Symplectic rational cuspidal curves

(joint w/ L. Starkston & F. Kütle)

Curves in the cx proj. plane $\mathbb{CP}^2$

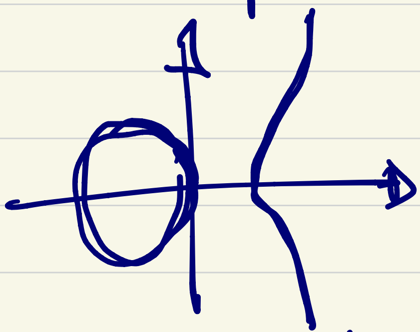
zero sets of hom. poly. in 3 variables.

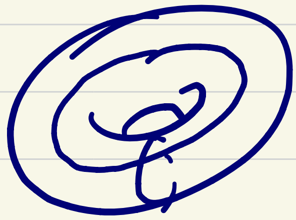
ex $\quad y^2 = x^3 + ax^2 + bx$

real part $\qquad$ "abstractly"

(a,b)

$(0,-1)$

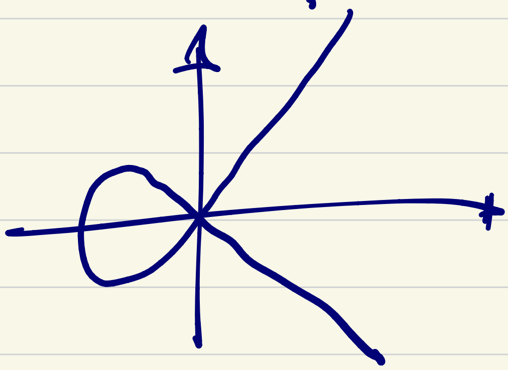


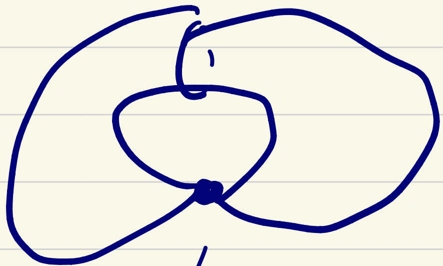


(smooth)

↘

$(-1,0)$



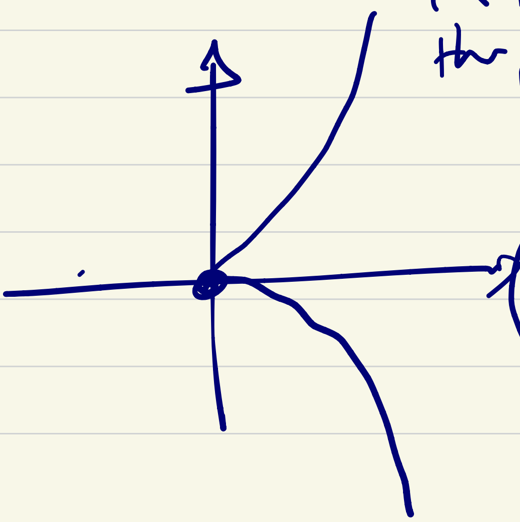


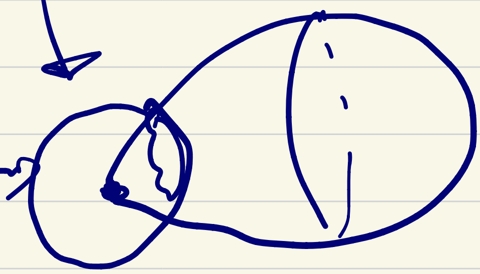
(sing.)

