1/2 3-manifolds

Goal: give some examples of:

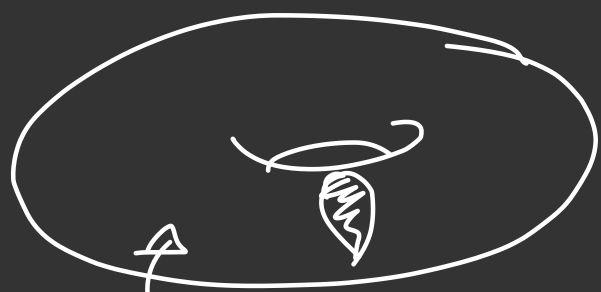
today + exos $\begin{cases} \text{- 3-mflds} \\ \text{- knots \& links } (\subset S^3) \\ \text{- branched covers of 3-mflds} \end{cases}$

next week { - $\Omega_3 = 0$

<u>def</u> A 3-manifold is a smooth mfld of dim = 3. i.e. it is locally <u>diffeo</u> to $\mathbb{R}^3$.

<u>ex</u> • $\mathbb{R}^3$, B^3 is a 3-mfld w/ $\partial = S^2$.
↳ closed 3-ball

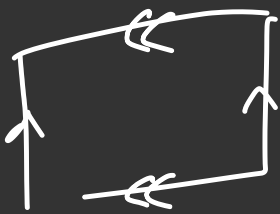
• $S^3 = \{ |z|^2 + |w|^2 = 1 \} \subset \mathbb{C}^2_{z,w}$

• $S^1 \times D^2 =$ 
↓
<u>solid torus</u>
$\mathbb{T}^3$

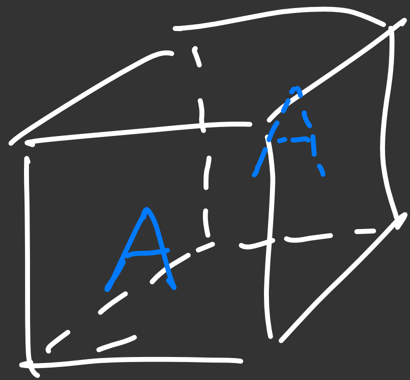
filled in

$\partial S^1 \times D^2 = T^2 = S^1 \times S^1$

- $T^3 = S^1 \times S^1 \times S^1$

$T^2 =$ 

3rd ax. of



$S^1 \times F$

↳ surface of genus g.

↳ can take F to be non-orientable

⇝ non-orientable 3-manifolds.

<u>def</u> A <u>lens space</u> is the quotient of S^3 by the action of $\boxed{C_p}$

cyclic gp of order p.

where the action is given by $\begin{pmatrix} \omega & 0 \\ 0 & \omega^q \end{pmatrix}$

ω is a primitive p-th root of unity

q another integer

$$G_{p,q} = \left\langle \begin{pmatrix} \omega & 0 \\ 0 & \underline{\underline{\omega^q}} \end{pmatrix} \right\rangle \subset GL(2,\mathbb{C})$$

<u>check</u>: ∘ $G_{p,q}(S^3) = S^3$

• acts <u>freely</u> on S^3 <u>if $(q,p)=1$</u>

• $|G_{p,q}| = p$.

$S^3/G_{p,q}$ is a 3-manifold

$$\pi_1(S^3/G_{p,q}) \cong G_{p,q} \cong C_p.$$

$$L(p,q) := S^3/G_{p,q}.$$

↳ lens space

q: when is $L(p,q) \cong L(p',q')$?

easy: • $L(p,q) \simeq L(p',q')$

$\Rightarrow p = p'$.

• $L(p,q) \cong L(p, p+q)$

• $L(p,q) \cong L(p,q^*)$,

where $qq^* \equiv 1 \pmod p$

• $L(p,q) \cong -L(p,p-q)$

i.e. there is an orientation-reversing diffeomorphism.

example In $\mathbb{C}^3$, consider

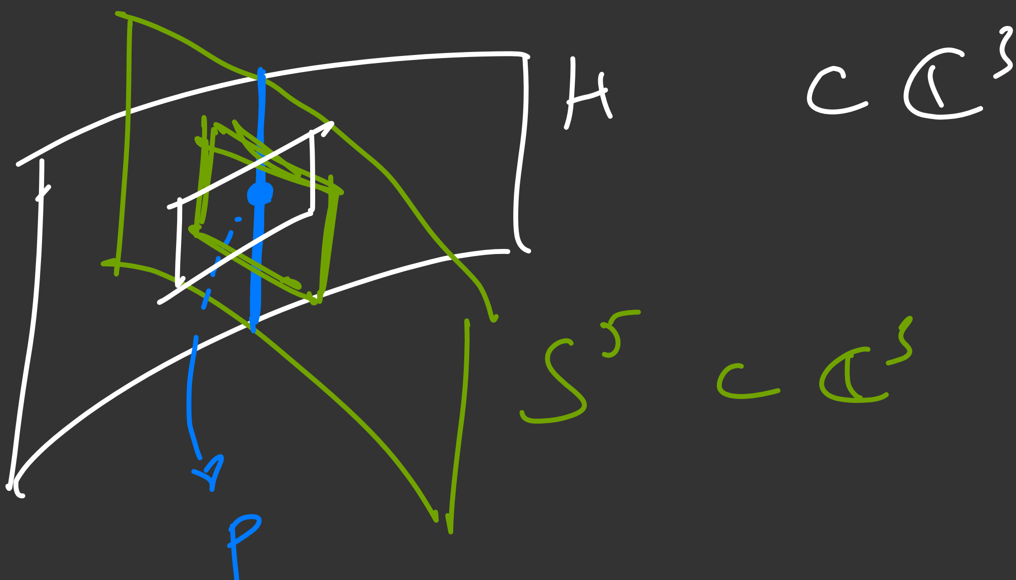
$$F(x,y,z) = x^a + y^b + z^c$$

$$H := \{F=0\} \cap S^5 = \left\{ (x,y,z) \in \mathbb{C}^3 \;\middle|\; \begin{array}{l} x^a + y^b + z^c = 0 \\ |x|^2 + |y|^2 + |z|^2 = 1 \end{array} \right\}$$

$\rightarrow$ this intersection is transverse. $\pitchfork$

that is $\langle T_p S^5 , T_p H \rangle = T_p \mathbb{C}^3$

$\rightsquigarrow$ 3-d picture of what's happening:



Whenever this happens,

$H \cap S^5$ is a submanifold of $\mathbb{C}^3$ around p.

$\rightsquigarrow$ consequence of the implicit function theorem.

$$F : \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

$\overset{\psi}{p}$ is a reg. value for F

$F^{-1}(F(p))$ is a submfld of dim $m-n$ in $\mathbb{R}^n$, near p.

Mantra: • smooth functions at reg. pts behave like linear funct.

