Lecture 6

examples of 4-manifolds.

- $S^4 = \{ x_1^2 + \ldots + x_5^2 = 1 \} \subset \mathbb{R}^5$

  has a cell dec. w/ one 0-cell

  & one 4-cell.

- $\mathbb{C}P^2 = \mathbb{C}^3 \setminus \{0\} / \mathbb{C}^*$ can reverse orient.

  $= S^5 / S^1$

  unit sphere in $\mathbb{C}^3 \simeq \mathbb{R}^6$

  unit circle in $\mathbb{C}$.

  $\overline{\mathbb{C}P}^2 \neq \mathbb{C}P^2$

- products: $F_g \times F_h$

  $S^1 \times Y$ ← 3-mfld

- $\mathbb{CP}^3 = \mathbb{C}^4 \setminus \{0\} / \mathbb{C}^* \cong S^7 / S^1$.

$\{F(x_0, \dots, x_3) = 0\} = V(F)$

hom. poly of degree d in $\mathbb{C}[x_0, \dots, x_3]$

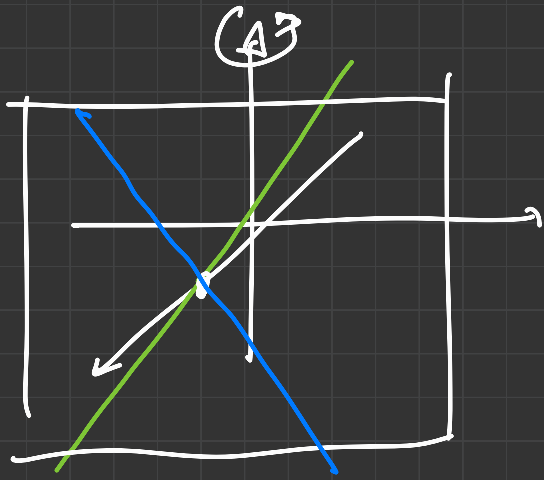
If F is generic ($\nabla F \neq 0$ on $\{F=0\}$) then (by IFT) $V(F) \subset \mathbb{CP}^3$ is a smooth 4-dim'l submanifold.

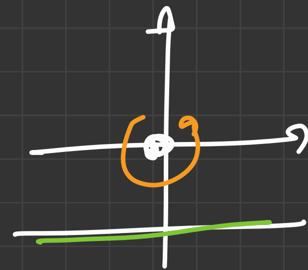
$d=1 \rightsquigarrow \mathbb{CP}^2$

$d=2$, quadric $\rightsquigarrow S^2 \times S^2$

$d=3$, cubic $\rightsquigarrow \mathbb{CP}^2 \# 6\overline{\mathbb{CP}}^2$

$d=4$, <u>K3 surface</u>.





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rmk $K3 \neq X \# X'$ X, X' not homeo to S^3.

$\uparrow$ hard.

Smooth 4-mflds behave very differently than top. 4-manifolds

e.g. $K3 \# \overline{\mathbb{CP}^2} \simeq_{C^0} 3\mathbb{CP}^2 \# 20 \overline{\mathbb{CP}^2}$
$\not\simeq_{C^\infty}$.

invariants of 4-manifolds:

π_1: for 3-mfds, π_1 almost determines the 3-mfd.
& the word problem & the isom. problem for 3-mfds is solvable.

any f.p. G is π_1 of a closed, smooth 4-manifold.

on the other hand: simply-connected 4-manifolds are already interesting.