↳ sing. double pt

$T(2,3)$ is the link

$(0,0)$





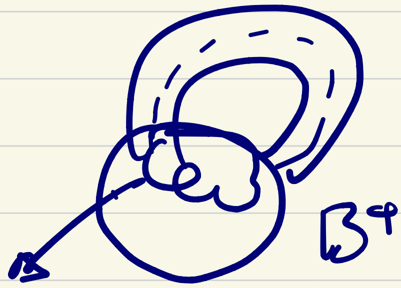
$\simeq S^2$

$T(2,3)$

↳ cone over a knot in S^3

def • A <u>complex</u> <u>rational cuspidal</u> curve in $\mathbb{CP}^2$ is a curve that is homeomorphic to S^2.

• A <u>PL sphere</u> in $\mathbb{CP}^2$ is an embedding of a <u>knot trace</u>.

$$X^4_m(K) =$$ 

framing

$K \subset S^3$

rmk If C is a rational cuspidal curve (w/ one sing), then there is a PL sphere with $m = (\deg C)^2$ & K = link of the sing of C.

Q Can we in any way classify these objects?

On $\mathbb{CP}^2$ we have a symplectic str
$\omega \in \Omega^2(\mathbb{CP}^2)$ • $d\omega = 0$
• $\omega \wedge \omega > 0$.

This form Fubini-Study form,
$\frac{i}{2} \partial\bar{\partial} \log |z|^2$.

property It is compatible w/ the complex structure on $\mathbb{CP}^2$.
$\Rightarrow$ if C is a complex curve then $\omega|_C$ is an area form.

$\leadsto$ gives us a notion of "positivity"

def A symplectic curve C is a 2-dim surface in $\mathbb{R}^4$ s.t. $\omega|_C > 0$.

q Can we classify symplectic rational cuspidal curves? (SRCC) (CRCC)

thm (joint w/ Starkston, Kütle)

If C is a sympl. rational cuspidal curve of degree ≤ 7, then it isotopic to a complex curve.

rmk : • degree : homology class.

• isotopic : isotopic through SRCC preserving the types of the singularities.

• singularities : of cx type.

Motivation: • these objects are not* classified in AG.

* Pelka & Petke: classifies the annoying "Negativity conjecture".

• there were conjectures:

  - a $\mathbb{C}$RCC can have at most four singularities. (Koras–Palka)

  - [Coolidge–Nagata] every curve is rectifiable (Koras–Palka)

• <u>symplectic isotopy problem</u>:

  Is every non-singular sympl. surface in $\mathbb{P}^2$ isotopic to a cx one? (true deg ≤ 17).

- the answer is known not to hyper if we drop any of the assumptions.

[Moishezon] sympl. cuspidal curves in $\mathbb{R}^2$ that are not isotopic to cx ones.

[Orevkov]: rational sympl. curve with non-cuspidal sing. that is not isotopic to any cx one.

<u>proof</u>: Case by case analysis.

finiteness: ensured by the adjunction formula: genus of a non-sing. alg-d curve

$$\to \sum g(k_i) = \frac{(d-1)(d-2)}{2}$$

links of the sing. of C

Seifert genus.

#deg:	3	4	5	6	7	8
#cones:	1	4	20	106	718	5612
# curves:	1	4	9	11	11	?

- We're only looking at curves s.t. they have a "complex-like" chart around each point.

  $\Rightarrow$ constrains the links of the sing.

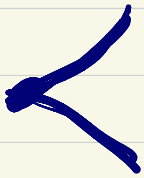
$\rightsquigarrow$ talk about the construction bit of the proof.

main input: birational transformations.

$\rightsquigarrow$ blow-ups & blow-downs.

