Branched covers in low dimensions
Example sheet 3

February 1, 2021

Solutions are accepted in English or French, and they are due on **February 8**. Please scan your solutions so that they're legible; pdf is the preferred format (there are apps to do that on your phone). They should be emailed to marco.golla(at)univ-nantes.fr.

You **can** work in groups, but solutions have to be **written up** and **submitted individually**.

If $i < j$, you can use the statement of problem i to solve problem j even if you haven't solved problem i. (Same for different parts within one problem, if there are more points in one problem, and you can solve later points even if you haven't solved earlier ones.)

Problems

1. Consider the torus knot $K := T(p, q) \subset S^3$.

   - Show that $\pi_1(S^3 \setminus K, x_0) \cong \langle x, y \mid x^p y^{-q} \rangle$.
   - Deduce that K is not isotopic to the unknot. (Hint: look for a group G and a homomorphism $\pi_1(S^3 \setminus K, x_0) \to G$ with non-cyclic image.)
   - (Bonus**) Can you show that K is not isotopic to $T(p', q')$ if $\{p', q'\} \neq \{p, q\}$ for some values of p, q, p', q'? Can you show it for all values?

2. Let $S \subset M$ be an embedded 2-sphere in a closed, orientable 3-manifold M. Show that either:

   - S bounds a 3-ball in M,
   - S exhibits M as a connected sum of two closed 3-manifolds $M_1, M_2 \neq S^3$, or
   - $M = M' \# (S^1 \times S^2)$ and S is a subset of the second summand of the form $\{\star\} \times S^2$.

   (Hint: what is a neighbourhood N of S? If S does not disconnect, choose a path γ connecting the two boundary components of N and consider a neighbourhood of $N \cup \gamma$.)

3. Let M be a compact 3-manifold whose boundary ∂M is a genus-g surface. Show that the inclusion $j \colon \partial M \hookrightarrow M$ induces a map $j_* \colon H_1(\partial M) \to H_1(M)$ of rank g. (Bonus*: can you find an example for which $\ker j_*$ is not a primitive subgroup of $H_1(\partial M)$?)