• transversal intersections behave like

a generic int. of linear subspaces.

in this case: $H \cap S^5$

"4-dim'l" subnfld

away from $(0,0,0) \in \mathbb{C}^3$ if $a,b,c \geq 2$.

5-dim'l submanifold of $\mathbb{C}^3$

3-dimensional submanifold of S^5

mantra: codimensions add up.

call $H \cap S^5 =: \Sigma(a,b,c)$

Brieskorn manifold.

examples (that we will not use)

hyperbolic 3-manifolds:

mflds that admit a complete

Riemannian metric w/

constant scalar curvature -1.

$\longleftrightarrow$ $\mathbb{H}^3 / \Gamma$, subgroup of $\mathrm{Isom}(\mathbb{H}^3)$
$\cong$
$\mathbb{R}^3$

<u>thm</u> (Perelman ~'04)
(<u>conj</u> Poincaré 1911,
Thurston 1970s)

If M is a closed, simply-conn. 3-mfld, $M \cong S^3$.

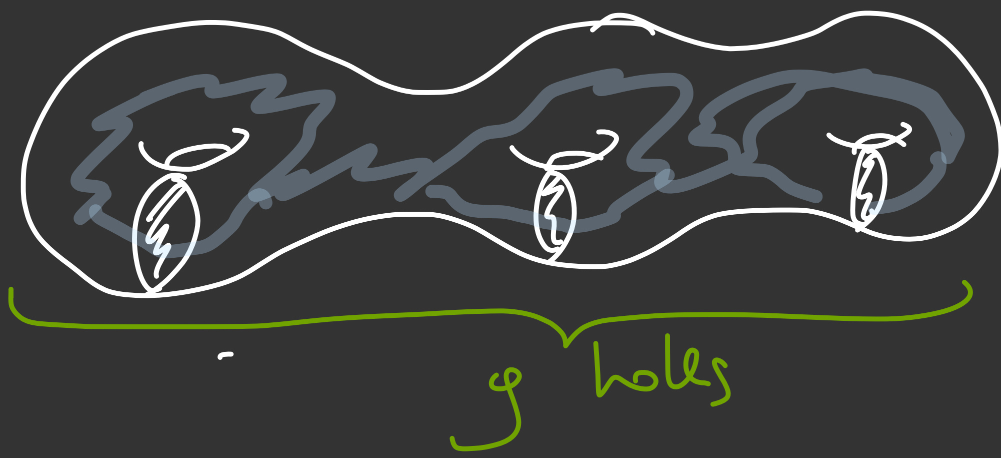
<u>def</u> A <u>Heegaard splitting</u> of a closed, orientable 3-mfld M is a decomposition

$$M = A \cup B$$

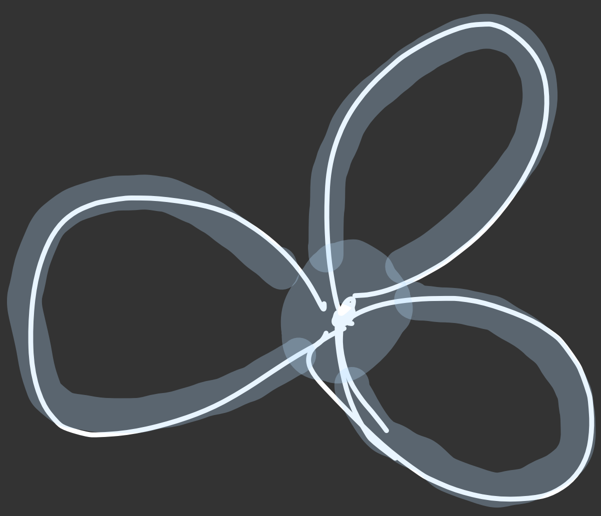
where $A, B \subset M$, A, B both

diffeomorphic to a genus-g handlebody

& $\partial A = \partial B \subset M$.

$H_g :=$ 

that is: $H_g \cong$ ~~the~~ reg. nbhd of a wedge of g circles in $\mathbb{R}^3$



$\rightsquigarrow \partial =$ genus-g surface.

$H_0 = B^3$, $H_1 = \mathbb{T}^3$

~~$\mathbb{S}^n$, $\mathbb{T}^n$~~

S^n, T^n ← I'm one of them ppl.

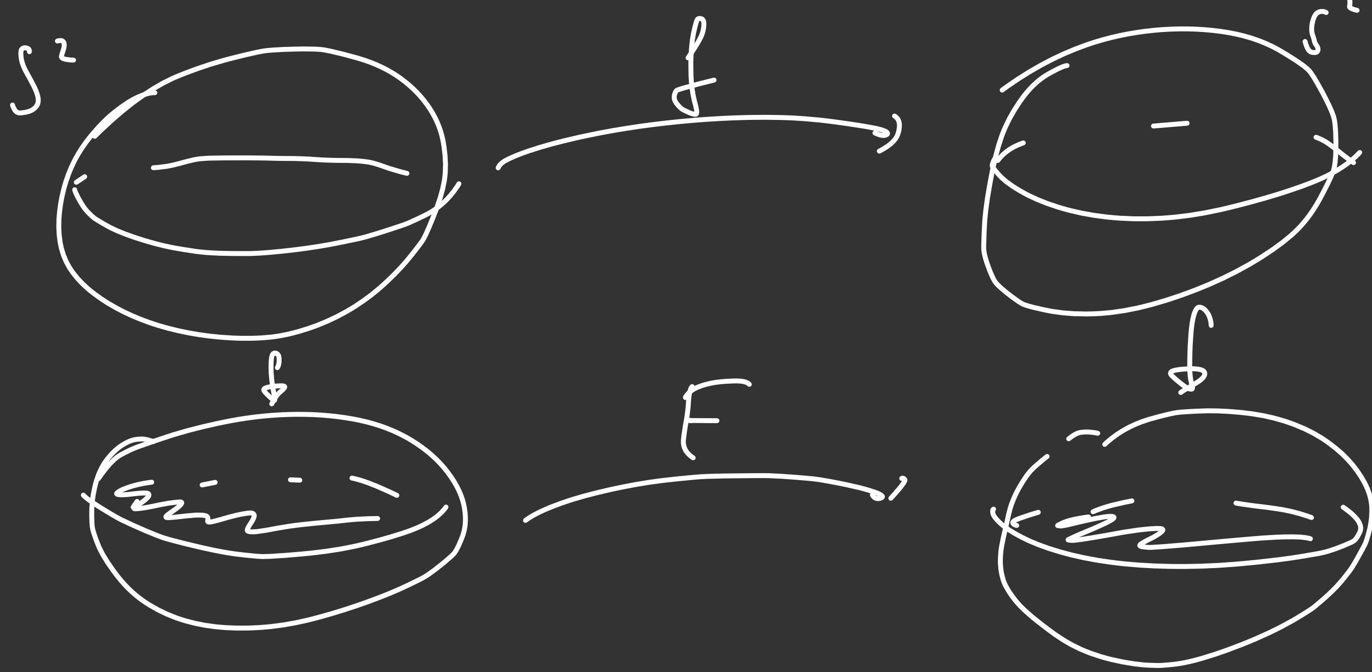
⇝ there's a way to represent M or $\partial A = \partial B$ ⇝ <u>Heegaard diagrams</u>: drawing a coll. of curves on a genus-g surface.

ex • $S^3 = B^3 \cup \underbrace{(\mathbb{R}^3 \cup \{\infty\} \setminus B^3)}_{\text{another 3-ball.}}$

↳ genus-0 Heegaard dec. of S^3.

thm (Alexander) If M has a genus-0 Heegaard decomposition then $M \cong S^3$.

