Lecture 7

Plan: talk about g-signatures & their properties; state the g-signature theorem.

+ appl. (of the G-sign. theorem to non-or. surfaces in S^4, Whitehead conjecture)

next two (?) lectures:

proof of the g-signature thm & two more applications.

Context: actions on mflds (by finite groups)

$$G \subset \mathrm{Diff}^{+}(X)$$

$\Downarrow$

g

preserves the orient.

X closed, oriented, smooth 4-mfld

g is a finite-order self-diffeo of X.

$\hookrightarrow$ ord $g = d$.

$H_*(X;\mathbb{C}) \circlearrowleft g_*$

g finite order $\Longrightarrow$ g is diagon/$\mathbb{C}$

& that its eigenvalues are roots of unity.

So $H_*(X;\mathbb{C})$ splits as a direct sum of eigenspaces of g_*.

$$H_k^{g,\eta}(X) = \omega^\eta\text{-eigenspace} \subset H_k(X;\mathbb{C}) \text{ of } g_*$$

sometimes I might drop this.

$\omega = \exp\left(\frac{2\pi i}{d}\right)$

We have a "quadratic form" on

$$H_2(X;\mathbb{C}) \otimes_{\mathbb{C}} H_2(X;\mathbb{C}) \to \mathbb{C}$$

by extending coefficients from $\mathbb{Z}$.

$$H_2(X;\mathbb{Z}) \otimes H_2(X;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

e.g. $g_* : H_2(\underset{A}{S^2} \times \underset{B}{S^2}) \longrightarrow H_2(S^2 \times S^2)$

rep. by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$g_*^2 = \mathrm{id}$, g_* has two eigenspaces:

$g_*(A+B) = A+B \to$ +1-eigenspace

$g_*(A-B) = A-B \to$ −1-eigenspace

$h \in \mathrm{Diff}^+(\underbrace{N\,\mathbb{CP}^2}_{X})$ $\quad e_1, \dots, e_N \in H_2(\ )$ generators

h that permutes the summands.

N-cycle. $\qquad \omega = \exp\left(\frac{2\pi i}{N}\right)$.

$$y = e_1 + \omega e_2 + \dots + \omega^{N-1} e_N \in H_2(X;\mathbb{C})$$

$$h_*(\text{———}) = e_2 + \omega e_3 + \dots + \omega^{N-1} e_1$$

$$= \omega^{-1} \cdot y$$

the "right" way of extending to $\mathbb{C}$

is not bilinearity but sesquilinearity,

$$Q_X(\lambda A, \mu B) = \lambda \bar{\mu}\, Q_X(A, B)$$

$\hookrightarrow$ Hermitian product on $H_2(X; \mathbb{C})$

<u>prop</u> The decomposition $\bigoplus H_2^{p,q}(X)$

is orthogonal wrt Q_X.

<u>proof</u> $g \in \mathrm{Diff}^+(X)$, $\hookrightarrow$ g

preserves the Q_X

If $A \in H_2^{r}(X)$, $B \in H_2^{s}(X)$

$$Q_X(A, B) = Q_X(g_* A, g_* B) =$$

$$= Q_X(\omega^r A, \omega^s B) =$$

$$= \omega^r \cdot \bar{\omega}^s \, Q_X(A, B) =$$

$$= \omega^{r-s} \, Q_X(A, B)$$

$$\Rightarrow \quad Q_X(A,B) = 0 \quad \underline{\text{if}} \quad r \neq s.$$

$\exp\left(\frac{2\pi i}{d}\right)$

<u>def</u> <u>g-signature of X</u>

$$\sigma(g, X) = \sum_{r=0}^{d-1} \omega^r \cdot \operatorname{sign}\left(H_2^{g,r}(X)\right)$$

$\in \mathbb{Z}$

makes sense: $Q_X|_{H_2^{g,r}(X)}$ which

i) a Hermitian form on $H_2^{g,r}(X)$

$\rightsquigarrow$ diagonalizable & has real eigenvalues.

e.g. • If $g = \mathrm{id}$, one eigenspace

all of $H_2(X;\mathbb{C})$

$\rightsquigarrow \sigma(\mathrm{id}, X) = \sigma(X)$.

