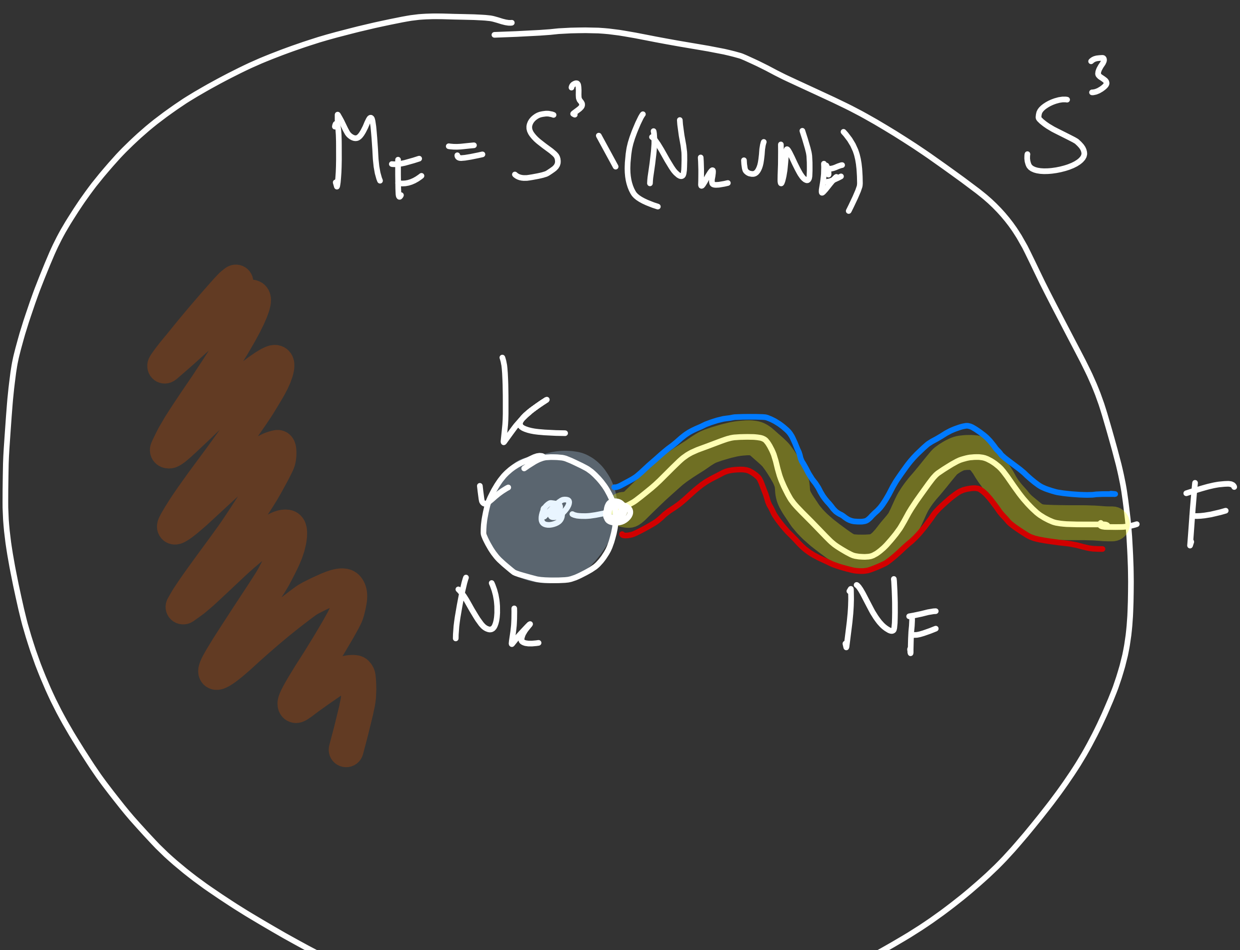
Lecture 4

Last time: we argued that $\forall d > 1$ integer there exists a d-fold cyclic cover of S^3 branched over a knot k.