Follows from the fact that every self-diffeomorphism of S^2 extends to a diffeo of the 3-ball

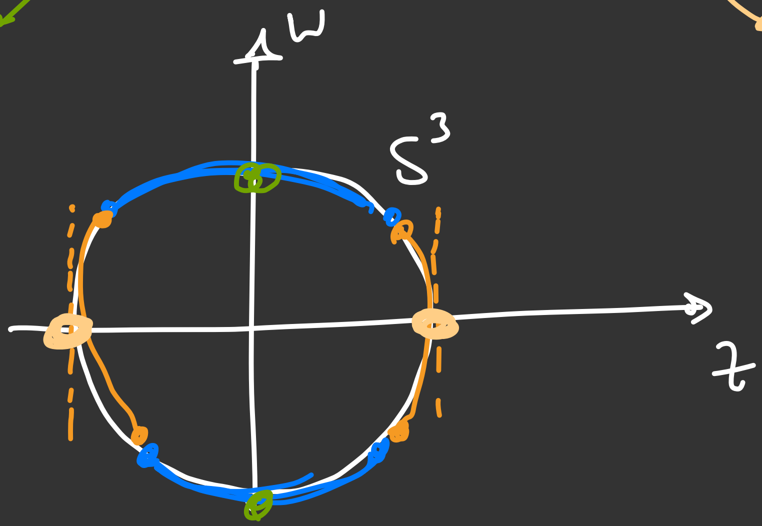


ex: $S^3 \subset \mathbb{C}^2$

$$S^3 = A \cup B, \qquad A = \{|z| \le 1/\sqrt{2}\}, \qquad B = \{|w| \le 1/\sqrt{2}\}$$

$$\{|z| \le 1/\sqrt{2}\} \cong S^1 \times D^2 \qquad \qquad \{|w| \le 1/\sqrt{2}\} \cong S^1 \times D^2$$

$|w| = 1$ $\qquad\qquad$ $|z| = 1$



$$A \cap B = \{|z| = |w| = 1/\sqrt{2}\} \cong T^2 = S^1_\theta \times S^1_\varphi$$

$$= \left\{\left(\frac{1}{\sqrt{2}} e^{i\theta}, \frac{1}{\sqrt{2}} e^{i\varphi}\right)\right\}$$

<u>ex</u> • the HD of genus-1 that we just saw is $G_{p,q}$-equiv.

$G_{p,q}$ preserves A & B

$$A/G_{p,q} \simeq \mathbb{T}^3$$

$$B/G_{p,q} \simeq \mathbb{T}^3.$$

$\rightsquigarrow$ $L(p,q)$ has a genus-1 Heegaard decomposition

$$A/G_{p,q} \cup B/G_{p,q}.$$

fact. If a closed, oriented 3-mfld has a HD of genus = 1, then it is either S^3, $L(p,q)$, or $S^1 \times S^2$.

$$\Downarrow$$

$(S^1 \times \text{equator}) \subset S^1 \times S^2$ exhibits a genus-1 HD of $S^1 \times S^2$.

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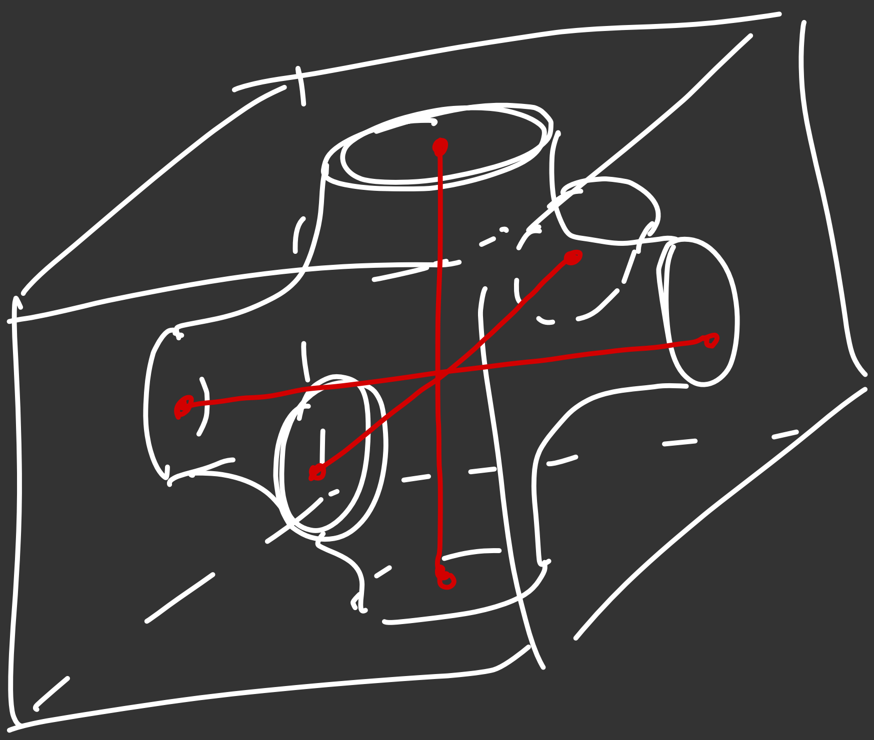
prop Every closed oriented 3-mfld M admits a HD.

proof (sketch): if you take triangulation of M (which exists) then a nbhd of the 1-skeleton of the triangulation is one half of a Heegaard decomp.



this is a genus-? HD of T^3.

ex?

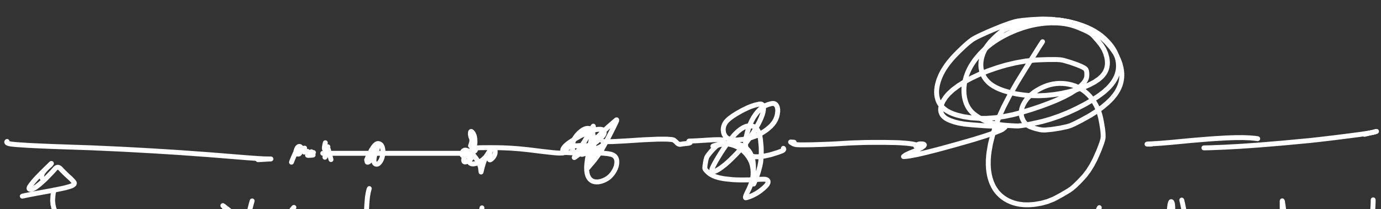
knots & links

def A **knot** in a 3-manifold M is an embedding $S^1 \hookrightarrow M$.

smooth, injective, immersion.

rmk Whatever we do w/ smooth emb. $S^1 \to M$, we can do with triangulated embeddings (PL embeddings). the imp. thing is the existence of a tubular nbhd that is homeo/diffeo to $\mathbb{T}^3$.

