Lecture 8

Plan:
- start proving the g-signature theorem
- give another application: min. genus problem in $\mathbb{CP}^2$.

thm (g-signature)
If $g \in \mathrm{Diff}^+(X)$, X closed, oriented, smooth 4-manifold, g of finite order, then

angles of rotation of g

$$\sigma(g, X) = \sum_F (F \cdot F) \frac{1}{\sin^2 \psi_F} - \sum_P \cot \theta_1^P \cot \theta_2^P$$

$F \subset \mathrm{Fix}(g)$ 2-dim^l cpts

$P \in \mathrm{Fix}(g)$ isolated pts

six steps to $\square$.

- prove that if g acts freely, $\sigma(g,X)=0$
  - → we will assume that g acts semi-freely
- we try to reduce to the case when g acts freely, by surgery along $\mathrm{Fix}(g)$

If When can we do these surgeries?

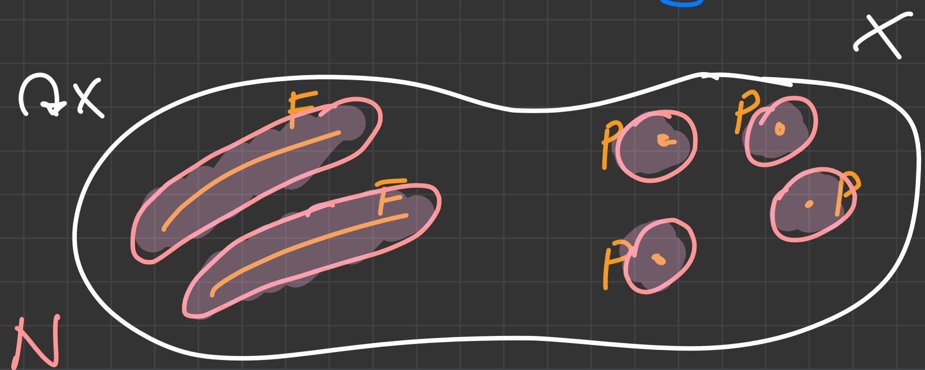
- we can do them when $\mathrm{Fix}(g)$ only contains isolated points (essentially, <u>if</u> we can do it for actions on surfaces)
- we can do it when $F\cdot F=0$ $\forall F$ 2-dim^l component of $\mathrm{Fix}(g)$ (& F is orientable)
- reduce to the case when $F\cdot F=0$ $\forall F$
- compute the local contributions given by

the surgeries.

semi-free: $\mathrm{Fix}(g) = \mathrm{Fix}(g^k)$

$\forall\ k = 1, \ldots, \underbrace{\mathrm{ord}\, g}_{d := \mathrm{ord}\, g} - 1$

