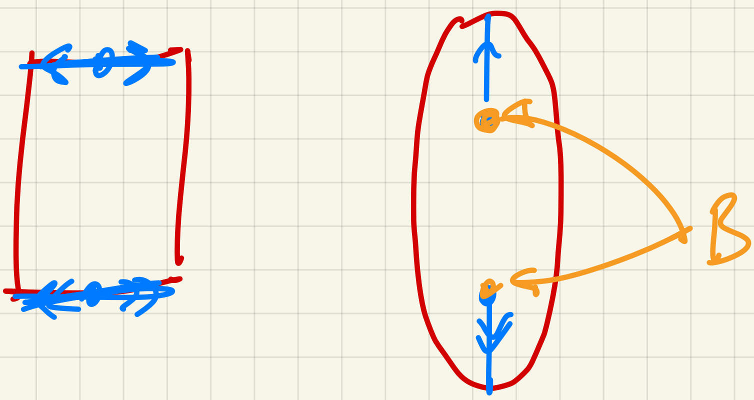
act $:= \tau$

A/τ

every eq. class in here has a rep. in the red half

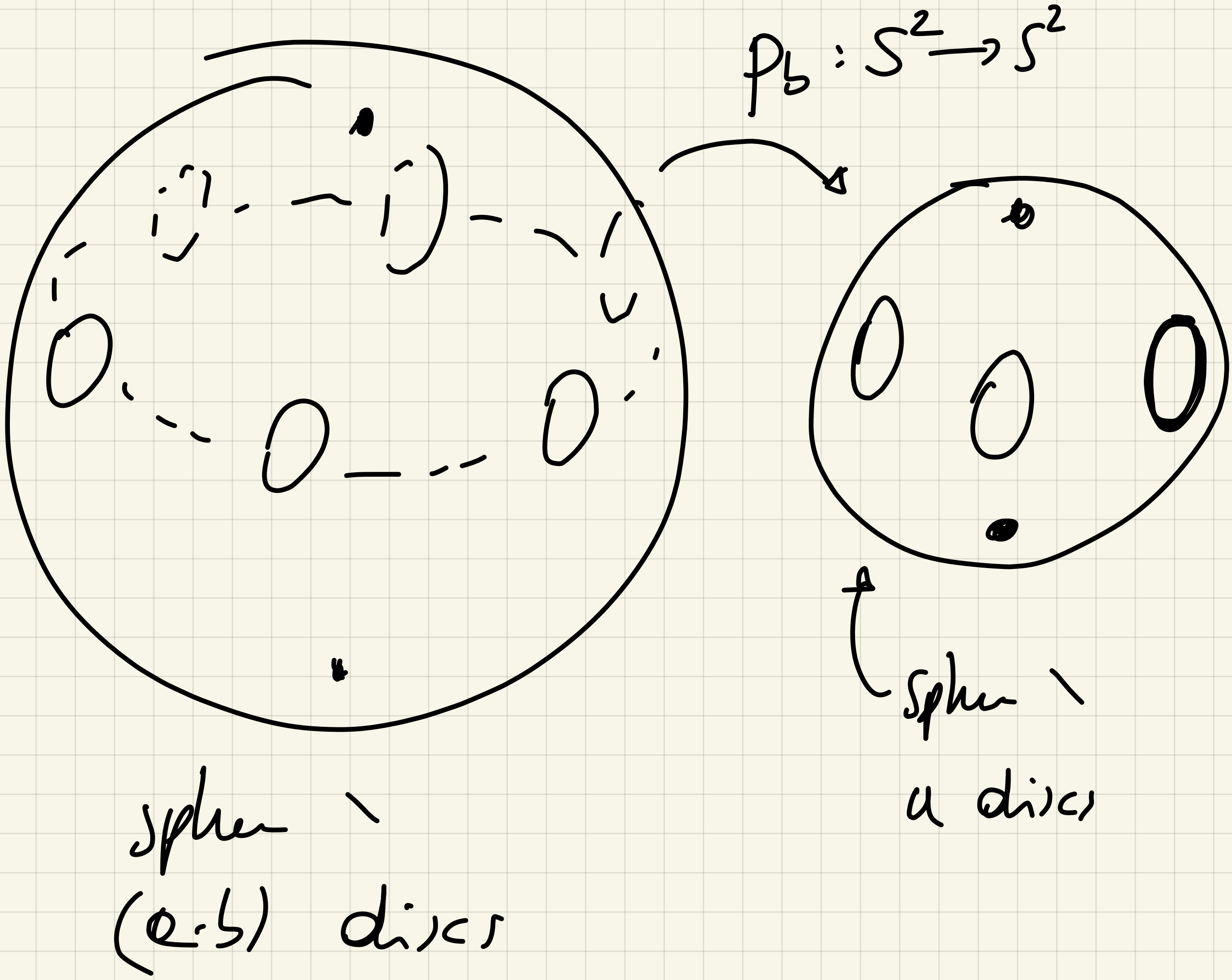
the rep. is unique except for points on the blue arcs