• (ex in the next ex. sheet)

$$\operatorname{sign}\Big(\underbrace{H_2^{g,0}(X)}_{(\omega^0=1)\text{-eigenspace}}\Big) = \sigma\big(X/\langle g\rangle\big)$$

$\sigma(g, X) =$ supertrace of g_*

$$H_2(X) = \overset{>0}{V^+} \oplus \overset{<0}{V^-}$$

$$\circlearrowright g_* \qquad \circlearrowright g_*$$

$$\sigma(g,X) = \operatorname{tr} g_*|_{V^+} - \operatorname{tr} g_*|_{V^-}.$$

recall: $G \subset \mathrm{Diff}^+(X^4)$ is

a finite subgroup, then

$\mathrm{Fix}(G)$ = coll. of surfaces $\cup$

coll. of points

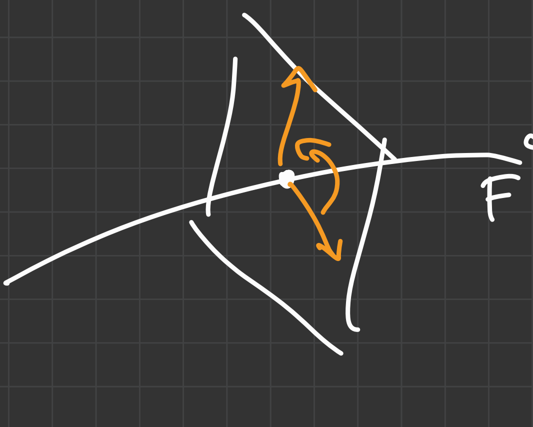
$F \subset \mathrm{Fix}(G)$.

& on each surface, $g \in G$

acts like rotations in the

normal bundle, rot. by ψ_F

$d \cdot \psi = 2\pi k$



$\subset$ fixed set

$g(F) = F$

$g(x) = x \ \forall x \in F$

X on each isolated pt

$p \in Fix(G)$, g acts like

$$\mathbb{C}^2 \longrightarrow \mathbb{C}^2$$
$$(x,y) \longmapsto (\theta_1^p x, \theta_2^p y)$$

$\theta_1^p, \theta_2^p \in S^1$

$\theta_1^d, \theta_2^d = 1 \in S^1$.

(g-signature theorem)

<u>thm</u> $G = \langle g \rangle \cong C_d$

$$\sigma(g,X) = \sum_{F \subset Fix(G)} \left(\operatorname{cosec}^2 \frac{\psi_F}{2}\right) \cdot (F \cdot F) - \sum_{p \in Fix(G)} \cot \frac{\theta_1^p}{2} \cot \frac{\theta_2^p}{2}.$$

$$\operatorname{cosec}\alpha = \frac{1}{\sin\alpha}$$

$$\operatorname{cotg}\alpha = \frac{1}{\operatorname{tg}\alpha} = \frac{\cos\alpha}{\sin\alpha}$$

<u>Rmk</u> What the thm says:

$\sigma(g, X)$ only depends on local data around $\mathrm{Fix}(g)$.

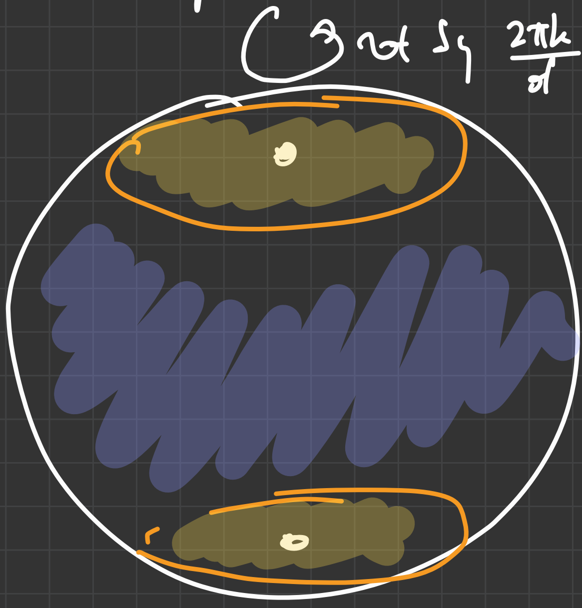
<u>Cor</u> (ex in this ex. sheet)