$g \curvearrowright X$

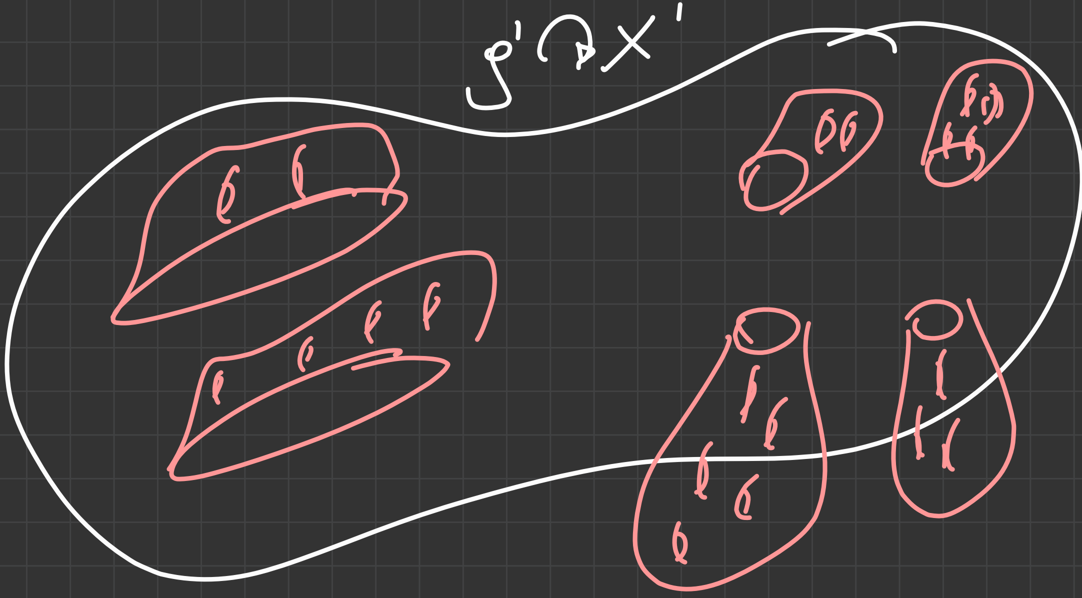


g acts $\underline{\underline{\text{freely}}}$ on $X \setminus \mathrm{Fix}(g)$

$$X \rightsquigarrow (X \setminus N) \cup N' =: X'$$

another manifold w/ $\partial N' = \partial N$

on which g extends <u>freely</u>.



essentially 2D.

$$\sigma(g', X') = 0$$

$$\shortparallel$$

$$\sigma(g, X \setminus N) + \sigma(g', N')$$

$$\shortparallel$$

$$\sigma(g, X) - \sigma(g, N) + \sigma(g', N')$$

Back to surfaces & actions on them.

Need to extend → def. of signatures to surfaces
→ def. of signatures to G-objects.

<u>def</u> A G-object is a formal linear combination $\lambda_1 X_1 + \dots + \lambda_n X_n \,/\, \mathbb{Q}$ of manifolds (surfaces/4-mflds) all of the same dimension, each coming with an action by a finite group G.

(from now on: $G := \mathbb{Z}/2\mathbb{Z}$).

if $G \circlearrowright X$ then we have

$$nX = \underbrace{X \sqcup X \sqcup .. \sqcup X}_{n \text{ times}} \quad \text{if } n>0$$

$$\underbrace{\bar{X} \sqcup -- \sqcup \bar{X}}_{-n \text{ times}} \quad \text{if } n<0.$$

where $\bar{X}$ is X with the or. reversed.

& we extend signature & g-signature linearly to G-objects.

$$\underset{\substack{\downarrow \\ g\text{-object}}}{\sigma}\Big(\sum \lambda_i X_i\Big) \overset{\text{def}}{:=} \sum \lambda_i\, \sigma(g, X_i)$$

If F is a surface

$Q_F^{\mathbb{Z}} : H_1(F) \otimes H_1(F) \longrightarrow \mathbb{Z}$

the intersection form; which is

$\mathbb{Z}$-bilinear, but anti-symmetric.

Go to $Q_F : H_1(F;\mathbb{C}) \otimes H_1(F;\mathbb{C})$

$$Q_F(x, y) := i\, Q_F^{\mathbb{Z}}(x, \bar{y})$$

$\hookrightarrow$ this is a Hermitian form on

$H_1(F;\mathbb{C})$, and as such it has

a well-defined signature

(if F is closed & oriented, then

Q_F is also non-degenerate)

check that Q_F is Hermitian.

$x \in H_1(F;\mathbb{C})$, $y \in H_1(F,\mathbb{C})$.

we can suppose that

$x = \lambda x_0$, where $\lambda \in \mathbb{C}$, $x_0 \in H_1(F;\mathbb{Z})$

$y = \mu y_0$, where $\mu \in \mathbb{C}$, $y_0 \in H_1(F;\mathbb{Z})$

want to verify that Q_F is

$\underbrace{\text{sesquilinear}}_{\text{immediate from the definition}}$ & $\underbrace{\text{symmetric}}_{Q_F(x,y) = \overline{Q_F(y,x)}}$

$$Q_F(y,x) = Q_F(\mu y_0, \lambda x_0) =$$
$$\mu\bar{\lambda}\, Q_F^{\mathbb{Z}}(y_0,x_0) = \mu\bar{\lambda}\,(i\, y_0 \cdot x_0)$$

