Lecture 9

We're halfway through the proof of the g-signature theorem.

$g \in \mathrm{Diff}^+(X)$, X closed, oriented, smooth 4-manifold

$$\sigma(g,X) = \sum (F_i \cdot F_i)\operatorname{cosec}^2 \psi_i - \sum \cot\frac{\theta_p^1}{2}\cot\frac{\theta_p^2}{2}$$

we have proven: that we can reduce to the free case by first reducing to $F_i \cdot F_i = 0 \ \forall i$ & then by surgering out $N(\mathrm{Fix}(g))$ & replacing it w/ a free object.

In the ex: we have computed the cost of this replacement.

What we need to do:

- to take sum of non-orientable cpts in $\mathrm{Fix}(g)$

Next here
- showing that if $\mathrm{Fix}(g^k) = \emptyset$, $\forall k$, then $\sigma(g, X) = 0$.

Application: non-existence of Fano plane in $\mathbb{C}P^2$. } more of an appl. of BC

In the ex: another application: towards the diffeo class. of $\{L(p,q)\}$

Taking care of non-orientable surfaces.

What is non-orientability?

F is non-or. $\iff$ $F \supset$ Möbius band.

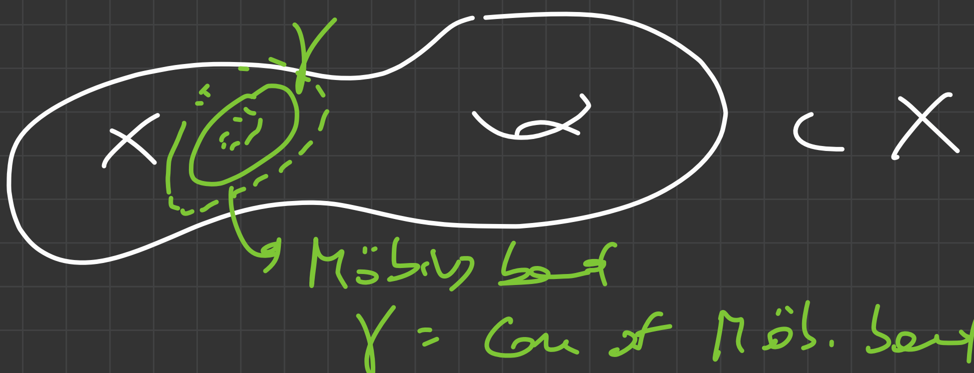
We can find $\boxed{2-\chi(F)}$ Möbius bands

in F s.t. $F \setminus \cup(\text{Möb. bands}) = \text{disc}$.

Won't We can be economical, and bring

this # down to 1 or 2

(depending on $\chi(F)$ mod 2).



Idea: remove a 4-dim'l nbhd of of the Möbius band, and replace it w/ something orientable instead.

