Motivation:

In the beginning instantons were studies by mathematical physicists. Donaldson discovered its spectacular applications to 3- and 4-manifold topology in the 80's. To mention some:

* If X is a smooth (negative) definite 4-mfld, then its intersection form is diagonalizable over $\mathbb{Z}$ (Donaldson)
  (Consequence: many top'l 4-mflds have no smooth structure)
* There are some topological 4-manifolds which admit infinitely countably many different smooth structures. (Donaldson, Freedman-Morgan, Okonek-van-de-Ven)
* The Brieskorn homology spheres generate a $\mathbb{Z}^\infty$ in the homology cobordism group (Furuta)
* Failure of the h-cobordism theorem (Donaldson)
* Uncountably many smooth str. on $\mathbb{R}^4$ (Taubes)
* "Property P" conjecture for knots (Kronheimer-Mrowka)
* Khovanov homology detects the unknot (Kronh.-Mr.)
* Every $Y = \mathbb{Z}$-homology-$S^3 \neq S^3$ has an irred. rep $\pi_1(Y) \to SL(2,\mathbb{C})$. (Z.)

X a compact Riemannian mfld, possibly with bdry

$G \subseteq \mathrm{Aut}(V)$ — compact Lie group; V a vector space

$\mathfrak{g} \subseteq \mathrm{End}(V)$ the Lie algebra of G

$\pi : P \to X$ a principal G-bundle — locally a product $G \times X$, has a free G-action

(convention: <u>right</u> action)

$E = P \times_G V$ the associated V-vector bundle

<u>Exercise</u>: P is globally trivial iff it has a global section.

$\mathrm{ad}(P) := P \times_{\mathrm{ad}} \mathfrak{g} \subseteq \mathrm{End}(V)$ from the adjoint rep'n $G \times \mathfrak{g} \to \mathfrak{g}$, $(g, \xi) \mapsto g \xi g^{-1}$

$\mathrm{Ad}(P) := P \times_{\mathrm{Ad}} G \subseteq \mathrm{Aut}(E)$ not a principal G-bundle although a locally trivial G-bundle

diff. of the conjugation action at identity Ad.

$\mathrm{ad} = \mathrm{Ad}_{*,e}$

Differential forms with value in ad(P):

$\wedge \otimes [\cdot,\cdot] \; : \; \Lambda^i(T^*X) \otimes ad(P) \otimes \Lambda^j(T^*X) \otimes ad(P)$

$\longrightarrow \Lambda^{i+j}(T^*X) \otimes ad(P)$

wedge product on forms

Lie algebra bracket
$[\xi,\eta] = \xi\eta - \eta\xi$
for matrix Lie groups

Notation:
$[\cdot \wedge \cdot]$

$$[a \wedge b] = (-1)^{|a||b|+1} [b \wedge a]$$

different from ordinary diff. forms.

If $a = \sum a_i \xi_i$

$b = \sum b_j \eta_j$, then

$$[a \wedge b] = \sum \underbrace{a_i \wedge b_j}_{= (-1)^{|a||b|} b_j \wedge a_i} \overbrace{[\xi_i, \eta_j]}^{= (-1) [\eta_j, \xi_i]}$$

So for ad(P)-valued one-forms a we don't have

$[a \wedge a] = 0$ necessarily.

$\pi : P \longrightarrow X$ has a vertical tangent space

$$VTP = \ker d\pi$$

There is a canonical isomorphism

$$\frac{d}{dt}\Big|_{t=0} p e^{t\xi} \longleftarrow\!\shortmid\ \xi$$

$$\begin{tikzcd}
\iota_p : VT_pP \arrow[r] \arrow[d, "g"] & \mathfrak{g} \arrow[d, "ad(g^{-1})"] \\
\iota_{pg} : VT_{pg}P \arrow[r] & \mathfrak{g}
\end{tikzcd}$$

$\xi \longmapsto \xi^{\#}$
denoted the vector field

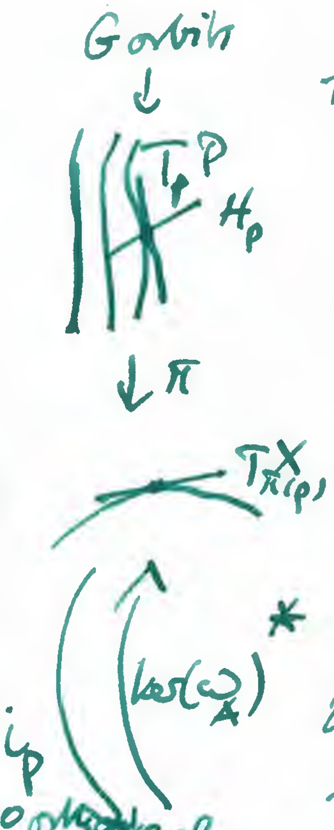
$\underline{\mathrm{Pf}}$: $$g_* \frac{d}{dt}\Big|_{t=0} p e^{t\xi} = \frac{d}{dt}\Big|_{t=0} pg\, g^{-1} e^{t\xi} g$$

acts on right

There is no canonical <u>complement</u> of $VTP \subseteq TP$.

Definition and Proposition:

A connection A in P is one of the following equivalent things:



* A G-invariant „horizontal" subbundle $H_A \subseteq TP$ transversal to the vertical tangent space $VTP \cong \ker(d\pi)$ (mapping isomorphically to TX via π)

$\circ$ orthogonal proj. from TP to VTP

* A $\mathfrak{g}$-valued 1-form ω_A on P which is ad-equivariant and which is i_p^* on the vertical tangent space (i.e. $\omega_A\left(\frac{d}{dt} p e^{t\xi}\right) = \xi$)

$d \circ H_A$
$\parallel$

* $d_A : \Omega^*_{ad}(P;\mathfrak{g}) \longrightarrow \Omega^{*+1}_{ad}(P;\mathfrak{g})$ $\boxed{\text{ad-equiv. and zero on } \cancel{\text{horizontal}} \text{ vertical vectors}}$
* A covariant derivative

$$d_A : C^\infty(X; ad(P)) \to C^\infty(X; ad(P) \otimes T^*X)$$

a cov. derivative mapping equiv. non-vertical forms to non-vertical forms

* A system of „parallel transport" in P („How to lift curves on X to P")

Curvature

proj. onto H_A

$$\tilde{F}_A \; \cancel{\Omega}_A := d\omega \circ H_A$$

~~Exercise~~: This is G-equivariant, and zero on vertical, hence defines

$$F_A \in \Gamma(\mathrm{ad}(P))$$

(maybe write F_A for all of these)

Interpretation: Ω_A measures the failure of H_A to be involutive

v, w horizontal vector fields

Then

$$\tilde{F}_A \; \cancel{\Omega}_A(v,w) = d\omega(v,w)$$

$$= \underbrace{v\,\omega(w) - w\,\omega(v)}_{\equiv 0} - \omega([v,w])$$

$$= -\omega([v,w])$$

So $\Omega_A \equiv 0 \iff H_A$ is involutive.

End of 1st lecture

Bundle automorphisms:

$$\mathrm{Aut}(P) = \left\{ u : \begin{array}{ccc} P & \xrightarrow{u} & P \\ & \searrow \quad \swarrow & \\ & X & \end{array} \quad \text{commuting with } G\text{-action} \right\}$$

$u \longmapsto \hat{u}$

$u(p) = p\hat{u}(p)$

$$\cong \left\{ \hat{u} : P \to G \quad G\text{-equiv'ly} \right\}$$

$$\cancel{\hat{u}(pg) = g\,\hat{u}(p)\,g}$$

$$u(pg) \stackrel{!}{=} pg\cdot\hat{u}(pg) \stackrel{!}{=} u(p)\cdot g = p\,\hat{u}(p)\cdot g$$

$$\Rightarrow \hat{u}(pg) = g^{-1}\hat{u}(p)g$$

$$\Longrightarrow \hat{u} \in \Gamma(\mathrm{Ad}(P))$$

Fact: $\mathrm{Map}^{\mathrm{ad}^{-1}}(P,G)$ $\quad \cancel{P\times_{\mathrm{ad}}G}$

$$\left(\cong \Gamma(\mathrm{Ad}(P))\right)$$

$$u \mapsto \check{u} \qquad \cong \Gamma(P\times_{\mathrm{ad}^{-1}}G)$$

via

$$\check{u}(x) = [p, \hat{u}(p)] \quad \text{is well-def.}$$

because $[pg, \hat{u}(pg)] = [pg, g^{-1}\hat{u}(p)g]$

$$= [p, \hat{u}(p)]$$

More gen'ly: $\rho : G \to \mathrm{End}(V)$

$$\{\alpha \in \Omega^*_{\mathrm{eq}}(P,V) \mid \alpha|_{TV} \equiv 0\} \cong \Omega^*(X; P\times_\rho V)$$

Cartan's formula:

$$\widetilde{F}_A = d\omega_A + \frac{1}{2}[\omega_A \wedge \omega_A]$$

Pf: Exercise: 3 cases:

$\widetilde{F}_A(v,w)$ where

- v vertical, w horizontal
- both vertical
- both horizontal

Defn $\boxed{\text{Ed}}$

If u is a gauge transformation, then $u(H_A)$ is a new connection

Exercise: $\widetilde{F}_{\tilde{u}(A)} = ad_{\tilde{u}^{-1}} \circ \widetilde{F}_A$

$$F_{u(A)} = ad_u \circ F_A = u \circ F_A \circ u^{-1}$$

↑ pointwise

If $s : U \to P$ is a section then for $a := s^* \omega_A$ one has

$$s^* \widetilde{F}_A = da + \frac{1}{2}[a \wedge a]$$

<u>Covariant derivative associated to A:</u>

$E = P \times_G V$

$e \in \Gamma(E)$ seen as $\underset{G\text{-equiv.}}{\tilde{e} : P \to V}$

$\widetilde{d_A e} := d\tilde{e} \circ H_A$, and project down.