↳ this means A/τ = red part$/\sim$ blue on the part



We have a cover $A \longrightarrow D^2$
of degree 2 branched over
two points, over each of them,
the local model is $z \mapsto z^2$.

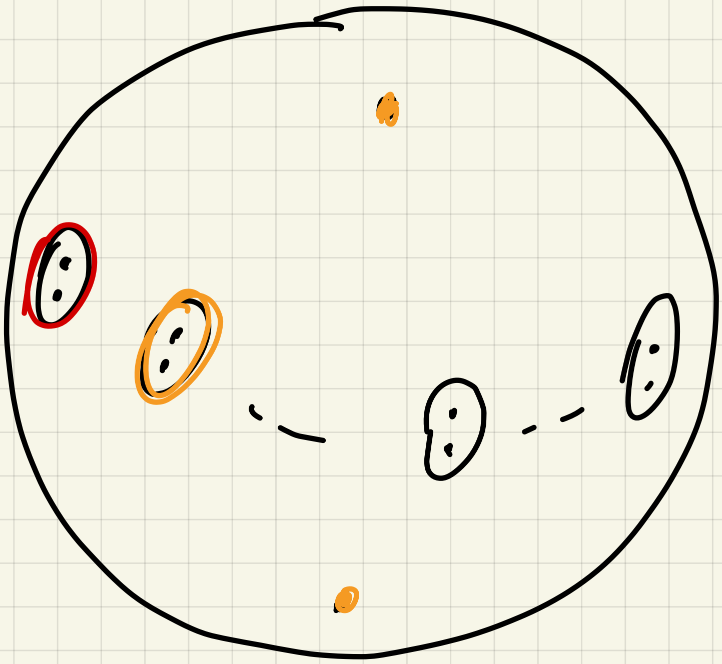
$\rightsquigarrow$ slightly generalize it



branched over two points, of degree
b.

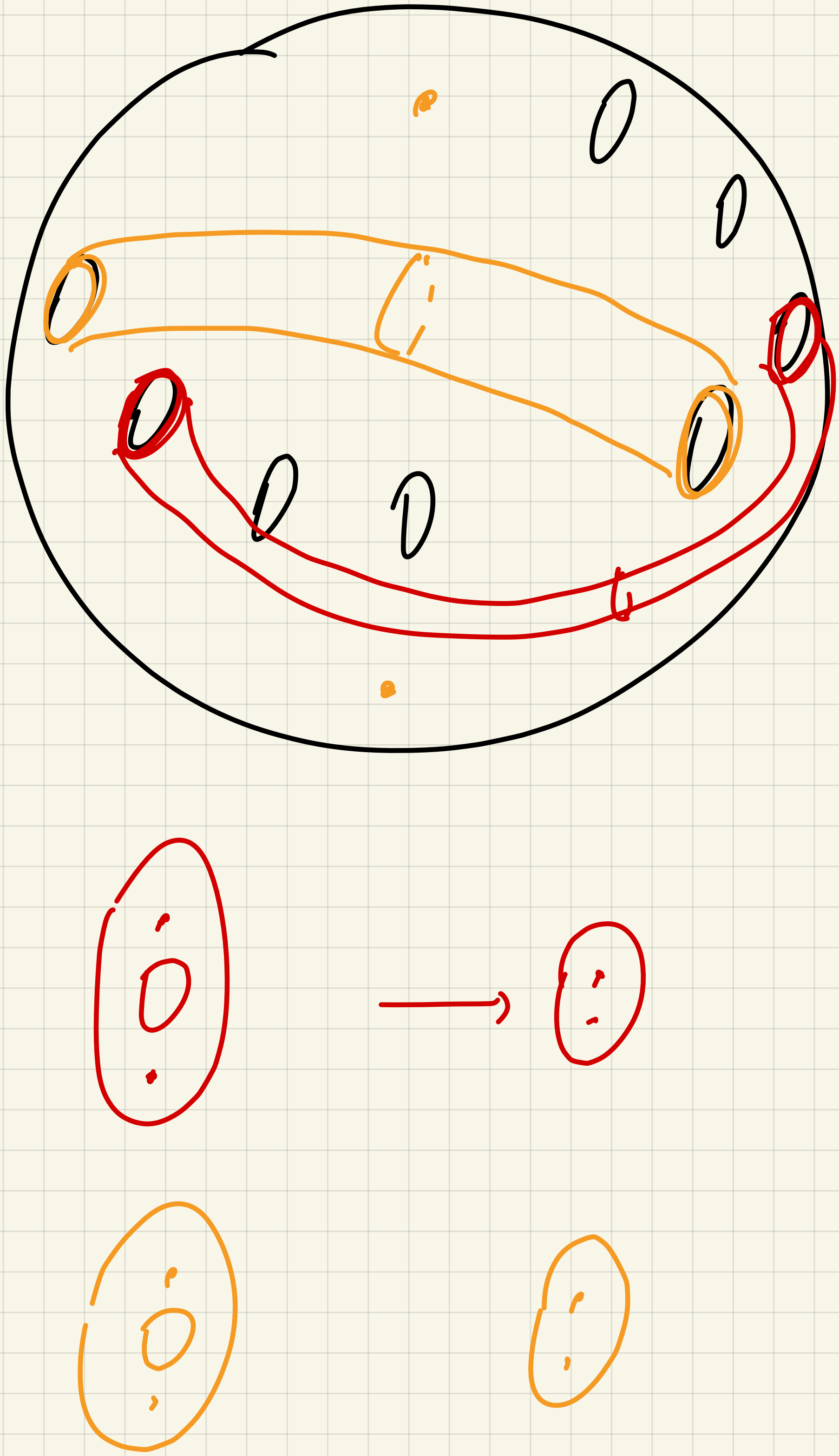
Prop Every orientable compact surface is a double cover of S^2.