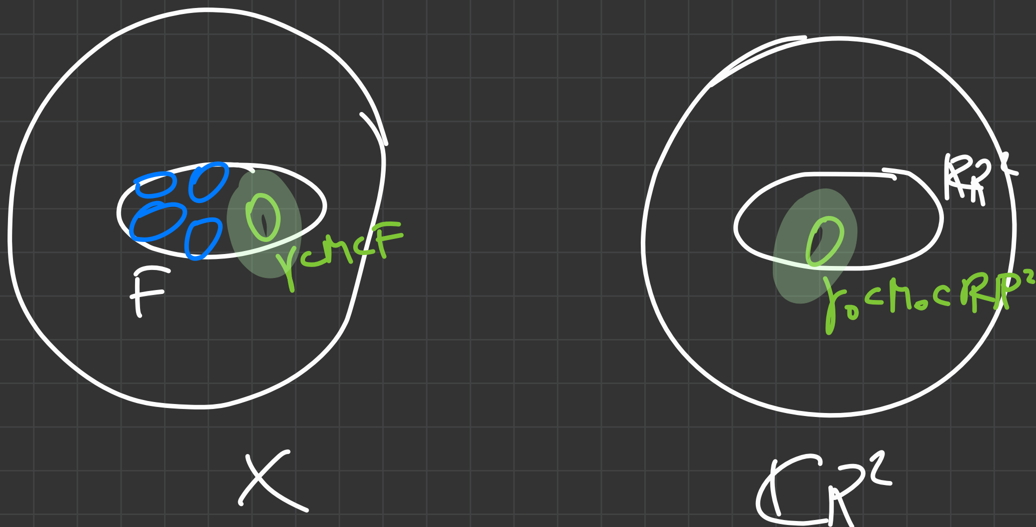
$\gamma \subset F \subset X$

$N_\gamma = \text{nbhd}(\gamma) \subset X$ equivariant

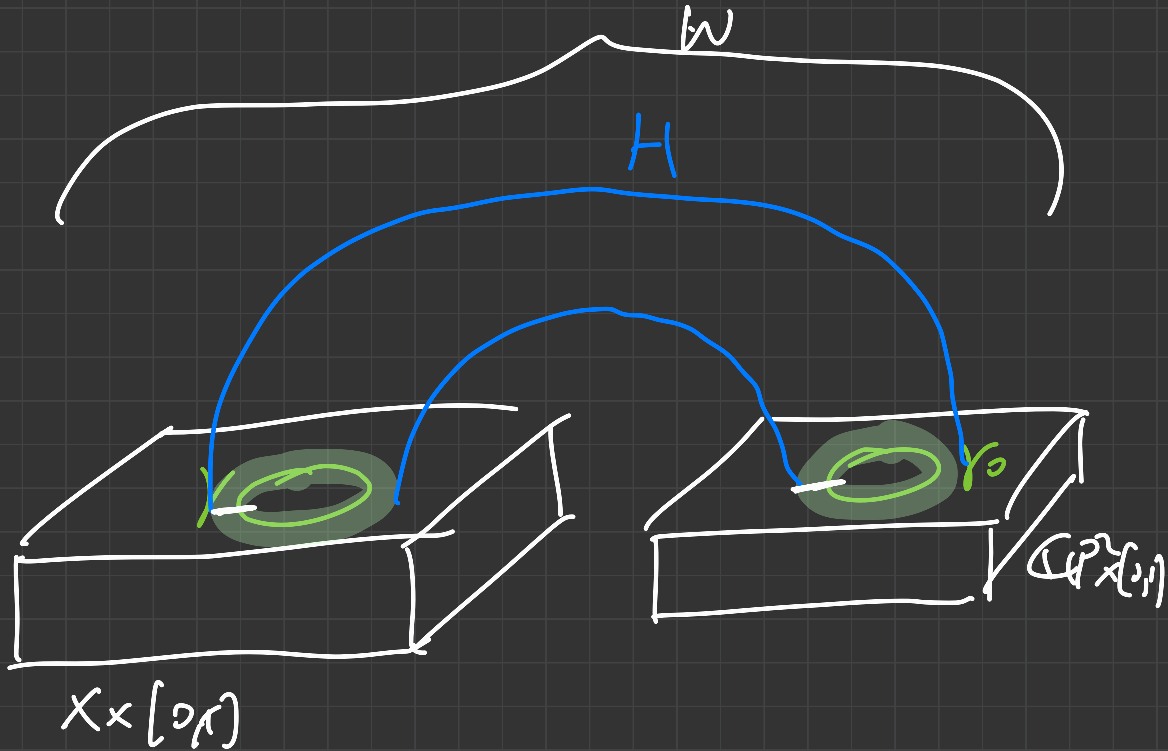
$X \setminus N_\gamma \cup \underbrace{\text{something else.}}_{\text{need to have the same } \partial \text{ as } N_\gamma, \text{ and an action that extends the given action on } \partial N_\gamma}$

The "something else" is a fixed object:

$$\mathbb{CP}^2 \supset \mathbb{RP}^2 = \mathrm{Fix}(\mathrm{conj})$$
$$\cup$$
$$\text{Möbius}$$



Need to
- check actions extend ✓
- compute the self-int of the new surface
- new g-signature ✓



$H = S^1 \times D^3 \times D^1$, round 5-dim'l 2-handle.

→ check that action extends

Constructed a 5-manifold $W \circlearrowleft g$

s.t. $\partial^- W = X \cup \mathbb{C}P^2 \circlearrowleft g$

$\partial^+ W = X'$ new 4-manifold $\circlearrowleft g$

$$\sigma(g, X') = \sigma(g, X) + \sigma(\mathrm{conj}, \mathbb{C}P^2)$$

$$\sigma(\mathrm{conj}, \mathbb{C}P^2) = -1$$

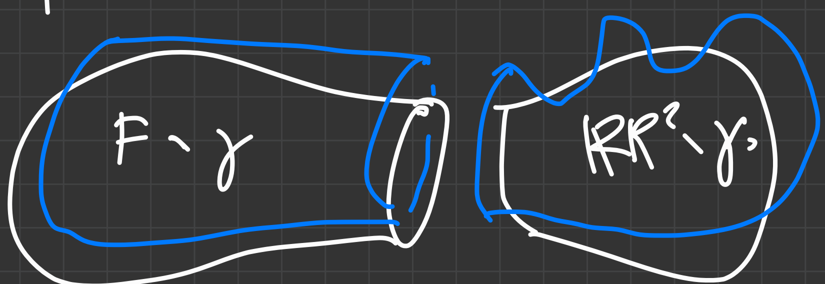
b/c $H_2(\mathbb{C}P^1) = (-1)$-eigenspace of $(\mathrm{conj})_*$.

<u>claim</u> $F' \cdot F' = F \cdot F - 1$

F' = new surface $\subset \mathrm{Fix}(g) \subset X'$

that is the "replacement" of F

<u>proof</u>:



self-int.: $\mathbb{RP}^2 \cdot \mathbb{RP}^2 = -1$

$\cap$

$\mathbb{CP}^2$

proof. $T\mathbb{RP}^2$ is "totally real"

im. unit ← $i(T\mathbb{RP}^2) \cap T\mathbb{RP}^2 = \{0\}$.

i induces an or. reversed isomorphism from

$T\mathbb{RP}^2$ to $N(\mathbb{RP}^2 \subset \mathbb{CP}^2)$

$\Rightarrow \mathbb{RP}^2 \cdot \mathbb{RP}^2 = -\chi(\mathbb{RP}^2)$

counting 0's of sections of $N(\mathbb{RP}^2) \subset \mathbb{CP}^2$

counting 0's of the tgt bundle

v, w real vectors in $\mathbb{C}^2$
$\quad \shortparallel \qquad\qquad \shortparallel$
$(1,0)$, $(0,1)$

(v, w, iv, iw) is a distinguished basis of $\mathbb{C}^2$

$$\big((1,0), (i,0), (0,1), (0,i)\big)$$

Back to $\Omega_j = 0$

In the course of that proof:

$$H^1(M; \mathbb{Z}) \longleftrightarrow [M, S^1]$$

$\{M \to S^1\}$ up to homotopy

M CW complex.