<u>Exercise:</u> $d\tilde{e} \circ H_A = d\tilde{e} + (\rho_* \omega_A) e$

or for forms $\Omega^*(E)$

$\widetilde{d_A \alpha} = d\tilde{\alpha} + \underbrace{\rho_* \omega_A \wedge \tilde{\alpha}}_{\text{hybrid notation}}$

<u>Pf:</u> $\xi \in \mathfrak{g}$

$$0 = (d\tilde{e} \circ H_A)(\xi)_p = d\tilde{e}(\xi^\#)_p + \underbrace{\rho_* \omega_A(\xi^\#)}_{=\rho_*(\xi)} \tilde{e}(p)$$

$$= \frac{d}{dt}\Big|_{t=0} \tilde{e}(p e^{t\xi}) + \rho_*(\xi)\tilde{e}(p)$$

$$= \frac{d}{dt}\Big|_{t=0} \rho^{-1}(e^{t\xi})\tilde{e}(p) + \rho_*(\xi)\tilde{e}(p)$$

$$= -\rho_*(\xi)\tilde{e}(p) + \rho_*(\xi)\tilde{e}(p) \qquad \boxed{}$$

Again, in a trivialization $s : U \to P$ (local section)

$s^* d_A e = de + [a, e]$

<u>Prop:</u> The space of ($\mathcal{C}^\infty$-) connections is an affine space over $\Omega^1(X; ad(P))$

$= \mathfrak{g}$-valued 1-forms on P which are zero vertically

$$= \Omega^1_{\substack{\text{equiv.}\\\text{non-vert}}}(P; \mathfrak{g})$$

<u>Pf:</u> $\omega_A - \omega_{A'}$ is equiv. and zero vertically $\square$

$$a \in \Omega^1_{\substack{\text{non-}\\\text{vert. equiv}}}(P; \mathfrak{g}) \cong \Omega^1(ad(P))$$

<u>Prop:</u> $\tilde{F}_{A+a} =$

$$= d(\omega_A + a) + \frac{1}{2}[(\omega_A + a) \wedge (\omega_A + a)]$$

$$= d\omega_A + da + [\omega_A \wedge a] + \frac{1}{2}[a \wedge a] + \frac{1}{2}[\omega_A \wedge \omega_A]$$

$$= \tilde{F}_A + d_A a + \frac{1}{2}[a \wedge a]$$

$\square$

<u>Bianchi-identity:</u>

$$0 = d_A F_A$$

<u>Proof:</u> Expand and get

$$d_A F_A = \frac{1}{2}[\omega_A \wedge [\omega_A \wedge \omega_A]]$$

$= 0$ by „Jacobi-identity" on $\Omega^*(P; \mathfrak{g})$

$$\left([a \wedge [b \wedge c]] + (-1)^{|a|(|b|+|c|)} [b \wedge [c \wedge a]] + (-1)^{|b|(|a|+|c|)} [c \wedge [a \wedge b]] = 0 \right)$$

„super Jacobi identity"

(skipped)

Chern-Weil theory

Φ a homogeneous polynomial
$\mathfrak{g} \to \mathbb{R}$ of deg k,
see as
$\Phi : \mathfrak{g}^k \to \mathbb{R}$ invariant under permutation

ad-invariant:
$\Phi(\mathrm{ad}_g \xi_1, \ldots, \mathrm{ad}_g \xi_k) \overset{!}{=} \Phi(\xi_1, \ldots, \xi_k)$
$\forall g$

Apply this to $t \mapsto e^{t\eta}$ and diff. at $t=0$

$$\Rightarrow 0 = \Phi([\eta, \xi_1], \xi_2, \ldots, \xi_k) + \Phi(\xi_1, [\eta, \xi_2], \ldots) + \ldots \quad (*)$$

A a connection on $P \to X$

$$c_\Phi(A) := \Phi(F_A, \ldots, F_A) \in \Omega^{2k}(P)$$

as form on P is equiv. and zero vertically $\Rightarrow \pi^*$ form on X

* This is a closed form:

$$dc_\Phi(A) = k \cdot \Phi(dF_A, F_A, \ldots, F_A)$$
$$\overset{(*)}{=} k \cdot \Phi(\underbrace{dF_A + [\omega_A \wedge F_A]}_{= d_A F_A \equiv 0 \text{ by Bianchi identity}}, F_A, \ldots, F_A)$$
$$= 0$$

$$\boxed{\pi^* d(\tilde{c}_\phi(A)) = d\,\pi^*\tilde{c}_\phi(A)}$$

$[c_\phi(A)]$

* does not depend on A:

A' another

$$A_t = A + ta$$

$$F_{A_t} = F_A + t\,d_A a + \frac{t^2}{2}[a \wedge a]$$

$$\Rightarrow \frac{d}{dt} F_{A_t} = d_A a + t\,[a \wedge a] = d_{A_t} a$$

$$\frac{d}{dt}\, c_\phi(A_t) = k\, c_\phi(d_{A_t} a, F_{A_t}, \dots, F_{A_t})$$

as above

$$= k \cdot d\, c_\phi(a, F_{A_t}, \dots, F_{A_t})$$

$$\Rightarrow c_\phi(A') - c_\phi(A) = \int_0^1 \left(\frac{d}{dt} c_\phi(A_t)\right) dt$$

$$= k \cdot d \underbrace{\left(\int_0^1 c_\phi(a, F_{A_t}, \dots, F_{A_t})\, dt\right)}_{=\ \text{Chern-Simons function of } A' \text{ rel. } A}$$

so the difference is exact. ☐

$$\rightsquigarrow c_\phi(P) \in H^{2k}(X)$$

Example: $\boxed{\det(1+t\xi) = \sum_{k=0}^{\infty} t^k \phi_k(\xi)}$ ← invariant polyn.

— 14 —

If one is given a complex vector bundle $E \longrightarrow X$ ($U(n)$ - principal bundle is the associated frame bundle),

then

$$c_k(E) = \left[\phi_k\left(\frac{i}{2\pi} F_A\right)\right] \in H^{2k}(X)$$

are the (real images) of the Chern-classes.

Exercise: $c_1(E) = \left[tr\left(\frac{i F_A}{2\pi}\right)\right]$

$c_2(E) = \left[\frac{1}{8\pi^2} tr(F_A \wedge F_A)\right]$

If $G = SO(3)$

$$k = -\frac{1}{4}\langle p_1(P), [X]\rangle$$

$$= -\frac{1}{8\pi^2}\int_X tr(F_A \wedge F_A)$$

End of 2nd lecture

(X, g) an oriented Riemannian mfld

$\hat{}$ to make it explicit

$g : TX \otimes TX \to \mathbb{R}$

induces

$g^\# : TX \xrightarrow{\simeq} T^*X$,

and hence get a metric

$g^* : T^*X \otimes T^*X \to \mathbb{R}$,

and this extends to higher tensor powers and differential forms

Hodge star:

$\Lambda^k T^*X^n \to \Lambda^{n-k} T^*X$

via the metric as follows:

If (e^i) is an orthonormal coframe $(e^i \in T^*X,\ g^*(e^i, e^j) = \delta^{ij})$, ~~s.th.~~ s.th.

$$vol_g = e^1 \wedge \dots \wedge e^n$$

compatible with orientation,

define $* e^I := e^J$ s.th. $e^I \wedge e^J = e^1 \wedge \dots \wedge e^n$

Here $I \subseteq \{1, \dots, n\}$, J complementary

I, J ordered tuples

Example: With $n = 4$ (e^1, e^2, e^3, e^4)

$$* e^2 = - e^1 \wedge e^3 \wedge e^4$$
$$* e^1 = e^2 \wedge e^3 \wedge e^4$$
$$* e^1 \wedge e^2 = e^3 \wedge e^4$$
$$* e^1 \wedge e^3 = - e^2 \wedge e^4$$

Exercise: Under conformal change of the metric

$$g \longmapsto t^2 g \qquad \text{(unit length scales to } t\text{)}$$

$$*|_{\Lambda^k} \longmapsto t^{n-2k} \cdot *|_{\Lambda^k}$$

In particular for $n = 2k$ the Hodge star is conformally invariant.

Exercise: $*^2 = (-1)^{k(n-k)}$

For $n=4$, $k=2$

$$*: \Lambda^2 \to \Lambda^2 \quad \text{has} \quad *^2 = id,$$

so get

$$\underset{\dim = 6}{\Lambda^2} = \underset{\dim = 3}{\Lambda^2_+} \oplus \underset{\dim = 3}{\Lambda^2_-}$$

In the frame (e^1, e^2, e^3, e^4) we have

$$\Lambda^2_+ = \langle e^1 \wedge e^2 + e^3 \wedge e^4,\ e^1 \wedge e^3 - e^2 \wedge e^4,\ e^1 \wedge e^4 + e^2 \wedge e^3 \rangle$$

$$\Lambda^2_- = \langle e^1 \wedge e^2 - e^3 \wedge e^4,\ e^1 \wedge e^3 + e^2 \wedge e^4,\ e^1 \wedge e^4 - e^2 \wedge e^3 \rangle$$

Exercise (or if you are too lazy take it as definition)

If $\omega \in \Lambda^k$, then

$$|\omega|^2 := g^*(\omega,\omega) = \frac{\omega \wedge *\omega}{vol_g} \qquad (\sout{some\ are}$$

i.e. $g^*(\omega,\omega) \cdot vol_g = \omega \wedge * \omega$

and

$$\| \omega \|_{L^2(X,g)} = \int_X \omega \wedge * \omega .$$

Exercise:

Observe that $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ is an orthonormal decomp.,

and also $\omega \wedge \eta = 0$ if ω is $\substack{ASD\\SD}$ and η is $\substack{SD\\ASD}$.

(Pf: $\underset{sd}{(\omega},\underset{asd}{\eta)}\, vol_g = \omega \wedge * \eta = - \omega \wedge \eta$
$= - * \omega \wedge \eta = - \eta \wedge * \omega$
$= -(\eta, \omega)\, vol_g$
$= -(\omega, \eta)\, vol_g$

$\omega \wedge \eta = - \omega \wedge * \eta = 0$ by above.)

The Yang-Mills functional

$$E(A) = -\int_X \mathrm{tr}(F_A \wedge * F_A)$$
$$= \|F_A\|^2_{L^2}$$

hybrid notation for
$-\mathrm{tr} \otimes (\_ \wedge * \_)$
which is the tensor product ~~for~~ of two metrics (inner products)

On matrices: $\xi, \eta \in M_n(\mathbb{C})$ adjoint

$$(\xi, \eta) = \mathrm{tr}(\xi \eta^*)$$
$$= \sum_{i,j} \xi_{ij} \cdot \overline{\eta_{ij}}$$

On $su(n)$ or $u(n)$ have
$$\eta^* = -\eta \quad , \text{ so}$$
$$(\xi, \eta) = -\mathrm{tr}(\xi\eta)$$

The Euler Lagrange equs for E are:

automatically satisfied by Bianchi id. $\longrightarrow$ $d_A F_A = 0$

"Yang-Mills eqn" $\longrightarrow$ $d_A^* F_A = 0$

$d^* = \pm * d *$ on forms

Here d_A^* is the ~~[illegible]~~ formal adjoint of d_A

Decompose the curvature:

$$F_A = \underset{\text{self-dual}}{F_A^+} + \underset{\text{anti-self dual part}}{F_A^-}$$

A connection A is called self-dual (SD) / anti-self-dual (ASD)

if $F_A^- = 0$ / $F_A^+ = 0$.

Solutions of $F_A^+ = 0$ are called <u>instantons</u>

<u>Observation:</u> If F_A is (anti-) selfdual,

then $d_A^* F_A = - * d_A * F_A \underset{\substack{\text{follows from} \\ \text{Bianchi identity}}}{=} 0$

$$\underset{\text{"instanton number"}}{8\pi^2 k} = \int_X tr(F_A \wedge F_A)$$

$$E(A) = -\int_X tr(F_A \wedge * F_A)$$

$$\Rightarrow \begin{cases} E(A) + 8\pi^2 k = \int_X tr(F_A \wedge \overbrace{(-*F_A + F_A)}^{=2F_A^-}) = +2\|F_A^-\|_{L^2}^2 \\ E(A) - 8\pi^2 k = -\int_X tr(F_A \wedge \underbrace{(F_A + *F_A)}_{=2F_A^+}) = 2\|F_A^+\|_{L^2}^2 \end{cases}$$

So $E(A) \geq 8\pi^2 |k|$ with equality iff $F_A^- = 0$ resp. $F_A^+ = 0$

Notice also that for an instanton

* $E(A)$ is a topological quantity

* $k \geq 0$

The basic instanton

$$S^7 \subseteq \mathbb{H}^2$$

↑ 2-dim. quaternionic space

$$SU(2) \cong Sp(1) = \{x \in \mathbb{H} \mid x\bar{x} = 1\}$$

$Sp(2)$ acts by isometries transitively on S^7.

