Branched covers in low dimensions
Example sheet 9

March 22, 2021

Solutions are accepted in English or French, and they are due on **March 29**. Please scan your solutions so that they're legible; pdf is the preferred format (there are apps to do that on your phone). They should be emailed to marco.golla(at)univ-nantes.fr.

You **can** work in groups, but solutions have to be **written up** and **submitted individually**.

If $i < j$, you can use the statement of problem i to solve problem j even if you haven't solved problem i. (Same for different parts within one problem, if there are more points in one problem, and you can solve later points even if you haven't solved earlier ones.)

Problems

1. We want to use the g-signature theorem to distinguish lens spaces. Recall that lens spaces are quotients of S^3 by linear cyclic actions of finite order. Consider two lens spaces $L = L(p,q)$ and $M = L(r,s)$, with $0 < q < p$ coprime and $0 < s < r$ coprime. We write $=$ to denote diffeomorphism.

   (i) Show that if $p = r$ and $qs \equiv 1 \pmod{p}$, then $L = M$

   From now on, suppose that $\phi\colon L \to M$ is a diffeomorphism.

   (ii) Show that $p = r$, so that $M = L(p,s)$.

   (iii) Show that there is a self-diffeomorphism Φ of S^3 such that

$$\begin{array}{ccc} S^3 & \xrightarrow{\Phi} & S^3 \\ \downarrow & & \downarrow \\ L & \xrightarrow{\phi} & M, \end{array}$$

   where the vertical arrows are the quotient maps.

   (iv) Deduce that there is an action on S^4 by $g \in \mathrm{Diff}^+(S^4)$ of order p with two fixed points; show that the angles of rotation of g at the two points are $(2k\pi/p, 2k\pi q/p)$ and $(-2\pi/p, -2s\pi/p)$ for some integer k.

   (v) Compute $\sigma(g, S^4)$ and deduce that

$$\cot\frac{k\ell\pi}{p}\cot\frac{k\ell q\pi}{p} = \cot\frac{\ell\pi}{p}\cot\frac{\ell s\pi}{p}.$$

Call now $\eta(p,q) = \sum_{\ell=1}^{p-1} \cot \frac{\ell\pi}{p} \cot \frac{\ell q\pi}{p}$.

(vi) Show that $\eta(p,q) = \eta(r,s)$.

(vii) Show that $\eta(p,-t) = -\eta(p,t)$ for every t.

(viii) Show that $\eta(p,1)$ is the *unique* maximum of $\{\eta(p,t)\}_{t=1,\dots,p-1}$ and deduce that if $L(p,t) = L(p,\pm 1)$ then $t \equiv \pm 1 \pmod p$.

(ix) Deduce from (i) and (viii) the classification of lens spaces for $p \leq 6$.

(x) Compute $\eta(7,2)$ and deduce from (i), (vii), and (viii) the classification of lens spaces with $p = 7$.

2. Let G be a finitely presented group. We want to show that there exists a closed oriented 4-manifold X whose fundamental group is isomorphic to G.

(a) Show that there exists a finite 2-dimensional CW complex C with one 0-cell x_0 such that $\pi_1(C, x_0)$ is isomorphic to G.

(b) Show that C can be thickened to a 4-manifold with boundary W that contains C as a deformation retract. This can be achieved by thickening each cell individually. (Hint: you need to use the assumption that you look for a 4-manifold. Where does the argument fail for 2- and 3-manifolds?)

(c) Show that W is constructed from ∂W by attaching 2-, 3- and 4-cells.

(d) Deduce that the inclusion $\partial W \hookrightarrow W$ induces a surjection $\pi_1(\partial W, x) \to \pi_1(W, x)$.

(e) Show that the *double* X of W, i.e. $X = W \cup_{\partial W} -W$ has fundamental group isomorphic to $\pi_1(W, x)$, so it is the required 4-manifold.