today: replace $\mathbb{Z}$ by $\mathbb{Z}/d\mathbb{Z}$

& S^1 by the corresponding

space $L^\infty(d)$

Why? because if we have a free

action $g \curvearrowright X$, then we have

a free action $G := \mathbb{Z}/d\mathbb{Z} \curvearrowright X$

so we have a covering map

$\varphi : X \longrightarrow X/G$

↳ manifold, since g free

$\Rightarrow$ a repr. of

$$\begin{tikzcd} \pi_1(X/G) \arrow[r] \arrow[d] & G \\ H_1(X/G) \arrow[ur] & \end{tikzcd}$$

abelian

$\Rightarrow$ a cohomology class in

$$\alpha \in H^1(X/G;\, \mathbb{Z}/d\mathbb{Z})$$

goal: associate to α a map

$$X/G \longrightarrow L^\infty(d),$$

prove that $[X/G] = 0 \in H_4(L^\infty(d))$

$$\Rightarrow X/G = \partial W^5 \subset L^\infty(d)$$

$$\Rightarrow \begin{array}{ccc} \partial\tilde{W} = X \subset & S^\infty \\ \downarrow & \downarrow \\ \partial W = X/G \subset & L^\infty(d) \end{array}$$

$$\partial\tilde{W} = X$$

$$\Rightarrow \sigma(g, X) = \sigma(g, \partial\tilde{W}) = 0.$$

Want to find a good target space T

maps i.e. that $[X/G, T] \cong H^1(X/G; \mathbb{Z}/d\mathbb{Z})$.

We care about the case when X/G

is a surface or a 4-manifold,

$\rightsquigarrow$ can choose $T = L^{11}(d)$, 11-dim^{l}

lens space: $S^{11} \subset \mathbb{C}^6$

& quotient by $\begin{pmatrix} \omega & & 0 \\ & \ddots & \\ 0 & & \omega \end{pmatrix}$

$\omega = \exp(2\pi i/d)$.

What do we know about T?

T is a 11-dim^{l} manifold, or.

$\pi_1(T) = \mathbb{Z}/d\mathbb{Z}$ & $\pi_k(T) = 0$

$k = 2, \dots, 10$.

Give $\varphi \in H^1(X/G; \mathbb{Z}/2\mathbb{Z})$ we want

to define $f_\varphi : X/G \longrightarrow T$

we choose a generator of $H^1(T; \mathbb{Z}/2\mathbb{Z})$

↳ dual generator γ_0

Corresponding to the universal cover $H_1(T; \mathbb{Z}/2\mathbb{Z})$

$$S^\infty \longrightarrow T.$$

Now we define a map $X/G \longrightarrow T$

Using a cell decomposition of X/G,

& going cell by cell.

$(X/G)^0$ = one cell. $\longrightarrow$ into T

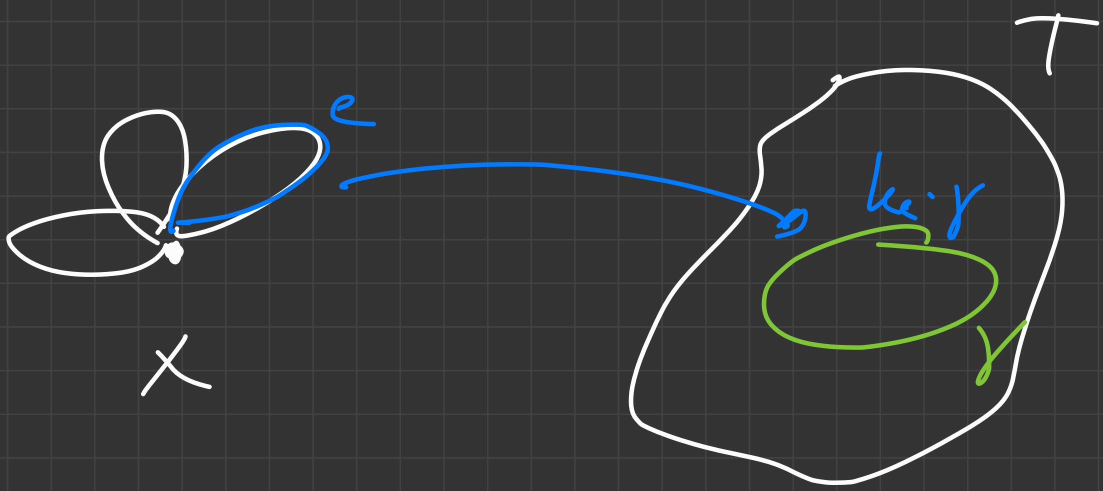
$(X/G)^1$ = bunch of 1-cells $\xrightarrow{!}$ into T

$e \subset X^1$ one cell $\rightsquigarrow$ $[e] \in H_1(X; \mathbb{Z}/d\mathbb{Z})$

$\varphi([e]) = k_e \in \mathbb{Z}/d\mathbb{Z}$

send e to $k_e \cdot$ ~~generator~~ $[\gamma]$

$\gamma \in \pi_1(T)$.



$e_2 \subset X^2$ 2-cell. $\to$ any disc whose ∂ is $\mathrm{im}(\partial e_2)$.

⇝ this gives a well-defined map $(X/G) \to T$ associated to $\varphi \in H^1(X/G; \mathbb{Z}/2\mathbb{Z})$

$$\begin{tikzcd}
X \arrow[r, "\mathbb{Z}/2\mathbb{Z}\text{ equiv.}"] \arrow[d, "\text{given}"'] & S^\infty \arrow[d, "\text{given}"] \\
X/G \arrow[r, "\text{we just}", "\text{constructed}"'] & T
\end{tikzcd}$$

<u>claim</u> $H_*(T;\mathbb{Z})$ is torsion for $* = 1, \dots, n$.

<u>proof</u> We can prove this for cohomology

$0 \neq \alpha \in H^k(T;\mathbb{Q})$.

