Branched covers in low dimensions
Example sheet 7

March 8, 2021

Solutions are accepted in English or French, and they are due on **March 15**. Please scan your solutions so that they're legible; pdf is the preferred format (there are apps to do that on your phone). They should be emailed to marco.golla(at)univ-nantes.fr.

You **can** work in groups, but solutions have to be **written up** and **submitted individually**.

If $i < j$, you can use the statement of problem i to solve problem j even if you haven't solved problem i. (Same for different parts within one problem, if there are more points in one problem, and you can solve later points even if you haven't solved earlier ones.)

Problems

1. Show that if a closed orientable 3-manifold Y with $b_1(Y) = 0$ is embedded in S^4, then $H_1(Y) = G \oplus G$ for some Abelian group G.

2. Let $p\colon X \to Y$ be a cyclic d-fold cover between two closed, oriented, smooth 4-manifolds, branched over B, a collection of closed surfaces in Y. Let $g \in \mathrm{Diff}^+(X)$ be a generator of the automorphism group of the cyclic cover (i.e. the restriction g to $X \setminus p^{-1}(B)$ generates the deck transformation group of the regular part of p).

   (a) Show that $\sigma^{g,0}(X) = \sigma(Y)$.

   (b) Express $\sigma^{g^k,r}(X)$ in terms of the collection $\{\sigma^{g,s}(X)\}_{s=0,\dots,d-1}$ and deduce that
   $$d\sigma(Y) - \sigma(X) = \sum_{k=1}^{d-1} \sigma(g^k, X).$$

   (c) Show that
   $$\sum_{k=1}^{d-1} \frac{1}{\sin^2 \frac{k\pi}{d}} = \frac{d^2-1}{3}.$$

   (d) Deduce that
   $$\sigma(X) = d\sigma(Y) - \frac{d^2-1}{3d} B \cdot B.$$

   (Hints. For (a), use a cell decomposition on X lifted from Y, and construct a map $C_*(Y) \to C_*(X)$ whose image is in the 1-eigenspace. For (c), use de Moivre's formula to find a polynomial whose roots are $\frac{1}{\mathrm{tg}^2 \frac{\pi}{d}}, \dots, \frac{1}{\mathrm{tg}^2 \frac{(d-1)\pi}{d}}$, and then express $\frac{1}{\sin^2 \frac{k\pi}{d}}$ in terms of $\frac{1}{\mathrm{tg}^2 \frac{k\pi}{d}}$.)