Lecture 2

last time: we looked at branched covers between surfaces

$$p : X \longrightarrow Y$$

closed oriented surfaces

p is a covering map away from finitely many pts.

thm (Riemann-Hurwitz formula)

$$\underbrace{2-2g(X)}_{\text{global}} = \underbrace{d(2-2g(Y))}_{\text{global}} - \underbrace{\sum_{x\in X}\left(e_p(x)-1\right)}_{\text{local}}$$

<u>thm</u> (degree-genus formula)

If $C \subset \mathbb{CP}^2$ is a non-singular plane projective curve, of degree d, then $g(C) = \frac{(d-1)(d-2)}{2}$.

$\mathbb{CP}^2 = \mathbb{C}^3 \setminus \{0\} \big/ (x,y,z) \sim \lambda(x,y,z)$

$\forall \lambda \in \mathbb{C}^*$.

$[(x,y,z)] = (x : y : z)$.

$\hookrightarrow$ is a complex surface (it is a 4-manifold) compact, without boundary.

$\mathbb{CP}^2 = \mathbb{C}^2 \cup \mathbb{CP}^1_\infty$

$\llcorner$ line at ∞

$\underline{\mathbb{C}P^1_\infty}$ is a copy of $\mathbb{C}P^1 \cong S^2$.

In $\mathbb{C}P^2$ we have zero-sets of polynomials

$F \in \mathbb{C}[x, y, z]$ <u>homogeneous</u>,

then $C = \{F = 0\} \subset \mathbb{C}P^2$

is a complex curve (usually, homeomorphic to a surface).

Implicit function thm $\Longrightarrow$

$\{F = 0\} =: C$ is a smooth surface

if $\nabla F \neq 0$ on C.

<u>def</u> We say that C is <u>non-singular</u> if $\nabla F \neq 0$ on C.

ex If F has degree 1;

If F has degree 2 & is not of the form

$$\underbrace{L_1(x,y,z)}\cdot \underbrace{L_2(x,y,z)}$$

linear functions in x,y,z

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prop (in the notes)

① The set of sing. curves is an algebraic submanifold of the space of all degree-d curves.

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rmk curves of degree d

$$\updownarrow$$

$$\{\text{polynomials of degree } d\} / \mathbb{C}^*$$

$\shortparallel$ projective space of dim $\binom{d+2}{2}-1$

② It is a proper submanifold

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or codim ({sing. curves} ⊂ {all curves})

is at least 2 $\Longrightarrow$ it does

not disconnect.

$\Longrightarrow$ {non-sing. curves} is connected.

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relevance to us:

it's enough to prove the formula

for one curve.