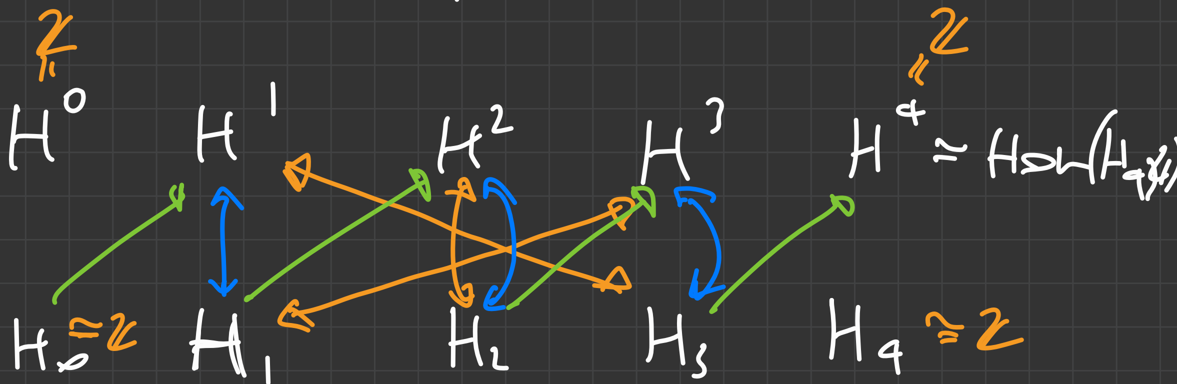
let X be 4-mfd, s.t. $\pi_1(X) = 1$. $\Longrightarrow$

$H_1(X) = 0$

If X is closed, oriented

then $H^3(X) = 0$, H_2 is torsion-free

& $H_3(X) = 0$.



$\longleftrightarrow$ PD

$\longleftrightarrow$ UCT, free parts

$\longrightarrow$ UCT, torsion parts

If $\pi_1 = 1$, then $H_2(X) \cong \mathbb{Z}^{b_2(X)}$

$b_2(X) = \operatorname{rk} H_2(X)$

Fun thing: how we have structure on H_2 or equivalently on H^2.

$$\begin{array}{ccccccc} Q_X : & H^2(X) & \otimes & H^2(X) & \longrightarrow & \mathbb{Z} \\ & \cup\!\!\!\shortmid & & \Psi & & \Psi \\ & \alpha & \otimes & \beta & \longmapsto & \langle \alpha \cup \beta, [X] \rangle \end{array}$$

$\alpha \cup \beta \in H^4(X) = H^{2+2}(X)$

$[X] \in H_4(X)$

Q_X is bilinear & symmetric.

Q_X is called the <u>intersection product</u> (or int. <u>form</u>) on $H^2(X)$

& I can Poincaré-dual it to $H_2(X)$

$$Q_X(A,B) = Q_X(PD(A), PD(B)).$$

<u>key point:</u> <u>IF</u> $A = [F]$, $F \hookrightarrow X$ oriented emb. surface

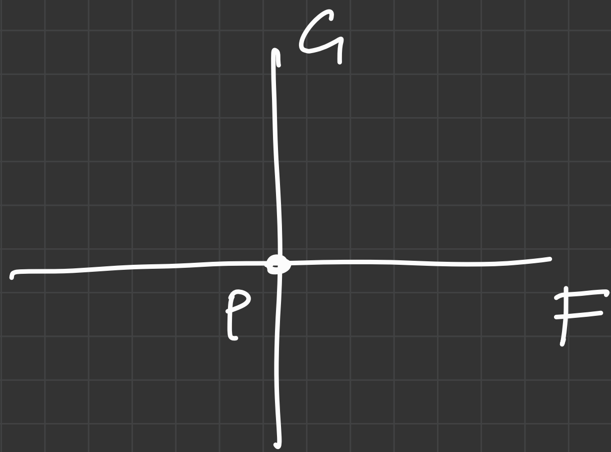
& $B = [G]$, $G \hookrightarrow X$

& $F \pitchfork G$ (transverse)

$$\Rightarrow Q_X(A,B) = \#(F \pitchfork G)$$

count of intersection pts w/ sign!!

Signs:



T_pF , T_pG , T_pX

oriented v.sp.

pick oriented basis v_1, v_2 & w_1, w_2 of T_pF T_pG

sign of p as $F \cap G$ is the sign of (v_1, v_2, w_1, w_2) as a basis of T_pX.

ex $\mathbb{C} \times 0$, $0 \times \mathbb{C} \subset \mathbb{C}^2$

$x_1 + iy_1$ $\qquad$ $x_2 + iy_2$

$(\partial_{x_1}, \partial_{y_1})$ $\qquad$ $(\partial_{x_2}, \partial_{y_2})$

ordered basis of $\qquad$ "

$T_{(0,0)}(\mathbb{C} \times 0)$ $\qquad$ $T_{(0,0)}(0 \times \mathbb{C})$

$(\partial_{x_1}, \partial_{y_1}, \partial_{x_2}, \partial_{y_2})$ ordered basis of $\mathbb{C}^2$

$\Longrightarrow$ this is a positive int. point.