(by the left or the right)

So get two different $SU(2)$-actions

$$(x,y)\cdot q = (xq, yq)$$
$$(x,y)\cdot q = (\bar{q}x, \bar{q}y)$$

$$S^7/Sp(1) \cong \mathbb{HP}^1 \cong S^4$$

get two different $SU(2)$-principal bundles

$$P_\pm \longrightarrow S^4$$

P_+ has right action

P_- has left action

Declare horizontal subspace $A_\pm$ of TS^7 to be <u>orthogonal</u> <u>complement</u> of $VTS^7 = \ker(d\pi)$
$\pi : S^7 \to S^4$.

<u>Später:</u>
$Sp(2)$ acts on S^7 isometrically, so preserves $A_\pm$, and it induces
<u>Exercise!</u> $SO(5)$-action on S^4.

Larger group $Sl(2,\mathbb{H})$ acts on S^7 by <u>conformal transformations</u>.

As the (A)SD equs are

<u>Exercise:</u> A_+ at $(x,y) \in S^7 \subseteq \mathbb{H}^2$ has connection 1-form

$$\omega_{A_+} = \operatorname{Im}(\bar{x}\,dx + \bar{y}\,dy)$$

Curvature formula is complicated, but

$$(F_{A_+})_{(0,1)} = d\bar{x} \wedge dx$$

$=$ pull-back from S^4 of an asd 2-form!

So A_+ is an ASD instanton!

$$x = x^1 + x^2 i + x^3 j + x^4 k ,$$

then the coefficients of

$$dx \wedge d\bar{x} = -2\Big[(dx^1 \wedge dx^2 + dx^3 \wedge dx^4)\, i + (dx^1 \wedge dx^3 - dx^2 \wedge dx^4)\, j + (dx^1 \wedge dx^4 + dx^2 \wedge dx^3)\, k \Big]$$

form a basis of self-dual 2-forms on $\mathbb{R}^4 \cong \mathbb{H}$, and those of

$$d\bar{x} \wedge dx = 2\Big[(dx^1 \wedge dx^2 - dx^3 \wedge dx^4)\, i + (dx^1 \wedge dx^3 + dx^2 \wedge dx^4)\, j + (dx^1 \wedge dx^4 - dx^2 \wedge dx^3)\, k \Big]$$

form a basis of anti-self-dual 2-forms.

---

inner product $((x,y),(z,w)) := x\bar{z} + y\bar{w}$

$\mathrm{Re}\,(-,-) =$ standard inner pr. on $\mathbb{R}^8 \cong \mathbb{H}^2$.

The ASD equs are conformally invariant, so the action of ~~the conformal~~ $Sl(2,\mathbb{H})$ produces other asd-instantons from A_+.

$Sp(2)$ acts isom. on S^7 $\Rightarrow$ preserves A_+
$Sl(2,\mathbb{H})$ acts on S^7 and produces other instantons because of conformal transformations

<u>Thm</u> [Atiyah-Hitchin-Singer]: All instantons on $S^7 = P_+$ are obtained this way.

induces $SO(5,1)$-action on S^4 = conformal transformations

<u>Thm</u> $\Rightarrow$ The moduli space is

$$B^5 = Sl(2,\mathbb{H}) / Sp(2)$$

preserves A_+

$$= SO(5,1) / SO(5).$$

~~minus the isometry~~

This ball is parametrized by the conformal transformations of S^4 which in stereographic projections are

$$\mathbb{R}_{>0} \times S^4 \longrightarrow SO(5,1)/SO(5)$$

$$(\lambda, b) \longmapsto T_{\lambda,b}$$

where $T_{\lambda,b}(x) = \lambda(x-b)$

~~[illegible]~~: translation by ~~by~~ $-b$
homothety by λ

if one identifies $T_{\lambda,b} \sim T_{1/\lambda, b^*}$

b^*: anti-podal of b

($T_{1,b}$ acts by identity $\forall b$)

The instanton

$T^*_{\lambda, b}$ (H) is said to have

<u>center</u> b,

<u>scale</u> λ if $\lambda \neq 1$.

A_+ is centerless.

Get a <u>section</u> of $S^7 \to S^4$ over $\mathbb{H} \cong \mathbb{R}^4 \subseteq S^4 \cong \mathbb{HP}^1$ by $x \mapsto [x,1] \in \mathbb{HP}^1$

$$s : x \longmapsto \frac{(x,1)}{\sqrt{1+|x|^2}}$$

<u>Exercise</u>:

$$s^* A_+ = \frac{\operatorname{Im}(\bar{x}\, dx)}{1+|x|^2}$$

and

$$s^* F_{A_+} = \frac{d\bar{x} \wedge dx}{(1+|x|^2)^2}$$

Now T_λ is rep. by $x \mapsto \lambda x$

$$T_\lambda^* s^* A_+ = \frac{\operatorname{Im}(\lambda \bar{x}\, \lambda dx)}{1 + |\lambda x|^2}$$

$$= \frac{\operatorname{Im}(\bar{x} dx)}{\frac{1}{\lambda^2} + |x|^2}$$

and $T_\lambda^* s^* F_{A^+} = \dfrac{d\bar{x} \wedge dx}{\left(\frac{1}{\lambda^2} + |x|^2\right)^2}$

Important
Observation:

* curvature concentrates at 0 as $\lambda \to \infty$ (and at ∞ as $\lambda \to 0$)

* In the limit $\lambda \to \infty$ one has

$$\lim T_\lambda^* s^* A_+ = \frac{\operatorname{Im}(\bar{x} dx)}{|x|^2}$$

away from zero,

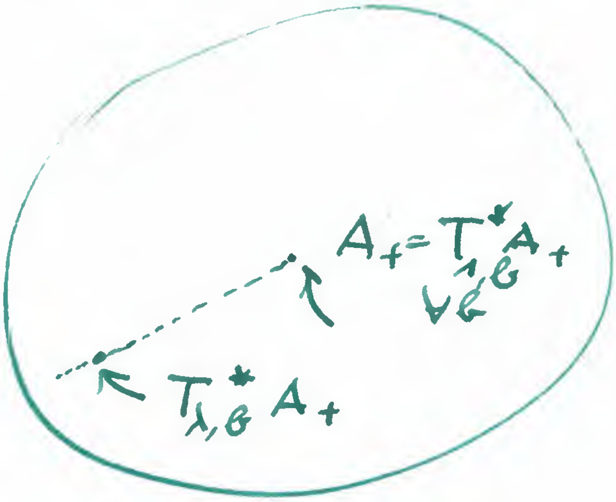
and thus is gauge equivalent to zero(!) by $g(x) = \frac{x}{|x|}$

* Idem with $T_{\lambda, b}^* s^* A_+$ at b

So the moduli space of (asd) instantons on $SU(2) \to S^7 \to S^4$

is

$\mathring{B}^5 \cong$



has a natural compactification equal to S^4, which can be interpreted as <u>concentration points</u>

Connections & gauge transformations

$\mathcal{A}$ = space of connections on $P \to X$

$\cong$ affine space over $\Omega^1(X; ad(P))$

$= \Gamma(X; \Lambda^1 T^*X \otimes ad(P))$

$$\begin{tikzcd} P \arrow[rr, "u"] \arrow[dr] & & P \arrow[dl] \\ & X & \end{tikzcd}$$

$u : P \xrightarrow{u} P$ bundle automorphisms

$u(pg) = u(p)g$

(commutes with g action)

$Aut(P) \cong \Gamma(X; Ad(P))$

= „group of gauge transformations"

$=: \mathcal{G}$

$\mathcal{G}$ acts on $\mathcal{A}$:

$$d_{u(A)} = u d_A u^{-1}$$
$$= d_A + u(d_A u^{-1})$$

$$u(A) = A + u(d_A u^{-1}),$$

where $d_A u^{-1}$ is the covariant derivative of u^{-1}, seen as a section of $End(E)$, where

$$E = P \times_G V$$

$\mathfrak{g} \in End(V)$

$G \subseteq Aut(V)$

$\mathfrak{g} \subseteq End(V)$.

Ultimately we will consider the moduli space of ASD instantons:

$$\mathcal{M}_p = \{ A \in \mathcal{A} \mid F_A^+ = 0 \} / \mathcal{G}$$
$$\subseteq \mathcal{A}/\mathcal{G}$$

($F_{u(A)} = u \circ F_A \circ u^{-1} = \mathrm{ad}_u (F_A)$

acts on the Lie algebra component

and so $F_A^+ = 0 \implies F_{u(A)}^+ = 0$,)

Lemma: $u(A) = A \iff u$ is an A-parallel section of $\mathrm{End}(E)$

Pf: $u(A) = A \iff u \, d_A u^{-1} = -(d_A u) u^{-1} \equiv 0$

$\iff d_A u \equiv 0$ $\boxtimes$

We have to work with Sobolev spaces and Fredholm operators (this is a convenient setup to do analysis)

For smooth sections f of some bundle ass. to $P \to X$ and a connection A we define

becomes tensor product with Levi-Civita connection

$$\|f\|^p_{L^p_{k,A}} = \int_X \sum_{i=0}^{k} |\nabla_A^i f| \, vol_g$$

$$L^p_{k,A} = \overline{C^\infty}^{\|\cdot\|_{L^p_{k,A}}}.$$

If $A - A' \in L^q_{l,A}$ and

$$L^p_k \times L^q_l \to L^p_{k-1}$$

is continuous, then the norms $L^p_{k,A}$ and $L^p_{k,A'}$ are equivalent.

<u>Embedding results:</u>

$$L^p_k(X) \hookrightarrow L^q_l \quad \text{if}$$

for the "weights"

"$w(L^p_k) := k - \frac{n}{p}$" we have

$$w(L^p_k) \geq w(L^q_l) \quad \text{and} \quad k \geq l.$$

And over a <u>compact mfld</u> the <u>embedding is compact</u> if $w(L^p_k) > w(L^q_l)$ and $k > l$ (Rellich Lemma)

We have

$$L^p_k(X) \times L^q_\ell(X) \longrightarrow L^r_m(X)$$

a bounded (continuous) multiplication

if $w(L^p_k) + w(L^q_\ell) \geq w(L^r_m)$

and $k, \ell \geq m$

Furthermore, if $w = k - \frac{n}{p} > 0$, then

L^q_ℓ is a module over L^p_k

Examples: $(n=4)$

$$L^2_1 \hookrightarrow L^4$$

$$L^2_2 \hookrightarrow L^4_1 \hookrightarrow L^p \qquad \forall p < \infty$$

$$L^2_2 \not\hookrightarrow C^0$$

$$L^2_1 \times L^2_1 \longrightarrow L^2$$

$$L^2_2 \times L^2_1 \longrightarrow L^p_1 \quad \text{or} \quad L^2_s$$

where $p < 2$
or $s < 1$.

„Stable range"

$$L^2_{k+1} \times L^2_k \longrightarrow L^2_k \qquad \text{for } k > 1.$$

<u>Rk</u> One can deal with ~~frac~~ real Sobolev indices s also L^2_s, interpolating the integer values.

<u>Advantage:</u> $L^2_s(X) \xrightarrow{\text{restr.}} L^2_{s-\frac{1}{2}}(\partial X)$

is bounded

Set this up
* by Fourier analysis
or * by Laplacian

$\Delta_{A_0} = \nabla^*_{A_0} \nabla_{A_0}$

has a spectrum ≥ 0

$\Rightarrow 1 + \Delta_{A_0}$ positive invertible

Spectral calculus

$\Rightarrow (1+\Delta_{A_0})^{-s/2}$ is well defined

and the image ~~is~~ ~~L^2_s~~

$(1+\Delta_{A_0})^{-s/2} : L^2 \to L^2$

is L^2_s.