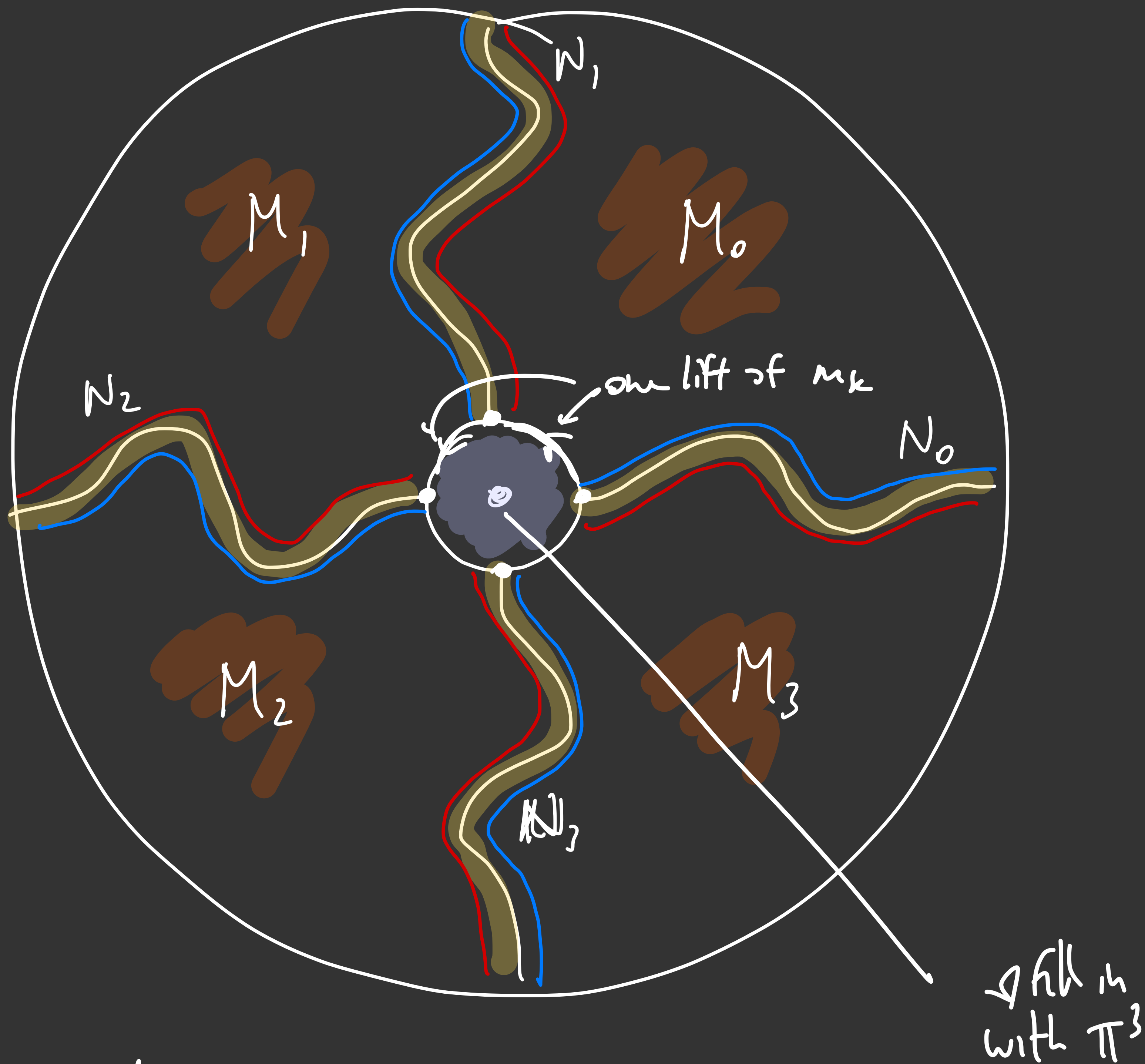
b/c $\exists (\sim !)$ homomorphism

$$\pi_1(S^3 \setminus k, x_0) \twoheadrightarrow \mathbb{Z}/d\mathbb{Z}.$$



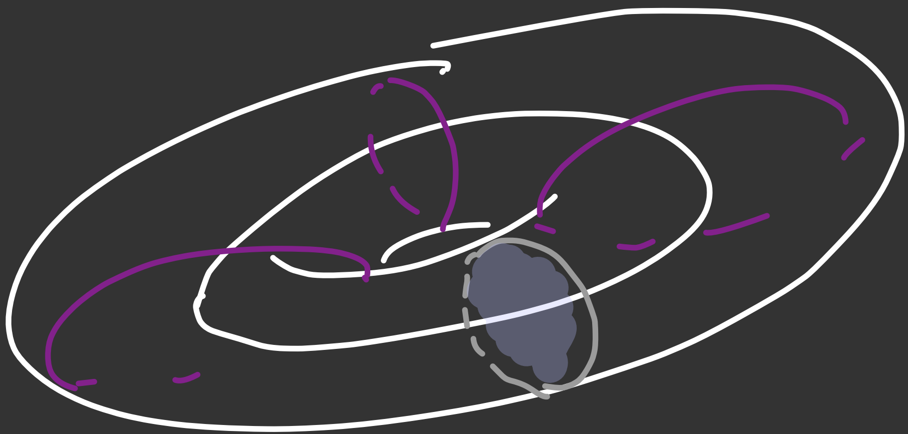
 = nbhd of k, $\simeq \Pi^3$

 = nbhd of F $\cong F \times I$

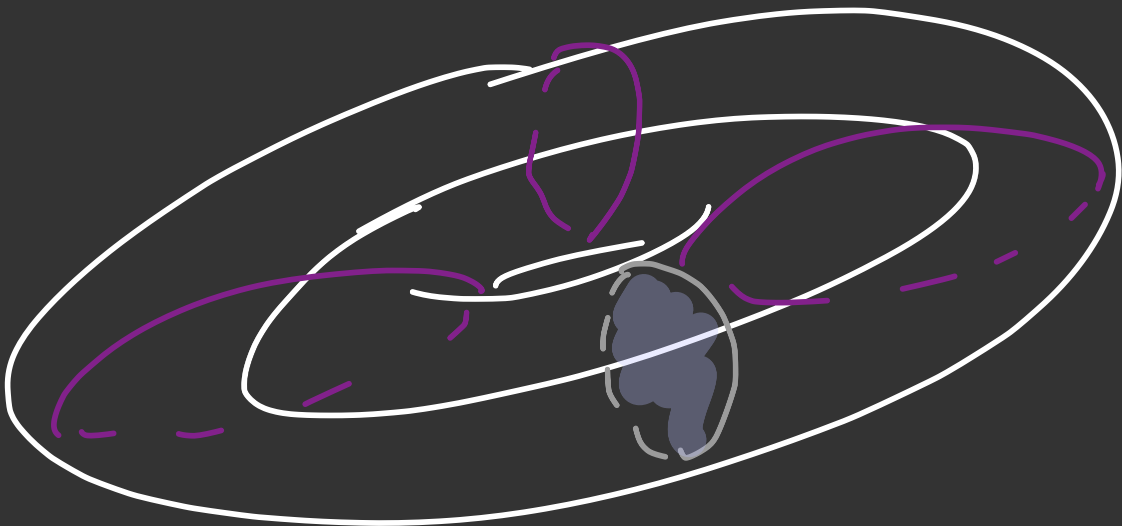


$d = 4$ as an example

The gluing is determined by the identifications of ∂N_F with the boundary of M_F.



wraps around d times.



↳ allows us to fill in the union $M_0 \cup F_0 \cup \dots \cup M_{d-1} \cup N_{d-1}$ with a solid torus & the projection map with $\mathrm{id}_{S^1} \times (z \mapsto z^d)$

Consider two knots $K_1, K_2 \subset S^3$ oriented & disjoint.

<u>def</u> The <u>linking number</u> between K_1 & K_2, $lk(K_1, K_2)$ is defined as follows:

$H_1(S^3 \setminus K_1) \cong \mathbb{Z}$, generated by the meridian of K_1.

Since K_2 is oriented, $[K_2] \in \underline{H_1(S^3 \setminus K_1)}$

$[K_2] = \ell \cdot [\text{meridian of } K_1]$.

$lk(K_1, K_2) := \ell$.

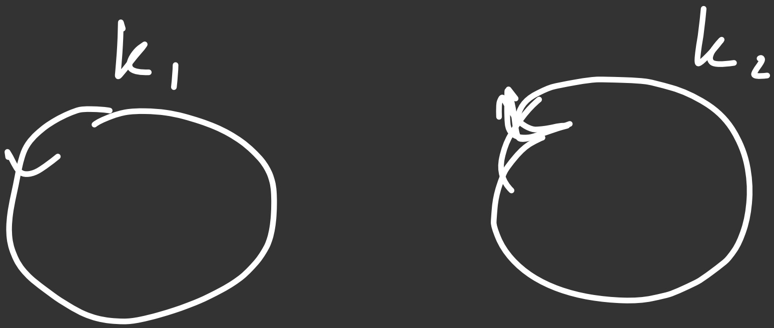
prop • $\mathrm{lk}(K_1, K_2)$ = signed count of intersections of K_2 with a Seifert surface for K_1.

• $\mathrm{lk}(K_1, K_2)$ is computed from a projection of $K_1 \cup K_2$ in S^3 by counting "mixed" crossings w/ sign. & divide by 2.

+ve crossing -ve crossing.

i.e. crossings that involve both components

<u>ex</u> : 0)



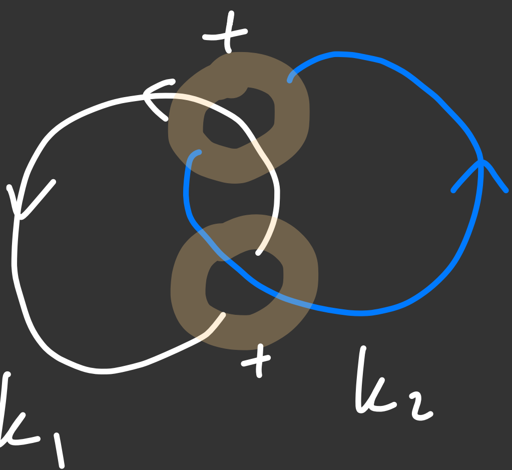
= unlink

unlink w/ two components

$lk(K_1, K_2) = 0.$

b/c there are no crossings

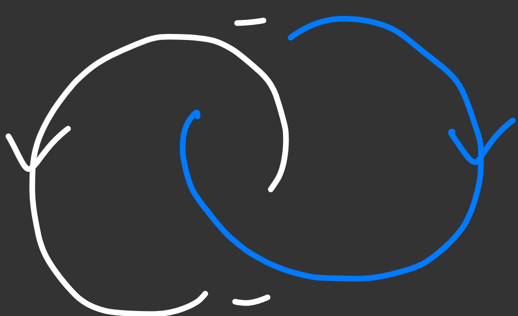
1)