$X \longrightarrow Y$ d-fold cover, cyclic branched on $F \subset Y$

$$\Rightarrow \sigma(X) - d\sigma(Y) = \frac{d^2-1}{3d} \cdot (F \cdot F).$$

<u>sketch of proof.</u> idea: separate X into $\mathrm{Nbhd}(\mathrm{Fix}(G))$ & the rest.

how you want to prove that "the rest" does not matter, & compute the info on $\mathrm{Nbhd}(\mathrm{Fix}(G))$



& want to prove $\sigma(g, \mathrm{Nbhd}(\mathrm{Fix}))$ is what you need,

$\sigma(g, X \setminus \mathrm{Nbhd}(\mathrm{Fix}))$
"
0

& that $\sigma(g,X) = \sigma(g,N) + \sigma(g,X \setminus N)$

separately: need to compute

$\sigma(g, N)$

surfaces

do something very short: prove it for surfaces w/ $F \cdot F = 0$

& then trade self-intersections for isolated fixed pts.

isolated fixed pts

prove multiplicativity: i.e. need to look at rotations on the disc.

i.e instead of looking at

$$F \subset \mathrm{Fix}(G) \subset X$$

with $F \cdot F = d$

&

replace it with

$$F' \cup pt \subset \mathrm{Fix}(G) \subset X'$$

$$F' \cdot F' = F \cdot F \pm 1.$$

that gives the same contrib.

thm If W^5 compact, oriented smooth 5-mfld, $\partial W^5 = X^4$

$g \in \mathrm{Diff}^+(W)$ $\quad (\Rightarrow g(X) = X)$

Then $\sigma(g, X) = 0$.

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cor If $g \in \mathrm{Diff}^+(X)$, $g \in \mathrm{Diff}^+(-X)$

$$\sigma(g, X) = -\sigma(g, -X).$$

In $\mathbb{CP}^2 \supset F_{\text{sing.}}$, F sing surface

that is topologically embedded,

$\simeq S^2$, but not smoothly em.

$$\{x^p z = y^{p+1}\} \subset \mathbb{CP}^2$$

$\uparrow$ homeo to $S^2 \cong \mathbb{CP}^1$

$$(t^{p+1} : t^p s : s^{p+1})$$

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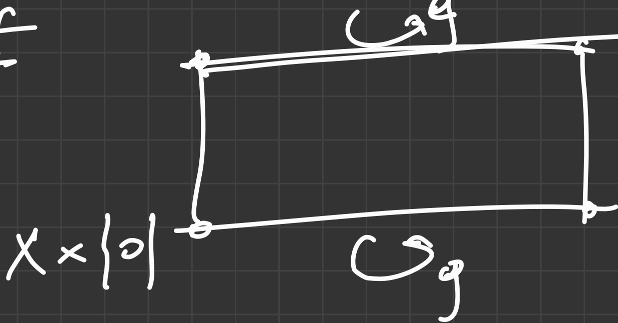
you can define a cyclic $(p+1)$-fold cover of

$X \longrightarrow \mathbb{CP}^2$ branched

over such a F.

Proof



$W = X \times I$

$\tilde{g} \curvearrowright W$

$\tilde{g}(x,t) = (g(x),t)$.

$$\sigma(g, X) + \sigma(g, -X)$$

$$= \sigma(\underbrace{g}_{\tilde{g}|_{\partial W}}, \underbrace{X \sqcup -X}_{\partial W})$$

$$= 0$$

$$H_2(X \sqcup -X) = \underset{\circlearrowleft \atop \tilde{g}}{H_2(X)} \oplus \underset{\circlearrowleft \atop \tilde{g}}{H_2(-X)}$$

$\square$

Ex (If $g = \mathrm{id}$,) if $X = \partial W$

$\Rightarrow \quad \sigma(X) = 0.$

Ex² $\quad \Omega_4 \neq 0$

$$\Omega_4 \xrightarrow{\sigma} \mathbb{Z} \longrightarrow 0$$

$$[\mathbb{CP}^2] \longmapsto 1$$

$$\Omega_4 = [\mathbb{CP}^2] \oplus \boxed{\ker \sigma}.$$

0 !!! ... Rokhlin

$$\underbrace{\mathbb{CP}^2 \# \mathbb{CP}^2}_{\chi = 4} \neq 0 \in \Omega_4.$$

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proof of thm

Look at $g = id_W$, $g|_X = id_X$.