By poincaré duality $\exists\, \beta \in H^{n-k}(T;\mathbb{Q})$ s.t. $\alpha \cup \beta \neq 0$

$$S^{11} \xrightarrow{\pi} T$$

$$\underset{\substack{\| \\ 0}}{\pi^*(\alpha)}, \underset{\substack{\| \\ 0}}{\pi^*(\beta)} \qquad \alpha, \beta \in H^*(T)$$

$$\cap$$

$$H^*(S^{11}) = 0$$

$$\pi^*(\alpha \cup \beta) = (\pi^*\alpha) \cup (\pi^*\beta) = 0$$

$$\|$$

$$\pi^*(\text{non-zero}) = d \cdot (\text{non-zero})$$

$$\cap$$

$$H^{11}(T) = \mathbb{Q}$$

$$X \xrightarrow{F} T$$

$$\alpha, \beta \in H^1(X) \in F^*(H^2(T))$$

$$\Rightarrow \alpha \cup \beta = F^*(\gamma \cup \delta)$$

where $F^*\gamma = \alpha$, $F^*\delta = \beta$.

<u>ex</u> $T^6 \xrightarrow{f} \mathbb{C}P^3$ cts,

then $\deg f$ is divisible by 6.

When are we: $X/G \xrightarrow{f_4} T$

$(f_4)_*[X/G] \in H_4(T)$ or

$\in H_2(T)$

& $H_2(T)$ is torsion & $H_4(T)$ torsion

$\Rightarrow$ $[X/G] = \partial$(rational chain k)

in T.

by some magic, we can replace

k by (almost) a manifold.

How to do this: work with simplicial homology ⇝ replace K by a simplicial chain K' that is homeo to a manifold (like σ is a sphere to simplex in K'), & then invoke some gen. thm that tells you that each such K' has a smooth structure.

How to turn K into a "simplicial manifold":

simplex by simplex

$K \longrightarrow T$ whose boundary

simpl cx

i) X/G.

$\cap$ embedded

T

$(\dim X/G = 4 \ll \dim T = 11)$

∂K is a manifold.

Let us pick a "bad" simplex $\sigma \subset K$,

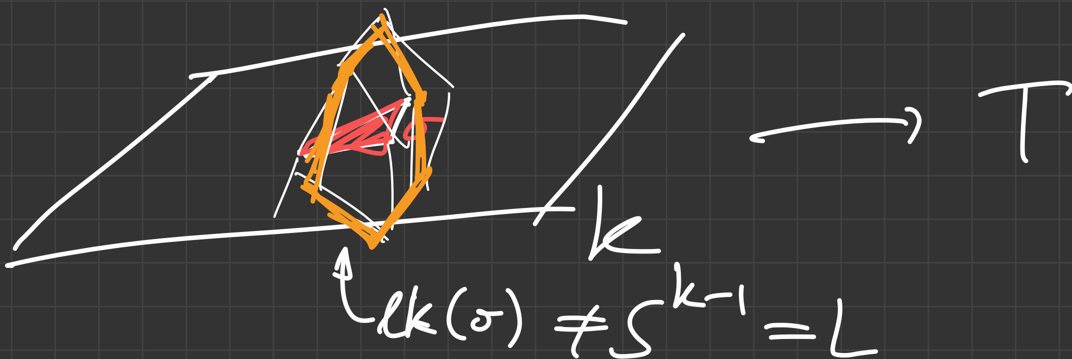
i.e a simplex s.t. $lk\, \sigma$ is not

a sphere. Let us pick it of

maximal dimension $n-k$

$n = \dim K \quad (5)$

We want to replace σ by something else, so that with the new replace. K has one fewer "bad" simplex.



however, since $n-k$ is maximal

$\Rightarrow$ $lk(\sigma)$ is a simplicial mfld.

What can k be?

lk (4-simplex) = 0-manifold

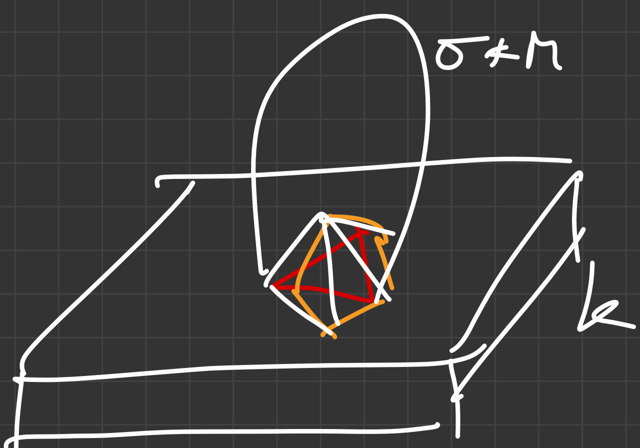
even # of pts, zero counted w/ sign

$\Rightarrow$ we can find a 1-manifold M

whose boundary is L

$\Rightarrow$ take $\sigma * L$ glue on top of it

$\sigma * M \to$ join of σ & M.



mfld

$$\partial(\sigma * M) = \underbrace{(\sigma * \partial M)}_{\text{we glue onto } L} \cup \underbrace{\boxed{(\partial\sigma * M)}}_{\text{left on } \partial}$$

& we extend the map

$$K \times I \longrightarrow X \quad \text{to all of}$$

$$K \times I \cup (\sigma * M) \longrightarrow X$$

(which we can do since $\sigma * L$ is a
retract of $\sigma * M$)

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we can do the same for both

3-simplices $\Rightarrow$ have 1-dim'l links L

$\Rightarrow$ bound surfaces M

2-simplices $\Rightarrow$ 2-dim'l links L

$\Rightarrow$ bound 3-mflds M

1-simpl. $\Rightarrow$ 3-dim'l links L

$\Rightarrow$ bound 4-mflds M (Ely??)