= the Hopf link

$$lk(K_1, K_2) = \frac{1+1}{2} = +1.$$

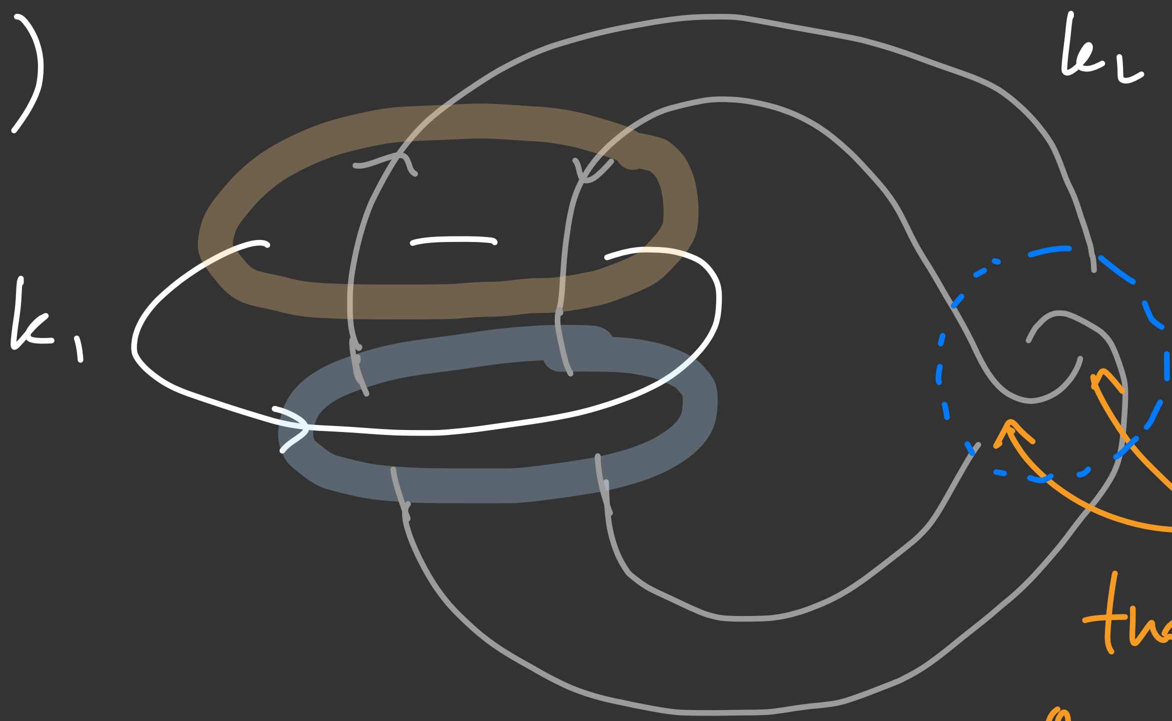
-1)



$lk(K_1, K_2) = -1$

−ve Hopf link.

0.1)



these two crossings are crossings of k_2 w/ itself.

Whitehead link

the two pairs of crossings that are highlighted give opposite contributions $\Rightarrow$

$lk(k_1, k_2) = 0$.

What do we learn?

unlink $\neq$ Hopf$^+$ $\neq$ Hopf$^-$ are all distinct.

Uncertain: Whitehead link $\overset{?}{=}$ unlink.

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prop The Whitehead link is linked.

proof let's check k_1 & consider the double cover of S^3 branched over k_1 → it's gonna be S^3 (b/c k_1 is unknotted)

$$\pi : S^3 \xrightarrow[\text{branched over } k_1]{2:1} S^3$$

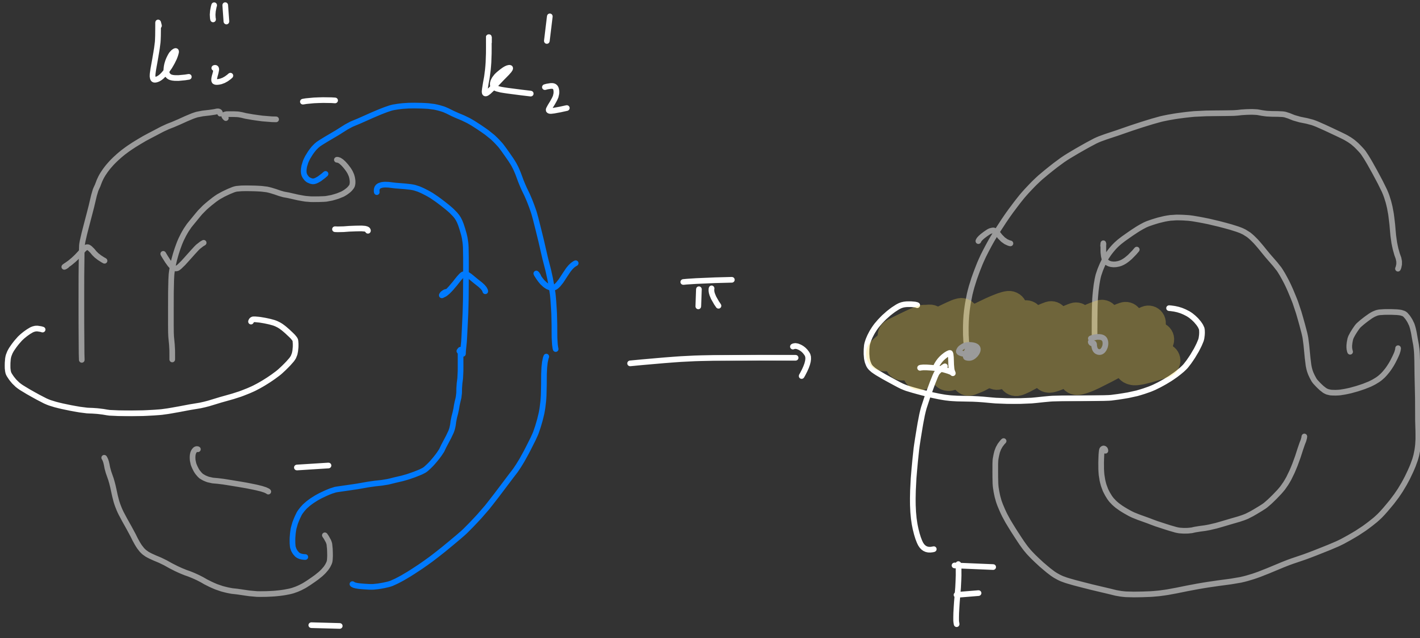
$$\begin{array}{ccc} \cup & & \cup \\ \underline{\pi^{-1}(k_2)} & \longrightarrow & k_2 \end{array}$$

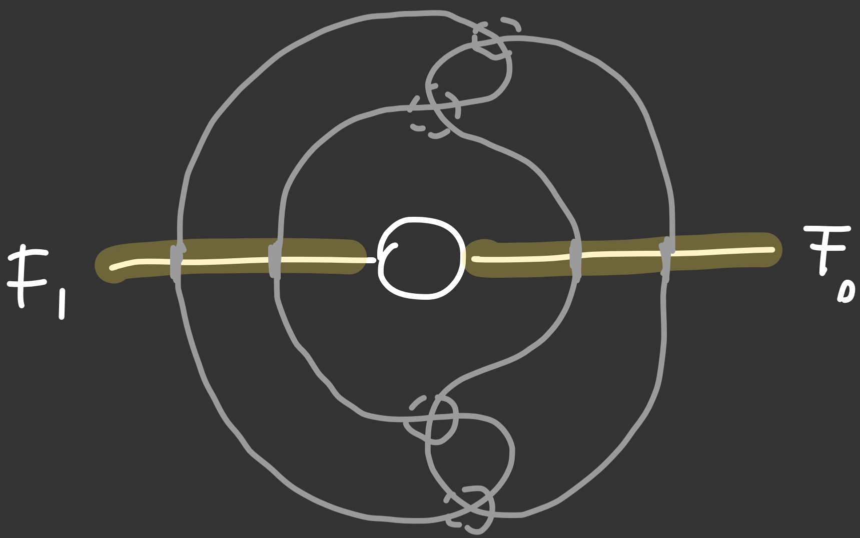
check on or.

↳ want to understand this.