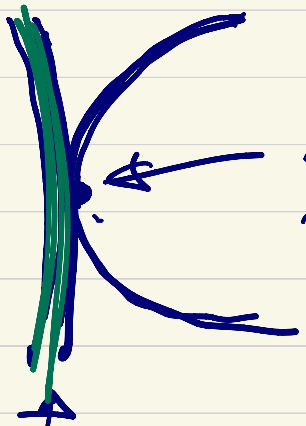
key: using birational transf to simplify a curve.

ex **local part:**

 $\{y^2 = x^3\} \subset \mathbb{C}^2.$

blow up at the origin

$$\mathbb{C}^2 \# \overline{\mathbb{CP}}^2$$



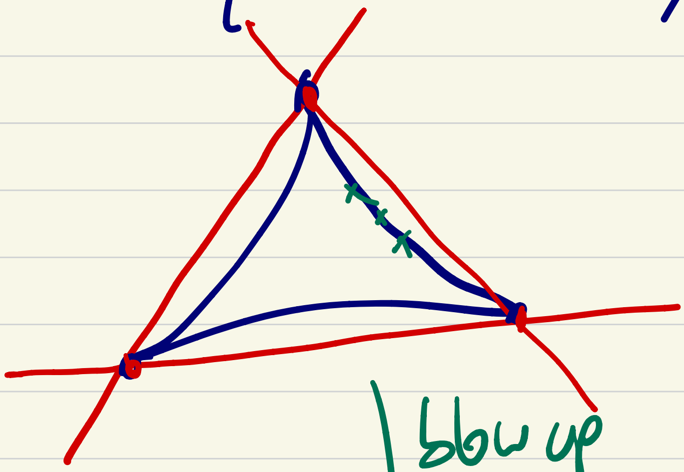
simple tangency

exceptional divisor, $E \simeq S^2$

(i.e. a line in here)

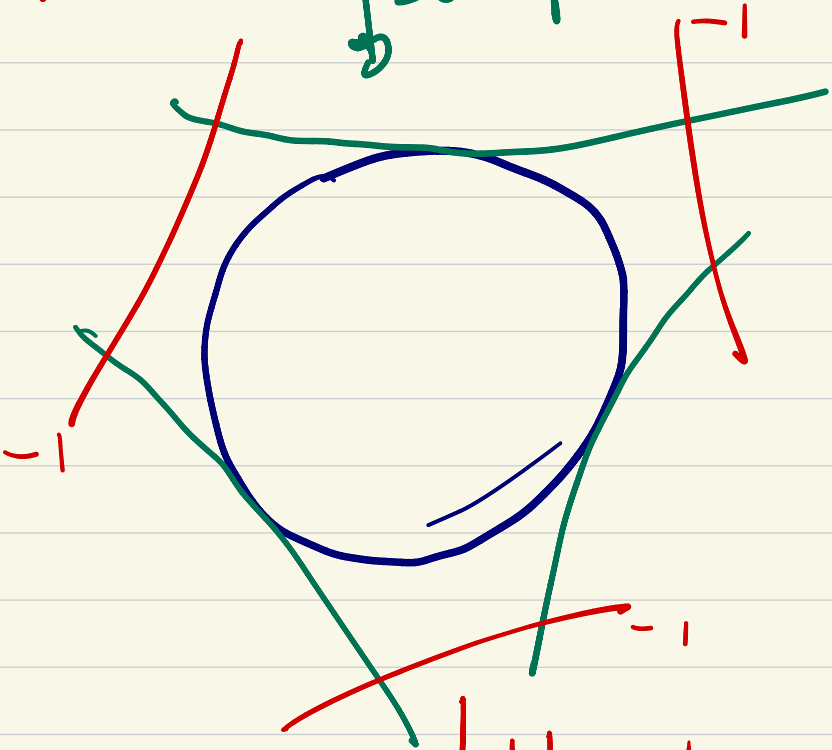
$E \cdot E = -1$

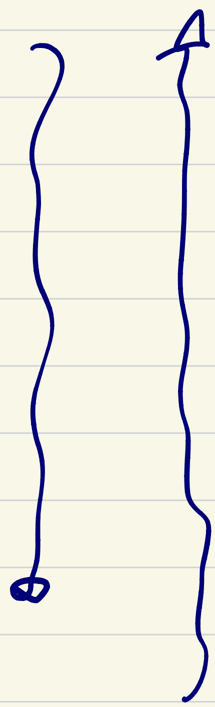
global part I want to create a quartic w/ three singularities