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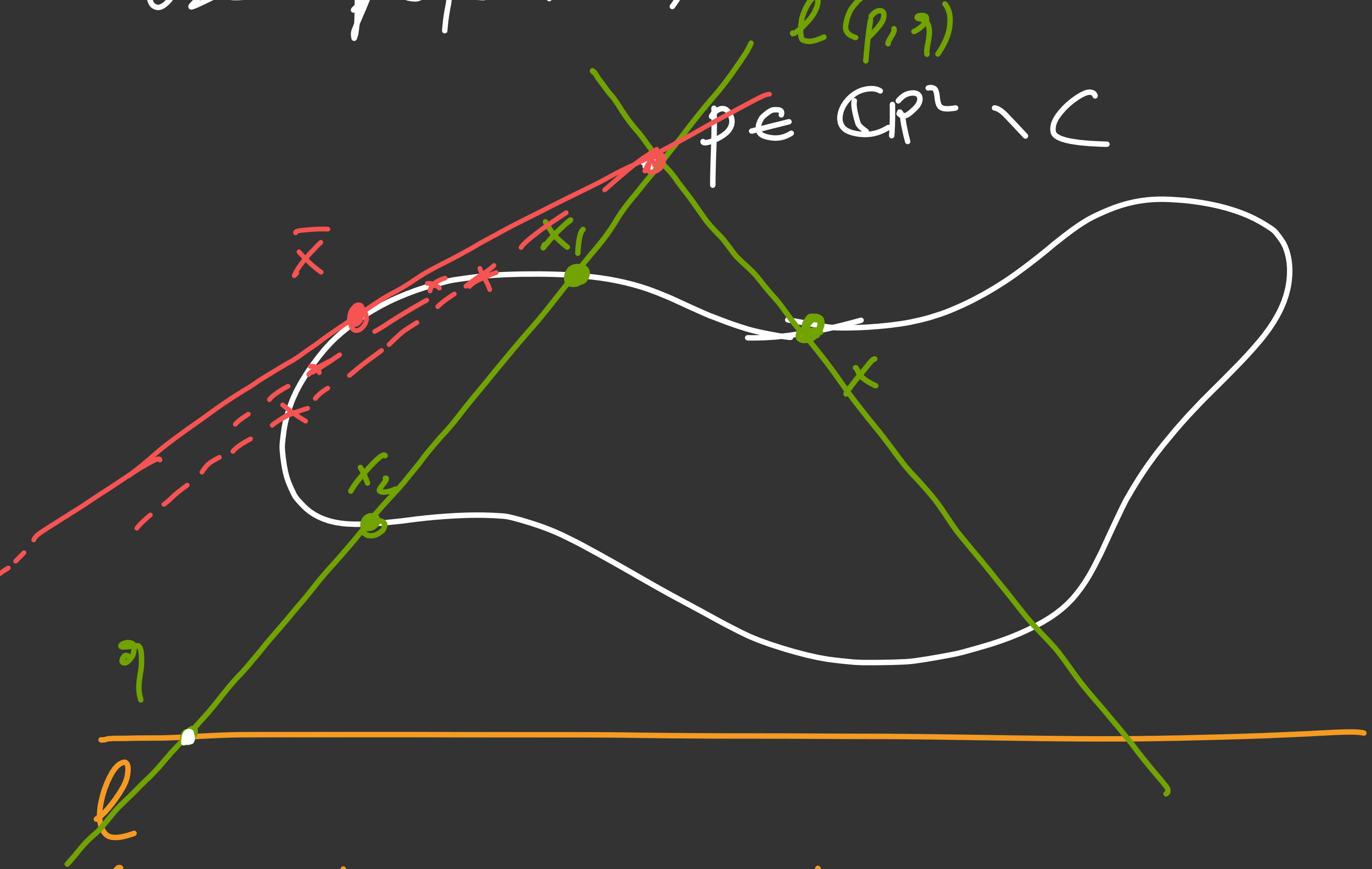
proof (degree-genus formula)

Use the Riemann-Hurwitz.

look for branched covers:

use projections onto lines.



↳ auxiliary complex line

$x_1, x_2 \longmapsto q$

$\pi : x \longmapsto \ell(x,p) \cap \ell$

magic: $\pi : C \longrightarrow \ell \simeq S^2$

is a branched cover.

Reason: π is a holomorphic map.

$\Rightarrow$ Locally $\underline{z \mapsto z^k}$ at each point.

$\Rightarrow$ this fits our local model description. $\Rightarrow$ it's a branched cover.

We're looking for $g(X)$ $(g(C))$

- we know $g(Y) = 0$
- we need to find d
- " " " " $e_\pi(x)$ $\forall x$.

- What is $\deg(\pi)$?

$\deg(\pi)$ = # of preimages of a generic point q on ℓ.

= # int. of the line through p & q with C.

= $\deg(C) = d$

- What is $e_\pi(x)$?

We know that $e_\pi(x)$ is 1 if $\pi(x)$ is generic.

$$\Big(\text{recall} \quad \sum_{x \in \pi^{-1}(y)} e_\pi(x) \equiv d\Big)$$

$\bar{x}$ on C is a point for which $e_\pi(\bar{x}) > 1$.

we can check ① $C = \{ x^d + y^d + z^d = 0 \}$

<u>claim</u> C is non-singular:

$$F(x,y,z) = x^d + y^d + z^d$$

$$\nabla F = \begin{pmatrix} d x^{d-1} \\ d y^{d-1} \\ d z^{d-1} \end{pmatrix} = 0 \Leftrightarrow x = y = z = 0.$$

$$\Rightarrow \{ \nabla F = 0 \} \cap \{ F = 0 \} = \{ (0,0,0) \}$$
$$\cap$$
$$\mathbb{C}^3$$

~ does not correspond to a point on C, $\Rightarrow$ C is non-singular.

② Choose $p = (0:0:1) \notin C$

③ Choose $\ell = \{ z = 0 \} \ni p$

Look at $\pi : C \longrightarrow \ell$

associated to this p we're done.