If we were looking at C^0-emb. i.e. C^0 + injective, then we could incur into trouble.



wild knots. not gonna talk about them.

ex The <u>unknot</u> is the only knot in M that bounds an embedded disc.

$$D^2 \hookrightarrow M$$

Rmk We will be looking at knots up to <u>isotopy</u>: two knots $\alpha, \beta : S^1 \longrightarrow M$ are <u>isotopic</u> if $\exists \, (\gamma_t)_{t \in [0,1]}$ of smooth embeddings $S^1 \longrightarrow M$ s.t.

$\gamma_0 \equiv \alpha$, $\gamma_1 \equiv \beta$,

γ_t varies smoothly w/ t.

$$\gamma : S^1 \times I \longrightarrow M.$$

s.t. $\gamma(S^1 \times \{t\})$ is a knot $\forall t$.

the def^n of the unknot makes sense b/c every two embeddings of $D^2 \hookrightarrow M$ are isotopic.

$S^3 \subset \mathbb{C}^2_{z,w}$

$C_{a,b} = \{ z^a = w^b \}$, a, b, coprime positive integers.

$C_{a,b} \pitchfork S^3 =$ 1-dim^l mfld compact.

$\hookrightarrow$ if connected, it's a knot.

$C_{a,b} \cap S^3 = \left\{ \frac{1}{\sqrt{2}} (\theta^b, \theta^a) \,\middle|\, \theta \in S^1 \subset \mathbb{C} \right\}$

note: I am implicitly using that $\gcd(a,b) = 1$.

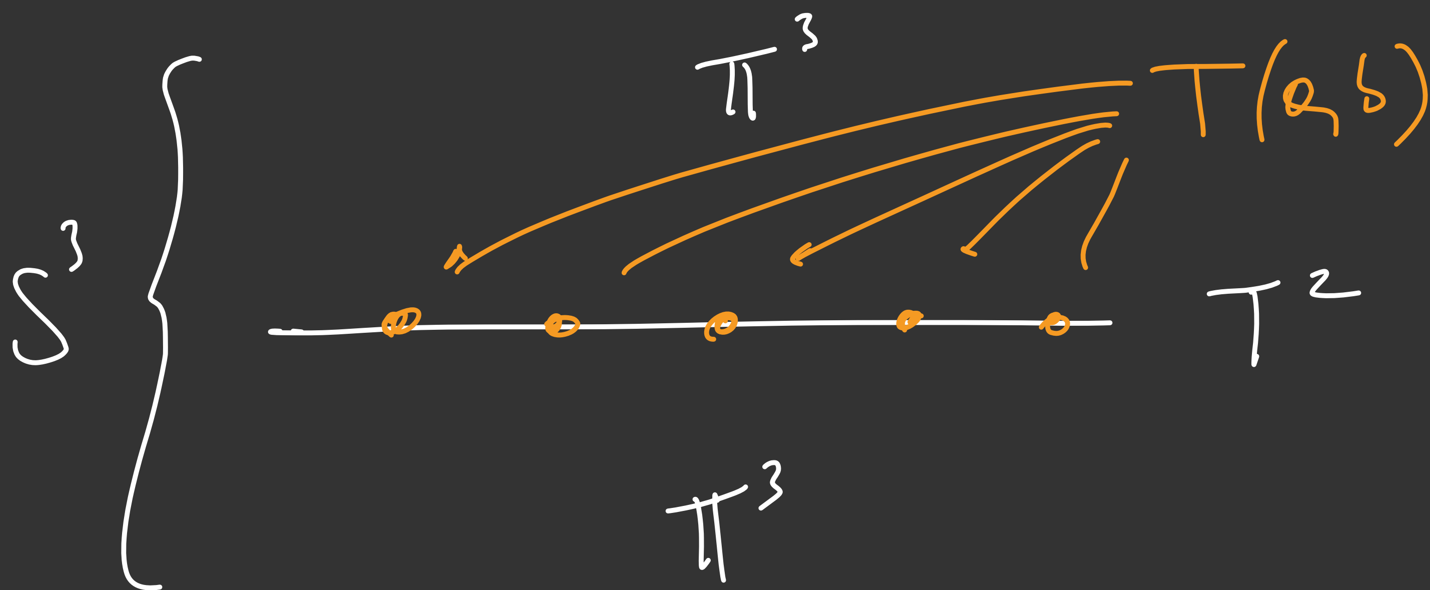
I will call $C_{a,b} \cap S^3$ a <u>torus</u> <u>knot</u>, $T(a,b)$

why?

b.c. it lies on the torus

$$|z| = |w| = \frac{1}{\sqrt{2}}.$$

useful observation:



$$\leadsto \quad S^3 \setminus T(a,b) = \mathbb{T}^3 \cup_? \mathbb{T}^3$$

glued along $T^2 \setminus T(a,b)$.

$\simeq S^1 \times$ (open int.).

sub-example $a=2,\ b=3$

(right-handed)

$\rightsquigarrow$ trefoil

This is called a knot projection

$\rightsquigarrow$ works for knots (& links) in S^3 or in $\mathbb{R}^3 = S^3 \setminus \{pt\}$.

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def A link in a 3-mfld M is an embedding of $\coprod_{\ell} S^1 \hookrightarrow M$.

i.e. a collection of pairwise disjoint knots.

<u>ex</u> torus links in S^3 which are $C_{a,b} \cap S^3$ when now we allow a,b to have a common divisor $\ell > 1$.

$\rightsquigarrow$ $C_{a,b} \cap S^3$ is a link with ℓ components, each isotopic to $T\left(\frac{a}{\ell}, \frac{b}{\ell}\right)$.

<u>ex</u> $a = b = 2$

(positive) Hopf link.

Prop Let $k \subset S^3$ be a knot.