$$= \mu\bar{\lambda}\left(i \, (x_0 \cdot y_0)\cdot(-1)\right) =$$

$$= \mu\bar{\lambda} \cdot \bar{i} \, (x_0 \cdot y_0) =$$

$$= \overline{\bar{\mu}\lambda \cdot i \cdot (x_0 \cdot y_0)} = \overline{i \cdot Q_F^{\mathbb{Z}}(\lambda x_0, \mu y_0)} =$$

$$= \overline{Q_F(x,y)} \qquad \square$$

<u>ex</u> $\sigma(\mathrm{id}_F, F) = 0.$

closed surface

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the proofs of the vanishing theorem & Novikov's additivity go through in exactly the same way for this "twisted" signature

$\Longrightarrow$ • If $F = \partial M^3$, $g \circlearrowright M$

compact, oriented 3-mfld,

$$\Longrightarrow \boxed{\sigma(g,F) = 0}$$

def As in the 4-d case:

$g \circlearrowright F \Longrightarrow$ eigenspace dec.

$$H_1(F;\mathbb{C}) = \bigoplus H_1^{g,s}(F)$$

$$\& \quad \sigma(g,F) = \sum \omega^s \cdot \underline{\sigma^{g,s}(F)}$$

sign. of Q_F restricted to $H_1^{g,s}(F)$.

• If g acts on F, g preserves

$F_1, F_2 \subset F$, $\mathring{F}_1 \cap \mathring{F}_2 = \phi$,

$F_1 \cup F_2 = F$

then $\sigma(g|_{F_1}, F_1) + \sigma(g|_{F_2}, F_2) = \sigma(g, F)$.

New thing: $\overset{g_1 \circlearrowright}{F_1}, \overset{g_2 \circlearrowright}{F_2}$ surfaces

$\underbrace{g_1 \times g_2}_{g} \circlearrowright \underbrace{F_1 \times F_2}_{X}$

then $\sigma(g, X) = \sigma(g_1, F_1) \cdot \sigma(g_2, F_2)$

morally speaking:

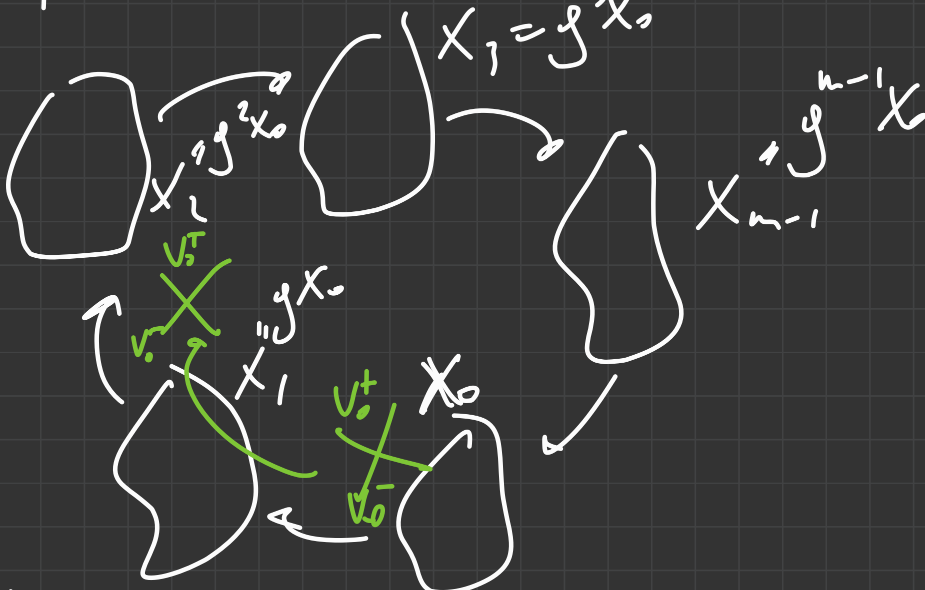
$(H_2(X), Q_X) = (H_1(F_1), Q_{F_1}) \otimes (H_1(F_2), Q_{F_2})$

Prop If $g \circlearrowright X$ (of finite order) without fixing any component of X, then $\sigma(g,X)=0$.

Proof It's enough to look at an orbit of g.