(in general, cx int. pts are true)

Gr Bezout's theorem on cx curves in $\mathbb{CP}^2$.

under the ass.

$F \pitchfork G$.

As a consequence of Poincaré duality, Q_X is <u>unimodular</u>, i.e. that

Q_X : two elements in $H_2 \rightsquigarrow \mathbb{Z}$

$H^2(X)$ if no torsion

$\wr\wr$

$$Q_X : H_2(X) \longrightarrow \mathrm{Hom}(H_2(X), \mathbb{Z})$$

$$A \longmapsto Q_X(A, \cdot)$$

Poincaré duality tells you that this map is an isomorphism, provided that there is no torsion in H_2.

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in practice, you pick a basis$/\mathbb{Z}$ $A_1, \dots, A_n$ for $H_2(X)$

& you represent Q_x as $n \times n$ matrix

$$M = \begin{pmatrix} A_1 \cdot A_1 & A_1 \cdot A_2 & \cdots & A_1 \cdot A_n \\ A_2 \cdot A_1 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_n \cdot A_1 & \cdots & & A_n \cdot A_n \end{pmatrix}$$

Q_x is unimodular $\iff |\det M| = 1$

<u>Fun</u>: Unimodular forms $/\mathbb{Z}$ are a mess

$\#$(isom. classes of unim. forms of rank n) grows exponentially with n

you can extract some invariants of Q_X that are useful:

- $\operatorname{rk} Q_X = \operatorname{rk} H_2(X) = b_2(X)$
- type of Q_X : even / odd
  - even if $Q_X(a,a) \equiv 0 \pmod 2$ $\forall a \in H_2(X)$
  - odd otherwise.

<u>ex</u>: (1), $\operatorname{rk}=1$, odd

$$\begin{pmatrix} \pm 1 & & 0 \\ & \pm 1 & \\ 0 & \ddots & \pm 1 \end{pmatrix}, \text{ any } \operatorname{rk}, \text{ odd.}$$

$$H := \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}, \operatorname{rk}=2, \text{ \underline{even}.}$$

$$(aA+bB)^2 = \cancel{a^2 \overset{0}{A^2}} + 2ab(A\cdot B) + \cancel{b^2 \overset{0}{B^2}}$$
$$= 2ab$$

ex $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus (-1) \cong \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$(\mathbb{CP}^1 \times \mathbb{CP}^1) \# \overline{\mathbb{CP}^2} = \mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}.$

- the signature $\sigma(Q_X)$:

$$
\begin{tikzcd}
Q_X^{\mathbb{R}} = Q_X \otimes \mathbb{R} : & H_2(X;\mathbb{R}) \otimes H_2(X;\mathbb{R}) \arrow[r] & \mathbb{R} \\
& H^2_{dR}(X) \otimes H^2_{dR}(X;\mathbb{R}) \arrow[r] \arrow[u, "\cong"] & \mathbb{R}
\end{tikzcd}
$$

$$\alpha \quad\quad \beta \quad\quad \int_X \alpha \wedge \beta$$

$Q_X^{\mathbb{R}} \cong_{\mathbb{R}}$ diagonal matrix $\begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$

$\sigma(X) = \sigma(Q_X) = (\#\text{ of } +1\text{s}) - (\#\text{ of } -1\text{s})$

in the diagonalization.

<u>equivalently</u> : $\sigma = \begin{pmatrix} \text{max dim of a} \\ \text{+ve definite subspace} \end{pmatrix} - \begin{pmatrix} \text{max. dim of a} \\ \text{-ve def. subspace} \end{pmatrix}$

<u>ex</u> $\sigma((1)) = 1 - 0 = 1$

$\sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \sigma\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0.$