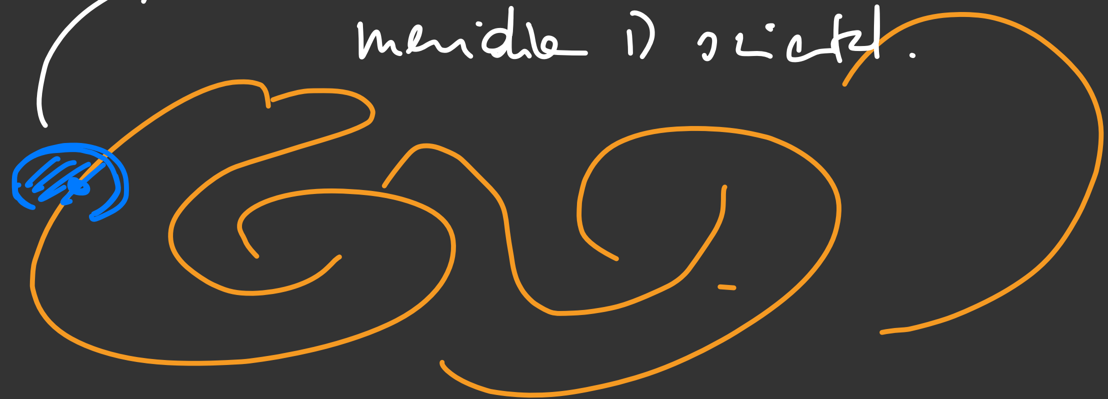
The homology of $S^3 \setminus k$ is

$$H_*(S^3 \setminus k) = \begin{cases} \mathbb{Z} & \text{if } *=0,1 \\ 0 & \text{otherwise.} \end{cases}$$

($\Rightarrow S^3 \setminus k$ is connected)

Moreover, $H_1(S^3 \setminus k)$ is generated by the <u>meridian</u> of k.

if k is oriented, then the meridian is oriented.

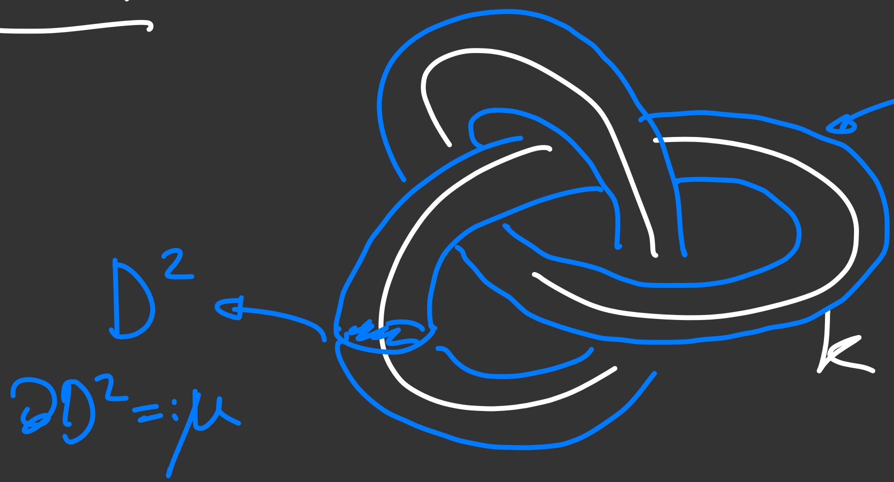


$k \subset M$ take $p \in k$, take V any complement of $T_p k$, take $\exp$ (small ball in V)

proof

$M := S^3 \setminus k$

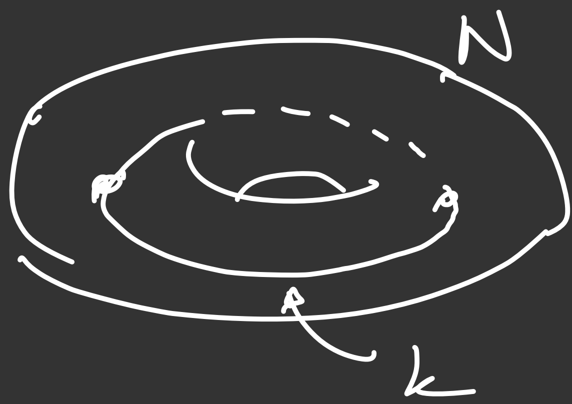
$\boxed{N} := \mathrm{nbhd}(k)$



so $N \cong S^1 \times D^2$

$\cup$

$k \hookrightarrow S^1 \times \{0\}$



$\mu = \{pt\} \times \partial D^2$, a meridian.

$$S^3 = M \cup N$$

& $M \cap N = N \setminus k = S^1 \times (D^2 \setminus \{0\})$

which retracts onto $S^1 \times S^1 = S^1 \times \partial D^2$.

$M \cap N$ is htpy equivalent to T^2.

this corresponds to μ.

$$H_*(S^3) \leftarrow \text{known}$$

$$\overset{\shortparallel}{M \cup N} \qquad H_*(M \cap N) \leftarrow \text{known.}$$

$$H_*(N) \leftarrow \text{known}$$

Write the Mayer-Vietoris long ex. sequence. (MV), keeping in mind $H_1(S^3) = H_2(S^3) = 0$,

$$H_2(N) = 0$$

$$0 \to H_3(S^3) \to H_2(M \cap N) \to H_2(M) \oplus \cancel{H_2(N)} \to 0$$

$\cancel{H_2(N)} = 0$

$$\begin{array}{ccccccc} H_3(S^3) & & H_2(M\cap N) & & & & \\ \wr\wr & & \wr\wr & & & & \\ \mathbb{Z} & \xrightarrow[\sim]{inj} & \mathbb{Z} & \longrightarrow & H_2(M) & \longrightarrow & 0 \end{array}$$

$\Downarrow$

at least torsion.

in fact $= 0$: one way is to use Poincaré-Lefschetz duality & univ. coeff. thm.

we know that $H_k(M)=0$ whenever $k\geq 3$: b/c M is a 3-manifold, $k=3$ b/c M is not a closed 3-mfld.

$$\cancel{H_1(S^3)} \to H_0(M\cap N) \to \boxed{H_0(M)} \oplus H_0(N) \to H_0(S^3)$$

$\cancel{H_1(S^3)} = 0$, $H_0(M\cap N) \cong \mathbb{Z}$, $H_0(N) \cong \mathbb{Z}$, $H_0(S^3) \cong \mathbb{Z}$, $H_0(S^3) \to 0$

squeezed : $\mathbb{Z}$.

Now, onto $H_1(M)$.

$$\cancel{H_2(S^3)} \to H_1(N\cap M) \to H_1(M)\oplus H_1(N) \to \cancel{H_1(S^3)}$$

$\cancel{H_2(S^3)} = 0$, $H_1(N\cap M) = \mathbb{Z}^2$, $H_1(M) = ?$, $H_1(N) = \mathbb{Z}$, $\cancel{H_1(S^3)} = 0$

$\mathbb{Z}^2 \to \boxed{\mathbb{Z}} \oplus \mathbb{Z}$

generated by a meridian of K. $\to$ dies in $H_1(N)$

μ survives in $H_1(M)$, & generates a free subgroup.

So, at least, in $H_1(M) \ni \mathbb{Z}\langle\mu\rangle$.

⇝ there can be nothing else:

b.c. $H_1(M) \oplus \mathbb{Z} \simeq \mathbb{Z}^2$.

(by MV).

$\Rightarrow$ $H_1(M) = \mathbb{Z}\langle\mu\rangle$.

⇝ there is a primitive class in $H_1(N \cap M)$

$\cong$

$H_1(\partial N)$

that dies in $H_1(M)$.