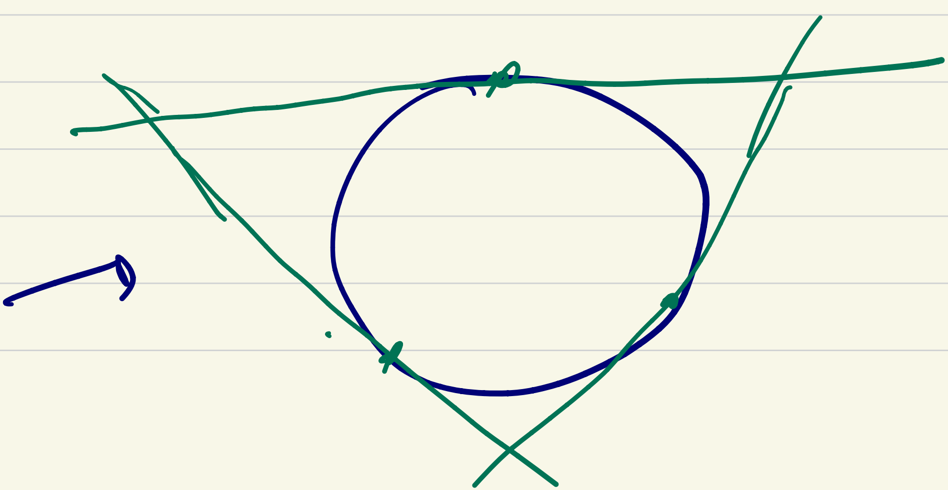
⇝ quartic + three lines

blow up





blow down



conic + circumscribed triangle.

main game: try to figure out how to get simpler config's from a ~~notion~~ SRCC, & use them to construct objects

<u>rectifiability</u>: it means that there is a birational transf. to a <u>line</u>.

Q How do I find these auxiliary config's?

<u>thm</u> (McDuff) If (X, ω) is any sympl. 4-mfld (closed) $\Sigma \subset X$ is a sympl. +1-sphere,

$$(\tilde{X}, \omega) \xrightarrow{\text{blow down}} (\mathbb{CP}^2, \omega)$$

$\tilde{\Sigma}$ ⟶ line

$\rightsquigarrow$ It gives you a way to construct birational transf.

Obstructions :

main observation : If $X_n^q(k) \hookrightarrow \mathbb{P}^2$, what can I say about the complement?

$$X_n^q(k) \simeq_{h.eq.} S^2$$

Prop Complement $W = \mathbb{P}^2 \setminus X_n^q(k)$ is a rational homology ball
↳ rational coeff.

$$\tilde{H}_*(W;\mathbb{Q}) = 0.$$

$$\rightsquigarrow \quad \tilde{H}_*(W;\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z} \text{ in degree } 1$$

degree of PL-sphere $\quad d^2 = n.$

Cor If d is odd $\Rightarrow$ W is spin.

Lots of obstructions kick in:

$\rightsquigarrow$ Heegaard Floer homology
Borodzik & Livingston

$\rightsquigarrow$ Involutive HF
Borodzik & Hom

In odd degrees: the Rokhlin invariant.

$$\partial X^4_n(k) = S^3_n(k)$$

$$\mu(S^3_n(k)) = 8\cdot \mathrm{Arf}(k) + n - 1 \in \mathbb{Z}/16\mathbb{Z}.$$

ex If $X_n(k) \hookrightarrow \mathbb{P}^2$, n odd

$\Rightarrow$ $S^3_n(k)$ bounds a spin $\mathbb{Q}HB^4$

$\Rightarrow$ $\mu(S^3_n(k)) = 0$.

quintic $(d=5, \ n=25)$

$T(2,7)$, $T(2,7)$, $T(2,7)$

$\mathrm{Arf}(T(2,7)) = 0$.