Exercise: Get multiplication results from this description and

$$f\cdot g \rightsquigarrow (f_3)*(g_3)$$

Take $\mathcal{A}^p_k = L^p_k$- connections

$$\mathcal{G}^p_{k+1} = \{u \in L^p_{k+1}(X; ad(P)) \mid u \in Ad(P) \text{ a.e.}\}$$

If $k+1 - \frac{n}{p} > 0$ this consists of continuous sections.

Lemma:

For $G = U(n)$ or $SO(n)$ and $k+1-\frac{n}{p}>0$ this is a Banach Lie group

Pf: Get from bounded multiplication

$$m : L^p_{k+1}(X; End(E)) \to L^p_{k+1}(X; End(E))$$

$$B \mapsto BB^*$$

$$\mathcal{G}^p_{k+1} = m^{-1}(1).$$

Under the condition m is a smooth map. m has surjective differential, and kernel of differentials admit complements

Implicit fct thm $\Rightarrow$ get ~~[illegible]~~ mfld structure on $\mathcal{G}^p_{k+1}$.

Lemma:

For $k+1-\frac{n}{p}>0$ the map

$$\mathcal{G}^p_{k+1} \times \mathcal{A}^p_k \to \mathcal{A}^p_k$$

is a smooth map of Banach manifolds.

In a trivialization this is

Pf: $(a,g) \longmapsto gag^{-1} + g\,dg^{-1}$ $\boxed{F}$

Sobolev-multiplication

Lemma For $k+1-\frac{n}{p}$

$$\mathcal{B}^p_k = \mathcal{A}^p_k / \mathcal{G}^p_{k+1}$$ is a

Hausdorff topological space.

The Sobolev norms have a nice description using Fourier analysis on T^n:

$$f(x) = \sum_{\xi \in \mathbb{Z}} f_\xi \, e^{i \xi \cdot x}$$

Fourier series

and

$$\text{Fourier series} \xrightarrow{L^2} f \,,$$

with

$$\| f \|^2_{L^2(T^n)} = \| (f_\xi)_{\xi \in \mathbb{Z}} \|_{\ell^2}$$

(Plancherel)

Now for smooth C^∞ sections have convergence in C^∞.

$$\frac{1}{i^{|\alpha|}} \frac{\partial^{|\alpha|}}{\partial x^\alpha} f = \sum_{\xi \in \mathbb{Z}} \xi^\alpha f_\xi \, e^{i \xi \cdot x}$$

So derivatives $\hat{=}$ multipl. with ξ

and

$$\| f \|^2_{L^2_s} = \sum_{\xi \in \mathbb{Z}^n} (1 + |\xi|^2)^{s/2} |f_\xi|^2$$

(equivalent norms)

and Fourier series converges to φ in $L^2(T^n)$.

Prop ~~It does the C^0 if~~
Fourier ~~series~~ converges to φ in C^0 if
$\varphi \in C^{2nk}$ for $k \geq [\frac{n}{2}]+1$

Pf: Integrate (*) by parts and get

$$|\varphi_\xi| \leq \frac{\|\varphi\|_{C^{2nk}(T^n)}}{(\prod \xi_i)^{2k}} \quad \forall \xi \neq 0$$

product of all non-zero ξ

$$\leq \frac{c_k}{(1+|\xi|^2)^k} \quad \forall \xi$$

where c_k depends on $\|\varphi\|_{C^{2nk}(T^n)}$

Recall
In C^0, if $|f_n| \leq g_n$ $\forall n$
and for $S_N := \sum_{n=1}^{N} g_n$ the $(S_N)_N$ is a Cauchy seq.,
then for $T_N := \sum_{n=1}^{N} f_n$ the $(T_N)_N$ is a Cauchy seq. in C^0

So the Fourier series converges in C^0 if
$\sum_{\xi \in \mathbb{Z}^n} |\varphi_\xi|$ converges.

How fast do balls in $\mathbb{Z}^n$ with the $\|\cdot\|_\infty$-norm grow?

$$S_j = \{ \xi \mid \max |\xi_i| = j \}$$

Then $|S_j| \leq 2n \, (2j+1)^{n-1}$

pairs of faces

and

$$\forall \xi \in S_j : \quad |\xi|^2 \geq j^2$$

So

$$\sum_{\xi \in \mathbb{Z}^n} |\varphi_\xi| = \sum_{j=0}^{\infty} \sum_{\xi \in S_j} |\varphi_\xi|$$

$$\leq \sum_{j=0}^{\infty} \sum_{\xi \in S_j} \frac{c_k}{(1+|\xi|^2)^k}$$

$$\leq \sum_{j=0}^{\infty} \frac{2n \, (2j+1)^{n-1}}{(1+j^2)^k}$$

$$\leq C \sum_{j=0}^{\infty} \frac{1}{j^{1+2k-n}}$$

and the latter series converges if $1+2k-n > 1$

$\iff \quad k > \frac{n}{2}$

$\iff \quad k \geq \lfloor \frac{n}{2} \rfloor + 1$

So if $\varphi \in C^{2nk}$ and $k \geq [\frac{n}{2}] + 1$, then Fourier series converges in C^0 to some function $\Phi_\leftarrow$ continuous

$\varphi = \Phi$ because Fourier series $\longrightarrow \varphi$ in L^2.

Observation:

If $\varphi = \sum_\xi \varphi_\xi e^{ix\cdot\xi}$ converges in C^ℓ,

then $D^\alpha \varphi = \sum_\xi \xi^\alpha \varphi_\xi \; e^{ix\cdot\xi}$ $\forall \alpha$ with $[\alpha] \leq \ell$,

and $\|D^\alpha \varphi\|_{L^2(T^n)} = \sum_\xi \xi^{2\alpha} |\varphi_\xi|^2$

(mult. with $\xi^\alpha \,\hat{=}\, D^\alpha$ in Fourier decomp.

Sobolev spaces:

For $s \in \mathbb{Z}$
(in fact $s \in \mathbb{R}$ is fine)

$$H_s := \Big\{ (u_\xi)_{\xi \in \mathbb{Z}^n} \;\Big|\; \sum_\xi (1+|\xi|^2)^s |u_\xi|^2 < \infty \Big\}$$

is a Hilbert space (check!) with

$$\langle u, v \rangle_s = \sum_\xi (1+|\xi|^2)^s \, u_\xi \cdot \bar{v}_\xi$$

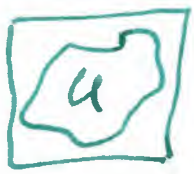
(even $|\langle u, v\rangle_s| \leq \|u\|_{s+t} \, \|v\|_{s-t}$ $\forall s,t$, We prove

* Rellich-lemma
* Sobolev-embedding result
* Fundamental elliptic estimate

on T^n. There we can use <u>Fourier series</u>.

Sufficiently general in that:

* For a general mfd M take charts mapping to bounded open sets $U \subseteq \mathbb{R}^n$, embed U in some torus.



* Work with a partition of unity subordinate to the cover

$\alpha = (\alpha_1, \dots, \alpha_n)$ multi-index
$\in (\mathbb{N} \cup \{0\})^n$

For $\eta \in \mathbb{R}^n$

$$\eta^\alpha = \eta_1^{\alpha_1} \cdot \dots \cdot \eta_n^{\alpha_n}$$

$$[\alpha] = \sum \alpha_i$$

Derivatives:

$$D^\alpha u := \left(\frac{1}{i}\right)^{[\alpha]} \frac{\partial^{[\alpha]} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

$\mathcal{P} = (2\pi)^n$ - periodic $\overset{\text{smooth}}{\text{fcts}}$ on $\mathbb{R}^n$

$Q = [0, 2\pi]^n$ n-cube

If $\varphi \in \mathcal{P}$, $\xi \in \mathbb{Z}^n$ inner pr. on $\mathbb{R}^n$

$$(*) \qquad \varphi_\xi = \frac{1}{(2\pi)^n} \int_{T^n} \varphi(x)\, e^{-ix \cdot \xi}\, dx$$

Fourier series:

$$\mathcal{F}[\varphi](x) := \sum_{\xi \in \mathbb{Z}^n} \varphi_\xi\, e^{ix \cdot \xi}$$

<u>Fact:</u>

$$L^2(T^n) \longrightarrow \ell^2(\mathbb{Z}^n)$$
$$\varphi \longmapsto (\varphi_\xi)_{\xi \in \mathbb{Z}^n}$$

is an isometry (Plancherel),

Above we have proved the

Sobolev lemma:

If $t \geq \left[\frac{n}{2}\right] + 1$, $u \in H_t$, then

$\sum u_\xi e^{ix\cdot\xi}$ converges uniformly in $C^0(T^n)$.

Exo: Hence get a bounded inclusion

$$H_t \hookrightarrow C^0$$

Cor: If $u \in H_t$ $t \geq \left[\frac{n}{2}\right] + 1 + m$, then $D^\alpha u = \sum \xi^\alpha u_\xi e^{ix\cdot\xi}$ converges uniformly for $[\alpha] \leq m$.

Exercise: If $t > \left[\frac{n}{2}\right] + 1$ then

$$H_t \hookrightarrow C^0 \text{ is compact}$$

For the proof, see next lemma:

<u>Thm</u> (Arzéla-Ascoli)

$(f_n)_{n\in\mathbb{N}} : \underset{\substack{\text{compact}\\ \text{interval}\\ \subseteq\mathbb{R}}}{I} \longrightarrow \mathbb{R}$ a family

of functions s.th.

* $\{f_n(x) \mid n\in\mathbb{N}\}$ is bdd $\forall x$
* (f_n) is <u>equicontinuous</u>,

then $\exists$ subsequence which converges uniformly

<u>Rk</u>: If (f_n) are C^1-functions which are Lipschitz-cont. with same constant $\forall n$ (implied if (f_n') are uniformly bounded), then (f_n) is equicontinuous

<u>Pf</u>: $(x_i)_{i\in\mathbb{N}} \subseteq I$ a countable dense sequence. $\forall x_i$

$(f_n(x_i))_{n\in\mathbb{N}}$ has a convergent subsequence (Bolzano-Weierstrass)

Diagonalization $\Rightarrow \exists$ subsequence $(n_k)_{k\in\mathbb{N}}$ s.th. $(f_{n_k}(x_i))_{k\in\mathbb{N}}$ converges $\forall i$.

Let $\varepsilon>0$.

$\forall i \quad \exists \quad N = N(\varepsilon, i)$ s.th.

$$|f_{n_k}(x_i) - f_{n_\ell}(x_i)| < \frac{\varepsilon}{3} \qquad \forall k, \ell \geq N.$$

$\exists$ open $U_i \ni x_i$ s.th.

$$x \in U_i \Rightarrow |f_n(x) - f_n(x_i)| < \frac{\varepsilon}{3} \quad \forall n$$

(equicontinuity)

I is compact $\Rightarrow$ cover I by finitely many of the $(U_i)_{i \in J}$ [finite J]

Take $N_\varepsilon := \max\{N(\varepsilon, i) \mid i \in J\}$ (J finite)

Let $x \in I$. Then $\exists\, i \in J$ mit $x \in U_i \ni x_i$.

$$|f_{n_k}(x) - f_{n_\ell}(x)|$$

$$\leq \underbrace{|f_{n_k}(x) - f_{n_k}(x_i)|}_{< \frac{\varepsilon}{3}} + \underbrace{|f_{n_k}(x_i) - f_{n_\ell}(x_i)|}_{< \frac{\varepsilon}{3}\ \forall k, \ell \geq N_\varepsilon} + \underbrace{|f_{n_\ell}(x_i) - f_{n_\ell}(x)|}_{< \frac{\varepsilon}{3}}$$

$$< \varepsilon$$

Rellich Lemma:

If $\left(\left(u_\xi^{(i)}\right)_{\xi\in\mathbb{N}}\right)^{i\in\mathbb{N}}$ is a sequence bounded in H_t, i.e.

$$\|u^{(i)}\|_t \le M \quad \forall i \text{ for some } M,$$

and if $s<t$, then $\exists$ subsequence which converges in H_s.