X (or Q_X) is <u>definite</u> if

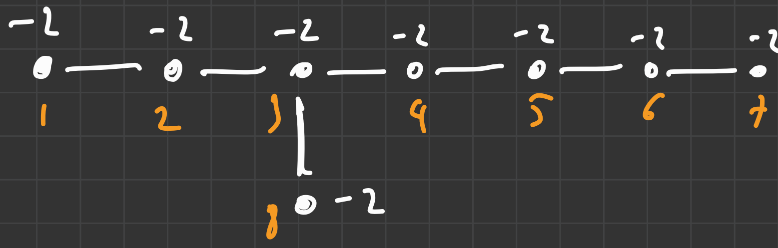
$$|\sigma(X)| = |b_2(X)|$$

$$|\text{sign}| = \text{rk}.$$

& <u>indefinite</u> o/w.

ex $-E_8$

$8 = \#\text{vertices}$



$\big\downarrow Q_{E_8}$ incidence matrix

$$\begin{pmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & & \\ & 1 & -2 & \cdots & & & & 1 \\ & & \vdots & -2 & 1 & 0 & & \\ & & & 1 & -2 & 0 & & \\ & & 1 & 0 & 0 & -2 & & \end{pmatrix}$$

$\longrightarrow$ definite

even

$$\det Q_{E_8} = 1, \left(\longrightarrow \frac{1}{2} + \frac{1}{3} + \frac{1}{5} = 1 + \frac{1}{30} \right)$$

thm (Milnor, Serre) If Q, Q' are indefinite unimodular forms $/\mathbb{Z}$,

$Q \cong Q' \iff$ same rk, same σ & same type.

every indef. Q is isom. to

either $\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}$ or

$$a H \oplus b(\pm E_8)$$

$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(If Q is even $\Rightarrow$ σ is div. by 8).

E_8 is even

$\begin{pmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}$ is odd.

E_8 gives the densest sphere packing in dim = 8. 2016.

geom. int. of Q_X : embedded surfaces

& their intersections.

prop $\forall A \in H_2(X;\mathbb{Z})$ is represented

by an embedded surface.

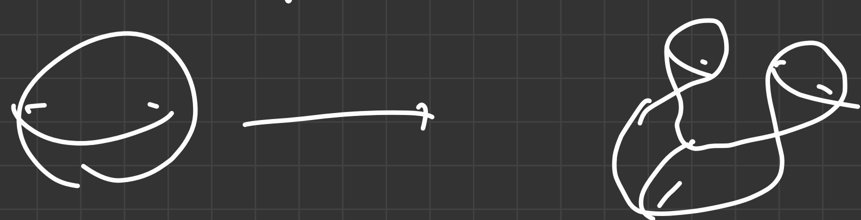
proof (if $\pi_1(X) = 1$)

$\Longrightarrow \pi_2(X) \cong H_2(X)$

$\Longrightarrow A = f_*([S^2])$ for some

$f : S^2 \longrightarrow X$.

can perturb f to be transverse



$f(S^2)$ has $\#$ double pts $\Rightarrow$

can smooth away double pts.

(do it in torically, $S^1 \times$ double pts)

$\{zw = 0\} \subset \mathbb{C}^2 \longleftrightarrow \{zw = \varepsilon\} \subset \mathbb{C}^2$

smooth,

"same" ∂.

$\rightsquigarrow$ can replace $f(S^2)$ with

an embedded surface F.

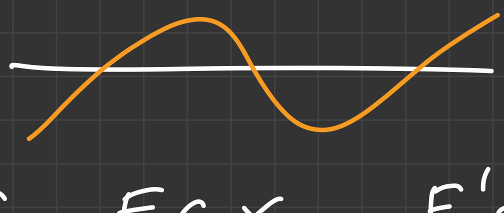
& $[F] = A = f_*([S^2])$. $\square$

Nice obs : $F \subset X$, can define

its self-intersection $F \cdot F$

$$Q_X(F,F) \neq \#(F \cap F)$$
$$\| \quad (F \pitchfork F')$$

F' pert. of F



If $F \subset X$, $F' \subset X'$ are two surf. of same genus, $F \cdot F = F' \cdot F'$, then they have diffeo neighbourhoods.

problem $A \in H_2(X)$
$\quad\quad\quad\quad\quad\quad\;\; \|$
$\quad\quad\quad\quad\quad [F]$ for some F.

What is the minimal genus of F?