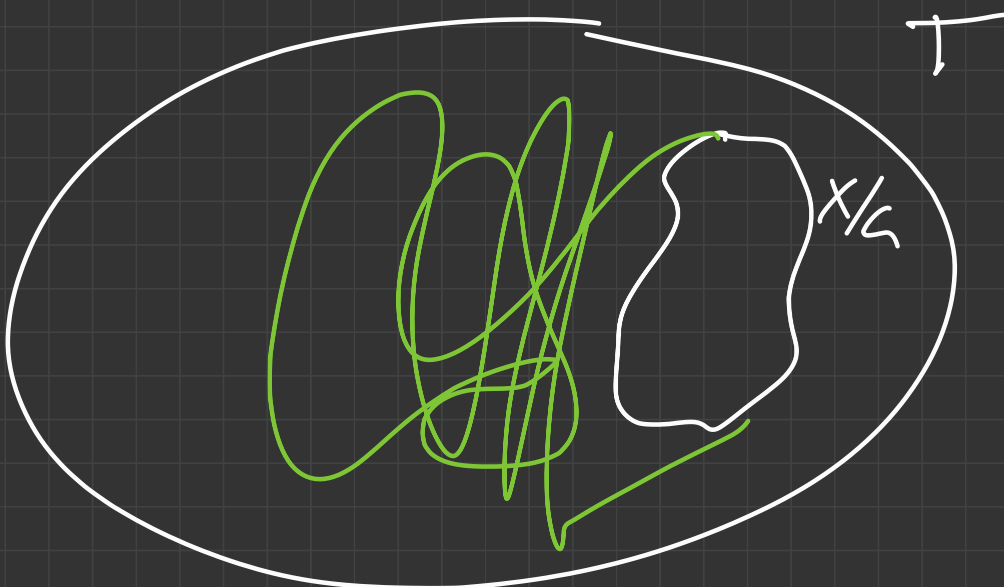
We are left with 0-simplices

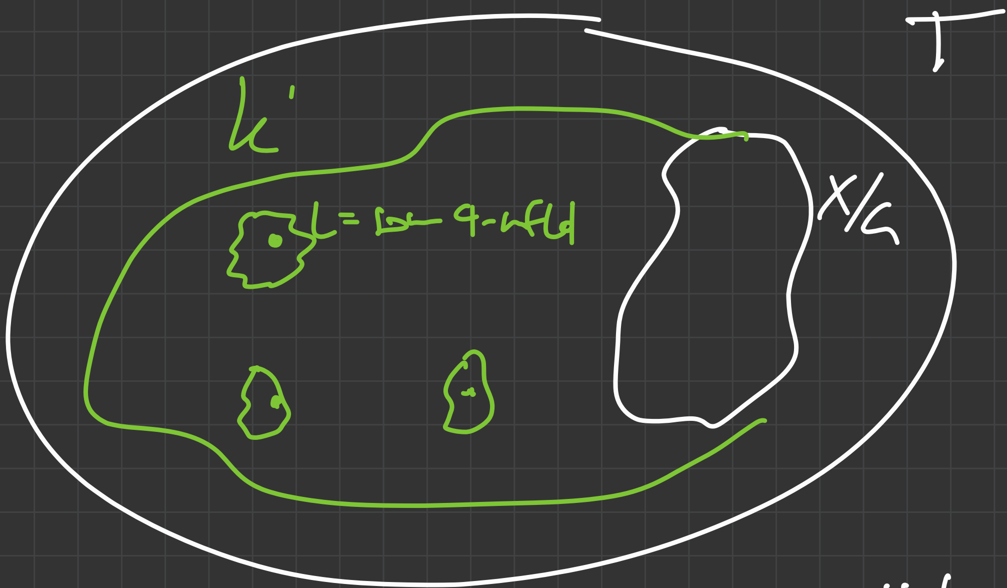
each 0-simplex is a 4-mfld,

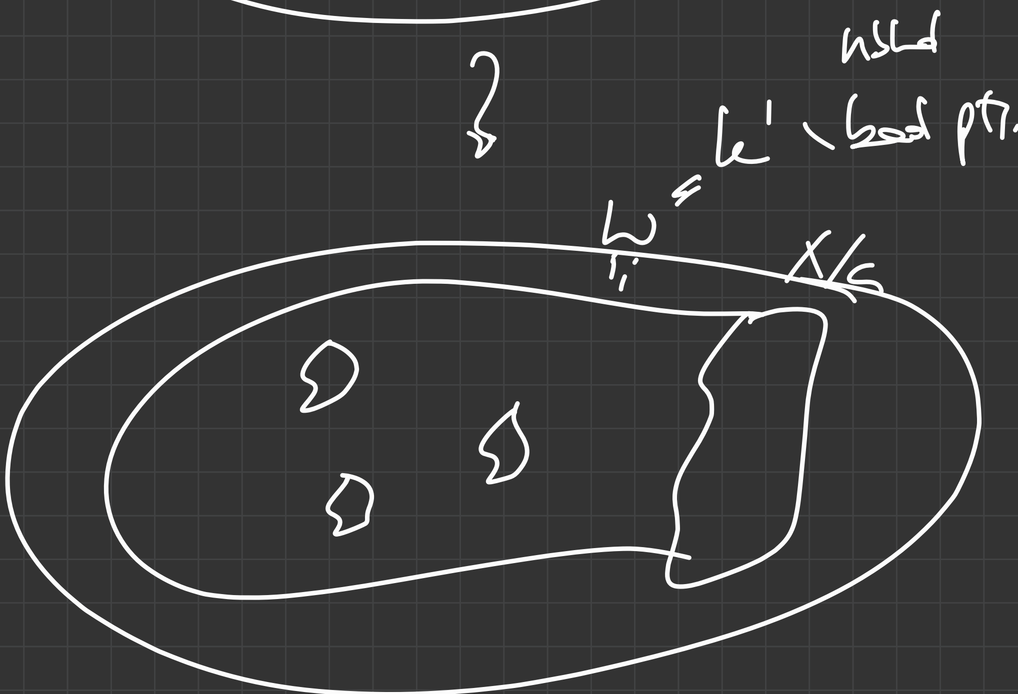
but not all 4-mflds bound

3-manifolds!