(b/c $X = \sum$ orbits of g)

& by additivity



$\Rightarrow$ all of these are diffeo.

suppose that X is $2d$-dimensional

find a g-inv. dec. of

$V^+ \oplus V^- = H_d(X;\mathbb{C})$ into a positive def.

& neg. definite part, &

take $\operatorname{tr} g|_{V^+} - \operatorname{tr} g|_{V^-}$

e.g. $V_0^{\pm} = \pm$ part of $H_d(X_0)$

$$V^{\pm} = \bigoplus g_*^k V_0^{\pm} \qquad g_* : V_0^+ \to V_1^+$$

$$\Rightarrow \quad g_*|_{V^+} \qquad \begin{array}{c|c|c} 0 & * & \\ \hline & 0 & * \\ \hline * & & 0 \end{array}$$

$\Rightarrow \operatorname{tr} g_*|_{V^{\pm}} = 0. \Rightarrow \sigma(g, X) = 0.$ $\square$ (almost)

We want to understand signatures, of actions on surfaces, & their relationship to 4-manifolds.

If $p \in X^4$ is an isolated fixed point:

we have coordinates $\mathbb{C} \times \mathbb{C}$ around p

such that $g \curvearrowright \mathbb{C} \times \mathbb{C}$

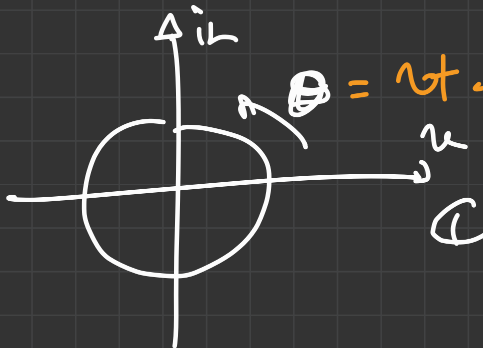
$$g \cdot (x, y) = (\omega^a x, \omega^b y)$$

for some complex numbers $\omega^a, \omega^b \in S^1$.

To excise this nbhd of p, we want to find a 2-d model that replaces the action on each component $\mathbb{C} \times \{0\}$ & $\{0\} \times \mathbb{C}$

by something free.

So the first thing that we want to do



θ = rot by a multiple of $\frac{2\pi}{d}$

$\mathbb{Z}/d\mathbb{Z}$

"G-object"

want to find a surface $\Sigma(\theta)$

s.t. $\partial\Sigma(\theta) = S^1$ $\exists\, g_\theta \curvearrowright \Sigma(\theta)$

that is free, $g_\theta|_{\partial\Sigma(\theta)}$ = rot. by

θ on S^1.

Idea: to the circle S^1, with rotation by θ, we can allocate a homology class γ in $H_1(L(d,1))$.

$\rightsquigarrow$ similar to what we had seen:

$$H^1(X;\mathbb{Z}) \longleftrightarrow [X, S^1]$$

S^1/θ, since

important props:

$\pi_1(S^1) = \mathbb{Z}$

& $\pi_k(S^1) = 0 \;\; \forall k > 1$

replace w/ $\mathbb{Z}/d\mathbb{Z}$

$S^1 \to S^1/\theta$ is a cover w/ deck transf. $\mathbb{Z}/d\mathbb{Z}$

$\rightsquigarrow H^1(S^1/\theta; \mathbb{Z}/d\mathbb{Z})$

Since $H_1(L(d,1)) = \mathbb{Z}/d\mathbb{Z}$

$\rightsquigarrow$ d copies of the hom class γ

vanish in $H_1(L(d,1))$

$\Rightarrow$ can be repr by a null-hom.

link w/ d components (each

in the hom. class $[\gamma]$) L

$L = \partial F$ & you can take

$\hookrightarrow$ compact oriented surface

(Seifert surface).

$$G \curvearrowright S^3 \xrightarrow{p} L(d,1)$$

$\tilde{F}/G = F$

$p^{-1}(F) = \tilde{F} \subset \tilde{F}$ & the action of

G on $\partial\hat{F}$ is by

rotation by θ.