What are lines through p?

$$\ell_{a,b} := \{ax + by = 0\}$$

$$(a:b) \in \mathbb{CP}^1.$$

If $x \in \ell_{a,b}$, where is it sent to by the map π?

$$x \mapsto \ell_{a,b} \cap \ell$$

$$\{ax + by = 0,\ z = 0\}$$

$$(b : -a : 0).$$

Q: If I fix $q \in \ell$, when is q a branching point of π?

$\Leftrightarrow$ when does q have fewer than d preimages?

$$q = (b : -a : 0) \rightsquigarrow$$

$\# \, \ell_{a,b} \cap C$?

$$\begin{cases} ax + by = 0 \\ x^d + y^d + t^d = 0 \end{cases}$$

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suppose $a \neq 0$, then

$$(b : -a : 0) = \left(-\frac{b}{a} : 1 : 0\right)$$

$\longrightarrow$ $\boxed{-\frac{b}{a} =: 2}$ $\longrightarrow$ I had made the a //. $a \neq 0$. If $a = 0$, need to check things separately.

$$\begin{cases} x = 2y \\ x^d + y^d + z^d = 0 \end{cases}$$

$$(2^d + 1)\, y^d + z^d = 0$$

looking for solutions in

$$(y : z) \in \mathbb{CP}^1.$$

$\rightsquigarrow$ has d distinct solutions

$$\Longleftrightarrow \quad \underset{\overset{\Vert}{-\alpha}}{(2^d + 1)} \neq 0.$$

if

$$\boxed{\alpha\, y^d = z^d}$$

$\to$ which has solutions

$$(\omega^k : \alpha_1)$$

where α_1 is any d-th root of α, ω is a d-th root of 1.

If $(2^d+1)=1$, then the only solution is

$$(1:0)$$

→ for the points for which

$$(2^d+1)=0\,,$$ there's

only one preimage in

$$\pi^{-1}(q) = x_q$$

$$\Rightarrow \quad e_\pi(x_q) = \boxed{d}\,.$$

# of points q s.t.

$$\boxed{(2^d+1)} = 0 \text{ is } d$$

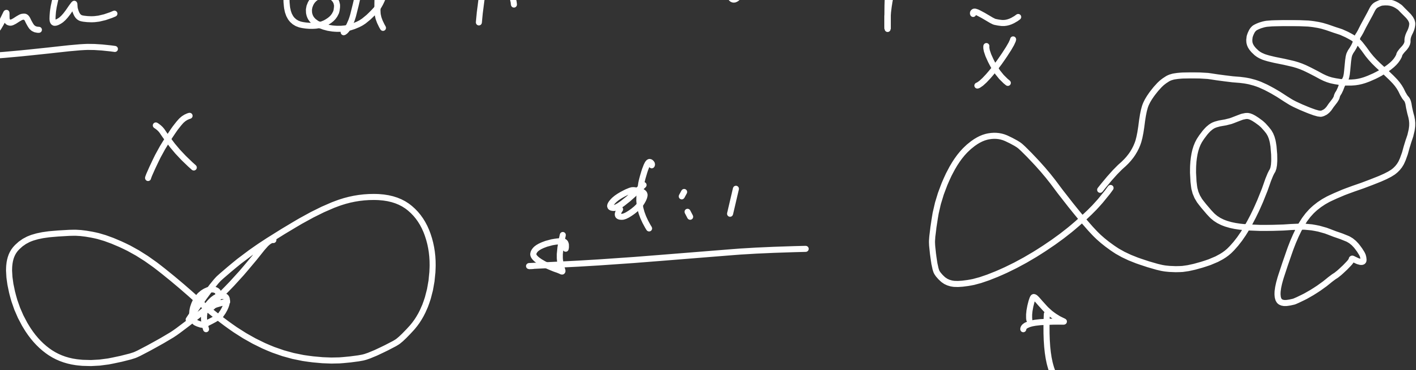
$$2-2g(C) = d\cdot(2-0) - \boxed{\sum_{x\in X} e_\pi(x)-1}$$

$$d\cdot(d-1)$$

$$2-2g(C) = 2d - d(d-1) = $$
$$= 3d - d^2$$

$$\Rightarrow g(C) = \frac{(d-1)(d-2)}{2}.$$

Rmk last time we proved that



another 1-cell cx with $b_1(\tilde{X}) =$

$d(b_1(X)-1)+1$

$\rightsquigarrow \pi_1(\tilde{X}) \underset{\text{index } d}{<} \pi_1(X)$

$\Rightarrow$ If $H < F_b$ ← free group on b gen.!

of index $(F_s : H) = d$,

then H itself is free

on $ds - d + 1$ generators

Enough about surfaces ...

