Branched covers in low dimensions
Example sheet 8

March 15, 2021

Solutions are accepted in English or French, and they are due on **March 22**. Please scan your solutions so that they're legible; pdf is the preferred format (there are apps to do that on your phone). They should be emailed to marco.golla(at)univ-nantes.fr.

You **can** work in groups, but solutions have to be **written up** and **submitted individually**.

If $i < j$, you can use the statement of problem i to solve problem j even if you haven't solved problem i. (Same for different parts within one problem, if there are more points in one problem, and you can solve later points even if you haven't solved earlier ones.)

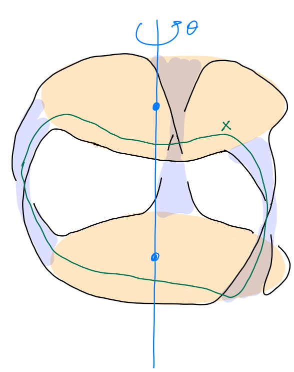
Problems

1. Let F_1 and F_2 be two compact oriented surfaces with non-empty boundary, and $g_j \in \mathrm{Diff}^+(F_j)$ be a finite-order diffeomorphism for $j = 1, 2$. Let $g = g_1 \times g_2 \in \mathrm{Diff}^+(F_1 \times F_2)$ be the product diffeomorphism, i.e. $g(x_1, x_2) = (g_1(x_1), g_2(x_2))$. Show that

$$\sigma(g, F_1 \times F_2) = \sigma(g_1, F_1) \cdot \sigma(g_2, F_2).$$

2. Let G be the cyclic group of order d, $0 < k < d$ an integer, $\theta = 2k\pi/d$, and (S^1, θ) the G-object generated by $g\colon S^1 \to S^1$, the rotation by the angle θ. The goal of this exercise is to compute the local contribution $\alpha(\theta) = \sigma(g, Q(\theta))$ of a *free* G-object $Q(\theta)$ whose boundary is (S^1, θ). We will do this indirectly.

   Consider the surface F_0 with boundary in the figure (where it is depicted for $d = 3$). It is obtained by taking two discs (in yellow) and attaching d bands (in blue). There's an obvious action by rotations by θ on it.



(i) Show that if d is even then F_0 can be capped off to a closed surface of genus $(d-2)/2$ with an action by G whose only fixed points are the centres of the two discs.

(ii) Show that if d is odd, then F_0 can be capped off to a closed surface of genus $(d-1)/2$ with an action by G with three fixed points, two of which are the centres of the two yellow discs. Show that at the third fixed point the angle of rotation is $\psi := \theta/2 - \pi$ (so that $2\psi = \theta$).

Call F the closed surface we obtain by capping off F_0 in either of the two cases and let x be the loop shown in the figure.

(iii) Show that $H_1(F)$ is spanned by $x, g \cdot x, \dots, g^{d-1} \cdot x$ (this will *not* be a basis if d is even!).

(iv) Determine the eigenspace decomposition of $H_1(F; \mathbb{C})$ and compute $\sigma(g, F)$.

Let F' be the G-object where we replaced a neighbourhood of each of the fixed points in $\mathrm{Fix}(g)$ by the corresponding free object.

(v) If d is even, use the fact that the action on F' is fixed-point–free and additivity of the signature to deduce that $\alpha(\theta) = -i\mathrm{cotg}\,\frac{\theta}{2}$.

(vi) If d is odd, deduce in a similar way that

$$2\alpha(\theta) - \alpha(\psi) = -2i\mathrm{cotg}\,\frac{\theta}{2} + i\mathrm{cotg}\,\frac{\psi}{2},$$

and this to conclude that $\alpha(\theta) = -i\mathrm{cotg}\,\frac{\theta}{2}$ (recall that $2\psi = \theta$ and that we know the above relation *for all θ simultaneously*).