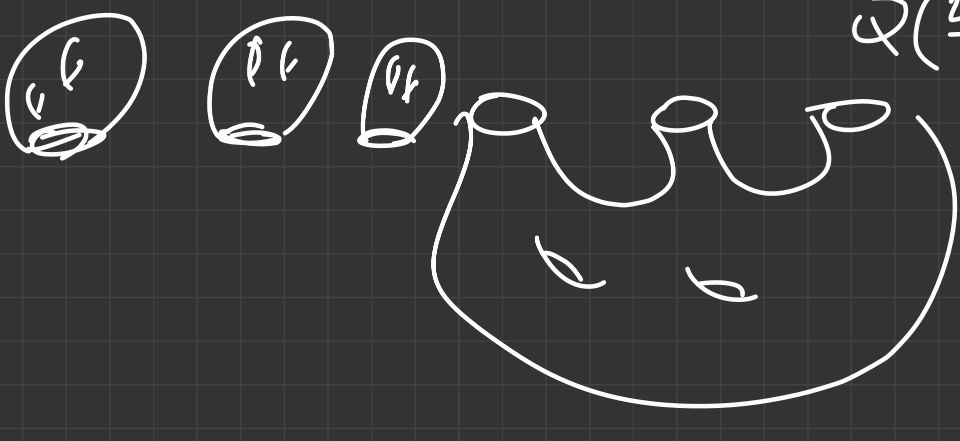
<u>key point</u>: If $g \curvearrowright F$ with one fixed point, p

Want to compute $\sigma(g,F)$

remove a nbhd of the fixed pt

$d(F \setminus N(p)) \cup d(Q(\theta))$ = F'

$Q\left(\frac{2\pi}{3}\right)$



If g acts freely, $\sigma(g, X) = 0$.

$$\sigma(g, F') = 0$$

$$\shortparallel$$

$$\sigma(g, F \setminus N(p)) + \sigma(g, Q(\theta))$$

$$\shortparallel$$

$$\sigma(g, F) - \underbrace{\overset{=0}{\cancel{\sigma(g, N(p))}}}_{\text{easy:}} + \sigma(g, Q(\theta))$$

$N(p) \cong B^2 \Rightarrow$ has no H_2

$\Rightarrow \sigma$ of a Herm. form

on a 0-dim'l space. $= 0$.

$$\Rightarrow \sigma(g, F) = -\sigma(g, Q(\theta))$$

⇝ what we proved: if the action is

semi-free ⇒

$$\sigma(g,F) = -\sum_{P \in \mathrm{Fix}(g)} \sigma(g, Q(P))$$

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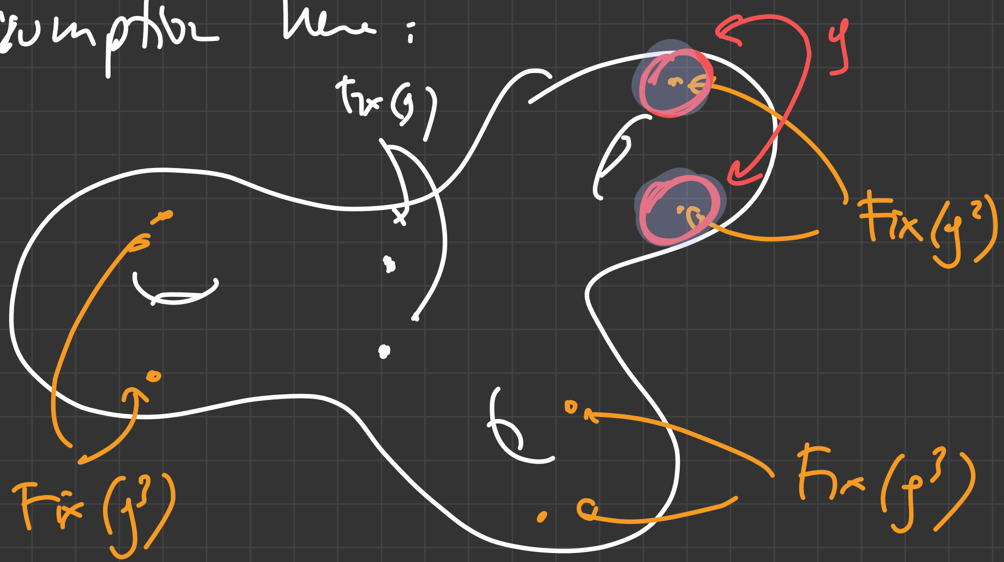
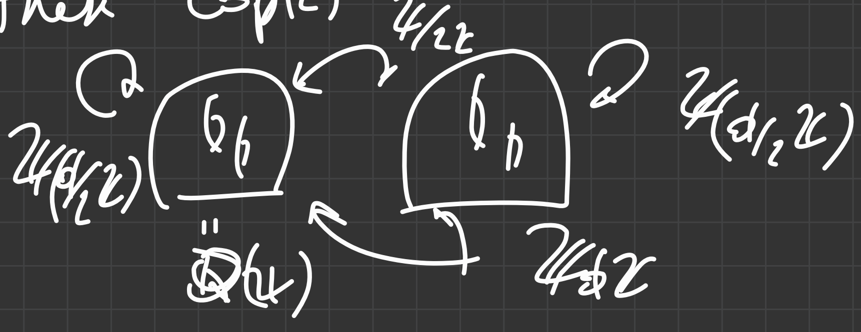
we can at least drop the semi-free