(open (Hurwitz problem ...))

2| Branched covers in higher dim.

Work in the context of smooth mfld & smooth submanifolds;

we're gonna assume that the branching set B is a smooth submanifold,

def A map $p: X^n \to Y^n$ that

i) smooth between smooth mflds of the same dim. n is a branched cover if

$\exists\, B \subset Y$ s.t. $p|_{p^{-1}(Y \setminus B)}$ is a cover onto $Y \setminus B$,

B smooth submanifold

- $\forall\, b \in B \ \exists\, U$ chart around b, $(U, \underbrace{U \cap B}_{B_U}) \cong (\mathbb{C} \times B_U, B_U)$

& $p^{-1}(U) = U_{x_1} \amalg \cdots \amalg U_{x_m}$

disjoint union of open sets,

$U_{x_i} \cong \mathbb{C}_z \times B_U$ &

s.t. $p|_{U_{x_i}} : (z, c) \mapsto (z^k, c)$

for some k (which depends on x_i)

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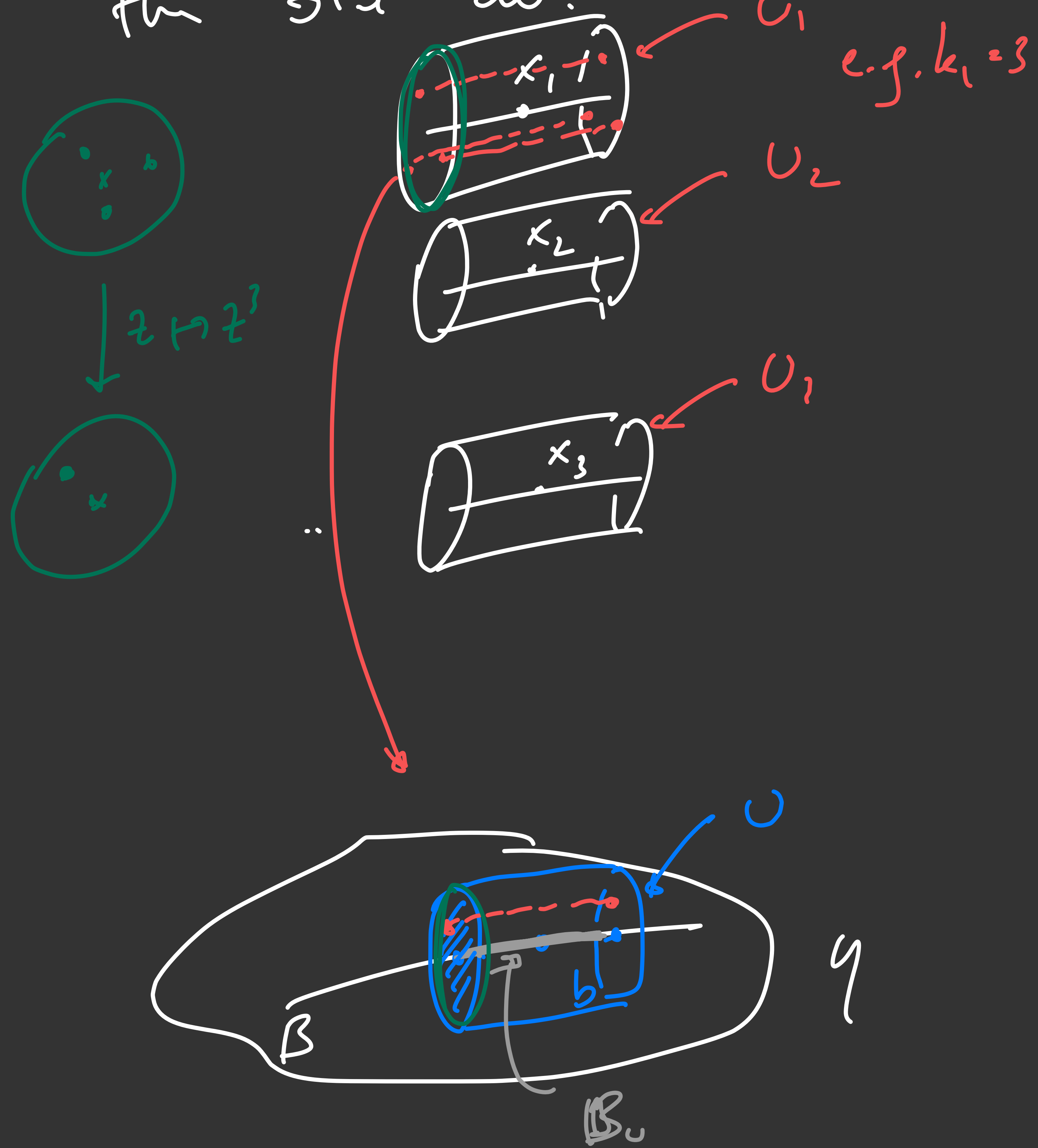
deg p is defined as in the 2D case: # of a $\underbrace{\text{generic}}$ preimage $=$ deg: $p|_{X \setminus p^{-1}(B)}$