& in fact it generates $H_1(N)$.

<u>def</u> Any simple closed curve in ∂N in this hom class is called a <u>Seifert longitude</u> of K.

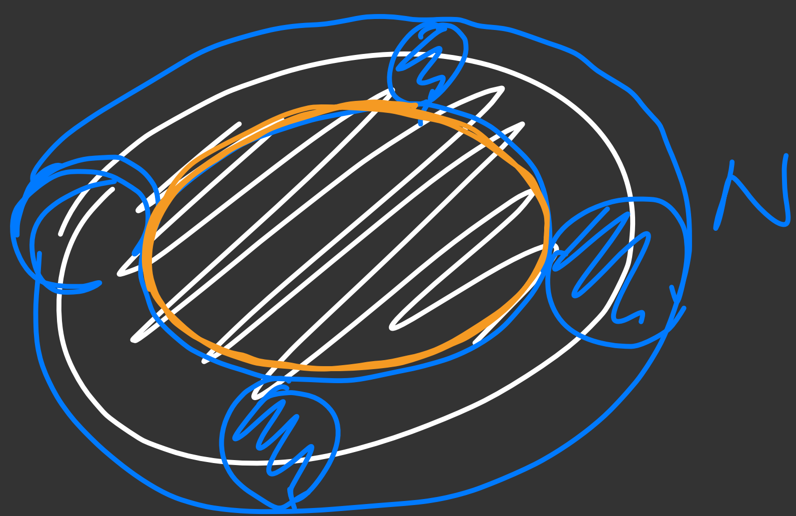
morally : dying in homology

$\Updownarrow$ (true in this dim.)

bounding a submanifold.

$\Longrightarrow$ K is the boundary of a embedded surface in S^3, that we call a Seifert surface.

e.g.



Cor For every $d \geq 1$ $\exists$ a d-fold cyclic cover $p : \Sigma_d(K) \longrightarrow S^3$ branched on K, has degree d, & for which the associated cover

$$\underbrace{\boxed{\Sigma_d(K) \setminus \tilde{K}}}_{Y_d} \longrightarrow \underbrace{\boxed{S^3 \setminus K}}_{Y}$$

$\tilde{K} := p^{-1}(K)$

is a cyclic cover

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this means that the deck transf. of $Y_d \longrightarrow Y$ is cyclic

& $Y = Y_d / \mathrm{Deck}(Y_d \to Y)$

& $\mathrm{Deck}(Y_d \longrightarrow Y) \cong C_d$.

<u>proof</u> Recall that if there exists a quotient $\tilde{X}/G \to X$ for some free action $G \curvearrowright \tilde{X}$, then $\pi_1(\tilde{X}, \tilde{x}_0) \triangleleft \pi_1(X, x_0)$

& $\pi_1(X, x_0) / \pi_1(\tilde{X}, \tilde{x}_0) \cong G$.

$\Longrightarrow$ $\exists$ surj homomorphism

$$\pi_1(X, x_0) \longrightarrow G.$$

Abelian

we're looking for

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[d, phantom, "\parallel" rotate=90] & \\
\pi_1(Y, x_0) \arrow[r, "\varphi"] \arrow[d] & C_d \\
\underbrace{\pi_1(Y, x_0)^{ab}}_{\text{abelianis. of } \pi_1} \arrow[ur, "\tilde{\varphi}"'] \arrow[r, phantom, "="] & H_1(Y)
\end{tikzcd}$$

$$H_1(Y) \xrightarrow{\tilde{\varphi}} C_d$$

$$\wr\wr$$

$$\mathbb{Z}$$

$\Rightarrow$ up to automorphism of C_d,

$\exists!$ surj hom $H_1(Y) \to C_d$

$\Rightarrow$ $\exists!$ " " $\pi_1(Y, x_0) \to C_d$.

$\Rightarrow$ $\exists!$ cyclic d-fold cover

$$Y_d \longrightarrow Y.$$

(Fox) $\Longrightarrow$ $\exists!$ cyclic d-fold

(last time)

branched cover

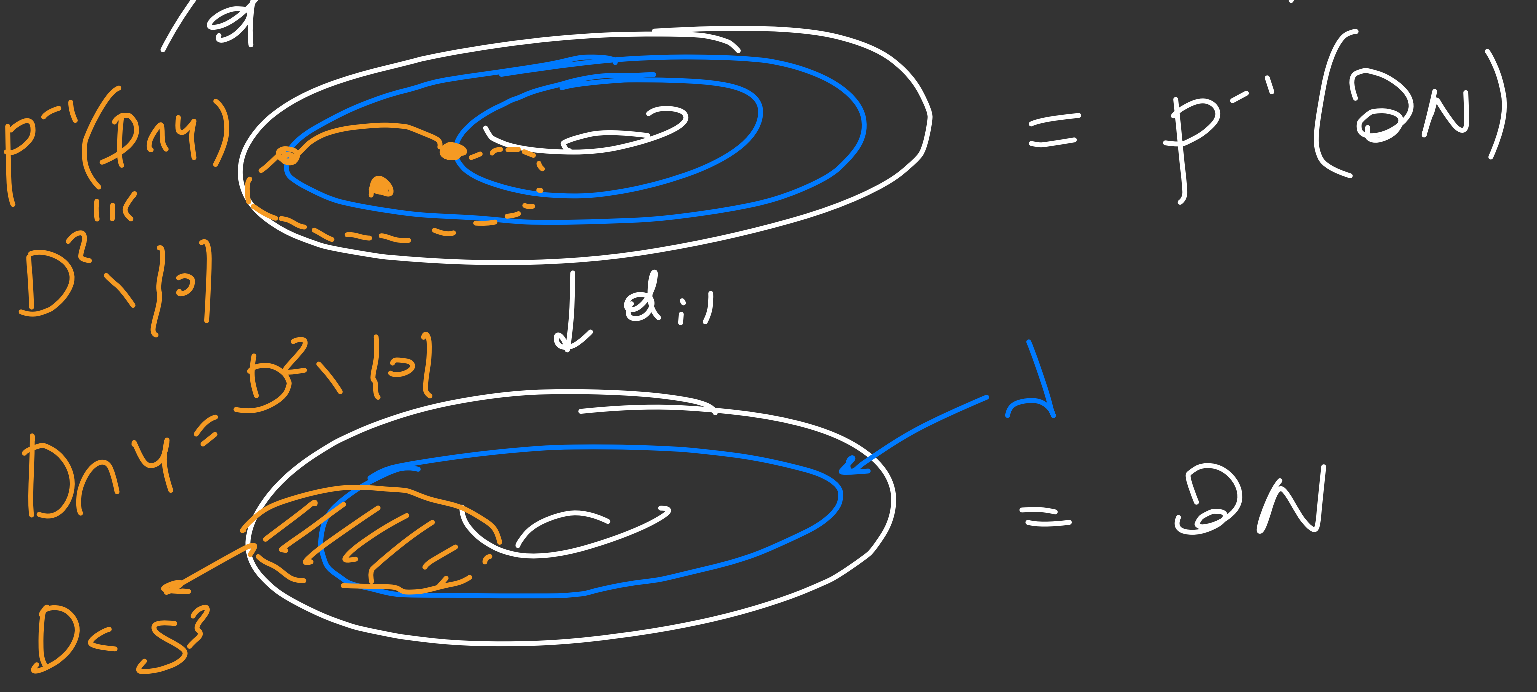
$$\Sigma_d(K) \longrightarrow S^3.$$

branched over K.