Proof Say that the surface has genus g.



choose g equally spaced discs along the equator of S^2 & two points inside each of them.

Claim: can double cover the sphere branching over these $2g+2$ pts & I will get a genus-g surface.



$\rightsquigarrow$ gives a decomp. of the double cover that we have constructed into g tori $\rightsquigarrow$ genus-g surface.

<u>prop</u> Every surface has a cover over S^2 branched over exactly three points. (but the degree is large).

<u>Notation - terminology</u> : If

$p: X \to Y$ is a branched cover

$x \in X$, either p is a homeo around x ($z \mapsto z'$) or there is a chart $z \mapsto z^{k_x}$ for some k_x.

This k_x is called the <u>index of p at x</u>, $e_p(x)$ or $e_x(p)$

<u>thm</u> (Riemann-Hurwitz formula)

If $p : X \longrightarrow Y$ is a branched cover between closed oriented surfaces,

$$2 - 2g(X) = d(2-2g(Y)) - \boxed{\sum_{x \in X} (e_p(x) - 1)}$$

where $d = \deg p$.

Looks very infinite

but for all but finitely many $x \in X$

$e_p(x) = 1$.

<u>ex</u>: $p_k : S^2 \longrightarrow S^2$

$g(S^2) = 0$

$\deg p_k = k$

two branching pts of index k.

$$2 = k \cdot 2 - 2(k-1)$$

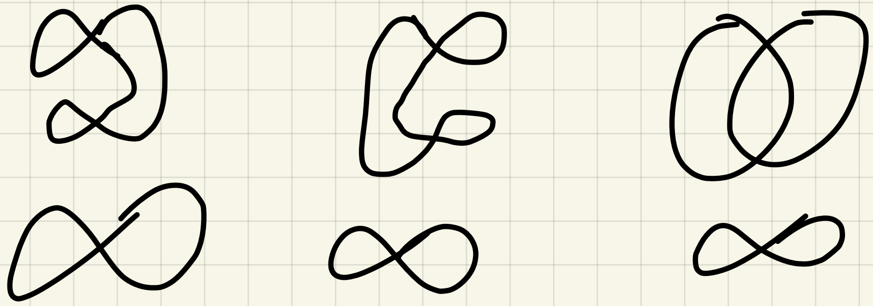
:)

ex $\begin{array}{c} F \longrightarrow S^2 \\ \text{genus} = g \end{array}$, deg $= 2$, branched over $2g+2$ pts w/ index 2 at each

$$2 - 2g = \overset{\deg}{2} \cdot (2 - 0) - (2g+2) \cdot (2 - 1)$$

☺

<u>Lemma</u> Suppose we have a 1-dim'l CW complex X connected, and has one 0-cell and b 1-cells. Let $\tilde{X}$ be any d-fold cover. Then $\tilde{X}$ is htpy eq. to a CW cx with one 0-cell and $db - d + 1$ 1-cells.