rmk • If $n=2$ $(\dim X = \dim Y)$

the new def. agrees w/

the old def.



<u>exa</u> – branched covers of surfaces;

– $p : X \to Y$ b.c. of surfaces

$q : A \to C$ any non-branched cover

$p \times q : A \times X \to C \times Y$

is a branched cover.

– $S^n \xrightarrow{d} S^n$.

what is $S^n = \{ x_1^2 + \dots + x_{n+1}^2 = 1 \} \subset \mathbb{R}^{n+1}$

polar coord. $= \{ x_1^2 + \dots + x_{n-1}^2 + r^2 = 1 \}$

when $\{\ \}$ using coordinates

$(x_1, \dots, x_{n-1}, r, \theta)$ on $\mathbb{R}^{n+1}$

define $p : S^n \longrightarrow S^n$

$$(x_1, \dots, x_{n-1}, r, \theta) \longmapsto (x_1, \dots, x_{n-1}, r, d\theta)$$

<u>claim</u> This is a branched cover.

"<u>proof</u>": two types of points:

$$r = 0, \quad r \neq 0.$$

$$\{r = 0\} \subset S^{n-2} \subset \mathbb{R}^{n-1} \times \{(0,0)\}.$$

standard unknotted $S^{n-2} \subset S^n$

- easy to see: if $x \notin S^{n-2}$, then x has exactly d preimages, if $x \in S^{n-2}$, then x has one preimage.

<u>exo</u> Verify the claim.