$$\pi^{-1}(k_1) \longrightarrow k_1$$

↳ this is unknotted

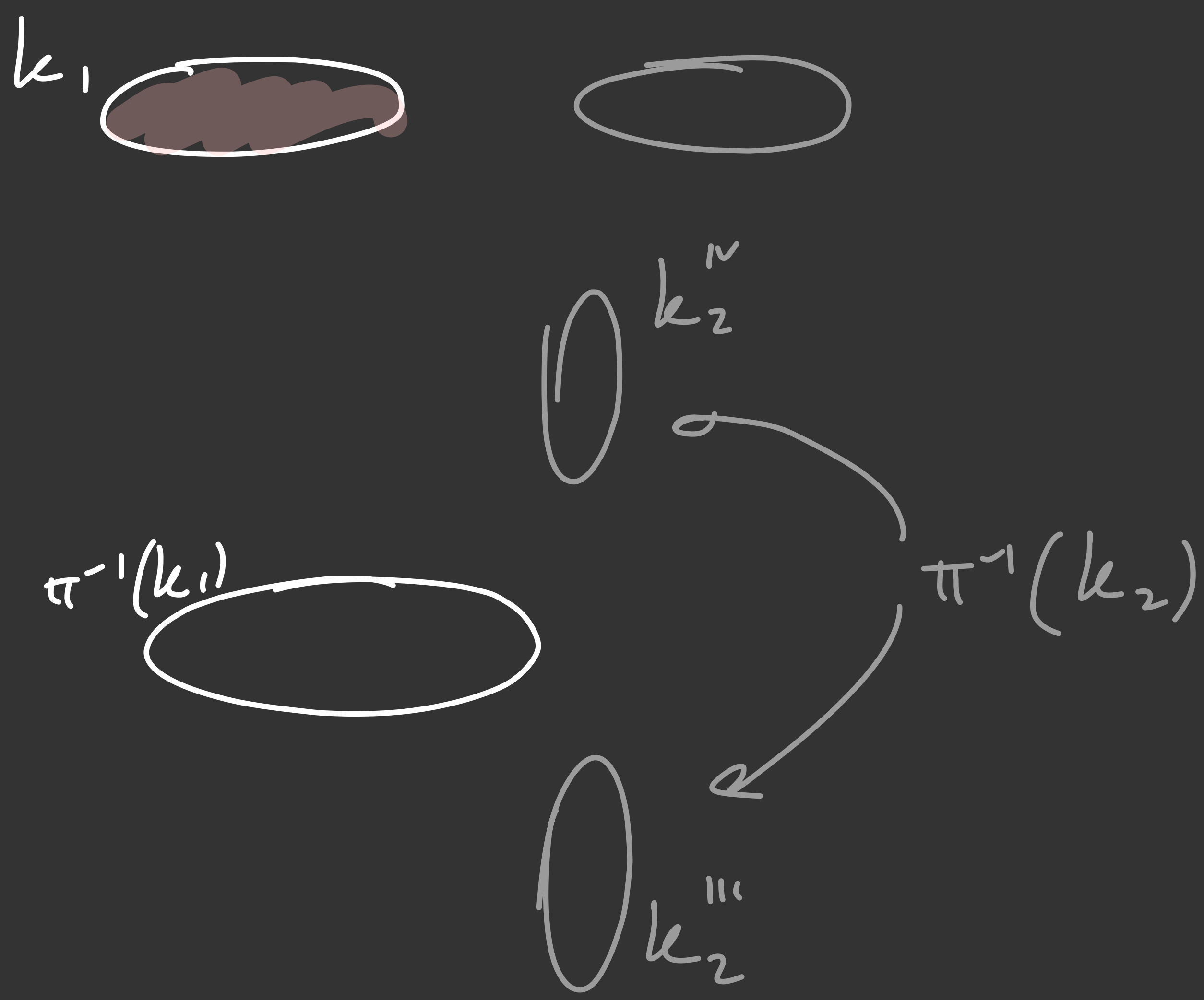




$$lk(k_2', k_2'') = \frac{-1-1-1-1}{2} = -2$$

$\Longrightarrow$ $k_2' \cup k_2''$ is not the unlink with two components.