Pf: Wlog $M=1$, then

$$\sum_\xi (1+|\xi|^2)^t |u_\xi^i|^2 \le 1 \quad \forall i$$

For a fixed ξ $\left((1+|\xi|^2)^t |u_\xi^i|\right)_i$ is a bounded seq $\Rightarrow$ admits a convergent subseq. Diagonalization $\Rightarrow$ $\exists\ (i_j)_{j\in\mathbb{N}}$ s.th.

$$\left((1+|\xi|^2)^t |u_\xi^{i_j}|\right)_j \quad \text{converges } \forall \xi.$$

Claim: (u^{i_j}) is Cauchy in H_s.

Pf:

$$\|u^{i_j}-u^{i_k}\|_s^2 = \sum_{|\xi|<N} (1+|\xi|^2)^{s-t}(1+|\xi|^2)^t |u^{i_j}-u^{i_k}|^2$$

fin. many terms each $\to 0$

make $< \frac{\varepsilon}{2}$ by $j,k>J_\varepsilon$ for some J_ε

$$+ \sum_{|\xi|\ge N} \underbrace{(1+|\xi|^2)^{s-t}}_{\le N^{2(s-t)}}(1+|\xi|^2)^t |u^{i_j}-u^{i_k}|^2$$

make $< \frac{\varepsilon}{2}$ by N large enough

$\le 4N^{2(s-t)}$

Elliptic operators

On $\mathbb{R}^n$:

L a partial diff. op. of order ℓ

$$L = P_\ell(D) + \dots + P_0(D)$$

diff. op. hom. of order j

$$P_j(D) = \sum_{[\alpha]=j} a_\alpha D^\alpha$$

$$P_j(\xi) := \sum_{j=[\alpha]} a_\alpha \xi^\alpha \qquad \text{for } \xi = (\xi^1, \dots, \xi^n)$$

j^{th} symbol

For (a_α) constant

$$\begin{tikzcd}
u \text{ function/section on } \mathbb{R}^n \arrow[r, "\text{Fourier series}"] \arrow[d, "\text{diff. } L"'] & (u_\xi)_\xi \arrow[d, "L(\xi)\ \text{multipl. with symbols}"] \\
Lu \arrow[r, "\text{Fourier series}"] & (L(\xi)u_\xi)
\end{tikzcd}$$

Def^n: L is elliptic at x

if $P_\ell(\xi)$ is invertible $\forall \xi \neq 0$

L is elliptic if elliptic at each point.

<u>Example:</u> $\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$

<u>Laplacian</u> is elliptic on $\mathbb{R}^n$

$$\Delta(\xi) = -|\xi|^2 \cdot id$$

<u>Wave-operator</u>

$$K = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \quad \text{on } \mathbb{R}^2$$

$$\left(\text{or } K = -\Delta \cancel{} - \frac{\partial^2}{\partial t^2}\right)$$

on $\mathbb{R}^n \times \mathbb{R}$

$$K((\xi^1, \xi^2)) = (\xi_1)^2 - (\xi_2)^2$$

$\Rightarrow K$ is <u>not elliptic</u>

<u>Equivalent</u>
<u>Definition:</u> $L : \Gamma(E) \to \Gamma(F)$

L of order l is <u>elliptic</u> at x

if the <u>endomorphism</u> $\sigma_\xi(L) : E_x \to F_x$

$$(L(f^l u))(x) =: \sigma_\xi(L)\, u(x),$$

is an iso

where $f \in C^\infty$ with $f(x) = 0$, $df_x = \xi$

<u>Exercise:</u> Check this is equivalent!

<u>Advantage:</u> Extends to other manifolds than T^n,

i.e. more intrinsic definition

Fundamental elliptic inequality

L elliptic of order l on T^n, $s \in \mathbb{R}$
then $\exists\, c > 0$ s.th.

$$\|u\|_{s+l} \leq c\left(\|Lu\|_s + \|u\|_s\right)$$

$$\forall u \in H_{s+l}$$

Pf: (Only in the special case of constant coefficients, and $L = P_l(D)$ (only leading term))

If $u \in \mathbb{C}^\infty$, $\underset{E_x}{\overset{\Downarrow}{u}} \neq 0$, $\xi \neq 0$, then by assumption

$$|P_l(\xi) u|^2 > 0 \quad .$$

More precisely $\exists\, c > 0$ s.th.

$$|P_l(\xi) u|^2 \geq c \qquad \forall u, \forall \xi \text{ with } |u| = 1,\ |\xi| = 1 .$$

$$\Longrightarrow |P_l(\xi) u|^2 \geq c\, |\xi|^{2l} |u|^2 \qquad \forall u, \forall \xi$$

For $\varphi \in \mathcal{C}^\infty(T^n; E)$

$$\|L\varphi\|_s^2 = \sum_\xi |P_\ell(\xi)\varphi_\xi|^2 (1+|\xi|^2)^s$$

constant coeff!

$$\geq c \sum_\xi |\xi|^{2\ell} |\varphi_\xi|^2 (1+|\xi|^2)^s$$

$$\Rightarrow \|L\varphi\|_s^2 + \|\varphi\|_s^2 \geq \sum_\xi (1+|\xi|^2)^s (1+c|\xi|^{2\ell}) |\varphi_\xi|^2$$

$$\geq c' \sum_\xi |\varphi_\xi|^2 (1+|\xi|^2)^{s+\ell}$$

$$= c' \|\varphi\|_{s+\ell}^2 \qquad \blacksquare$$

$$u^h := \frac{T_h(u) - u}{|h|}$$

where $(T_h(u))(x) = u(x+h)$

$$\left((u^h)_\xi\right) = \left(\left(\frac{e^{ih\cdot\xi} - 1}{|h|}\right) u_\xi\right)$$

On voit que $\|T_h u\|_{H_s} = \|u\|_{H_s}$

(translation is an isometry in L^2)

$\Rightarrow u^h \in H_s$ if $u \in H_s$.

Now

$$\left|\frac{e^{ih\cdot\xi} - 1}{|h|}\right|^2 = \frac{2(1 - \cos(h\cdot\xi))}{|h|^2}$$

$$= \frac{4\sin^2(\frac{1}{2}h\cdot\xi)}{|h|^2}$$

$$\leq \frac{|h\cdot\xi|^2}{|h|^2}$$

$$\leq 1 + |\xi|^2 \qquad \forall h$$

So $\|u^h\|_s \leq \|u\|_{s+1}$.

The converse is true:

Lemma: Let $u \in H_s$, and suppose that $\|u^h\|_s \leq M \ \forall\, 0 \neq h \in \mathbb{R}^n$.
Then $u \in H_{s+1}$

Proof: For $N \in \mathbb{N}$ denote $u_N \in H_s$ the series truncated at N:

$$(u_N)_\xi = \begin{cases} u_\xi & \text{if } |\xi| < N \\ 0 & \text{otherwise} \end{cases}$$

Show that $\|u_N\|_{s+1}$ is uniformly bounded (in N).

Take $h = t \cdot e_i$
$\uparrow$ ith basis vector in $\mathbb{R}^n$
$(e_1, \dots, e_n)$ standard

$$(*) \quad \left|\frac{e^{ih\cdot\xi} - 1}{|h|}\right|^2 = \left|\frac{e^{it\xi_i} - 1}{t}\right|^2 \to |\xi_i|^2 \quad \text{as } (t \to 0)$$

$$M^2 \geq \|u_N^h\|_s^2 \Rightarrow \ \geq \sum_{|\xi|<N} (1+|\xi|^2)^s |u_\xi|^2 \underbrace{\left|\frac{e^{ih\cdot\xi} - 1}{|e_i|}\right|^2}_{\leq 2|\xi_i|^2}$$

$$\cancel{(*) \underset{\text{\& fin. many terms}}{\Longrightarrow} \sum_{|\xi|<N} (1+|\xi|^2)^s |u_\xi|^2 \, |\xi_i|^2}$$

$M^2 \geq$ ← ass

$$\| u_N^h \|_s^2 = \sum_{|\xi|<N} (1+|\xi|^2)^s |u_\xi|^2 \left| \frac{e^{ih\cdot\xi}-1}{|h|} \right|^2$$

~~finite~~ bounded distance from $|\xi_i|^2$ for $|h| \leq 1$

$$\geq \frac{1}{2} \sum_{|\xi|<N} (1+|\xi|^2)^s |u_\xi|^2 |\xi_i|^2$$

$$\Rightarrow \quad \| u_N \|_{s+1} \leq 2M^2 \cdot n + \| u \|_s$$

$\forall N$

$\blacksquare$

Theorem (Elliptic regularity)

Let L be elliptic of order l.

Assume $u \in H_{-\infty}$, $v \in H_t$ and

$$Lu = v .$$

Then $u \in H_{t+l}$

Proof (Only when L has constant coefficients)

Enough: $u \in H_s$ and $v = Lu \in H_{s-l+1}$
$\Rightarrow u \in H_{s+1}$

$$L(u^h) = (Lu)^h \quad \text{in this case}$$

$$\overset{\text{Fund. elliptic ineq.}}{\Longrightarrow} \|u^h\|_s \leq c\left(\|L(u^h)\|_{s-l} + \|u\|_{s-l}\right)$$

$$\leq c\left(\|v^h\|_{s-l} + \|u\|_{s-l}\right)$$

$$\overset{\text{above}}{\leq} c^*\underbrace{\left(\|v\|_{s-l+1} + \|u\|_{s-l}\right)}_{\text{is indep. of } h}$$

Lemma $\Rightarrow u \in H_{s+1}$.

<u>Lemma</u>: For $k+1-\frac{n}{p} \geq 0$

$$\mathcal{B}^p_k = \mathcal{A}^p_k \big/ \mathcal{G}^p_{k+1}$$ is a Hausdorff space

<u>Proof</u>: Do it for L^2_2 gauge transform. $w(L^2_2) = 0$

(general case similar)

~~Proof~~ Exercise: This is equivalent to saying that

$$\{(A, gA) \mid A \in \mathcal{A}^2_1, g \in \mathcal{G}^2_2\} \subseteq \mathcal{A}^2_1 \times \mathcal{A}^2_1$$ is <u>closed</u>.