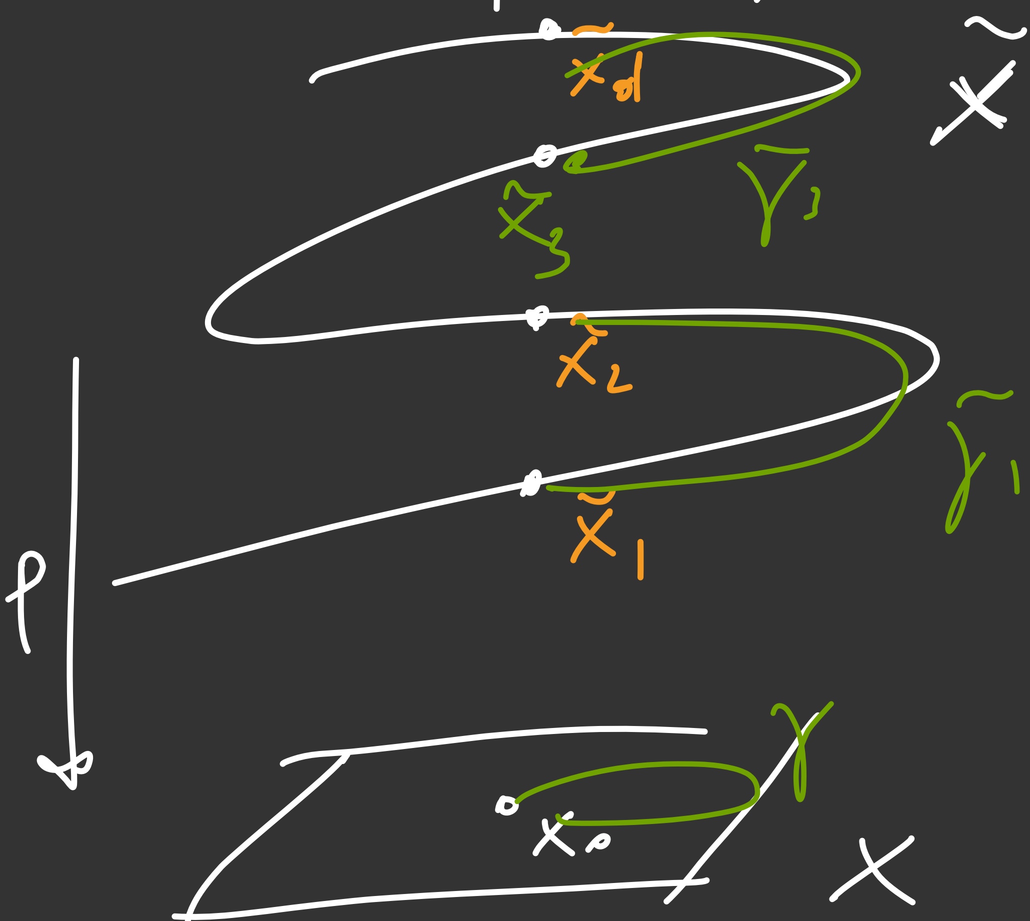
(idea: this is the same as

$$\mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$$
$$z \longmapsto z^{-1} \quad)$$

Something about covering

Suppose $p: \tilde{X} \to X$ is a covering (no branching), suppose that it's finite, $d := \deg p$.

In this situation we can define an action of $\pi_1(X, x_0)$ onto $p^{-1}(x_0)$



Choose a htpy class $[\gamma]$

$[\gamma] \in \pi_1(X, x_0)$ & let

$\tilde{\gamma}_1, \dots, \tilde{\gamma}_d$ be its lifts

from $\tilde{x}_1, \dots, \tilde{x}_d$, respectively

The endpoint of $\tilde{\gamma}_1$ is

another point on $p^{-1}(x_0)$

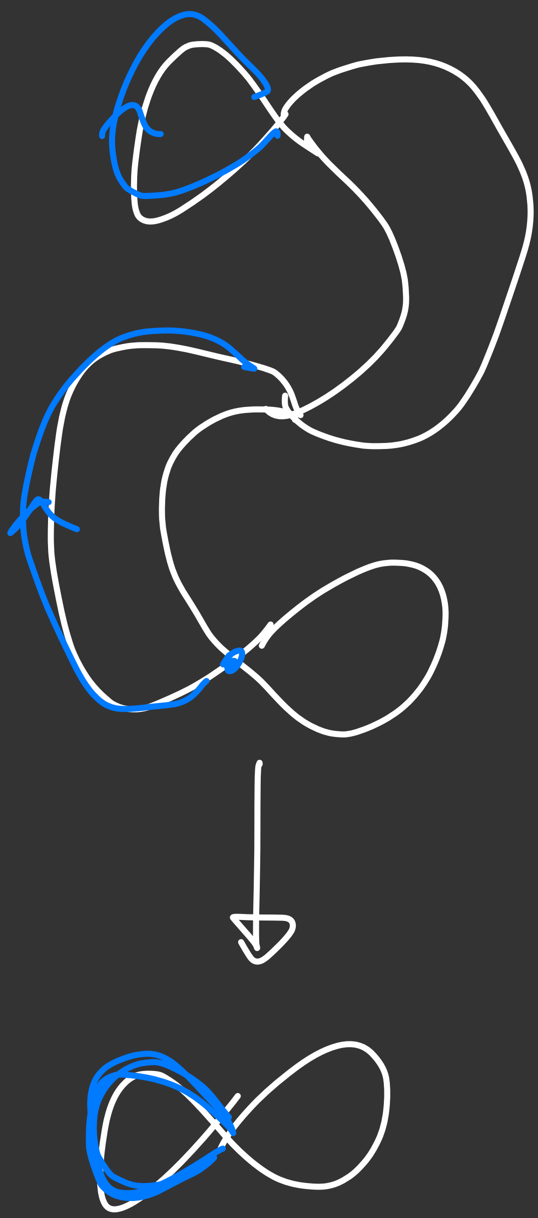
<u>rmk</u> • The endpoint of $\tilde{\gamma}_1$ is

$\tilde{x}_1$ iff $[\gamma] \in p_*(\pi_1(\tilde{X}, \tilde{x}_1))$

$$p_*(\pi_1(\tilde{X}, \tilde{x}_1)) \subset \pi_1(X, x_0)$$

• It could happen that

the endpoint of $\tilde{\gamma}_1$ is $\hat{x}_1$, but the endpoint of $\tilde{\gamma}_2$ is not $\tilde{x}_2$



This defines an action:

$$\gamma \cdot \tilde{x}_i := \text{endpoint of } \tilde{\gamma}_i$$

$\rightsquigarrow$ γ is sent to a permutation of $\{\tilde{x}_1, \dots, \tilde{x}_d\}$

~~exa~~ / ex

$p : S^1 \xrightarrow{z \mapsto z^d} S^1$

the action induced by p on $p^{-1}(1)$ is by d-cycles.

set of d-th roots of unity

$\langle (1\ 2\ 3\ \dots\ d) \rangle$

this representation

$\pi_1(X) \longrightarrow \boxed{\text{permutations of } p^{-1}(x_0)}$

is called the <u>monodromy</u>

(mon. permutation) of p.