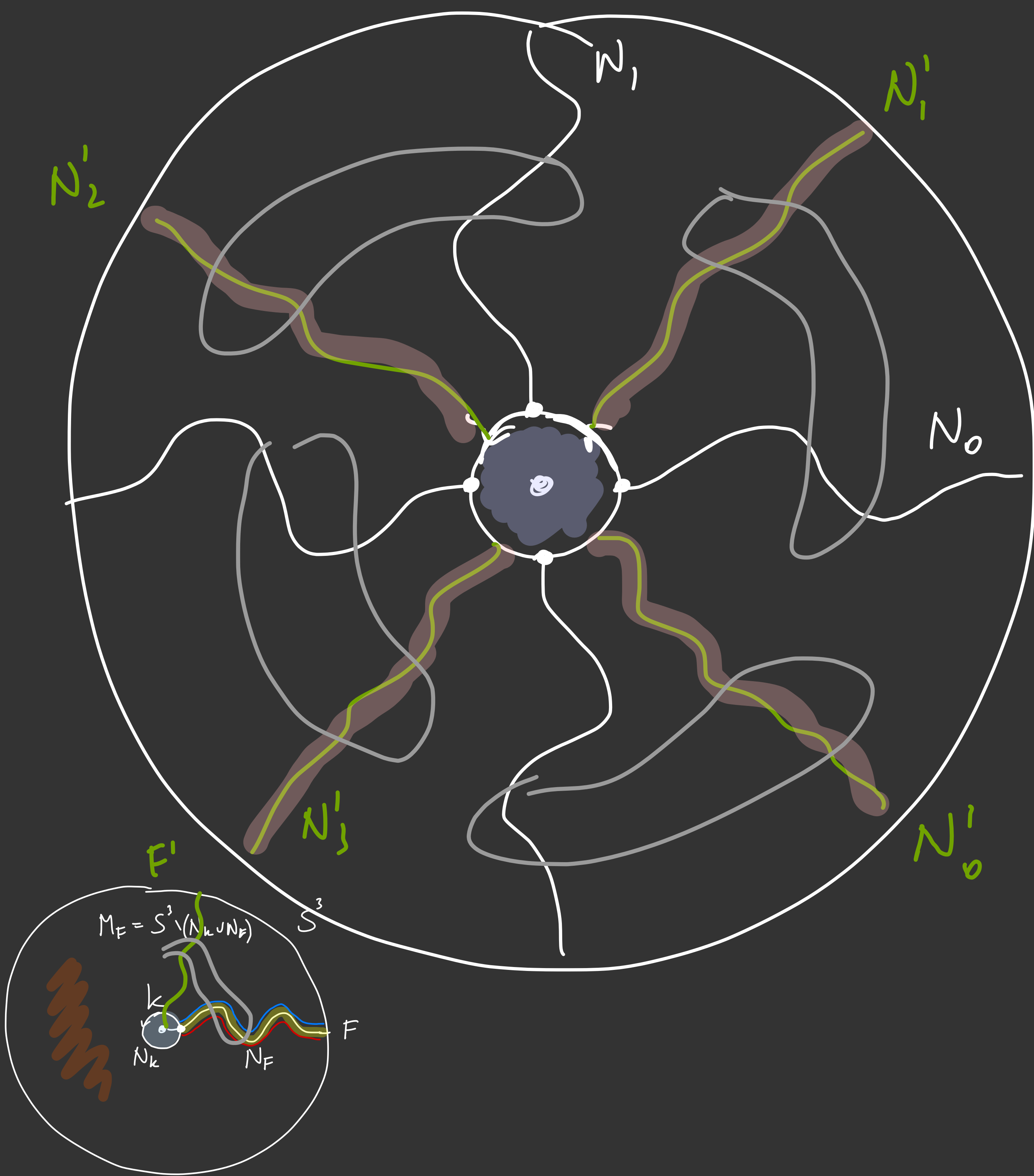
with the unlink



$$lk(k_2''', k_2^{iv}) = 0.$$

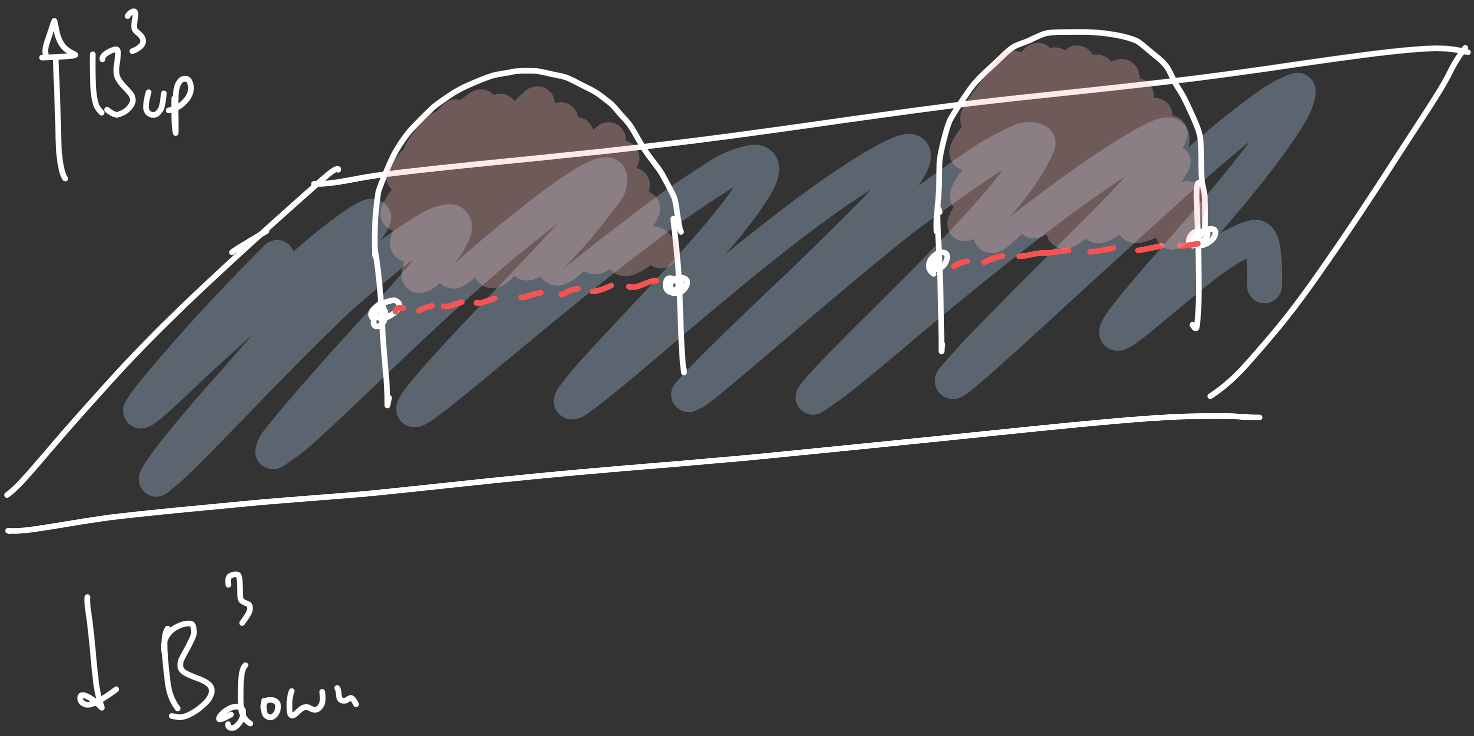
$\Rightarrow$ Whitehead link is not the unlink. $\square$

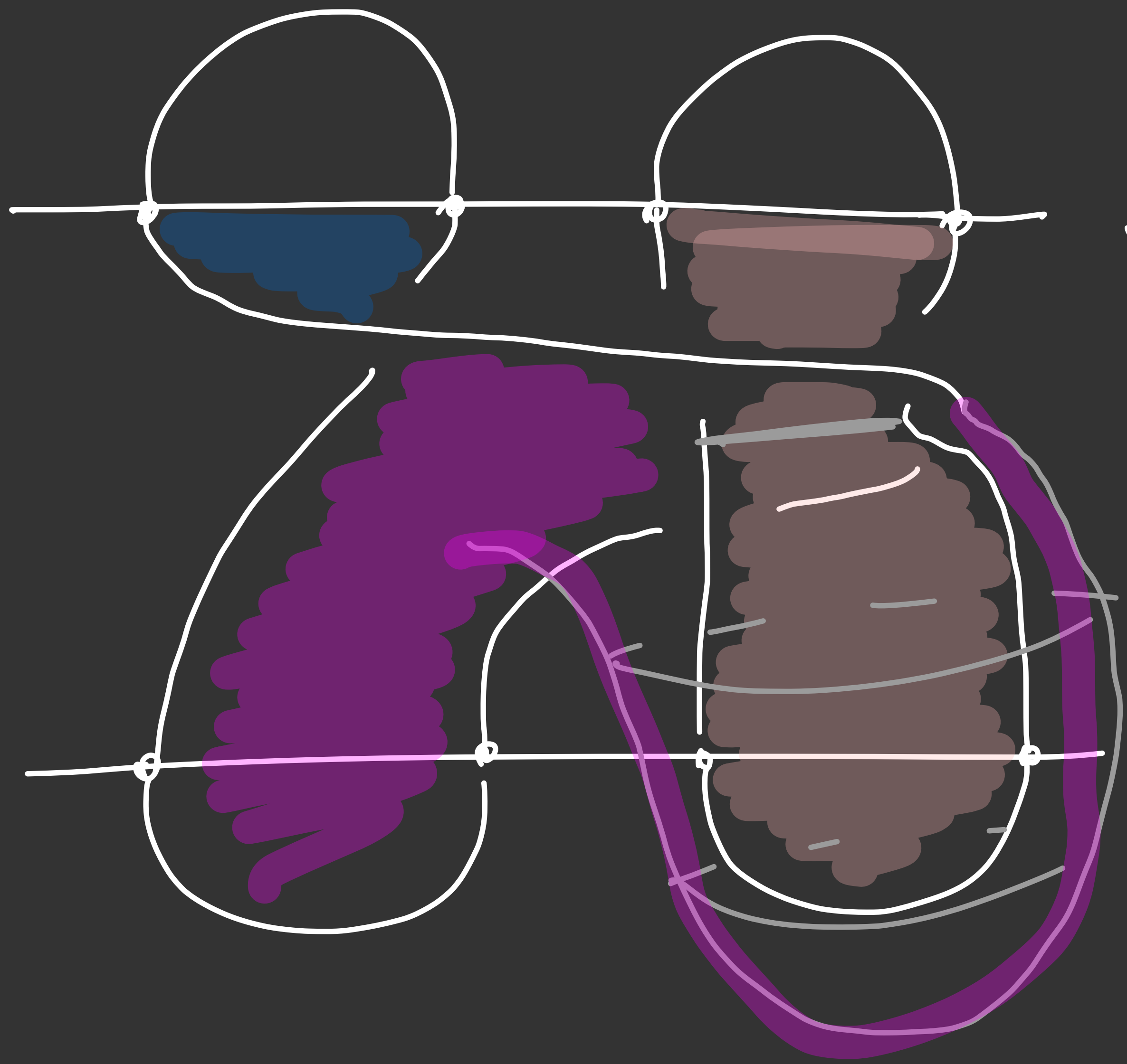
Rmk The constr. of the cycle C_K does not depend on the Seifert surface