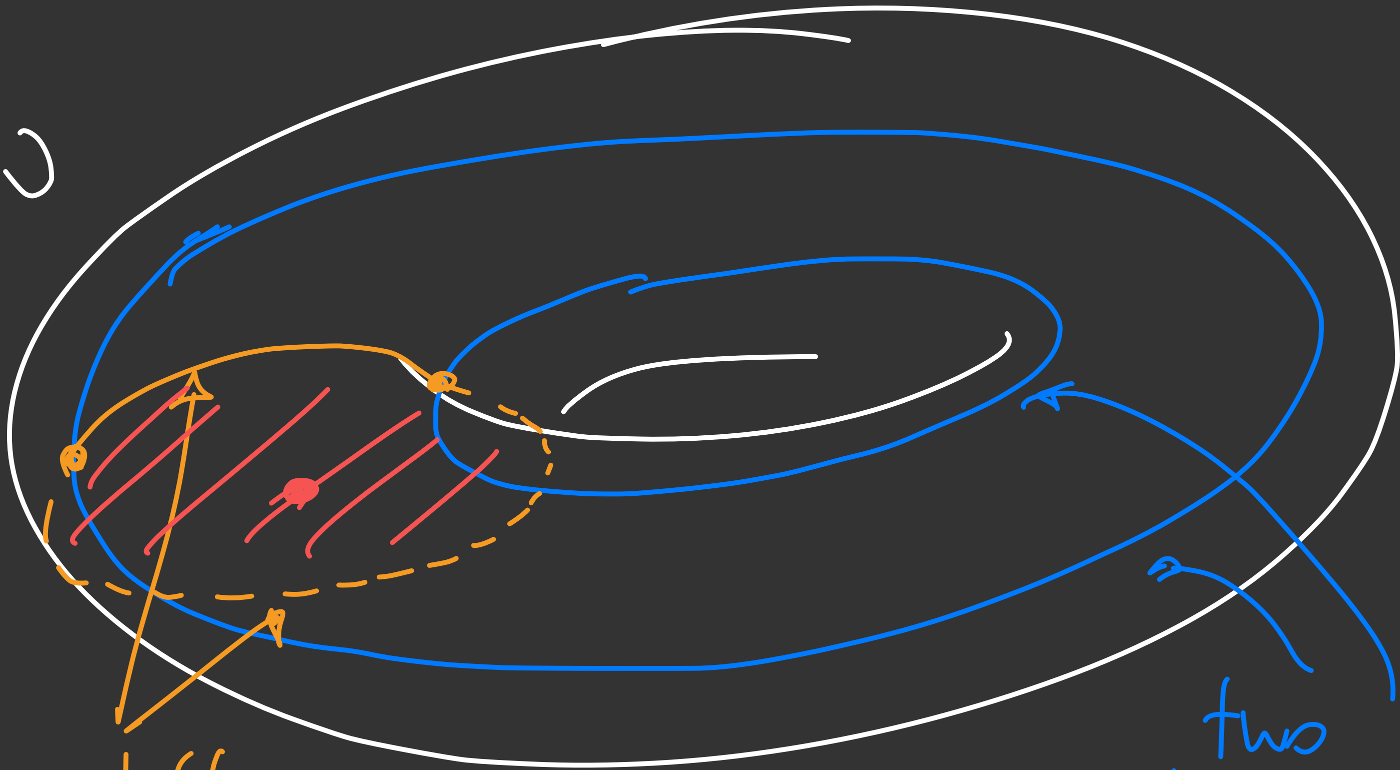
idea of proof by hand of the implication in orange:

$\boxed{N \text{ nbhd of } k}$, $\partial N \simeq T^2$

$p^{-1}(\partial N) \subset Y_d$ is again a torus, and a Seifert longitude of k lifts to d parallel curves, and the meridian of k unwraps (each lift is $1/d$-th of a curve on $p^{-1}(\partial N)$)



$Y_d \supset$



two lifts of the meridian

$\tilde{N} := p^{-1}(N \cap Y) = T^2 \times [0,1)$, two lifts

$\Uparrow$

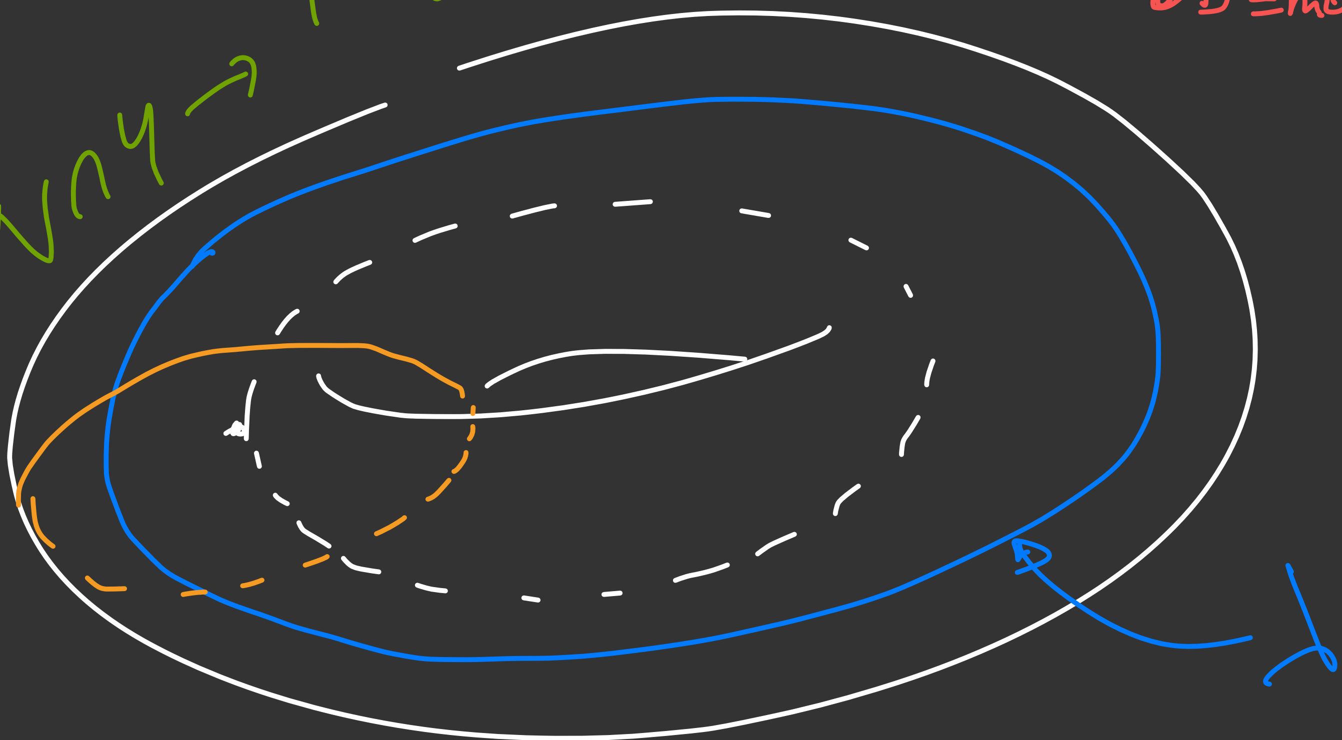
$S^1 \times (D^2 \setminus \{0\})$

∂ = union of d lifts of m.