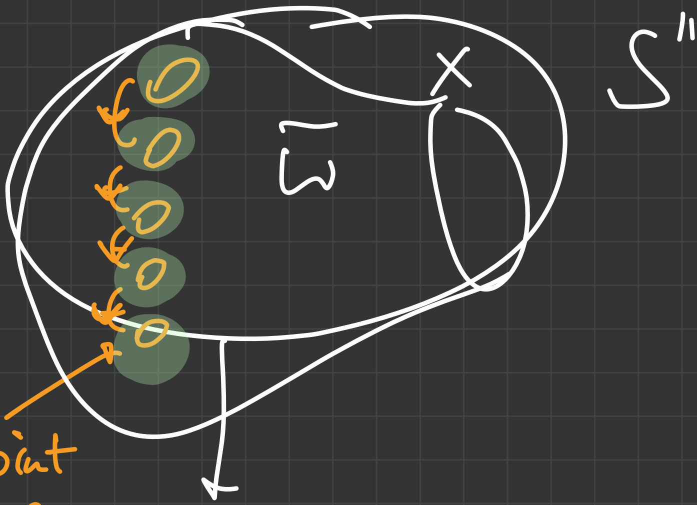
The picture has become



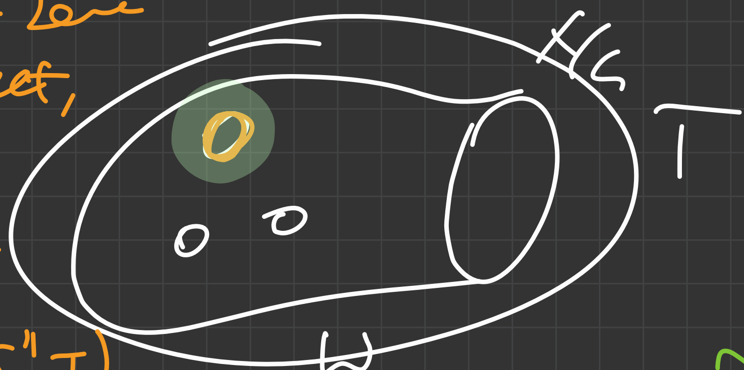




W is a 5-dim'l cobordism from closed 4-mfld to X/G in T



d disjoint copies of the same object, on which $\mathrm{Aut}(S'',T) \cong \mathbb{Z}/d\mathbb{Z}$ acts freely



$\partial - \widetilde{W}$

Upshot: $\widetilde{W}$: $\underbrace{\text{some 4-mfld on which } \mathbb{Z}/d\mathbb{Z} \text{ acts without fixing pts}}$ $\rightsquigarrow X$

w/ our given action of G.

$$\Rightarrow \quad \sigma(g, \partial \tilde{W}) = 0$$

b/c no component is fixed.

& $\sigma(g, \partial^{+}\tilde{W}) = \sigma(g, X)$

by cobordism invariance.

$$\Rightarrow \quad \sigma(g, X) = 0.$$

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from semi-free to general case:

idea: • purge out all isolated fixed points of g^k

(~ to what we did to show that

fixed-point-free actions on surfaces
have signature 0),

this leaves us only with fixed comp.
of g^k for some k.

- to prove that we can do equivalent conn.-sums with $\pm(\mathbb{CP}^2, L, \psi)$
& make all of their surfaces
of self-int 0.

- reduce to the case when
fixed points of g^k make a
surface on which g acts.

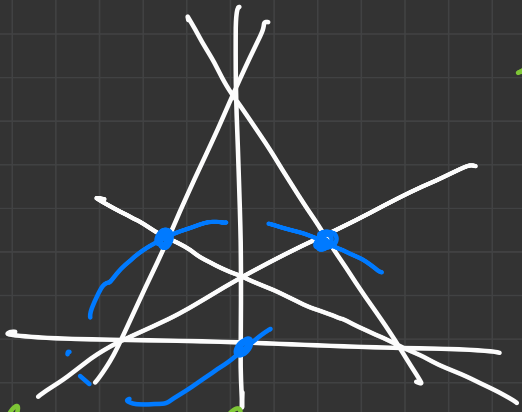
$\rightsquigarrow$ reduces to the case of

$$(F, g) \times (D^?, \text{rotation})$$

we use the same arguments of surgering out & replacing with sthg free.

(See GORDON's notes in "À la recherche de la topologie perdue".)

On to FANO.



$\ell_1 \,--\, \ell_7$

$=:$ Fano

$= \mathbb{F}_2 \mathbb{P}^2.$

easy to see: you cannot find seven complex lines in $\mathbb{CP}^2$ that intersect in the same way as the seven lines in $\mathbb{F}_2 \mathbb{P}^2$.

$(0 \neq 2 \in \mathbb{C})$

thm (Ruberman – Starkston)

$F_{a,0}$ cannot be realized by embedded 2-spheres in $\mathbb{CP}^2$ intersecting transversally.

Translate: $\nexists\ F_1 \sqcup \dots \sqcup F_7 \overset{C^\infty}{\hookrightarrow} \mathbb{CP}^2$

$F_i \cong S^2\ \forall i$, s.t.

$\emptyset \neq F_i \cap F_j \cap F_k \Longleftrightarrow \ell_i \cap \ell_j \cap \ell_k \neq \emptyset$.