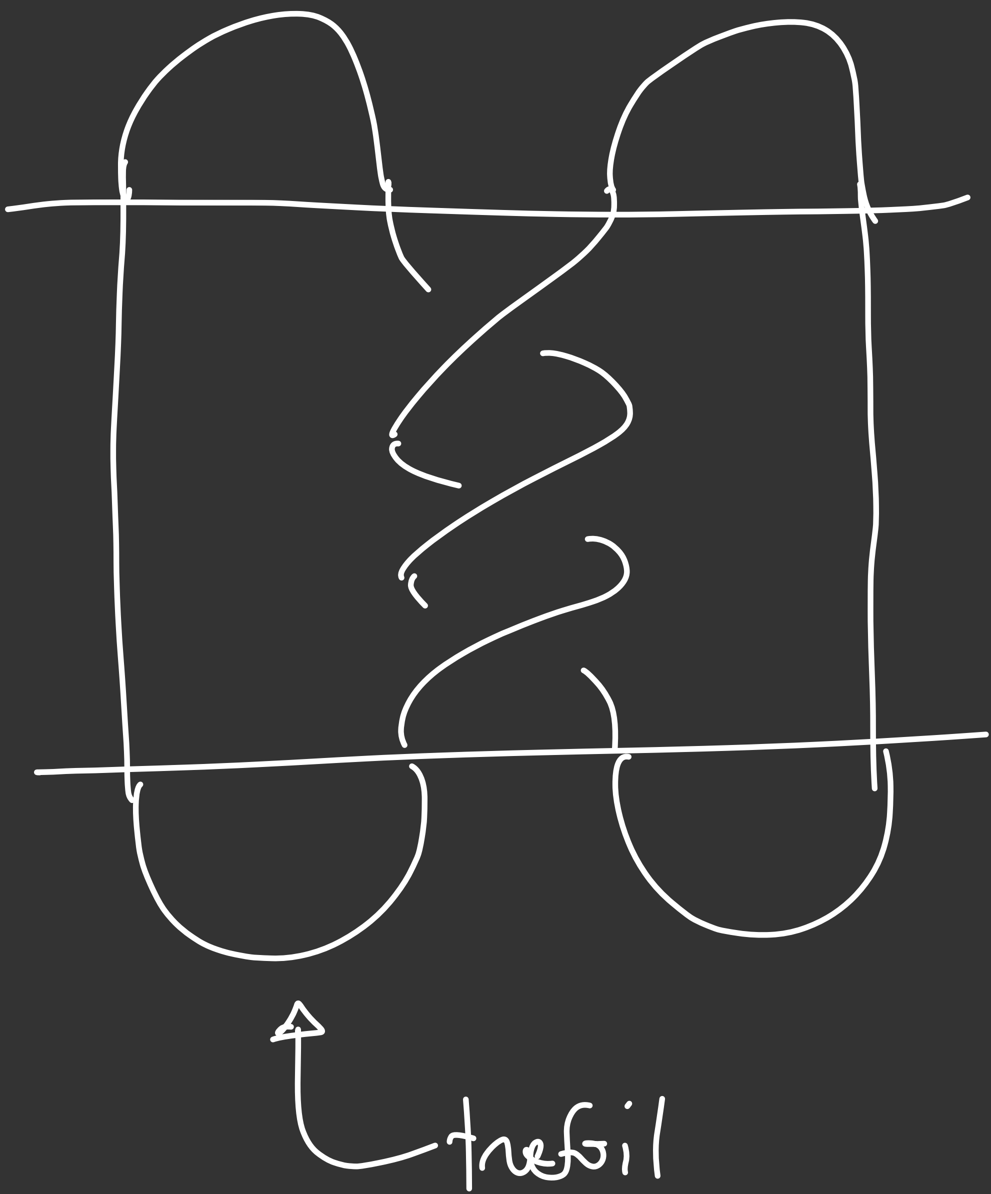
$$S^3 = B^3_{down} \cup B^3_{up}$$

if $k \subset S^3$, I can hope that k sits nicely wrt this decomposition of S^3





S^2
S^2



<u>in general</u> : a knot is in <u>bridge position</u> with respect to $S^3 = B_{down} \cup B_{up}$ if it's the union of some # of arcs in B_{up} & in B_{down}

each of which bounds a disc in B_{up} / B_{down}.

a knot/link is a 2-bridge knot/link if we can use two arcs.

e.g. • The trefoil is a 2-bridge knot.

• The Hopf link(s) are 2-bridge links

• The 2-unlink is a 2-bridge link.

Why?

Balls are simple &

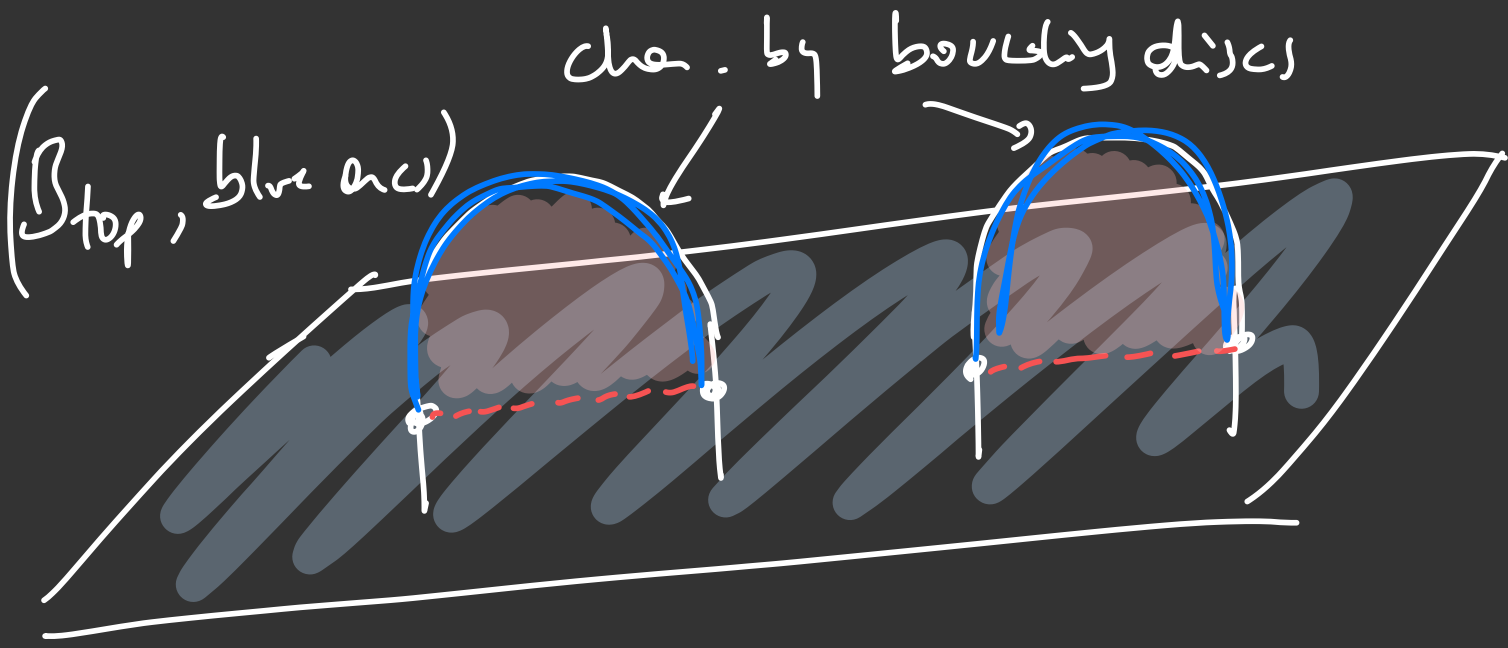
caps & cups are simple

"links" (tangles) in

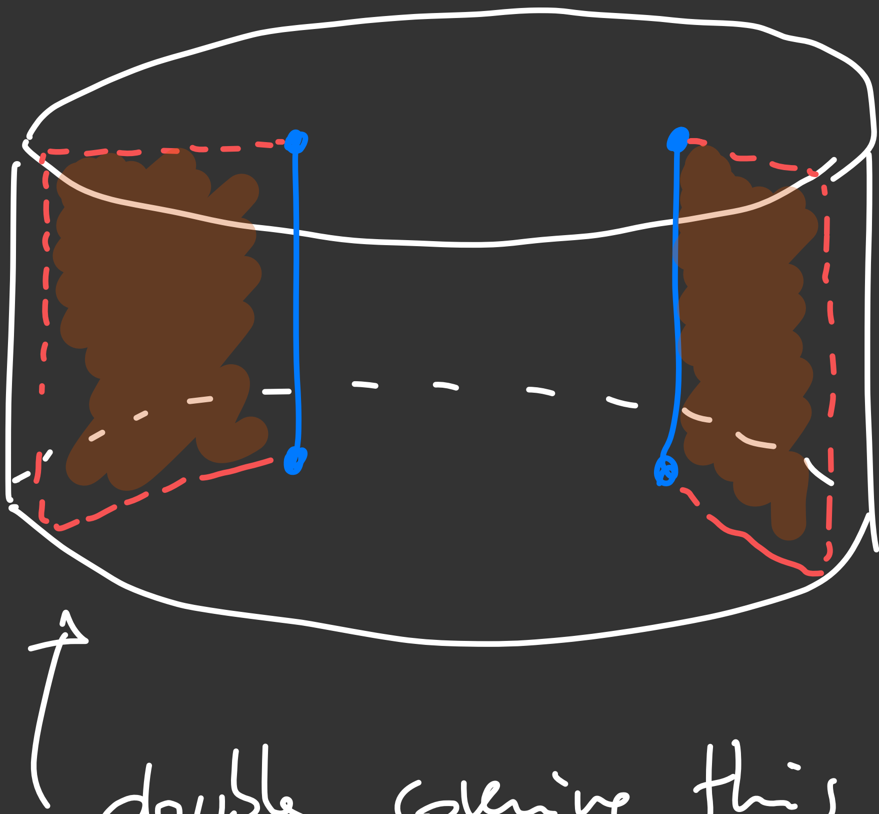
a very simple mfld.

$X = X_1 \cup X_2$ any reg. decomp.

$$\begin{array}{ccccc}
\tilde{X} & = & \tilde{X}_1 & \cup & \tilde{X}_1 \\
\Big\downarrow \text{\small 2:1 cover over } B \subset X & & \Big\downarrow & & \Big\downarrow \\
X & = & X_1 & \cup & X_2 \\
 & & \cup & & \cup \\
 & & B_1 = B \cap X_1 & & B_2 = B \cap X_2
\end{array}$$



$$\cong$$



$B^2 \times I$

double covering this is easy!

