Introduction to knot theory
Example sheet 1

January 28, 2019

These exercises can be done in groups or individually; please refrain from looking/asking for solution online. There will be some hints on the course webpage:

`http://www.math.sciences.univ-nantes.fr/~golla/docs/courses/intro-to-knots.htm`

Homework can be handed in in English, French, or Italian; the deadline is **Feb 4, 2019**.

1. Let M be a compact, orientable 3-manifold, whose boundary ∂M is a surface of genus g. Prove that the map $H_1(\partial M;\mathbb{Q}) \to H_1(M;\mathbb{Q})$ induced by the inclusion $\partial M \hookrightarrow M$ has rank g. (Note that this generalises the case when M is a knot complement.)

2. An *integral homology sphere* is a 3-manifold Y with the same homology of S^3; i.e. $H_*(Y;\mathbb{Z}) = H_*(S^3;\mathbb{Z})$. For $i = 1,2$, let K_i ($i = 1,2$) be an oriented knot in S^3, with regular open neighbourhoods N_i, meridian μ_i, and Seifert longitude λ_i, and let $M_i = S^3 \setminus N_i$. Let $h : \partial M_1 \to \partial M_2$ be a diffeomorphism that sends μ_1 to λ_2 and λ_1 to μ_2 *as oriented curves.*

   (i) Show that such an h exists, and that it is orientation-reversing.

   (ii) Show that $M_1 \cup_h M_2$ is an oriented 3-manifold, and that it is an integral homology sphere.

3. Show that all torus knots are prime.

4. Let D be a knot diagram. A 3-colouring of D is a way of colouring the arcs of D with three colours, such that at every crossing either the three arcs have the same colour, or they have three different colours. A knot is said to be 3-colourable if there exists a diagram that admits a non-monochromatic 3-colouring. Prove the following assertions:

   (i) if a knot K is 3-colourable, then *every* diagram representing K has a non-trivial colouring; in particular, 3-colourability can be checked on any diagram;

   (ii) the unknot is not 3-colourable, while the trefoil is;

   (iii) the *number* of 3-colourings of a diagram is an invariant of the knot;

   (iv) the number of 3-colourings (including the three monochromatic ones) of any knot is a power of 3;

   (v) every 3-colouring of a knot K corresponds to a map $\pi(K) \to S_3$ such any loop whose homology class is a meridian is sent to a transposition.