Can A be repr. by a 2-sphere?

answer. no, not always. $g(F) = 0$ is possible.

we're gonna prove this.

thm (Kronheimer–Mrowka '94)

If F repr. the class $d h \in H_2(\mathbb{CP}^2)$,

then $g(F) \geq \underbrace{\frac{(d-1)(d-2)}{2}}_{\text{genus of a cx curve of degree } d}$.

$$H_*(\mathbb{CP}^2) = \begin{array}{ll} \mathbb{Z} & *=4 \\ 0 & 3 \\ \mathbb{Z}=\mathbb{Z}h. & 2 \\ 0 & 1 \\ \mathbb{Z} & 0 \end{array}$$

$$Q_{\mathbb{CP}^2}(h,h) = \#(L \cap L') = 1.$$

h: hom. class of a line in $\mathbb{CP}^2$

if $h = k \cdot h'$

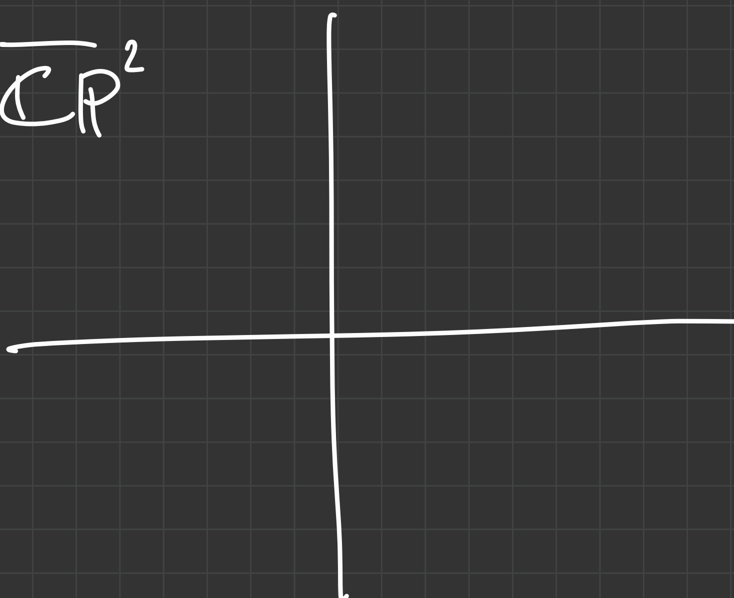
$$Q(h',h') = Q\left(\frac{1}{k}h, \frac{1}{k}h\right) =$$

$$= \frac{1}{k^2} Q(h,h) = \frac{1}{k^2}$$

Rmk $\quad H_*(\overline{\mathbb{CP}^2}) = \begin{array}{l} \mathbb{Z} \\ 0 \\ \mathbb{Z}\,\bar{h} \\ 0 \\ \mathbb{Z} \end{array}$

- what is $Q_{\overline{\mathbb{CP}^2}}(\bar{h}, \bar{h}) = -1$

$\overline{\mathbb{CP}}^2$



$$\Rightarrow \quad Q_{\mathbb{CP}^2} = (1) \Rightarrow \sigma(\mathbb{CP}^2) = +1$$

$$Q_{\overline{\mathbb{CP}^2}} = (-1) \Rightarrow \sigma(\overline{\mathbb{CP}^2}) = -1$$

if they were orient.-pres. diffeo,

then same signature:

$\varphi : X \longrightarrow X'$ pres. the or.

$\varphi^* : H^*(X') \longrightarrow H^*(X)$

isom. of rings.

$$\Rightarrow \mathbb{CP}^2 \neq \overline{\mathbb{CP}^2}.$$

Branched covers

Surfaces are codim-2, so they are (maybe) branching sets of covers.

<u>ex</u> • $F_g \times F_h \xrightarrow{p \times id} F_{g'} \times F_h$ is a b.c.