(double cover of B^2 branched over two points) $\times I$

$\rightsquigarrow A \times I = S^1 \times I \times I = S^1 \times D^2$

$\mathbb{T}^3$ is a solid torus.

$$\Longrightarrow \Sigma_2(K) = \mathbb{T}^3_{up} \cup_{\partial} \mathbb{T}^3_{down}$$

if K is a 2-bridge knot or link

all the complexity lies here

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the map gluing $\partial\mathbb{T}_{up}$ & $\partial\mathbb{T}_{down}$ is obtained by lifting the map that glues ∂B_{up} with ∂B_{down} (keeping track of the endpoints of the two arcs)

What we proved is that

Σ_2(2-bridge link) is

a 3-mfld with Heegaard

genus ≤ 1.

<u>ex</u> If M is a closed orientable

3-mfld with Heegaard genus = 1,

then it's either $S^1 \times S^2$

or a lens space.

$\Rightarrow$ Σ_2(2-bridge) is a

lens space, or $S^1 \times S^2$, (or S^3.)

<u>rmk</u> The converse is also true.

There's a third way in which branched covers of 3-mflds arise.

<u>recall</u>: $\Sigma(p,q,r) = \{x^p + y^q + z^r = 0\} \cap S^5_1$

thought of as an abstr. 3-mfld

$T(p,q) = \{x^p + y^q = 0\} \cap S^3_1$

↳ as a knot/link $\subset S^3$.

<u>prop</u> $\Sigma(p,q,r)$ is the r-fold cyclic cover of S^3 branched over $T(p,q)$.

<u>proof</u> (sketch) look at the projection $\mathbb{C}^3 \longrightarrow \mathbb{C}^2$

$(x,y,z) \longmapsto (x,y)$

& restrict it to $\Sigma(p,q,r)$.

$\leadsto$ the projection

$$\pi : \Sigma(p,q,r) \longrightarrow \mathbb{C}^2$$

is a cover over its image, branched over $x^p + y^q = 0$.

$$(x, y, \omega z_0) \longmapsto (x, y) \text{ s.t. } x^p + y^q \neq 0$$

where z_0 is any r-th root of $-(x^p + y^q)$, & ω is any r-th root of 1.

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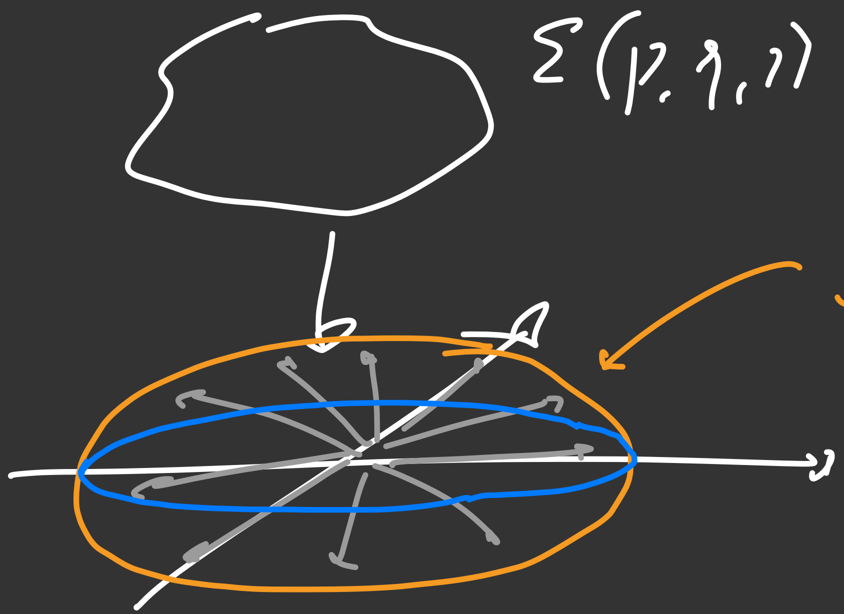
problem: $\pi(\Sigma(p,q,r)) \neq S^3$

$\Sigma(p,q,r) \subset \mathbb{C}^3$



S^3 (see notes)

ex $\Sigma(2,3,5)$ is the 5-fold cyclic cover of S^3 branched on the trefoil.

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prop If $d = p^e$ prime power (p prime, $e \geq 1$ integer), $K \subset S^3$ knot, then $\Sigma_d(K)$ has

$$H_1(\Sigma_d(K); \mathbb{Q}) = 0.$$

(equivalently: $H_1(\Sigma_d(K))$ finite).

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def A closed, oriented 3-mfld with $H_1(Y)$ finite is called a rational homology sphere ($\mathbb{Q}HS^3$).

e.g. lens spaces are $\mathbb{Q}HS^3$.

Proof Three parts to the proof:

1) Reduce to showing that

Algebra $H_2(\Sigma_d(k); \mathbb{F}_p) = 0.$

2) Reduce to showing that

MV $H_2(\Sigma'; \mathbb{F}_p) = 0.$

where Σ' is $\Sigma_d(k) \setminus \pi^{-1}(k)$

(i.e the d-fold cover of $S^3 \setminus k$)

meat here.

3) Proving that $H_2(\Sigma'; \mathbb{F}_p) = 0.$

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exe $S^1 \times S^2$. $H_2(S^1 \times S^2) = \mathbb{Z}$

but if you remove $S^1 \times \{pt\}$

& call M that you get $\Rightarrow H^2(M) = 0.$

rmk: $H_*(X)$ I mean $\mathbb{Z}$-coeff.

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step 1: key pt: H_2(closed 3-mfld $\hookrightarrow M$) is a free abelian group

why?

$$H_2(M) \underset{PD}{\cong} H^1(M) \underset{\text{nat. can.}}{\overset{\text{univ. coeff. thm}}{\cong}}$$

$$\underbrace{\mathrm{Hom}(H_1(M;\mathbb{Z}),\mathbb{Z})}_{\substack{\text{free b/c} \\ \mathrm{Hom}(A,\mathbb{Z}) \text{ are free}}} \oplus \underbrace{\mathrm{Tor}(H_0(M))}_{\substack{= 0 \\ \text{b/c there's} \\ \text{no torsion} \\ \text{in } H_0}}$$

Univ. Coeff thm

$$H_2(X;\mathbb{F}_p) = H_2(X)\otimes\mathbb{F}_p \oplus$$

"p-torsion" in $H_1(X)$.

If $H_2(X;\mathbb{F}_p)=0 \Longrightarrow$

$\mathrm{Free}(H_2(X))=0$ & no p-torsion in $H_1(X)$.

$$\Longrightarrow H_2(\Sigma_\alpha(K))=0.$$

& by the univ. coeff thm &

Poincaré duality $\Longleftrightarrow$

$H_1(\Sigma_\alpha(K))$ finite

$$\Longleftrightarrow H_1(\Sigma_\alpha(K);\mathbb{Q})=0.$$

step 2 $\quad \Sigma' = \Sigma_d(k) \setminus \pi^{-1}(k)$ $\;\overset{\Sigma}{\shortparallel}$

$$\underline{H_2(\Sigma'; \mathbb{F}_p) = 0} \Rightarrow H_2(\Sigma; \mathbb{F}_p) = 0$$