assumption here:



we want to replace neighbourhoods of $\mathrm{Fix}(g^2) \setminus \mathrm{Fix}(g)$ by G-objects that are free.

These pts are paired up by g, & g^2 acts by rotation by an angle ψ: what we do is replace $N(\text{pts})$ by 2 copies of $Q(\psi)$ & let g permute these copies



$(Q(\theta), g_\theta)_1 \amalg (Q(\theta), g_\theta)_2$

↳ the one we constructed

let's define $g \curvearrowright Q(\theta)$

$g \cdot x_1 = x_2$

$x_1 \in$ first copy

the same el't in the second copy

$g \cdot x_2 = (g_\theta \cdot x_1)$

$\sigma(g, Q(\theta) \amalg Q(\theta)) = 0.$

we proved: free & semi-free on "the same" from the g-signature perspective.

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by using the local model, $\alpha(\theta)$ we can work on 4-manifolds.

If F has self-int. $\Rightarrow$, orientable.

$F \subset \mathrm{Fix}(g)$,

then $N(F) = F \times D^2$

& the action of g is by rotation by ψ_F in the D^2-direction.

Now we replace $F \times D^2$ by

$$F \times Q(\psi)$$

on which g acts trivially on the F-component & by rot. by g_ψ in the $Q(\psi)$-component.

& by mult. of the signature,

$$\sigma(g, F \times Q(\psi)) =$$

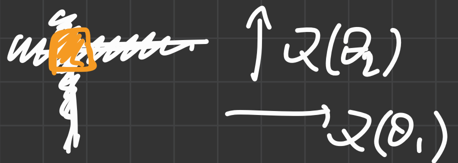
$$\sigma(g, F) \cdot \sigma(g_\psi, Q(\psi))$$

$$\sigma(g, F) = \sigma(\mathrm{id}_F, F) = 0$$

$$\Rightarrow \sigma(g, F \times Q(\psi)) = 0.$$

The isolated fixed points are replaced not by $Q(\Theta^1) \times Q(\Theta^2)$, but by

$$\left(Q(\Theta^1) \cup D^2\right) \times \left(Q(\Theta^2) \cup D^2\right) \Big/ \boxed{D^2 \times D^2}$$



Getting rid of self-intersections

trick: idea: if F is a surface of self-intersection $F \cdot F$ in X^4, then by blowing up I can lower the self int. of F.

$F \to$ by blow-up $\# \overline{\mathbb{C}P^2}$

$\rightsquigarrow$ by anti-holo. blow-up $\# \mathbb{C}P^2$

$$\begin{tikzcd} F \arrow[r] \arrow[d, phantom, "\cap"] & F \# E \arrow[d, phantom, "\cap"] \\ X \arrow[r] & X \# \overline{\mathbb{CP}}^2 \end{tikzcd}$$

we need to define an action on $\overline{\mathbb{CP}}^2$ or $\mathbb{CP}^2$ that fixes E/H & has a given angle of rotation around E/H, so that we can glue F & E/H equivariantly.