Fix some $A_0 \in \mathcal{A}^p_k$

Suppose $(A_i, g_i A_i) \xrightarrow[\text{in } L^2_1]{} (A, B)$ $(i \to \infty)$

$A_i =: A_0 + a_i$ $\qquad A = A_0 + a$

$g_i A_i =: A_0 + b_i$ $\qquad B = A_0 + b$

Note $g_i A_i = A_0 + g_i d_{A_0} g_i^{-1} + g_i a_i g_i^{-1}$

$$\Rightarrow \begin{cases} g_i \underline{d_{A_0} g_i^{-1}} + g_i a_i g_i^{-1} = b_i \\ = -g_i^{-1} d_{A_0} g_i (g_i)^{-1} \end{cases}$$

$0 = d_{A_0}(g_i g_i^{-1}) = (d_{A_0} g_i) g_i^{-1} + g_i d_{A_0}(g_i^{-1})$

as sections of $\mathrm{End}(E)$

$$\Rightarrow \boxed{d_{A_0} g_i = g_i a_i - b_i g_i} \quad (*)$$

$a_i \longrightarrow a$ in L^2_1

$b_i \longrightarrow b$ in L^2_1

Need to show: $A = A_0 + a$ is gauge equiv to $B = B_0 + b$

$\boxed{-53-}$

G is compact $\Longrightarrow$ a_i bdd in L^4

b_i —— L^4

(Need to decompose here

$g_i = g_i^{\ker d_{A_0}} + g_i^{\perp}$

smooth, hence L^∞-bdd

controlled by symbol of $\nabla_{A_0} g_i$

$\Rightarrow \begin{cases} g_i a_i & \text{bdd in } L^4 \\ b_i g_i & \text{—— } L^4 \end{cases}$

(*) $\Longrightarrow$ d_{A_0} is injective

(g_i) is bounded in L^4_1

Differentiate (*) again and get

L^2 bdd

$$\nabla^2_{A_0} g_i = (\nabla_{A_0} g_i)a_i - b_i \nabla_{A_0} g_i + g_i(\nabla_{A_0} a_i) - (\nabla_{A_0} b_i) g_i$$

bdd in L^4, L^4, L^4; bdd in L^4; bdd in L^2; is L^2-bdd by Sobolev-mult.; bdd in L^2, in L^4_1; bdd in L^2

$L^4_1 \times L^2 \to L^2$

So (g_i) is L^2_2-bounded.

Pass to a subseq. where $g_i \longrightarrow g$ weakly and $g_i \to g$ in L^2_1 strongly (Rellich-Lemma). For a further subseq. g_i converges pointwise $\Rightarrow g^* g = 1$

$g_i \in Ad(P)$ a.e. $\Rightarrow g$ is a gauge transf.

Now

$$d_{A_0} g_i = g_i a_i - b_i g_i$$

conv. weakly in L^2_1

$w = 1 - \frac{4}{2} = -1$

$$d_{A_0} g_i \xrightarrow{\text{in } L^2} d_{A_0} g \qquad g_i \xrightarrow{\text{in } L^4} g \qquad a_i \xrightarrow{L^4} a$$

$w(L^r) = -\frac{4}{r}$

So in L^2 we have

$$d_{A_0} g = g a - b g$$

$\blacksquare$

Slices for the gauge group action

$\mathcal{A}_{3/2}$ $\qquad$ for completeness

$\mathcal{G}_{5/2}$ $\qquad$ (sufficient because $\mathcal{G}_{5/2} \hookrightarrow C^0$)

Slices: Take orthogon. complement of tangent spaces to orbits

$T_A(\mathcal{G}(A))$

$= d_A \Omega^0(X; ad(P))$

$L^2_{5/2}$ - completed

because

$$\frac{d}{dt}\Big|_{t=0} e^{t\xi}(A) = \frac{d}{dt}\Big|_{t=0}\left(A + e^{-t\xi} d_A e^{t\xi}\right)$$

$$= d_A \xi$$

gen. image of σ

$$d\, tr(\sigma \wedge *a) = tr(d_A\sigma \wedge *a) + tr(\sigma \wedge d_A * a)$$

$$\Rightarrow \quad -\int_X tr(d_A\sigma \wedge *a) = \int_{\partial X} tr(\sigma \wedge *a)$$

$$\langle d_A\sigma, a\rangle_{L^2(X)} \qquad + \langle \sigma, d_A^* a\rangle_{L^2(X)}$$

should hold $\forall \sigma$ $\qquad$ $d_A^* = - * d_A *$

Thus if $\partial X = \emptyset$ we want a to satisfy

$$d_A^* a = 0$$

and in addition $*a\big|_{\partial X} = 0$ if $\partial X \neq \emptyset$.

For some $A \in \mathcal{A}^2_{3/2}$

$$S_{s,A}(\varepsilon) = \{ A + a \mid a \in L^2_{3/2,A}(X; T^*X \otimes \mathrm{ad}(P)),$$

$$\# \; d_A^* a = 0$$

$$\|a\|_{L^2_{1,A}} < \varepsilon \}$$

$\mathrm{Stab}(A) \subseteq \mathcal{G}$ is a Lie group, finite dim'l

<u>Prop:</u> (Slice theorem)

$\forall A \in \mathcal{A}^2_{3/2}$ $\exists$ $\varepsilon > 0$ s.th.

$$\left(S_{3/2,A}(\varepsilon) \times \mathcal{G}_{5/2}\right) / \mathrm{Stab}_A \longrightarrow \mathcal{A}_{3/2}$$

$$[A + a, g] \longmapsto A + gag^{-1} - (d_A g) g^{-1}$$

is a $\mathcal{G}_{5/2}$-equiv. diffeo onto its image

<u>Big slices</u> works with

$$S_{1,A}(\varepsilon) \times \mathcal{G}_{5/2} / \mathrm{Stab}A \longrightarrow \mathcal{A}_{3/2}$$

and image contains an L^2_1-ball around A

Uhlenbeck's fundamental lemma

Let B = a geodesic ball in a Riemannian 4-manifold. Let Γ be the (flat) connection coming from a trivialisation $P = B \times G$ for a principal G-bundle (G compact) matrix Lie group. Then there are constants C, ε_0 s.th. for any connection $A = \Gamma + a$ with $a \in L^2_1(B, T^*B \times \mathfrak{g})$ and

$$\underbrace{\int_B tr(F_A \wedge * F_A)}_{(= \| F_A \|^2_{L^2(B)})} \leq \varepsilon_0$$

there is a $g \in L^2_2(B, M_n(\mathbb{C}))$ with $g \in G$ a.e. so that for

$$b = g a g^{-1} + g dg^{-1} \quad (= g(\Gamma + a) g^{-1})$$

we have

$$* d^* b = 0 \quad \text{and} \quad * b|_{\partial B} \equiv 0$$

and $*$ $$\underbrace{\int_B (|\nabla_\Gamma b|^2 + |b|^2) * 1}_{(= \|b\|^2_{L^2_1(B; \nabla_\Gamma^{\text{Levi-Civita}})})} \leq C \underbrace{\int_B |F_A|^2 * 1}_{(= \|F_A\|^2_{L^2})}$$

Proof: $\mathcal{B} = \exp_x(\underbrace{B_0(\delta)}_{\subseteq T_xX})$,

change metric to pull-back metric on $\mathcal{B}$ via $\exp_x^{-1}$.

One obtains equivalent norms, so up to adjusting the constants may suppose to have the metric from T_xX.

$\tilde{*}$ Hodge star for this metric.

The proof follows a 'continuity argument'

$$V_\varepsilon = \left\{ a \in L^2_{3/2}(\mathcal{B}; T^*\mathcal{B} \otimes \mathfrak{g}) \,\middle|\, -\int_{\mathcal{B}} tr(F_A \wedge \tilde{*} F_A) \leq \varepsilon \right\}$$

$W_\varepsilon \subseteq V_\varepsilon$ subset where the conclusion holds for some $C > 0$.

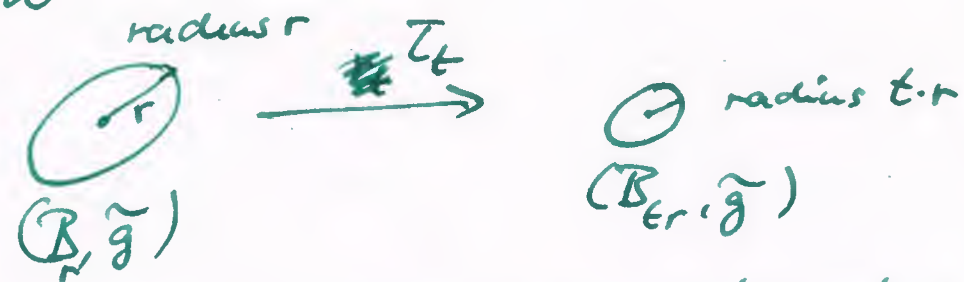
Show that for ε small enough:

* V_ε is connected
* W_ε is closed
* W_ε is open
* W_ε is non-empty

Prove for $L^2_{3/2}$ conn.
& $L^2_{5/2}$ gauge transf.,
then the other case by a continuity argument (all bounds will only be L^2_1-bounds!).

(1.) V_ε is connected:

Exploit the advantage that now



(in tangent space $r_t(x) = t \cdot x$)
is a conformal map:

$$\tau_t^* \tilde{g} = t^2 \cdot \tilde{g} .$$

(scaling along geodesics in image of $\exp_x$)

$$A_t := \tau_t^* (A|_{B_x(t\delta)})$$

L^2-norm of curvature is invariant

~~$-\int_B tr(F_{A_t}$~~

$$\int_B |F_{A_t}|^2_{\tilde g} \, vol_{\tilde g}$$

$$= \int_B r_t^* \left(|F_A|^2_{\tilde g} \, vol_{\tilde g} \right)$$

conformal invariance

$$\left\{ \begin{array}{l} r_t^* |F_A|^2_{\tilde g} = t^{-4} |F_{r_t^* A}|^2_{\tilde g} \\ \quad \{\text{2-forms scale like } t^{-2} \\ r_t^* \, vol_{\tilde g} = t^4 \, vol_{\tilde g} \end{array} \right.$$

transformation ~~rule~~ formula

$$= \int_{r_t(B)} |F_A|^2_{\tilde g} \, vol_{\tilde g} \qquad (\to 0 \ (t \to 0))$$

$$\leq \varepsilon \qquad \forall t \quad \text{if} \quad A \in V_\varepsilon .$$

Check: A_t is a continuous path in $L^2_{3/2}$ $\square$

(2.) W_ε is closed in V_ε :

$A_i = \Gamma + a_i$ a seq. in W_ε which converges in $L^2_{3/2}$ - top. to $A = \Gamma + a \in V_\varepsilon$

By ass. $\exists\, g_i$ so that

(*) $g_i \cdot A_i = \Gamma + b_i$, where the b_i are in Coulomb gauge and satisfy the estimate

(b_i) unif bounded in L^2_1

$\Rightarrow$ It has a subsequence converging weakly in L^2_1 to some $b \in L^2_1$.

Now

$$(*) \Leftrightarrow \quad dg_i = g_i a_i - b_i g_i \qquad (**)$$

$$\Gamma\; g_i \cdot A_i = g_i(\Gamma + a_i) g^{-1}$$
$$= g_i\, dg^{-1} + g_i a_i g^{-1} + \Gamma$$
$$= \Gamma + b_i$$

$\lrcorner$

Now a_i converges in $L^2_{3/2}$, hence in L^2_1, hence is bounded in L^2_1, hence in L^4 by $L^2_1 \hookrightarrow L^4$

So dg_i is bounded in L^4

$(\|b_i\|_{L^2_1}$ is bounded, $\|g_i\|_{L^\infty}$ is bounded$)$,

hence g_i $\overbrace{\text{is in}}^{\text{bounded}}$ L^4_1.

Differentiate $(**)$ and get

$$\nabla d g_i = \overbrace{(\nabla g_i)}^{\text{bdded in } L^4}\, \overbrace{a_i}^{\text{bdded in } L^4} + \overbrace{g_i}^{\substack{L^\infty \\ \text{bdded in}}}\, \overbrace{(\nabla a_i)}^{\text{bdded in } L^2} \qquad \overbrace{dg_i}^{\text{bdded in } L^4}$$

$$- \underbrace{(\nabla b_i)}_{\substack{\text{bdded} \\ \text{in } L^2}}\, \underbrace{g_i}_{\substack{L^\infty \\ \text{bdded}}} - \underbrace{b_i}_{\substack{\text{bdded} \\ \text{in } L^2_1 \\ \hookrightarrow L^4}}\, (\nabla g_i)$$

RHS is bounded in L^2

$\Rightarrow$ g_i is bounded in L^2_2.