<u>proof of the lemma</u>:

the 0-cell in X lift to

d 0-cells in $\tilde{X}$.

Each 1-cell in X lifts to

d 1-cells in $\tilde{X}$.

$\tilde{X}$ will have d 0-cells and

$d \cdot b$ 1-cells.

& $\tilde{X}$ is connected by assumptions,

so $\tilde{X}$, up to homotopy, has one

0-cell & $db-(d-1)$ 1-cells.

$\square$

$\rightsquigarrow$ that is to say, $\chi(\tilde{X}) = 1 - (d_b - d + 1) = d - d_b$.

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proof of the R-H formula

- If there is no branching at all, the formula is a consequence of the mult. of the Euler ch. under covering maps $\rightsquigarrow$ ex.
- If there is branching, let us look at the principal part

$$p' : X' \longrightarrow Y'.$$

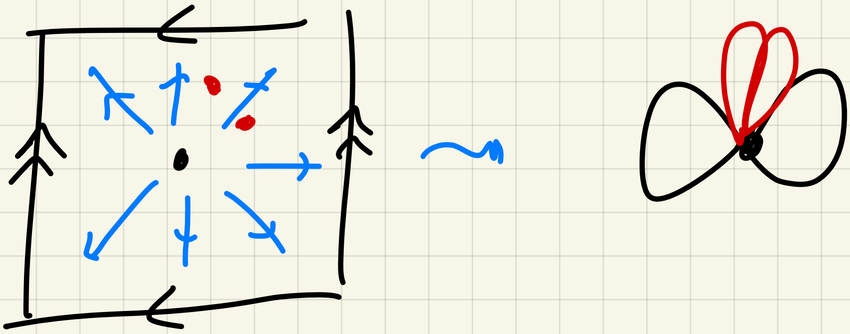
$$Y' = \underset{\substack{\uparrow \\ \text{surface}}}{Y} \setminus B$$

finite, non-empty set

<u>fact</u> A compact surface \ a few points

retracts onto a CW cx

of dim 1.