& (F_i & F_j intersect $\pitchfork$ once)

& $[F_i] = h \in H_2(\mathbb{CP}^2)$

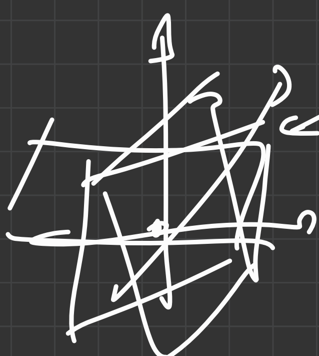
$\hookrightarrow$ generator of $H_2(\mathbb{CP}^2)$.

how to prove this:

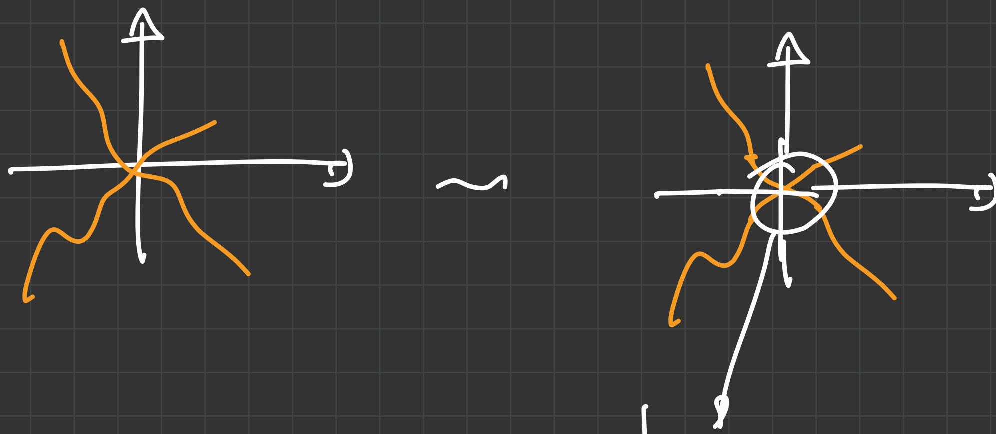
first step: making the F_i disjoint.

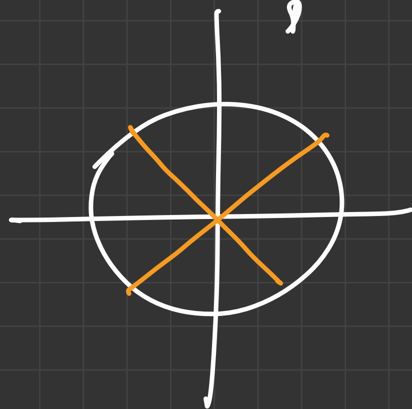
if they were ~~cx~~ locally cx, then we could blow-up.

& we can arrange for them to be locally cx, since they intersect transversally:

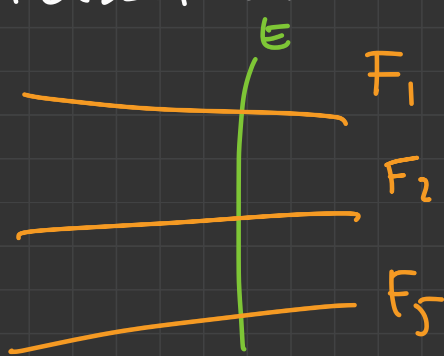


three 2-planes in $T_p^* \mathbb{CP}^2$, we can deform the three planes so that they are all sympl. cx.

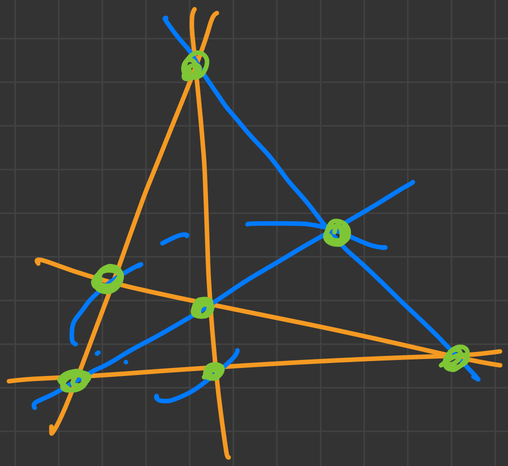




and now we blow up at their intersection.



E is a (-1)-sphere embedded in $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$



do the same at each of the seven points $\Longrightarrow$ you've made all surfaces disjoint.

we pay a price in terms of:

- homology classes
- signatures
- self-intersections

We're no longer in $\mathbb{CP}^2$, but

rather in $\mathbb{CP}^2 \# 7\overline{\mathbb{CP}^2} =: X$

blowups, correspond to $\#\overline{\mathbb{CP}^2}$

$$H_*(X) = \begin{cases} \mathbb{Z} & \text{if } * = 0,4 \\ 0 & \text{if } * = 1,3 \\ \mathbb{Z} \oplus \mathbb{Z}^{\oplus 7} & \text{if } * = 2 \end{cases}$$

$\ni$ "h" $\quad H_2(\mathbb{CP}^1)$

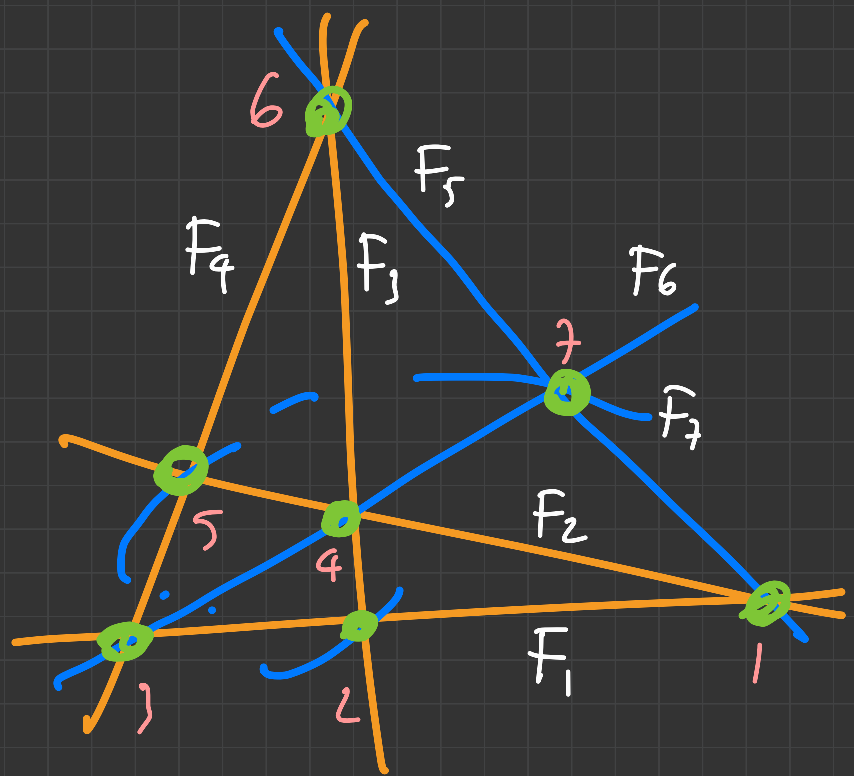
"$\langle e_1, \dots, e_7 \rangle$" $\quad H_2(7\overline{\mathbb{CP}^2})$