Hence $\exists$ subsequence s.th.

$$g_i \longrightarrow g \quad \text{in } L^2_2$$

A subsequence converges strongly in $L^2_1 \hookrightarrow L^2$

$\Rightarrow$ a subseq. converges a.e. $\Rightarrow g \in G_{a.e.}$

$\|a_i\|_{L^2_1}$, $\|b_i\|_{L^2_1}$, $\|g_i\|_{L^2_2}$ bded

$$L^2_1 \hookrightarrow L^4 \text{ is bded}$$
$$L^2_1 \hookrightarrow L^{4-\varepsilon} \text{ is compact } \forall \varepsilon$$
$$L^2_2 \hookrightarrow L^4_1 \xhookrightarrow[\text{compact}]{} L^p \quad \forall p$$

$$g_i \xrightarrow{\text{in } L^2_1, \text{ in } L^p} g \quad \forall p$$
$$a_i \longrightarrow a \quad \text{in } L^{4-\varepsilon}$$
$$b_i \longrightarrow b \quad \text{in } L^{4-\varepsilon}$$

So in

$$\begin{array}{ccccc} dg_i & = & g_i a_i & - & b_i g_i \\ \big\downarrow \text{in } L^2 & & \big\downarrow \text{in } L^2 & & \big\downarrow \text{in } L^2 \\ dg & = & ga & - & bg \end{array}$$

$\leftarrow$ this equ holds in L^2,

so $\Gamma + a$ is gauge equiv to $\Gamma + b$ via g. Show: b satisfies the estimate and $b \in L^2_{3/2}$

The estimate:

$$\int_B |F_{\Gamma+b}|^2 \,\mathrm{vol} = \int_B \langle db + b\wedge b, db + b \wedge b \rangle$$

Have $b_i \to b$ in L^2_1,

$d^* b_i \to d^* b$ in L^2, hence

$$\underbrace{\langle d^* b_i , d^* b \rangle_{L^2}}_{\substack{=0 \\ \forall i}} \longrightarrow \langle d^* b , d^* b \rangle_{L^2}$$

$$\| d^* b \|^2_{L^2}$$

Hence $d^* b \equiv 0$

$b_i \to b$ in L^2_1

$\Rightarrow * b_i|_{\partial B} \to * b|_{\partial B}$ in $L^2_{1/2}(\partial B)$

$\Rightarrow \underbrace{* b_i|_{\partial B}}_{\equiv 0 \ \forall i} \to * b|_{\partial B}$ in $L^2(\partial B)$ for some subsequence

Hence $* b|_{\partial B} \equiv 0$

$$\left(\int_B |F_{\Gamma+b}|^2 \, vol\right)^{1/2} \quad \left(= \|F_{\Gamma+b}\|_{L^2(B)}\right)$$

$$= \left(\int_B (db + b\wedge b,\, db + b\wedge b)\right)^{1/2}$$

$$\begin{aligned}
&= \|db + b\wedge b\|_{L^2} \\
&\geq \|db\| - \|b\|^2_{L^4(B)} \\
&= \|(d+d^*)b\|_{L^2(B)} - C^2 \varepsilon^{1/2} \|b\|_{L^2_1}
\end{aligned}$$

$$\left\{ \begin{array}{l} L^2_1 \hookrightarrow L^4 \\ \|b\|_{L^4} \leq C \|b\|_{L^2_1} \\ \|b\|_{L^2_1} \leq c' \|F_{\Gamma+b}\|_{L^2} \end{array} \right.$$

bdry terms vanish

$$\begin{aligned}
&= \|\nabla b\|_{L^2} - C^2 c' \varepsilon^{1/2} \|b\|_{L^2_1} \\
&\geq \underbrace{\left(\frac{\lambda_1}{1+\lambda_1} - C^2 c' \varepsilon^{1/2}\right)}_{\text{positive for some small enough } \varepsilon} \|b\|_{L^2_1}
\end{aligned}$$

Here have used $*$ $\|\nabla b\| \geq \lambda_1 \|b\|_{L^2}$ for $b \perp \ker \nabla$ and $* \; b|_{\partial B} \equiv 0 \Rightarrow b$ not constant $\neq 0$

$$\overset{*}{\Longrightarrow} \|b\|_{L^2_1} = \|b\|_{L^2} + \|\nabla b\|_{L^2} \leq \left(\frac{1}{\lambda_1} + 1\right) \|\nabla b\|_{L^2}$$

Finally, $b \in L^2_{3/2}$:

$$dg = \underbrace{\underset{\in L^2_2}{g}\,\underset{\in L^2_1}{a}}_{} - \underbrace{\overbrace{b}^{\in L^2_1}\,\underset{\in L^2_2}{g}}_{\in L^2_1} \qquad w(L^2_2) = 2 - \frac{4}{2} = 0$$

Take d^* of this eqn

$$\Delta g = -*(dg \wedge *a) + g\, d^* a + *(*b \wedge dg)$$

$$\Leftrightarrow \quad \underbrace{\Delta g}_{\in L^2_2} - \underbrace{\boxed{*(*b \wedge dg)}}_{\in L^2_{1/2}}$$

$$= \underbrace{- * (\underset{\in L^2_1}{dg} \wedge \underset{\substack{\in L^2_{3/2} \\ \text{by ass.}}}{*a})}_{\substack{\in L^2_{1/2} \\ \text{by Sobolev} \\ \text{multiplication}}} + g\, d^* a$$

$$\left\{ \begin{aligned} & *dg|_{\partial B} \\ & = g|_{\partial B} \, \underbrace{*a|_{\partial B}}_{\in L^2_1} \end{aligned} \right.$$

$$g \longmapsto *(*b \wedge dg) \quad \text{seen as}$$

$$L^2_{5/2} \longrightarrow L^2_{1/2}$$

has operator norm controlled by $\|b\|_{L^2_1}$

Now:

$$L^2_{5/2}(B) \longrightarrow L^2_{1/2}(B) \times L^2_1(\partial B)$$
$$h \longmapsto \left(\Delta h, *dh\big|_{\partial B}\right)$$

has kernel and cokernel the constant sections

ind = 0

<u>Lemma</u> (curvature is proper)

ε_0 as in Uhlenbeck's lemma

$$SV_{\varepsilon_0/2} = \Big\{ A = \Gamma + a \in L^2_1 \;\Big|\; \int_B |F_A|^2 \operatorname{vol} \le \frac{\varepsilon_0}{2}, \; d^* a = 0,\; *a|_{\partial B} = 0 \Big\}.$$

Then the curvature map

$$F_\cdot : \begin{cases} SV_{\varepsilon_0/2} \longrightarrow L^2(B; \Lambda^2(T^*B) \otimes \operatorname{ad}(P)) \\ A \longmapsto F_A \end{cases}$$

is <u>proper</u>

(i.e. preimages of compact sets are compact) a ball

A (reflexive?) Banach space

$$B_{\cancel{X}} = \{ x \in H \mid \|x\| \leq 1 \}$$

is closed in the weak topology:

<u>Pf</u>: Sei $\underset{\in B}{x_i} \longrightarrow y$

Hahn-Banach $\Rightarrow$ $\exists\ f \in H^*$

with $f(y) = \|y\|$ and $\|f\| = 1$

Nun ist

$$f(x_i) \longrightarrow f(y) = \|y\|$$ wg. schwacher Konvergenz.

Andererseits ist

$$|f(x_i)| \leq \|f\| \, \|x_i\| \leq 1 \quad \forall i$$

Also ist $\|y\| \leq 1$. $\blacksquare$

<u>Chern-Simons</u>

common Y

$A|_X = B$

$$\boxed{CS(B) = \int_X tr(F_A \wedge F_A)}$$

mod $8\pi^2\mathbb{Z}$ well def

B_0, B_1 pair of connections on $E \to Y$

$(B_t)_{t\in[0,1]}$ interpolating $\rightsquigarrow$ A on $[0,1]\times Y$

$$CS_{B_0}(B_1) = \int_{[0,1]\times Y} tr(F_A \wedge F_A)$$

$$B_1 = B_0 + b$$
$$B_t = B_0 + tb$$

$$\Rightarrow F_A = \underbrace{F_{B_0}}_{\substack{=0 \text{ because} \\ \text{from trivialization}}} + d(tb) + \frac{t^2}{2}[b\wedge b]$$

$$= dt\wedge b + t\,db + \frac{t^2}{2}[b\wedge b]$$

$$\Rightarrow tr(F_A\wedge F_A) = 2\,dt\wedge tr\Big(tb\wedge db + \frac{t^2}{2} b\wedge[b\wedge b]\Big)$$

$$\Rightarrow \int_{[0,1]\times Y} tr(F_A\wedge F_A) = \int_{[0,1]\times Y} 2\,dt\wedge tr\Big(t\,b\wedge db + \frac{t^2}{2} b\wedge[b\wedge b]\Big)$$

$$= \int_Y tr\Big(b\wedge db + \frac{1}{3} b\wedge[b\wedge b]\Big)$$

Prop: $CS(B_1) - CS(gB_1) = 8\pi^2 \deg(g)$

__Mean value property:__

X connected, smooth with boundary. A a connection in $P \to X$. Then

$$-\int_X \mathrm{tr}(F_A \wedge * F_A) \geq + CS(A|_{\partial X})$$

with equality iff A is ASD.

__Pf:__

$$CS(A|_{\partial X}) = \int_X \mathrm{tr}(F_A \wedge F_A) = \int_X \mathrm{tr}(F_A^- \wedge F_A^-) + \int_X \mathrm{tr}(F_A^+ \wedge F_A^+)$$

$$= -\int_X \mathrm{tr}(F_A^- \wedge * F_A^-) - \int_X \mathrm{tr}(F_A^+ \wedge * F_A^+) + 2\int_X \mathrm{tr}(F_A^+ \wedge * F_A^+)$$

$$= + \|F_A\|_{L^2}^2 - 2 \|F_A^+\|_{L^2}^2$$

$$\Longrightarrow \quad -CS(A|_{\partial X}) = -\|F_A\|_{L^2}^2 + 2\|F_A^+\|_{L^2}^2$$

$\blacksquare$

Uhlenbeck's local compactness theorem:

Let $U \subseteq X$ geodesic ball — bundles are trivial

$$M^{\varepsilon}(U) = \{[A] \mid F_A^+ = 0 \text{ and } \int_U |F_A|^2 \mathrm{vol} \leq \varepsilon\}$$

<u>Thm</u>: There is some $\varepsilon_0 > 0$ s.th. for any proper subball $U' \subseteq U$ the restriction map

$$M^{\varepsilon_0}(U) \longrightarrow M(U')$$

is compact in the sense: Any seq. $([A_i]) \subseteq M^{\varepsilon_0}(U)$ has a L_1^2-conv. subseq. (given L_1^2-topology)

<u>Pf</u>: By the fundam. Lemma $\exists\, \varepsilon_0 > 0$ s.th. $[A_i]$ is rep. by $A_i = \Gamma + a_i$ with uniform L_1^2-estimates on a_i.

$\exists$ subseq. $A_i \longrightarrow A$

Let's say difference in radius in U, U' is δ. There is some $M > 0$

~~... and a set~~

$U = B_R$
$U' = B_{R-\delta}$

such that the following holds:

$\forall i \quad \exists$ set $\quad I_i \subseteq [R-\delta, R]$

<u>of positive measure</u> s.th.

$$\left\| a_i|_{\partial B_r} \right\|_{L^2_1(\partial B_r)} < M$$

$$\forall r \in I_i .$$

(By the Fubini theorem)

$\exists$ subsequence (for notation (i) itself) s.th.

