Branched covers in low dimensions
Example sheet 1

January 18, 2021

Solutions are accepted in English or French, and they are due on **January 25**. Please scan your solutions so that they're legible; pdf is the preferred format (there are apps to do that on your phone). They should be emailed to marco.golla(at)univ-nantes.fr.

You **can** work in groups, but solutions have to be **written up** and **submitted individually**.

If $i < j$, you can use the statement of problem i to solve problem j even if you haven't solved problem i. (Same for different parts within one problem, if there are more points in one problem, and you can solve later points even if you haven't solved earlier ones.)

Problems

1. Let X be a finite simplicial complex (or a finite CW complex) with c_k simplices (cells) of dimension k for each k. Show that $\sum(-1)^k c_k = \chi(X) := \sum(-1)^k \dim H_k(X;\mathbb{F})$. (Here $\mathbb{F}$ is any field.) This justifies the fact that one can count simplices instead of homology ranks when computing Euler characteristics.

2. Let $p\colon X \to Y$ be a degree-d cover between finite simplicial complexes (or finite CW complexes).

   (a) Show that $\chi(X) = d \cdot \chi(Y)$.

   In particular, point (a) implies that if $\chi(Y) = 0$ then $\chi(X) = 0$.

   (b) Give an example where this latter statement fails if the cover is not finite.

3. (a) Let $p\colon X \to Y$ and $q\colon Y \to Z$ be branched covers between surfaces. Show that the composition $q \circ p\colon X \to Z$ is a branched cover.

   (b) Use point (a) and the branched covers we constructed in the lectures to show that there is branched cover from any closed (i.e. compact and without boundary) oriented surface to S^2, branched over three points. (Hint: the three points can be chosen as the north and south poles, and one point on the equator. You can email me for more hints.)