orth. decomp. $h^2 = 1$

$e_i^2 = -1$.

obs $[F_i] = h - e_p - e_q - e_r$

when p, q, r are the three pts of the conf. that lie on F_i.



$$[F_1] = h - e_1 - e_2 - e_3$$

$$[F_2] = h - e_1 - e_4 - e_5$$

$$[F_3] = h - e_2 - e_4 - e_6$$

$$[F_4] = h - e_3 - e_5 - e_6$$

$$\Rightarrow [F_i]^2 = 1 - 1 - 1 - 1 = -2$$

$$[F_1] + [F_2] + [F_3] + [F_4] =$$

$$2(2h - e_1 - e_2 - e_3 - e_4 - e_5 - e_6)$$

$\Longrightarrow$ $\exists$ double cover

$$\begin{array}{c} \tilde{X} \\ \downarrow \\ X \end{array}$$

branched over $F_1 \cup \dots \cup F_4$

We can compute:

$$\chi(\tilde{X}) = 2\chi(X) - \chi(F_1 \cup \dots \cup F_4)$$

$$= 2 \cdot 10 - 4 \cdot 2$$

$$= 12$$

$b_1(\tilde{X}) = 0 \Longrightarrow$

$b_2(\tilde{X}) = 12 - b_0(\tilde{X}) - b_4(\tilde{X}) = 10.$

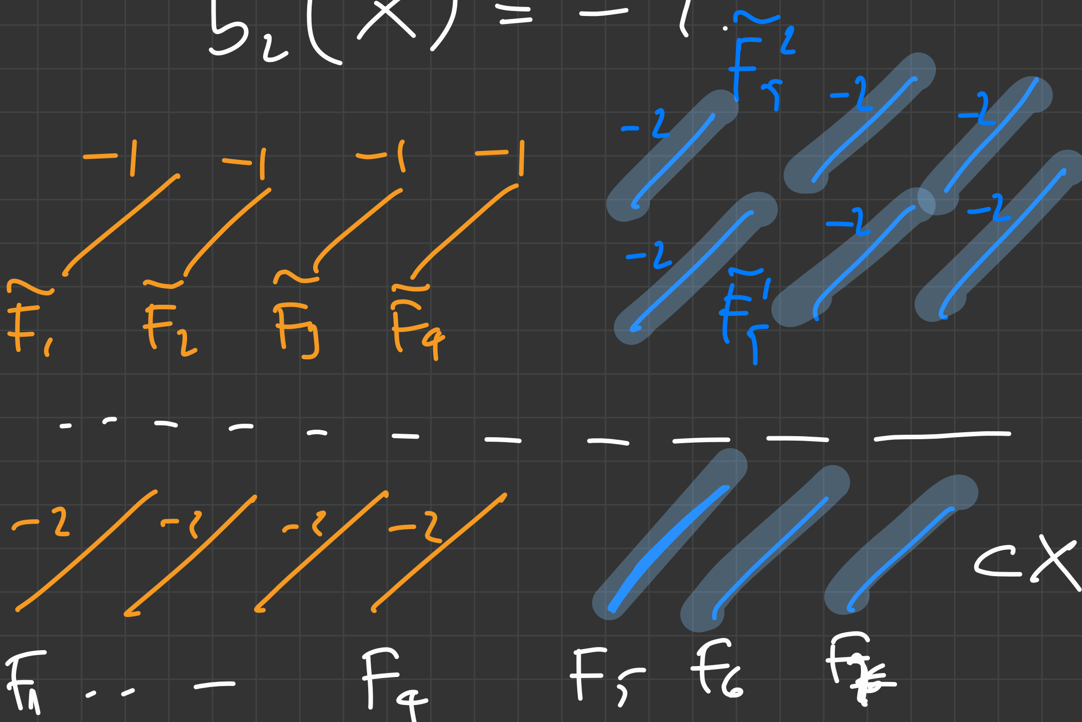
$$\sigma(\tilde{X}) = 2\sigma(X) - \frac{1}{2}(B\cdot B)$$

$$= 2\cdot(-6) - \frac{1}{2}(4\cdot(-2)) =$$

$$= -12 + 4 = -8$$

$$\Rightarrow \quad b_2^+(\tilde{X}) = 1$$

$$b_2^-(\tilde{X}) = -9.$$



$b_2^-(\hat{X}) = 9 \Longrightarrow$ the max.

neg. def. subspace of $H_2(\tilde{X})$

is 9-dim'l.

But the subspace gen. by

$\tilde{F}_1 \ldots \tilde{F}_4, \tilde{F}_5^1, \tilde{F}_5^2, \ldots, \tilde{F}_7^1, \tilde{F}_7^2$

comprises ten neg., pairwise

orth. hom. classes. $\lightning$.