$p: F_g \to F_{g'}$

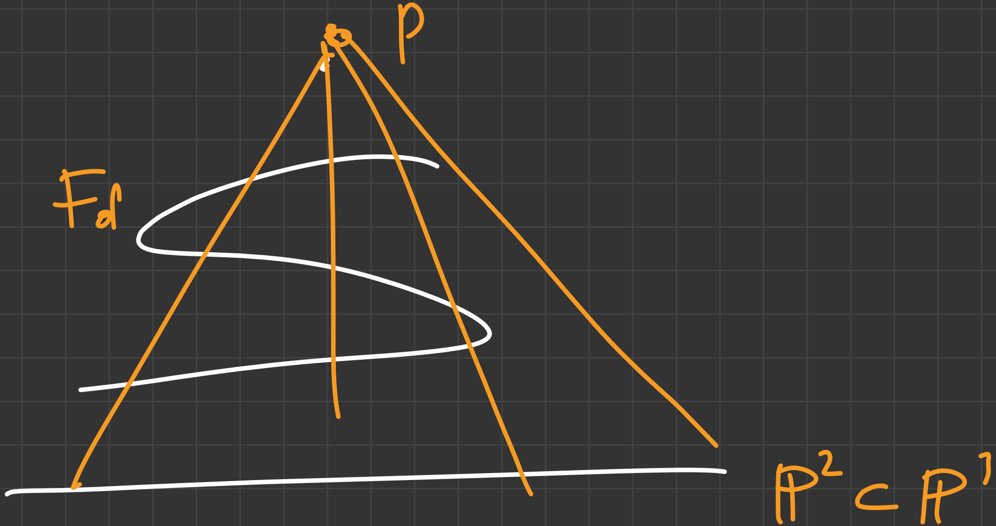
• $M \times S^1 \xrightarrow{p \times id} M' \times S^1$ is a b.c.

& $p: M \to M'$ is a b.c. of 3-mflds

• $F_d = \{ X_0^d + \dots + X_3^d = 0 \} \subset \mathbb{CP}^3$

$\downarrow$

$\mathbb{CP}^2$



For the appr. choice of p, $\mathbb{P}^2 \subset \mathbb{P}^3$,

the cover is cyclic of degree d

branched over $\{x_0^d + x_1^d + x_2^d\} \subset \mathbb{CP}^2$.

e.g. $S^2 \times S^2 \xrightarrow{2:1} \mathbb{CP}^2$

ex (w/o proof) $\mathbb{CP}^2 \circlearrowleft \mathrm{conj} =: \tau$

$$\tau : (z_0 : z_1 : z_2) \mapsto (\bar{z}_0 : \bar{z}_1 : \bar{z}_2)$$

τ <u>preserves</u> the orientation of $\mathbb{CP}^2$!

$$\mathbb{CP}^2/_{\tau} = S^4 \rightsquigarrow \quad \mathbb{CP}^2 \xrightarrow{2:1} S^4$$

branching set of the cov.:

Upstairs: $\mathrm{Fix}(\tau) = (z_0, z_1, z_2 \in \mathbb{R})$

$$\begin{array}{ccc} \mathbb{RP}^2 & \subset & \mathbb{CP}^2 \\ \downarrow & & \downarrow \\ \mathbb{RP}^2 & \subset & S^4 \end{array}$$

, call it R

$$\begin{array}{c}\overline{\mathbb{CP}^2} \\ \downarrow \\ S^4\end{array}$$ a 2-fold cover.

$S^4 \supset \mathbb{RP}^2$, call R_+

__claim__ R_+ & R_- are __not__ isotopic! $\square$

$$\begin{tikzcd}
(\mathbb{CP}^2, \mathbb{RP}^2) \arrow[r, "\text{or. rev.}", "\text{diff.}"'] \arrow[d] & (\overline{\mathbb{CP}^2}, \overline{\mathbb{RP}^2}) \arrow[d] \\
(S^4, R_-) \arrow[r, "\text{or. rev.}", "\text{diffeo.}"'] & (-S^4, R_+)
\end{tikzcd}$$

$$S^4 \underset{\text{diffeo}}{\cong} -S^4$$

? When do we have a d-fold cyclic cover $X \longrightarrow Y$ branched over $B \subset Y$.

$\underbrace{X}$ want to know if it exists.

$\underbrace{B \subset Y}_{\text{fixed}}$

d fixed, but arbitrary.

B: collection $B_1 \dots B_m$ of emb. mfds.

Answer: should involve $Y \setminus B$

& maps $H_1(Y \setminus B) \longrightarrow \mathbb{Z}/d\mathbb{Z}$

$\wr\wr$ rel. to.