Long ex. sequence of the pair $(W, X)/\mathbb{C}$

$$H_3^\wedge(W) \xrightarrow{A} H_3^\wedge(W,X) \to H_2^\wedge(X) \to H_2^\wedge(W) \xrightarrow{{}^tA} H_2^\wedge(W,X)$$

$$H_3(W,X) \underset{\text{PLD}}{\cong} H^2(W) \underset{\text{UCT}/\mathbb{C}}{\cong} H_2(W)^\vee$$

respect maps

transpose

$$H_3(W) \cong H^2(W,X) \cong H_2(W,X)^\vee$$

$$\Rightarrow 0 \to B \to H_2(X) \to C \to 0$$

B = coker A, C = ker ${}^tA$

same dimension!

$$H_2(X) \cong B \oplus C$$

same dim.

<u>claim:</u> $Q_X\big|_B \equiv 0.$

claim $\Longrightarrow$ thm:

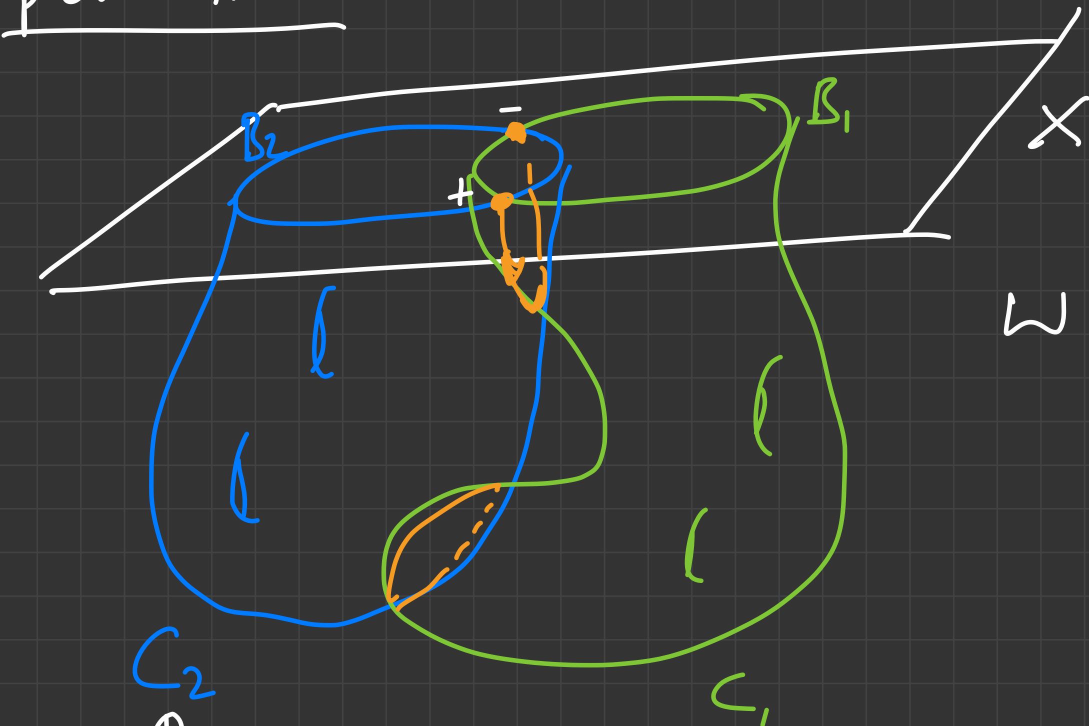
$$\begin{array}{c|cc} & B & C \\ \hline B & 0 & *^{I} \\ C & *^{I} & * \end{array} = M$$

$\downarrow$

repr. Q_X

$$\det M \neq 0 \Longrightarrow \sigma(M) = \sigma(Q_X)$$
$$= \sigma(X) = 0.$$

prove the claim:



$C_2 \in H_3(W,X)$ $\qquad$ $C_1 \in H_3(W,X)$

$C_1 \pitchfork C_2 =$ cpct 1-dim'l subm. of W

↳ intervals in here pair up int. points of $B_1 \cap B_2$.