<u>fact</u> the monodromy repres. determines p.

<u>rmk</u> If you change x_0 you get a conjugate monodromy:

$$\pi_1 \circlearrowright p^{-1}(x_0)$$

$$\pi_1 \circlearrowright p^{-1}(y_0)$$

Key result (Fox)

If $B \subset Y$ is a codim-2 smooth submanifold, and we choose a monodromy representation (up to conjugacy)

$$\pi_1(Y \setminus B) \longrightarrow S_d$$

perm. group,

then there exists a branched cover $p : X \to Y$ branched over B, inducing the given monodromy (coming from

$$p|_{X \setminus p^{-1}(B)} : X \setminus p^{-1}(B) \to Y \setminus B.$$

& this branched cover is unique.

<u>rmk</u> X is connected iff

the monodromy of $X \to Y$

acts transitively on $\{1, \dots, d\}$.

<u>Where to find covers & branched covers "in nature"?</u>

<u>key</u>: group actions.

mantra: group actions on smooth manifolds often induce branched covers

$$X \longrightarrow X/G$$

when G acts on X.

Prop (from covers)

If G is a finite group acting on X without fixed points $\Longrightarrow$

$$X \longrightarrow X/G$$

is a covering map.

& in this case

$$\pi_1(X) \triangleleft \pi_1(X/G)$$

& $\pi_1(X/G)/\pi_1(X) \cong G$.

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the converse also holds:
if $p: X \to Y$ is a covering s.t. $p_*(\pi_1(X)) \triangleleft \pi_1(Y)$,

then $\exists\, G \curvearrowright X$ s.t.

the quotient is Y &

$$\begin{tikzcd}
& X \arrow[dl, "\pi"'] \arrow[dr, "p"] & \\
X/G \arrow[rr, "\sim"] & & Y
\end{tikzcd}$$

<u>spoiler</u> under certain assumptions

if $G \curvearrowright X$, X/G is

a branched cover.

Looking at G cyclic

$G = C_d$, cyclic of order d.

<u>thm</u> If G acts on X^n oriented in an orientation-preserving way, G is finite cyclic, X smooth n-dim. mfld

then $\mathrm{Fix}(G)$ is a disjoint union of even-codimensional submflds. (possibly empty).

$$\phi : G \longrightarrow \mathrm{Diff}^+(X)$$
$$g \longmapsto \phi_g$$

$$\{ x \in X \mid \phi_g(x) = x \ \forall g \in G \}$$

Proof

1st step: show that there is a G-invariant Riemannian metric on X

$\to$ to see this, pick an arbitrary Riemannian m. $\bar{h}$ on X

$$h := \frac{1}{d}\left(\bar{h} + \phi_g^*\bar{h} + \phi_{g^2}^*\bar{h} + \dots + \phi_{g^{d-1}}^*\bar{h}\right)$$

where g is a gen. $\in G$ & $d = |G|$.

this is a Riemann. metric on X, since Rie. m. are convex

& it is G-invariant

$$\phi_g^* h = \phi_g^* \left(\frac{1}{d} \sum_{k=0}^{d-1} \phi_{g^k}^* \bar{h} \right)$$

$$= \frac{1}{d} \left(\sum_{k=0}^{d-1} \phi_g^* \phi_{g^k}^* \bar{h} \right)$$

$$= \frac{1}{d} \left(\sum \phi_{g^{k+1}}^* \bar{h} \right) =$$

$$= h.$$

$\rightsquigarrow$ Can assume that G is a cyclic group of isometries of (X, h), Rie. metric.