On $\mathbb{CP}^2$, consider the following action

by $\mathbb{Z}/d\mathbb{Z}$, generated by $\bar{g}$

$$\bar{g}^k \cdot (x : y : z) = (\omega^k x : \omega^k y : z)$$

where $\omega = \exp\left(\frac{2\pi i}{d}\right)$.

$$\underset{\substack{\| \\ P}}{\mathrm{Fix}(\bar{g}) = \{(0:0:1)\}} \cup \underset{\substack{\| \\ H}}{\{z=0\}}$$

claim: the angles of rotation of

$\bar{g}$ at P : (ω^k, ω^k)

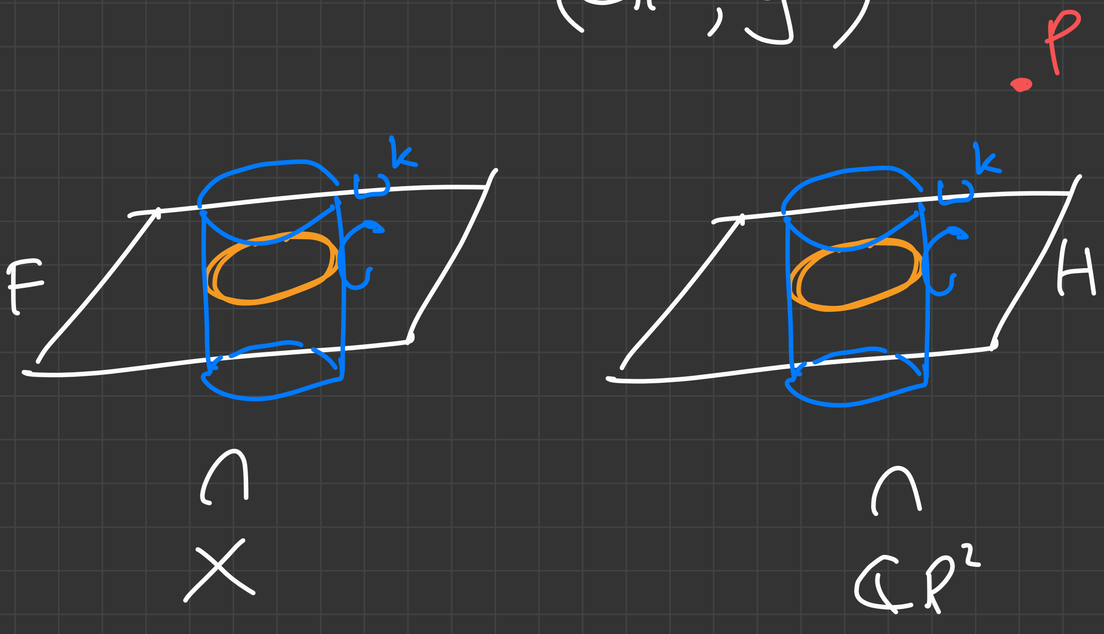
and at H : ω^k

ω

Now, if $F \subset X$ is a fixed comp. of $g \curvearrowright X$, with angle of rotation

$$\psi_F = \exp\left(\frac{2\pi k i}{d}\right)$$

$\rightsquigarrow$ can do an equiv. connected sum with $(\mathbb{CP}^2, \overline{g}^k)$



get a action on $X \# \mathbb{CP}^2$, fixing $F \# H$

& w/ the same angle of rotation.