Sub-claim: $p \in B_1 \cap B_2$ & is

an endpoint of an interval

in $C_1 \cap C_2$

$$\underbrace{\varepsilon(p, \overset{\text{2-dim'l}}{B_1 \cap B_2})}_{\text{in } X} = \underbrace{\varepsilon(p, \overset{\text{1-dim'l}}{I} \cap \overset{\text{4-dim'l}}{X})}_{\text{in } W}$$

$\Rightarrow \forall p \in B_1 \cap B_2 \; \exists$ interval I

$I \subset C_1 \cap C_2$ of which p is an endpoint

$$\sum \varepsilon(p, B_1 \cap B_2) = \sum_{\partial I = p \cup p'} \varepsilon(p) + \varepsilon(p') =$$

$$= \sum_I 0 = 0.$$

$\square$

What we proved $\sigma(X) \Rightarrow$

$\sigma^{g,2}(X) \Rightarrow$ $\forall\, 2$ $\square$

<u>thm</u> (Novikov additivity)

If $X = X_1 \cup X_2$, $g \in \mathrm{Diff}^+(X)$

compact, oriented, smooth 4-mfld

codim-0, submfld,

& $g(X_1) = X_1$, $g(X_2) = X_2$

& $X_1 \cap X_2$ = union of components of $\partial X_1, \partial X_2$

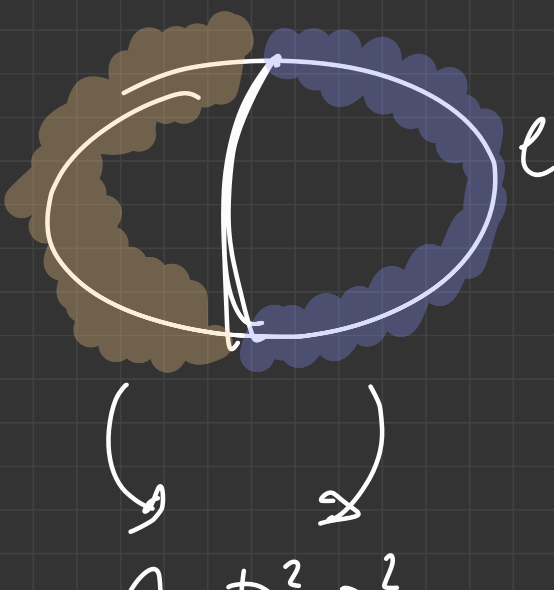
$$\Longrightarrow \sigma(g, X) = \sigma(g, X_1) + \sigma(g, X_2)$$

Rmk: If you glue only partially, the additivity does not hold.

e.g. $\mathbb{C}P^2 \setminus (\text{open ball})$

$\shortparallel$

nbhd of a cx line ℓ

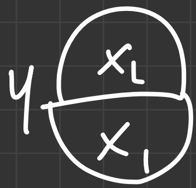


$\rightsquigarrow \sigma = +1$

$\simeq D^2 \times D^2 \longrightarrow \sigma = 0$

proof make two simplifying assumptions

1) $\partial X = \emptyset$



, 2) $g = id$

Mayer-Vietoris: $Y = \partial X_1 = -\partial X_2$

$$H_2(Y) \to H_2(X_1) \oplus H_2(X_2) \to H_2(X) \xrightarrow{\partial} H_1(Y) \to H_1(X_1) \oplus H_1(X_2)$$

$$P_1 \oplus N_1 \oplus Z_1 \qquad P_2 \oplus N_2 \oplus Z_2$$

$Q_{X_1}|_{P_1}$ is positive def.

$Q_{X_1}|_{N_1}$ " neg. "

$Q_{X_1}|_{Z_1}$ " 0.

by def. $\sigma(X_1) = \dim P_1 - \dim N_1$

$\sigma(X_2) = \dim P_2 - \dim N_2$

~~ideal sections:~~ too optim. $H_2(X) = P_1 \oplus P_2 \oplus N_1 \oplus N_2$

& no mix.

