Introduction to knot theory
Example sheet 2

February 11, 2019

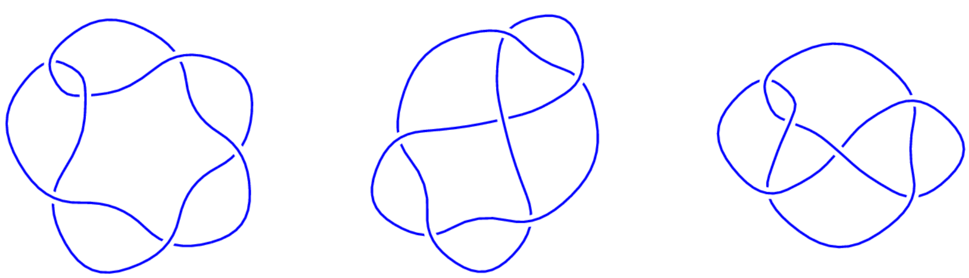
These exercises can be done in groups or individually; please refrain from looking/asking for solution online. Upon request, there will be some hints on the course webpage:

`http://www.math.sciences.univ-nantes.fr/~golla/docs/courses/intro-to-knots.htm`

Homework can be handed in in English, French, or Italian; the deadline is **Mar 4, 2019**.

Notes: (1) Exercise 3(iv) is just a bonus, for your entertainment/general culture.
(2) Exercise 4 is long and hard. It is broken up into three parts: one algebraic, one topological, and one of applications to knots. Each part consists of several points, some of which are consequences of previous points; however, even if you get stuck and—say—can't prove (A) or point (T2), you can use their statements to go on with the remaining points. (The same holds for exercise 3, in fact.)

1. Compute the Alexander polynomials of the 6-crossing knots, 6_1, 6_2, and 6_3, below. Use at least two different methods (i.e. compute it once for each, with the method you prefer, but do not use the same method for all three). Pictures are taken from `http://www.indiana.edu/~knotinfo`.



2. Compute the signature function $\sigma_K\colon S^1 \to \mathbb{Z}$ for the following knots:
   - $K_n := T(2, 2n+1)$, $n \geq 1$;
   - the figure-8 knot, 4_1;
   - $K_0 := T(3, 4)$.

   Use these computations to prove that the set $\{K_0, K_1, K_2, \dots\}$ is linearly independent in the concordance group $\mathcal{C}$; concretely, this means that if

$$(\#^{a_0} K_0)\#(\#^{a_1} K_1)\# \dots \#(\#^{a_n} K_n)\# m\big((\#^{b_0} K_0)\#(\#^{b_1} K_1)\# \dots \#(\#^{b_n} K_n)\big)$$

   is slice for some $(a_0, \dots, a_n), (b_0, \dots, b_n) \in \mathbb{Z}_{\geq 0}^{n+1}$, then $(a_0, \dots, a_n) = (b_0, \dots, b_n)$. (As usual, $m(J)$ denotes the mirror of the knot J.)

3. Recall that the branched double cover of a knot $K \subset S^3$ is a 3-manifold $\Sigma(K)$ with a knot $\tilde{K} \subset \Sigma(K)$ and a projection $p : \Sigma(K) \to S^3$, such that $p|_{\Sigma(K)\setminus\tilde{K}}$ is a (connected) 2-fold cover of $S^3 \setminus K$, and $p|_{\tilde{K}}$ is a diffeomorphism onto K.

   (i) Mimicking our computation of the presentation of the Alexander polynomial (or otherwise), show that $|H_1(\Sigma(K);\mathbb{Z})| = \det(V + {}^tV)$, where V is a Seifert matrix associated to some Seifert surface F of K.

   (ii) Deduce that $|H_1(\Sigma(K);\mathbb{Z})| = \Delta_K(-1)$, and that this is an odd integer.

   (iii) Using point (K2) in exercise 4 below, show that $\Sigma(T(p,q))$ is an integral homology sphere (i.e. $H_1(\Sigma(T(p,q));\mathbb{Z}) = 0$) if and only if p and q are odd.

   (iv)* Show that $\Sigma(T(p,q))$ is the link of the singularity $B(p,q) = \{x^2 + y^p + z^q = 0\} \subset \mathbb{C}^3$; concretely, it is the intersection of a sphere of radius $\sqrt{3}$ with $B(p,q)$.

4. Given H a group, let $H' = [H,H]$ be the commutator subgroup, i.e. the subgroup of H *generated by* commutators. Let also $\mathbb{Z}[H]$ be the group ring of H, i.e. $\mathbb{Z}[H]$ is a ring whose underlying additive group is the set of (finite) linear combinations of elements of H, and whose multiplication linearly extends multiplication on H. (E.g. if $H = \mathbb{Z}$, then $\mathbb{Z}[H]$ is the ring of Laurent polynomials in one variable.)

   Let $G = \langle x_1,\dots,x_m \mid r_1,\dots,r_n\rangle$ be a finitely presented group; that is, we have an exact sequence

$$F_1 \to F \xrightarrow{\phi} G \to 0$$

   with F_1 and F free. We view F as the free group on $x_1,\dots,x_m$, and r_i as an element of F. Suppose also that there is an isomorphism $G/G' \to \mathbb{Z}$, and let $\tilde{X}$ be the covering of X whose fundamental group is $G' < G$. Call $\alpha : G \to \mathbb{Z}$ the composition of the quotient map and the isomorphism above.

   Define the following operators d_{x_i} on $\mathbb{Z}[F]$, called *Fox derivatives* or *free derivatives*, by using the following two rules, and then extending by linearity:

$$d_{x_i}x_j = \delta_{ij} \qquad d_{x_i}uv = d_{x_i}u + u d_{x_i}v \quad \forall u,v \in F,$$

   (A1) Show that G' is a normal subgroup of G.

   (A2) Prove that $d_{x_i}1 = 0$ and $d_{x_i}x_j^{-1} = -\delta_{ij}x_j^{-1}$.

   (T1) Construct a 2-dimensional cell complex X with one 0-cell x_0, m 1-cells, and n 2-cells, such that $\pi_1(X,x_0) = G$. (Recall m and n are the numbers of generators and relations in the presentation for G above.)

   (T2) Construct a 2-dimensional cell complex structure on $\tilde{X}$ that is equivariant under the action of $\mathrm{Aut}(\tilde{X} \to X) \cong \mathbb{Z}$.

   (T3) Prove that the boundary of the lift of the cell corresponding to the relation r_i is the 2-chain $\sum_j \alpha(\phi(d_{x_j}r_i)) \in C_2(\tilde{X})$.

   (T4) Deduce that the first homology group $H_1(\tilde{X};\mathbb{Z})$, viewed as a $\mathbb{Z}[G/G']$-module, is presented by the matrix $(\alpha(\phi(d_{x_j}r_i)))_{i,j}$.

   We can apply this method of computing homology to the case of knot complements. In particular, since $H_1(X_\infty(K))$ only sees the knot group $\pi(K)$, it can be computed by the procedure above.

   (K1) Deduce that the Alexander polynomial of K is the greatest common divisor of all maximal minors of the matrix $(\alpha(\phi(d_{x_j}r_i)))_{i,j}$.

   (K2) Using the presentation of $\pi(T(p,q))$ given in the first lecture (or any other), show that $\Delta_{T(p,q)} = \dfrac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}$.

   (K3) Show that $g(T(p,q)) = \frac{(p-1)(q-1)}{2}$ and that $T(p,q)$ is not slice.