<u>philosophy</u> : isometries of

Riem. mflds are determined

by what happens at a point

& on its tangent space.

$$F : X \xrightarrow{\text{isom}} X$$

connected

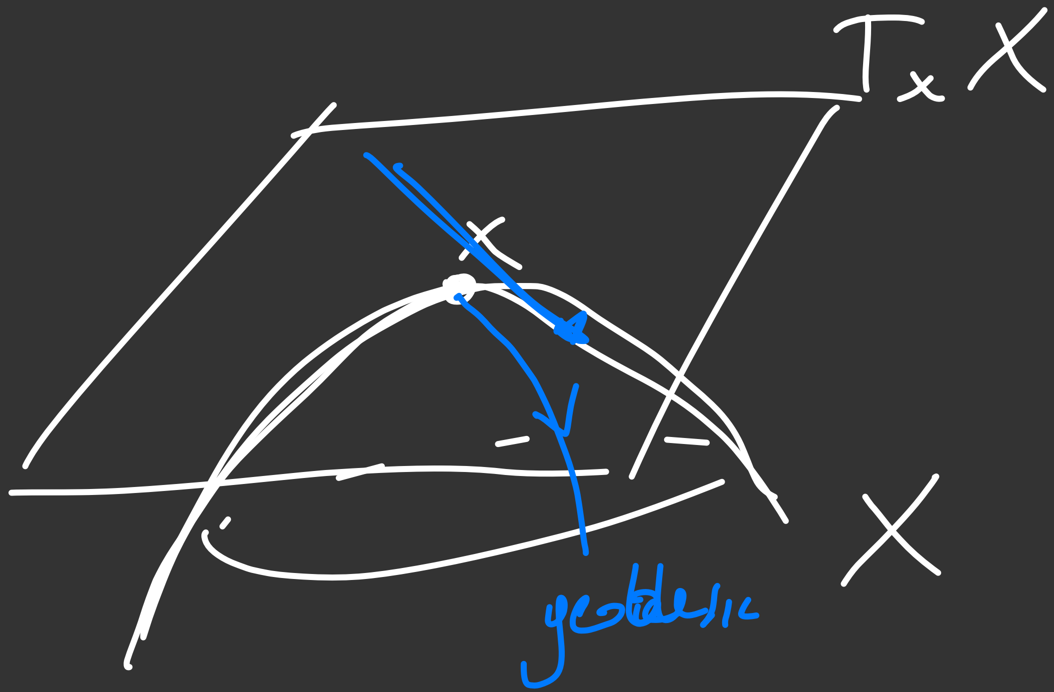
is determined by

knowing $F(x)$ for any x

& dF_x for that x.

Why? This is because we have the exponential map

$$T_x X \supset U \longrightarrow X$$



What about fixed points?

If $x \in \mathrm{Fix}(G)$ is a fixed point, then ϕ_g is determined by $(d\phi_g)_x$.

Since ϕ_g is an isometry, then $(d\phi_g)_x$ is an isometry of $(\boxed{T_x X}, h_x) \underset{\text{isom.}}{\cong} (\mathbb{R}^n, \text{eucl.})$

$\rightsquigarrow$ can we think of $(d\phi_g)_x$ as an element of $SO(n)$.

or.-preserving.

this is b.c. $\phi_g^* h = h \iff$
$h(d\phi_g v, d\phi_g w) = h(v, w)$

we know that elements in $SO(n)$ are diagonalisable & that their eigenvalues are of norm 1.

& their product $\left(\det\left((d\phi_g)_x\right)\right)$ is 1.

- $\omega, \bar{\omega}$, $\omega \in S^1 \setminus \{\pm 1\}$ → come in pairs
- -1 → come in pairs
- $+1$

$\Longrightarrow$ multi. of $+1$ as an eigenvalue is $\equiv n \ (2)$

$$\Rightarrow \quad \mathrm{Fix}\big((d\phi_g)_x\big) =: \underline{\underline{E_{+1}}}$$

$$= \text{+1-eigenspace of } (d\phi_g)_x$$

$$= \text{has even codimension}$$

If $v \in \mathrm{Fix}\big((d\phi_g)_x\big)$

$\exp_x(tv)$ ~~invariant~~ pointwise fixed under g

$$\begin{tikzcd}
T_xX \arrow[r, "(d\phi_g)_x"] \arrow[d, "\exp_x"'] \arrow[dr, phantom, "\circlearrowright"] & T_xX \arrow[d, "\exp_x"] \\
X \arrow[r, "\phi_g"] & X
\end{tikzcd}$$

Now we're done: since

$E_{+1} \subset T_x X$ has

even codim &

$\exp_x(E_{+1})$ is an

even-codim'l submanifold

made of fixed points. $\square$

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a bit more can be squeezed out: namely we a local model for any cyclic action around any fixed point.