$$\underline{\underline{S^2 \times S^2}} = D^2_+ \times S^2 \cup D^2_- \cup S^2$$

$$\sigma = 0 \qquad \sigma = 0$$

$$P = N = 0$$

$$Z = \mathbb{C}$$

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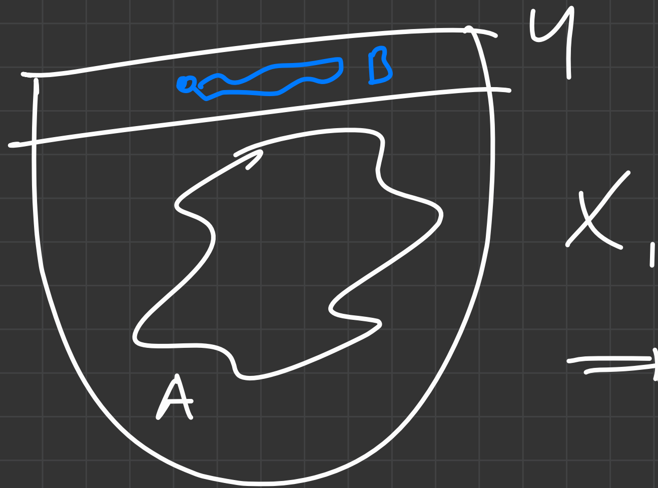
rmk: $\operatorname{Im}\left(H_2(Y) \to H_2(X_1)\right) \subset Z_1$

this is because if $A \subset H_2(X_1)$ is any cycle, then I can make it disjoint from any cycle B coming from Y.



$\Longrightarrow B \in Z_1$

$$P_1 \oplus P_2 \oplus N_1 \oplus N_2 \subset H_2(X)$$

but there is also

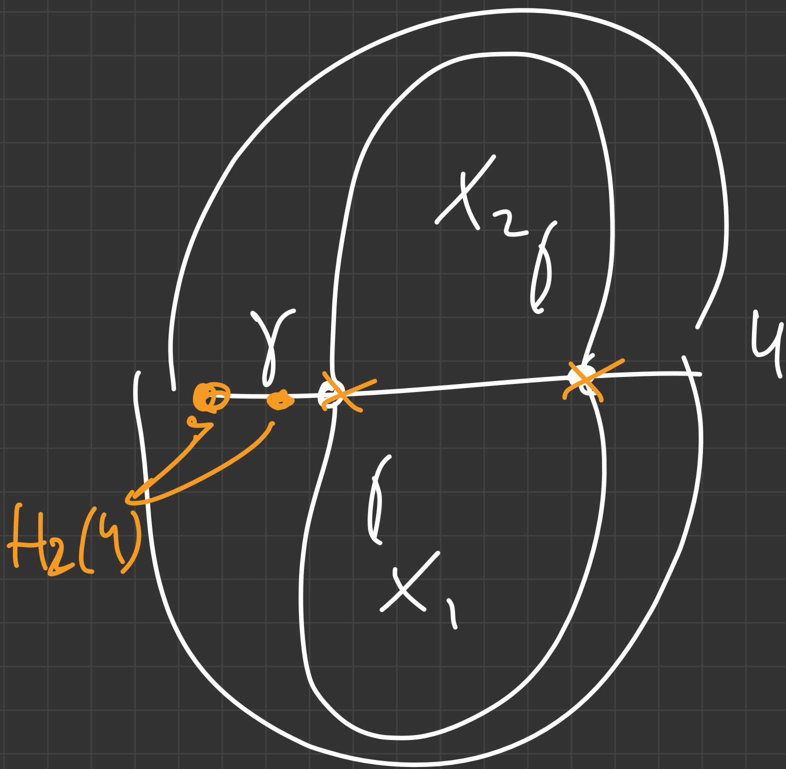
$$\boxed{Z_1 \oplus Z_2 / \mathrm{Im}\, H_2(Y)}$$

$\downarrow$

a null subspace of $H_2(X)$

now there is another subspace of $H_2(X)$, coming from $H_1(Y)$

what is this space



$$\gamma \in \ker \left(H_1(Y) \to H_1(X_1) \oplus H_1(X_2) \right)$$

iff $\gamma = \partial A_1 , \partial A_2$

2-chains in X_1 & X_2.