$T^2 \times [0,1) \simeq S^1 \times \underbrace{(D^2 \setminus \{0\})}$

$\partial D^2 = m$

$Y \supset N \cap Y \to$



λ

The map $p|_{\tilde{N}} : \tilde{N} \longrightarrow N \cap Y$

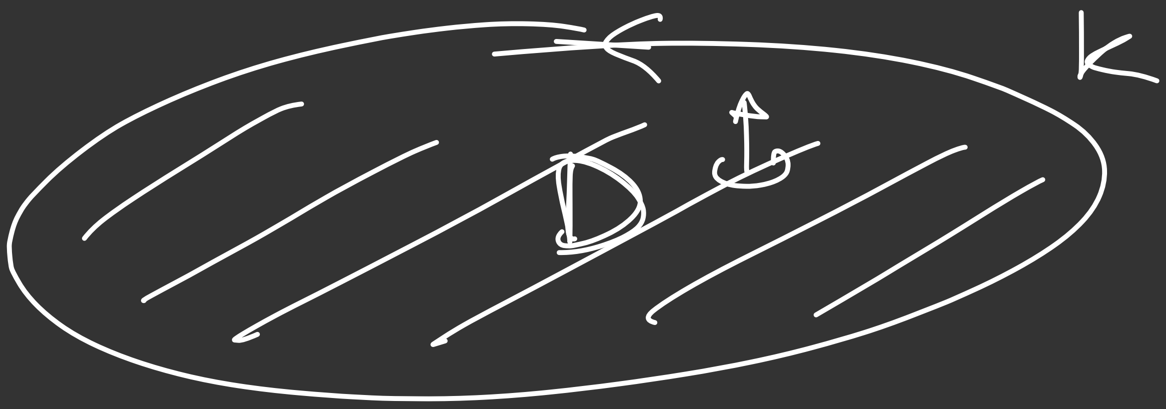
in the coordinates

$$S^1 \times D^2 \setminus \{0\} \longrightarrow S^1 \times D^2 \setminus \{0\}$$

is $(\theta, z) \longmapsto (\theta, z^d)$

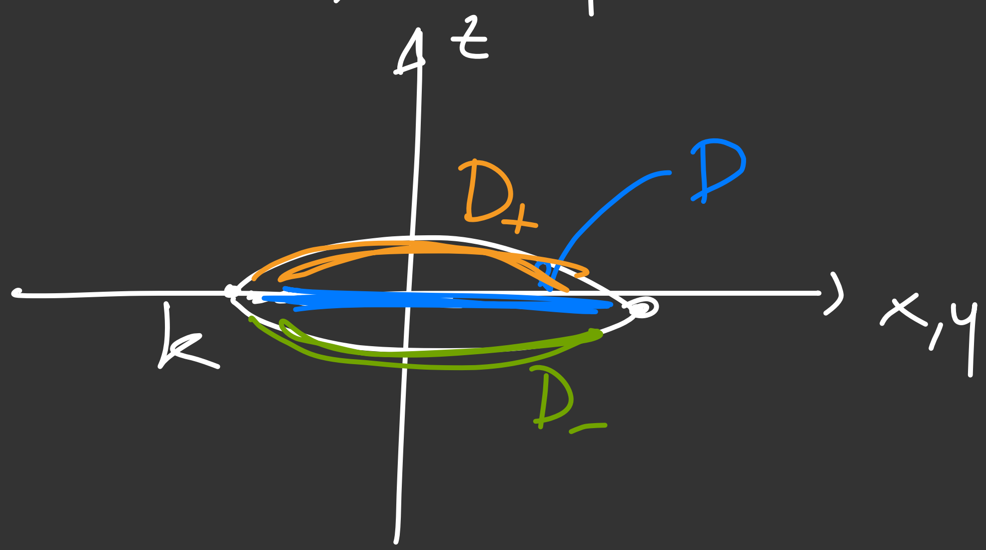
A concrete way of constructing branched covers of knots is by cut & paste.

e.g. take k = unknot



I want to construct its double cover: to do that, I cut S^3 along a disc that k bounds, say D.

$X = S^3 \setminus D$



take two copies of X, X_1, X_2

and glue $X_1 \cup X_2$

by identifying $D_+ \subset X_1$ with

$D_- \subset X_2$, & $D_- \subset X_1$ with

$D_+ \subset X_2$

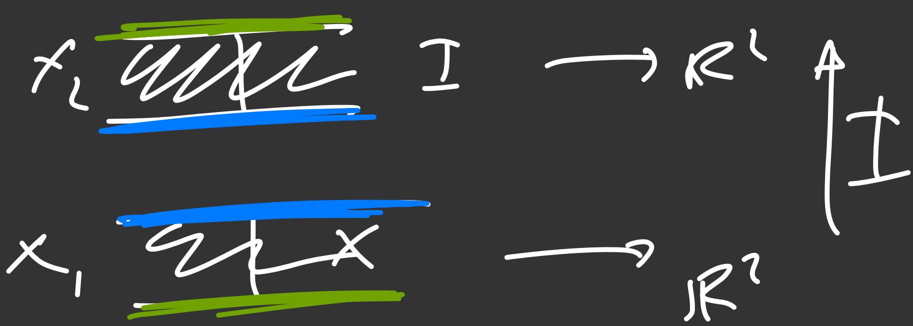
$$S^3 \setminus k \cong S^1 \times \mathbb{R}^2$$

$$(S^3 \setminus k) \setminus D \cong S^1 \times \mathbb{R}^2 \setminus \{pt\} \times \mathbb{R}^2$$

$$\cong \cancel{\mathbb{R}} \times \mathbb{R}^2 \qquad D_+ = \{0\} \times \mathbb{R}^2$$

$$[0,1] \times \mathbb{R} \qquad D_- = \{0\} \times \mathbb{R}^2$$

$X_1 \cup X_2 =$



$$\rightsquigarrow S^1 \times \mathbb{R}^2 \rightsquigarrow S^3.$$