$H^1(Y \setminus B; \mathbb{Z}/d\mathbb{Z})$.

Prop Suppose that each comp. of B is orientable or that $d = 2$.

Then $\exists\, p: X \to Y$ d-fold cyclic cov branched over each B_i $\iff$

$$\sum \lambda_i [B_i] = 0 \in H_2(Y; \mathbb{Z}/d\mathbb{Z})$$

w/ $\lambda_i \in (\mathbb{Z}/d\mathbb{Z})^*$ $\forall i$.

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e.g. If $m = 1$ (B conn.)

$\implies [B_i] = d \cdot A \in H_2(X; \mathbb{Z})$.

<u>proof</u>: we look at long. exact sequence of the pair (Y, B) & relate $H_3(Y, B)$ with $H^1(Y \setminus B)$.

Rmk $H_3(Y,B) \cong H_3(Y,N)$

(N nbhd of B)

$\cong$ excision

$H_3(Y \setminus \mathring{N}, \partial N)$

$\cong$ PL-duality

$$H^1(Y \setminus B) \cong H^1(Y \setminus \mathring{N}).$$

$$H_3(Y,B) \to H_2(B) \to H_2(Y)$$

$\rightsquigarrow$ if you chase the isomorphisms around: $\exists\, \varphi : H_1(Y \setminus B) \to \mathbb{Z}/d\mathbb{Z}$ that sends $\mu_i \longmapsto \lambda_i$

μ_i: mer. to B_i

λ_i: gen. of $\mathbb{Z}/d\mathbb{Z}$

$\Rightarrow$ cyclic cover $\square$

Prop In the not. of the previous prop,

$$\begin{tikzcd} X \arrow[d] & \supset & \tilde{B}_i = p^{-1}(B_i) \arrow[d, "2"] \\ Y & \supset & B_i \end{tikzcd}$$

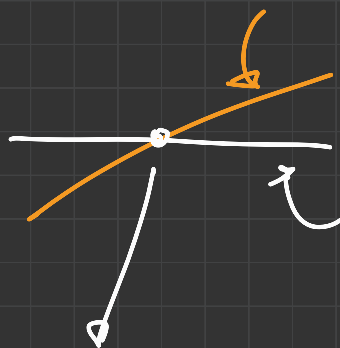
$$\underbrace{\underbrace{\tilde{B}_i \cdot \tilde{B}_i}_{\text{in } X} = \frac{1}{d} \underbrace{B_i \cdot B_i}_{\text{in } Y}.}$$

this makes sense for non-orient. surfaces as well.

$$F \cdot F = F \cdot F' := \#(F \cap F')$$

w/ sign.

induces an or. of F' near p.



pick any or. near p

has a well defined sign.

<u>Thm</u> (G-signature theorem, easy version)

If $p: X \to Y$ is a d-fold cyclic cover, branched on $B \subset Y$, then
$$\sigma(X) = d\,\sigma(Y) - \frac{d^2-1}{3d}\, B \cdot B.$$

Ex X_d degree-d hyp. in $\mathbb{CP}^3$, d-fold b.c. of $\mathbb{CP}^2$ branched over a surface of self-int. d^2.

$$\Rightarrow \sigma(X_d) = d \cdot \sigma(\mathbb{CP}^2) - \frac{d^2-1}{3\cancel{d}} \cdot d^{\cancel{2}}$$

$$= d - \frac{d^3-d}{3} = \frac{4d-d^3}{3}.$$

$\Rightarrow$

d	$\sigma(X_d)$
1	1
2	0
3	-5
4	-16 ← div. by 16

important

<u>ex</u> $\mathbb{CP}^2 \xrightarrow{2:1} S^4$ $\exists n$ over R_-

$$1 = \sigma(\mathbb{CP}^2) = 2\sigma(S^4) - \frac{1}{2} R_- \cdot R_-$$

$$= -\frac{1}{2} R_- \cdot R_-$$

$$\Longrightarrow \quad R_- \cdot R_- = -2$$

<u>ex'</u> $\overline{\mathbb{CP}^2} \xrightarrow{2:1} S^4$ br. over R_+

$$\Longrightarrow \quad R_+ \cdot R_+ = +2.$$