& $\gamma = \partial \big(\underbrace{[A_1 \cup -A_2]}_{A_\gamma} \big)$

pick a basis $\gamma_1, \dots, \gamma_m$

for $\ker\left(H_1(Y) \to H_1(X_1) \oplus H_1(X_2)\right)$

& a dual set $B_1, \dots, B_n \subset H_2(Y)$

such that $B_i \cdot \gamma_j = \delta_{ij}$

$$\begin{array}{ccc} B_i & \cdot & \gamma_j \\ \cap & & \cap \\ H_2(Y) & & H_1(Y) \end{array}$$

in Y

<u>claim</u>: $\{A_{\gamma_i}\}$ & $\{B_i\}$

are dual to each other:

in $H_2(X)$, $A_i \cdot B_j = \delta_{ij}$

$$H_2(X) = \begin{array}{c|c}
 & \begin{array}{cccc} P_1 & N_1 & P_2 & N_2 \end{array} \\
\begin{array}{c} P_1 \\ N_1 \\ P_2 \\ N_2 \\ A_{\gamma_i} \\ B_i \end{array} &
\left(\begin{array}{c|c|c|c}
\begin{matrix} +1 & & 0 & \\ & -1 & & \\ 0 & & +1 & \\ & & & -1 \end{matrix} & 0 & 0 & \\
\hline
0 & * & I & \phi \\
\hline
0 & I & 0 & 0 \\
\hline
 & 0 & 0 & 0
\end{array}\right)
\end{array}$$

$$\rightsquigarrow \quad \sigma(X) = \sigma(X_1) + \sigma(X_2)$$

not: "annoying eigenvalue by eigenvalue"

$$\sigma^{g,2}(X) = \sigma^{g,2}(X_1) + \sigma^{g,2}(X_2)$$

$$\Rightarrow \sigma(g,X) = \sigma(g,X_1) + \sigma(g,X_2).$$

Fun part of the day:

thm (Massey–Whitehead thm)

$F \subset S^4$ is a non-orientable surface, then

$$F \cdot F \in \{-2h, -2h+4, \dots, 2h-4, 2h\}$$

$$h := h(F) = b_1(F; \mathbb{F}_2)$$

$$\rightsquigarrow \quad F = \#^h \mathbb{RP}^2$$

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rmk: $R_\pm \subset S^4$, embedded $\mathbb{RP}^2$ with $R_\pm \cdot R_\pm = \pm 2$ $(h=1)$

$$F_{a,b} := \underbrace{R_+ \# \dots \# R_+}_{a} \# \underbrace{R_- \# \dots \# R_-}_{b} \subset S^4$$

$$F_{a,b} \cdot F_{a,b} = 2a - 2b$$

$$h = a + b$$

$$F_{a,b} \cdot F_{a,b} = 2h - 4b$$

$\rightsquigarrow$ recover all possible values.

proof $H_1(S^4 \setminus F) = \mathbb{Z}/2\mathbb{Z}$,

gen. by the meridian of F.

$\Rightarrow$ can take the double cover of S^4 branched over F.

$\rightsquigarrow$ we obtain a 4-mfld X_F

$$\chi(X_F) = 2\chi(S^4) - \chi(F)$$
$$= 4 - (2-h) = 2+h$$

claim $H_1(X_F) =$ finite.

(follows from $H_1(S^4 \setminus F)$ is cyclic & that 2 is a prime power)

$$\Rightarrow \quad \chi(X_F) = 1 - 0 + b_2(X_F) - 0 + 1$$

$$\chi(X_F) = 2 + h \Rightarrow b_2(X_F) = h.$$

G-sign. $\Rightarrow$

$$\sigma(X_F) = 2\sigma(S^4) - \frac{1}{2} F \cdot F =$$

$$= -\frac{1}{2} F \cdot F$$

<u>key obs:</u> $|\sigma| \leq b_2$

$$\Downarrow$$

$$\left| -\frac{1}{2} F \cdot F \right| \leq h$$

$$\Rightarrow |F \cdot F| \leq 2h$$

Key obs II: If q non-deg,

$$\sigma(q) \equiv \operatorname{rk} q \pmod 2$$

$$\#1s - \#(-1)s = \left(\#1s + \#(-1)s\right) - \left(2\#(-1)s\right)$$

$$\Rightarrow -\frac{1}{2}(F \cdot F) \equiv h \pmod 2$$

$$\Rightarrow F \cdot F \equiv 2h \pmod 4 \quad \square$$