$n = 25$, $\mu(S^3_n(k)) =$

$$0 + 25 - 1 \equiv 8 \ (16)$$

no such genus can exist. $\lightning$

Branched covers

main idea: If C is a degree-d curve, then exists a m-fold branched cover of $\mathbb{CP}^2$ branched over C. $\forall\, m$ dividing d.

ex $\Sigma_2(\mathbb{P}^2, \text{sextic}) = K3$.

point: you can do it for singular curves as well.

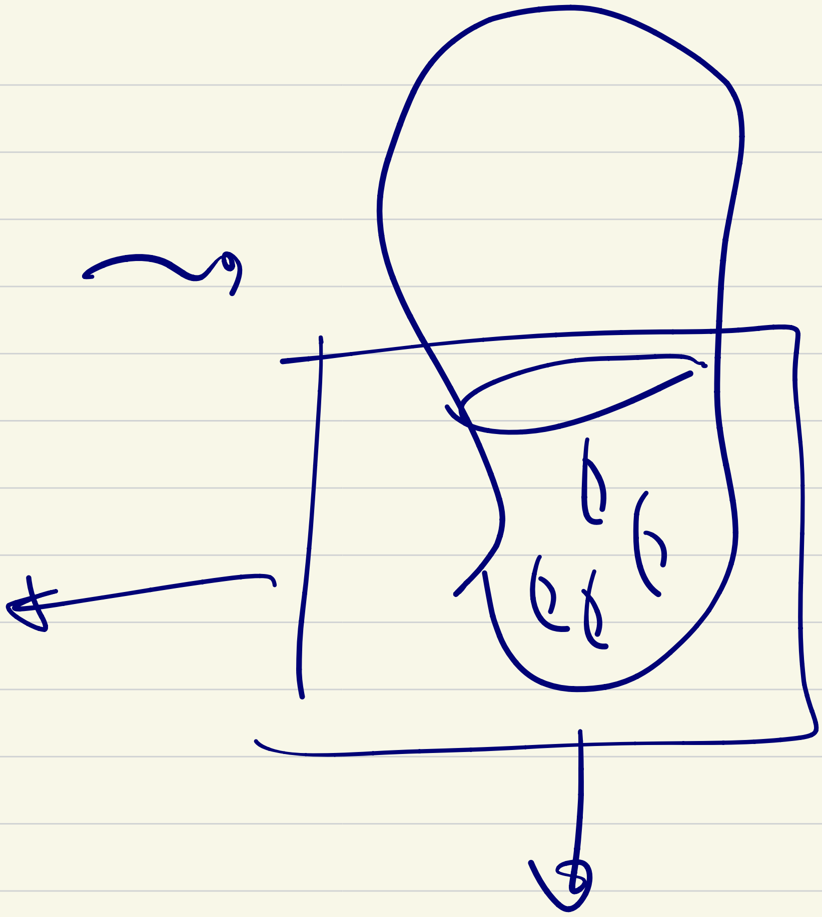
Idea: use singularities & their resolutions / smoothings against RCC. (PL-spheres).

here we can use global ideas
(G-signature theorem &
bounds on Betti numbers)
against local ideas
(branched covers of B^4
& Milnor fibres ...)

to get contradictions.

ex C degree 6, w/ one sing. of type $T(2,21)$

K :



$b_2^- = 20$

$K3$ = double cover

has $b_2^- = 19$. $\square$

$X_n(T_{a,b}) \hookrightarrow \mathbb{P}^2$

what can we say about n, a, b?

If this comes from a SRBC

$(a, b, n) \in (d, d+1, d^2)$

$(d, 4d-1, 4d^2)$

$(3, 22)$

$(\cancel{5}6, 43)$

(F_{odd}, F_{odd+4})

(F^2_{odd}, F^2_{odd+2})

Fernandez de Bobadilla, Luengo, Melle-Hernández, Némethi (2006)