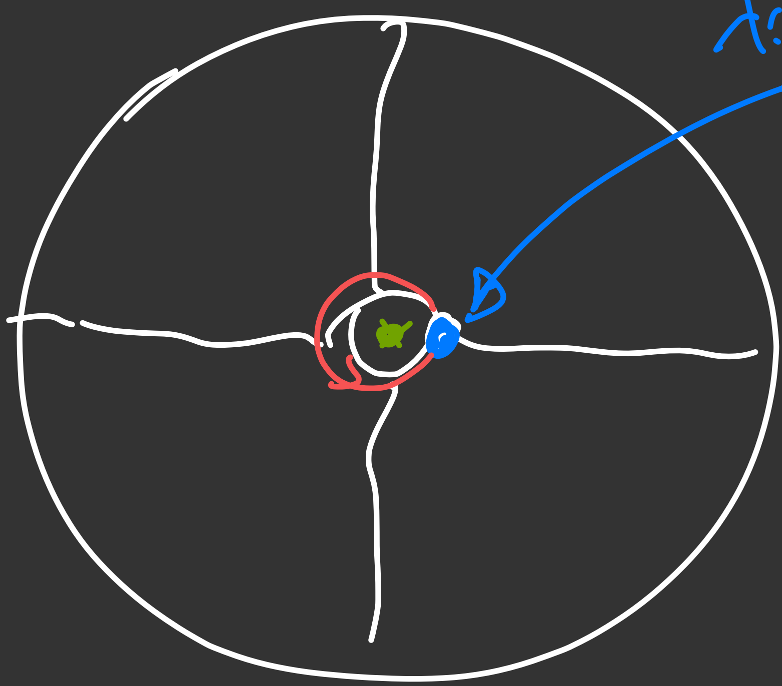
Mayer-Vietoris:

$$\Sigma = \Sigma' \cup \mathbb{T}^3 \qquad \partial \mathbb{T}^3 = T$$

$$\cancel{H_2(\Sigma')}^{\,0} \oplus \cancel{H_2(\mathbb{T}^3)}^{\,0} \to H_2(\Sigma) \to H_1(T) \to$$

$$\to H_1(\Sigma') \oplus H_1(\mathbb{T}^3)$$

$$0 \to H_1(\Sigma) \to H_1(T) \to H_1(\Sigma') \oplus H_1(\mathbb{T}^3)$$



$\lambda := \partial F_1$ = a longit. of $\pi^{-1}(\underline{k})$

$m_{\tilde{k}}$ = lifts of m_k into Σ'.

key pt: $m_{\bar{k}}$ & λ generate $H_1(T)$ &

- $\lambda = \partial F_1 \Rightarrow [\lambda] = 0 \in H_1(\Sigma')$ & $[\lambda]$ generates $H_1(\mathbb{T}^3)$

- $[m_{\bar{k}}] = 0 \in H_1(\mathbb{T}^3)$

$m_{\bar{k}}$ intersects F_1 transversally once

$\rightsquigarrow$ $m_{\bar{k}}$ & F_1 are dual to each other

$\in H_1(\Sigma')$ $\quad$ $\in H_2(\Sigma', T)$

$\Rightarrow$ each of the generates a free subgroup.

$\Rightarrow a \cdot [m_{\bar{k}}] \neq 0$ in $H_1(\Sigma')$

$\& \; a \cdot [\lambda] \neq 0$ in $H_1(\mathbb{T}^3)$

$$\Rightarrow \quad H_1(T) \hookrightarrow H_1(\Sigma') \oplus H_1(\mathbb{T}^3)$$

$$\Rightarrow \quad H_2(\Sigma) = 0 .$$

<u>step 3</u> lifting a CW structure from $S^3 \setminus k$ to Σ'.

the catch: the CW structure you have lifted comes <u>for free</u> with an action by $\mathbb{Z}/d\mathbb{Z}$.

let's pick a CW str. on $S^3 \setminus k$

& lift it.

Every cell in Σ' corresponds

to a cell in $S^3 \setminus k$ +

a label

- $x_0^d = t^d \cdot x_0 \equiv x_0$.

$\vdots$

- $x_0^1 = t \cdot x_0$

- x_0

do this for every cell.

$\leadsto$ $C_*^{cw}(\Sigma')$ has a structure of $\mathbb{F}_p[t]/(t^d-1)$-module.

& the fun thing is that if we now let $t=1$, we recover $C_*^{cw}(S^3, k)$

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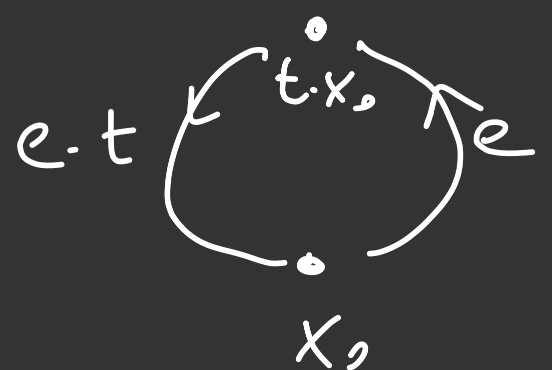
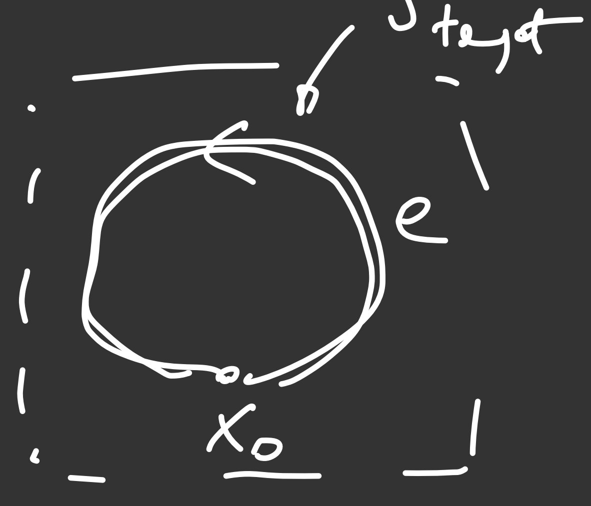
ex: $S^1_{\text{source}} \to S^1_t$
$z \mapsto z^2$





$$C_*^{CW}(S^1_{\text{[illegible]}}) =$$

dim = 1   dim = 0

e x_0

$t \cdot e$ $t \cdot x_0$

$$\partial e = t x_0 - x_0$$

$$\partial(t \cdot e) = x_0 - t x_0 = t^2 x_0 - t x_0$$

$$= t(t x_0 - x_0) = t(\partial e)$$

If let $t = 1$

$$\partial e = x_0 - x_0 = 0.$$

$$0 \to C_*^{cw}(\Sigma') \xrightarrow{t-1} C_*^{cw}(\Sigma') \to C^{cw}(S^3 \setminus k) \to 0$$

short exact sequence of chain complexes

$\Rightarrow$ long exact sequence in homology

$$\to H_k(\Sigma') \to H_k(\Sigma') \to H_k(S^3 \setminus k) \to$$

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this works for infinite covers as well.

if you take the infinite cyclic cover of $S^3 \setminus k$

(associated to $H_1(S^3 \setminus k) \to \mathbb{Z}$).

we get the same thing:

$$\tilde{\Sigma} = \text{infinite cyclic cover}$$

$C_*^{cw}(\tilde{\Sigma})$ is a $\mathbb{F}_p[t]$-module

& get a long exact sequence

① $H_k(\tilde{\Sigma}) \xrightarrow{t-1} H_k(\tilde{\Sigma}) \to H_k(S^3 \setminus K)$

② $H_k(\tilde{\Sigma}) \xrightarrow{t^d-1} H_k(\tilde{\Sigma}) \to H_k(\Sigma')$.

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "t \mapsto e^{2\pi i t}"] \arrow[dr, "t \mapsto e^{2\pi i t/d}"'] & S^1 \\
& S^1 \arrow[u, "z \mapsto z^d"']
\end{tikzcd}$$

from ①

$$\underbrace{H_2(\tilde{\Sigma}) \xrightarrow[\text{onto}]{t-1} H_2(\tilde{\Sigma}) \to H_2(S^3\setminus k)}_{}\qquad H_2(S^3\setminus k) = 0$$

$$H_2(S^3\setminus k) \to H_1(\tilde{\Sigma}) \xrightarrow[\text{inj}]{t-1} H_1(\tilde{\Sigma})$$

$$H_2(S^3\setminus k) = 0$$

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from ②

$$H_2(\tilde{\Sigma}) \xrightarrow[\text{onto}]{t^d-1} H_2(\tilde{\Sigma}) \to H_2(\Sigma') \to H_1(\tilde{\Sigma}) \xrightarrow[\text{inj}]{t^d-1} H_1(\tilde{\Sigma})$$

$$H_2(\Sigma') = 0\,!$$

in char $= p$, $\quad t^d - 1 = (t-1)^d$

& comp. of surj maps is onto

" " inj " is inj.

$\square$