$\rightsquigarrow$ $\mathrm{Fix}(g') = \mathrm{Fix}(g) \cup \underline{\underline{P}}$

replace F by $F \# H$

What's the upshot of all of this?

we traded one self-int of F with an isolated fixed point

What happens to the g-signature?

claim: $\sigma(g', X \# \mathbb{CP}^2) =$

$= \sigma(g, X) + 1$

(if $\sigma(g', X \# \overline{\mathbb{CP}^2}) = \sigma(g, X) - 1$)

this is b/c by additivity

$$\sigma(g', X \# \mathbb{C}P^2) =$$

$$\sigma(g', X \setminus B^4) + \sigma(\bar{g}^k, \mathbb{C}P^2 \setminus B^4) =$$

$$= \sigma(g, X) + \sigma(\bar{g}^k, \mathbb{C}P^2)$$

$$\bar{g}_* = \mathrm{id}_{H_2(\mathbb{C}P^2)} \Longrightarrow \mathrm{tr}\, \bar{g}_* = 1$$

$$\Longrightarrow \sigma(\bar{g}^k, \mathbb{C}P^2) = \sigma(\mathbb{C}P^2)$$

All in all:

$g \curvearrowright X$

$F \subset X$ of self-int $F \cdot F = 1$

repl. X with $X' = X \# \overline{\mathbb{C}P^2}$

$\quad\quad\quad\quad\quad\quad\quad\quad\cup$

$\quad\quad\quad\quad\quad\quad\quad\quad F'$ of self-int 0

$$\sigma(g, X) = \sigma(g, X') + 1$$

→ we assume that all surfaces have self-int 0, and this will get us

$$\sigma(g, X) = \sigma(g', X') + \sum_{F \subset Fix(g)} (F \cdot F)$$

only surfaces of self-int 0 & isolated fixed pts.

coming from X (θ_p^1, θ_p^2)

coming from $\mathbb{CP}^2$ & $\overline{\mathbb{CP}}^1$ (ψ_F, ψ_F)

$$\sigma(g', X') = -\sum \cot \theta_p^1 \cot \theta_p^2$$

$$\pm \sum \cot^2 \psi_F$$

$$\sigma(g, X) = -\sum \cot \theta_p^1 \cot \theta_p^2 + \sum \underbrace{(1 + \cot^2 \psi_F)}_{\frac{1}{\sin^2 \psi_F}} \cdot (F \cdot F).$$

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conclude w/ an application

showing that $g(dh) \simeq d^2$,

where $h \in H_2(\mathbb{CP}^2)$ is the hyperplane class, i.e. hom. class of a cx line, $d \in \mathbb{Z}$.

let's start w/ $d=3$.

Suppose that $F \subset \mathbb{CP}^2$ is a genus-g surface in the hom. class $3h$.

We want to estimate g.

3 is div. by 3, so $[F]=3[h]$

$\Longrightarrow$ $\exists$ 3-fold cover of $\mathbb{CP}^2$, branched on F.

We can compute $H_1(\mathbb{CP}^2 \setminus F) = \mathbb{Z}/3\mathbb{Z}$

$\Longrightarrow$ • $\left| H_1(\Sigma_3(\mathbb{CP}^2, F)) \right| < \infty$

• $\sigma(X) = 3\sigma(\mathbb{CP}^2) - \dfrac{3^2-1}{\cancel{3\cdot 3}} \cdot \cancel{(F\cdot F)}$

$= 3\cdot 1 - 8 = -5$

- What you proved in the exercise:

$$\chi(X) = 3\chi(\mathbb{CP}^1) - 2\chi(F)$$

$$= 9 - 4 + 4g = 5 + 4g$$

$b_1(X) = 0$, $b_3(X) = 0$ $\Rightarrow$ $b_2(X) = 3 + 4g$.

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claim: $|\sigma(\tilde{X})| \leq b_2(X)$

$$|b_2^+ - b_2^-| \leq |b_2^+| + |b_2^-|$$

$$\Rightarrow \quad +5 \leq 3 + 4g$$

$$\Rightarrow g \geq 1$$

In general; given d, fix the smallest prime $q \mid d$.

& Consider the $X = \Sigma_q(\mathbb{CP}^2, F)$

$$\sigma(X) = q - \frac{q^2-1}{3q} \cdot d^2$$

$$= q - \frac{q^2-1}{3} \cdot d \cdot \frac{d}{q}$$

$$b_2(X) = 3q - (q-1)(2-2g)$$

$$|\sigma(X)| \leq b_2(X)$$

$$d^2 \cdot \frac{q}{3} \leq 2q \cdot g$$

ii $|\sigma(g,x)| \geq |\sigma(x)|$