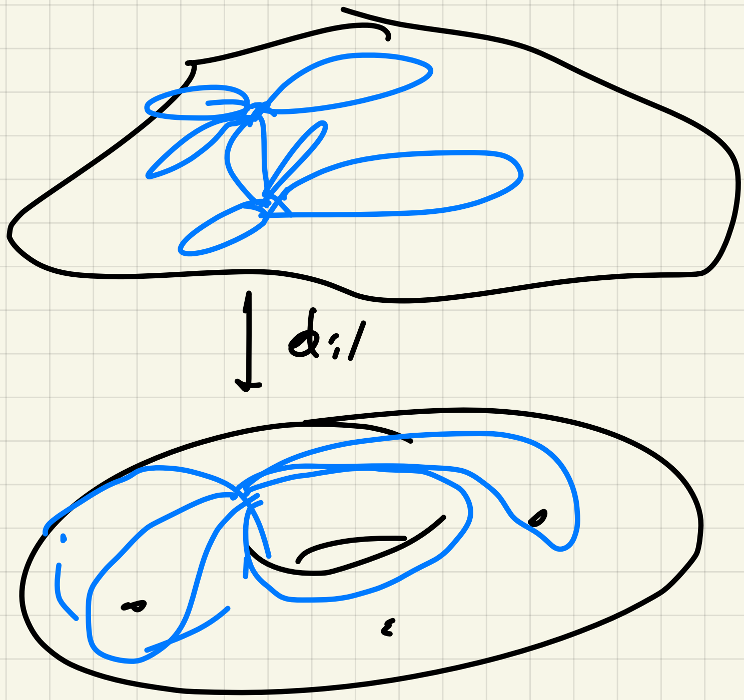


$p' : X' \longrightarrow Y'$ can be seen

as a thickening of a d-fold

cover of a 1-dim'l CW cx

A



The lemma tells you that

$$\chi(p^{-1}(A)) = d - db$$

We need to compare

$$\boxed{\chi(A) \longleftrightarrow \chi(Y)}$$

$$\chi(A) = \underset{\substack{\downarrow \\ 2-2g(Y)}}{\chi(Y)} - |B|$$

$$\boxed{\chi(p^{-1}(A)) \longleftrightarrow \chi(X)}$$

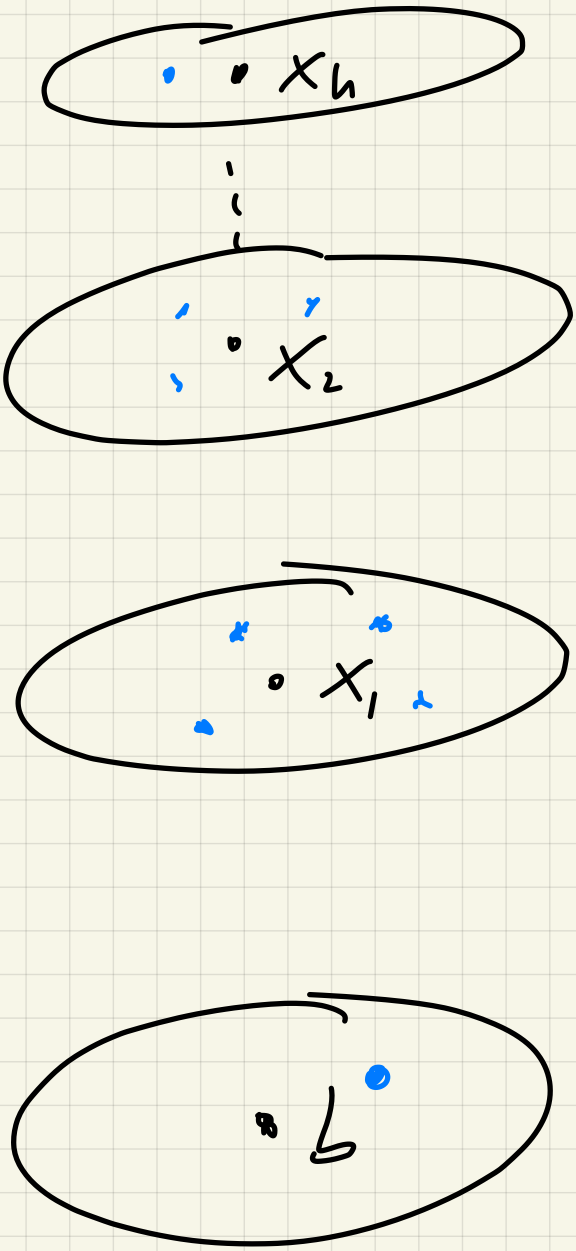
$$\underset{\substack{\Downarrow \\ d\,\chi(A)}}{\chi(p^{-1}(A))} = \underset{\substack{\downarrow \\ 2-2g(X)}}{\chi(X)} - |p^{-1}(B)|$$

Main pf: removing a pt from a surface decreases χ by 1.

Fix $b \in B$, $p^{-1}(b) \subset X$

$x_1^b, \dots, x_{h_b}^b$ (h depends on b)

Around x_j: coord. $p: z \mapsto z^{k_j}$



$z \mapsto z^{k_h}$

$z \mapsto z^{k_2}$

$z \mapsto z^{k_1}$

$\forall b$, $\sum_{x \in p^{-1}(b)} e_p(x) = d$

$|p^{-1}(B)| = \sum_{b \in B} h_b$

$$\chi(X) - |p^{-1}(B)| = d\chi(A) =$$

$$= d(\chi(Y) - |B|)$$

$$2 - 2g(X) = d(2 - 2g(Y)) -$$

$$\boxed{(d|B| - |p^{-1}(B)|)}$$

$$d|B| - |p^{-1}(B)| =$$

$$\sum_{b\in B} (d - h_b) = \sum_{b\in B} \Big(\sum_{x\in p^{-1}(B)} e_p(x) - \sum_{x\in p^{-1}(B)} 1 \Big)$$

$$d \to h_b$$

$$\sum_{i=1}^{h_b} k_i^b = \sum_{x\in p^{-1}(b)} e_p(x)$$

$$= \sum_{b\in B} \Big(\sum_{x\in p^{-1}(b)} (e_p(x) - 1) \Big)$$

$$\sum_{b \in B} \sum_{x \in p^{-1}(b)} \left(e_p(x) - 1 \right) =$$

$$= \sum_{x \in X} \left(e_p(x) - 1 \right). \qquad \square$$