$\bigcap I_i$ has positive measure also, and is non-empty in particular.

For this subsequenc there is therefore some $r \in [R-\delta, R]$ s.th.

$$\| a_i|_{\partial B_r} \|_{L^2_1(\partial B_r)} < M$$

Now $L^2_1 \xrightarrow{(\partial B_r)} L^2_{1/2}(\partial B_r)$ is compact, so there is a subseq which converges

in $L^2_{1/2}$ $\quad a_i \longrightarrow a \quad$ in $L^2_{1/2}$

Recall:

$$CS(a_i|_{\partial B_r})$$
$$= \int_Y tr\left(a_i \wedge da_i + \frac{1}{3} a_i \wedge [a_i \wedge a_i]\right)$$

Claim: If $a_i \to a$ $\boxed{\text{in } L^2_{1/2}(\partial B_r)}$, then

$CS(a_i) \to CS(a)$ because:

$a_i \in L^2_{1/2}$
$da_i \in L^2_{-1/2}$ $\Rightarrow a_i \wedge da_i \in L^1$

Also:

$w(L^2_{1/2}) = \frac{1}{2} - \frac{3}{2} = -1$

$\Rightarrow 3 \cdot w(L^2_{1/2}) = -3 = w(L^1)$

Now: $A_i \to A$ in L^2_1 $\boxed{\begin{matrix} L^2_1 \times L^2_1 \\ \to L^2 \end{matrix}}$

$\Rightarrow F_{A_i} \to F_A$ in L^2 because

Mean value property

$\Longrightarrow \|F_{A_i}\|_{L^2} = CS(a_i|_{\partial B_r})$ converges.

Hence $\|F_{A_i}\|_{L^2(B_r)}$ converge also.

Exercise:

Weak convergence & convergence of the norms

$\Longrightarrow$ strong convergence.

So $F_{A_i}|_{B_r} \longrightarrow F_A|_{B_r}$ in $L^2(B_r)$!

Looks like:

„curvature is proper" implies $A_i|_{B_r} \xrightarrow{L^2_1} A|_{B_r}$.

[Little problem:

$A_i|_{B_r}$ no longer satisfy

$$*a_i|_{\partial B_r} \equiv 0,$$

but up to a further gauge transfo this can be arranged]

∎

Alternative approach:

ASD equ & gauge fixing $d^*a = 0$ give an elliptic PDE:

$$\begin{cases} d^+a_i = -(a_i \wedge a_i)^+ \\ d^*a_i = 0 \end{cases}$$

~~F_{A_0}~~ difficulty: make the bootstrapping started

$P := d^+ \oplus d^*$ is elliptic. get bounds (fund. elliptic ineq.) when passing to interior domains

elliptic bootstrapping

<u>One can show the following:</u>

A an L^2_1-connection and $\|F_A\|_{L^2}$ suff. small. Then $\exists$ L^2_2-gauge transf. g s.th.

$$g(A) \in C^\infty$$

<u>Prop:</u> get C^∞-convergence in the Uhlenbeck's local compactness thm.

<u>Thm:</u> G a compact Lie group. $P \to X$ principal G-bundle. Let $[A_i]$ be a sequence of ASD-connections with

$$\int_X |F_{A_i}|^2 \, vol \leq M .$$

Then after passing to a subsequence

* $\exists$ $x_1, \dots, x_n \subset X$
* a G-bundle $P' \to X$
* a seq. of bundle isom.

$$g_i : P|_{X \setminus \{x_1, \dots, x_n\}} \longrightarrow P'|_{X \setminus \{x_1, \dots, x_n\}}$$

* an ASD-connection A in $P' \to X$

s.th. on $X \setminus \{x_1, \dots, x_n\}$ $\quad g_i(A_i) \to A$ subsets in C^∞ on cpt

Proof:

$\exists$ subsequence where

$|F_{A_i}|^2 \mathrm{vol}$

converges as measure in the weak-$*$-topology to a measure ν, i.e.

$$\int_X f\, |F_{A_i}|^2 \mathrm{vol} \longrightarrow \int_X f\, d\nu$$

$\forall f \in C^0(X)$

↑ dual Banach space to measures

"Banach-Alaoglu": Unit ball in a dual to a Banach space is weak-$*$-compact".

$$\int_X d\nu = 8\pi^2 k$$

$x_1, \dots, x_n$

$\Rightarrow$ At most $n_0 \left[\frac{8\pi^2 k}{\varepsilon^2}\right]$ ~~balls~~ points which do not lie in a geodesic ball with ν-measure $< \varepsilon^2$.

Local compactness results: Get a sequence of ~~balls~~ geodesic balls & $\{B_\alpha\}_{\alpha\in\mathcal{N}}$ where

$$\int_{B_\alpha} |F_{A_i}|^2 vol \leq \varepsilon^2 \qquad \forall \alpha,$$

and may have chosen the sequence so that a ~~the~~ proper subballs $\{B'_\alpha\}_{\alpha\in N}$ still cover $X \setminus \{x_1, \dots, x_4\}$

* On the pairs (B_α, B'_α) use the local compactness results:
For a subsequence can have

$g_{i\alpha}$ L^2_2 - gauge transform.

so that $g_{i\alpha}(A_{i\alpha})|_{B'_\alpha}$ converges to a connection $\Gamma + a_\alpha$ in C^∞ - top'ly.

* Patch these local connections and gauge transformations:

$$\eta_{i,\alpha\beta} = g_{i\alpha}|_{B_\beta}\, g_{i\beta}^{-1}|_{B_\alpha}$$

satisfy

$$d\eta_{i,\alpha\beta} = \underbrace{\underbrace{\eta_{i\alpha\beta}}_{\in G \text{ a.e.}} a_{i,\beta} - a_{i\alpha} \underbrace{\eta_{i,\alpha\beta}}_{\in G\text{ a.e.}}}_{\text{converges in } L^2}$$

$\Rightarrow$ $\eta_{i\alpha\beta}$ converges in L^2_1. Eqn $\Rightarrow$ converges in L^2_k $\forall k$

Bootstrapping $\Rightarrow$ $\eta_{i,\alpha\beta}$ converges in C^∞

....

Get an ASD connection A^∞
on ~~the~~ $P \to X \setminus \{x_1, \dots, x_n\}$

<u>Thm</u> (Removable singularities, Uhlenbeck)

If A is an ASD connection on P on a punctured ball B, and if $\| F_A \|_{L^2(B \setminus \{0\})} < \infty$, then $\exists$ ASD connection A', possibly on a different bundle $P' \to B$, s.th. $\exists$ bundle isom.

$$P|_{B \setminus \{0\}} \cong P'|_{B \setminus \{0\}}$$

which maps $A|_{B \setminus \{0\}}$ to $A'|_{B \setminus \{0\}}$

<u>Analogy</u>: For holomorphic functions on punctured balls: If $|f(z)|$ stays bounded and is holom. on $B \setminus \{0\}$, then f extends as a holom. fct on B. $\square$

~~that~~
Notice that

$$\int_{X \setminus \{x_1, \dots, x_n\}} |F_A|^2 \, \mathrm{vol} < 8\pi^2 k$$

if ~~more than~~ at least one point has ν-measure $\geq \varepsilon^2$ for all geodesic balls and ~~for~~ it all large enough i

$$\nu = |F_A|^2 \mathrm{vol}_g + 8\pi^2 \sum n_r \delta_{x_r}$$

(not deep in terms of measure theory:

A' is ASD on X and
$|F_{A'}|^2 = |F_A|^2$ on $X \setminus \{x_1, \dots, x_n\}$

$k' = \frac{1}{8\pi^2} \int_X |F_{A'}|^2 \mathrm{vol}_g \Rightarrow \sum n_r$ is an integer.

$\in \mathbb{Z}$

$= c_2(P')$

In fact: Every $n_r \in \mathbb{N}$.
Choose disjoint balls B_r around x_r's.

$$cs(A_i|_{\partial B_r}) \quad \text{~~[illegible]~~}$$

$$\to cs(A'|_{\partial B_r}) \quad (i \to \infty)$$

Otoh:

$$\cancel{n_r = \frac{1}{8\pi^2}\int_{Z_r}}$$

$$\int_{Z_r} \nu = -\int_{Z_r} tr(F_A \wedge F_A) + n_r \cdot 8\pi^2$$

$$\uparrow \ (i \to \infty)$$

$$\int_{Z_r} |F_{A_i}|^2 vol = -\int_{Z_r} tr(F_{A_i} \wedge F_{A_i})$$

$$\Rightarrow \ n_r = \frac{1}{8\pi^2} \lim_{i\to\infty} \int_{Z_r} tr(F_A \wedge F_A) - tr(F_{A_i} \wedge F_{A_i})$$

$$= \frac{1}{8\pi^2} \left(cs(A|_{\partial Z_r}) - \lim_{i\to\infty} cs(A_i|_{\partial Z_r}) \right)$$

as

$A_i|_{\partial Z_r} \to A|_{\partial Z_r}$ in C^∞

$$\equiv 0 \pmod{8\pi^2 \mathbb{Z}}$$

Therefore $n_r \equiv 0 \pmod{\mathbb{Z}}$

$\blacksquare$

Moduli space:

If $F_A^+ = 0$, then we have
an ~~complex~~ elliptic complex

$$0 \to \Omega^0(ad(P)) \xrightarrow{d_A} \Omega^1(ad(P)) \xrightarrow{d_A^+} \Omega^2_+(ad(P)) \to 0$$

Proof: $u_t = \exp(t\xi)$.

Then

$$u(A) = A + u d_A u^{-1}$$

$$0 = \frac{d}{dt}\Big|_{t=0} F^+_{u_t(A)}$$

$$= \frac{d}{dt}\Big|_{t=0} \left(F_A^+ + d_A^+ \left(u d_A u^{-1}\right) + \frac{1}{2}\left[u d_A u^{-1}, u d_A u^{-1}\right]^+ \right)$$

$$= (d_A^+ \circ d_A)\xi$$

(Shorter: $F_A = d_A \circ d_A$
Check this!)

$\square$

Therefore

$$d_A^* \oplus d_A^+ : \Omega^1(ad(P)) \longrightarrow \Omega^0(ad(P)) \oplus \Omega^2_+(ad(P))$$

is elliptic, and it is Fredholm in the corr. Sobolev completions.

The index

$$\mathrm{ind}(d_A^* \oplus d_A^+) = \dim(\ker(d_A^* \oplus d_A^+)) - \dim(\mathrm{coker}(d_A^* \oplus d_A^+))$$

can be computed with the Atiyah-Singer index thm. (or by excision from trivial bdle case & S^4-case)

$$\mathrm{ind} = 8c_2(P) + 3(b_1(X) - b_2^+(X) - 1)$$

Notice: For the trivial bundle

$$\mathrm{ind} = 3(b_1 - b_2^+ - 1)$$

($1 = b_0$)

can be computed explicitly by hand (exercise!)

$$\ker(d^* \oplus d^+) \overset{\text{Exo}}{=} \ker(d^* \oplus d) = \mathcal{H}^1(X) \leftarrow \text{harmonic forms}$$

$$\mathrm{coker}(d^*) = \text{constant fcts (Hodge decomp)} = \mathbb{R}^{b_0}$$

$$\mathrm{coker}(d^+) = \mathcal{H}_2^+ \leftarrow \text{self-dual harmonic forms.}$$

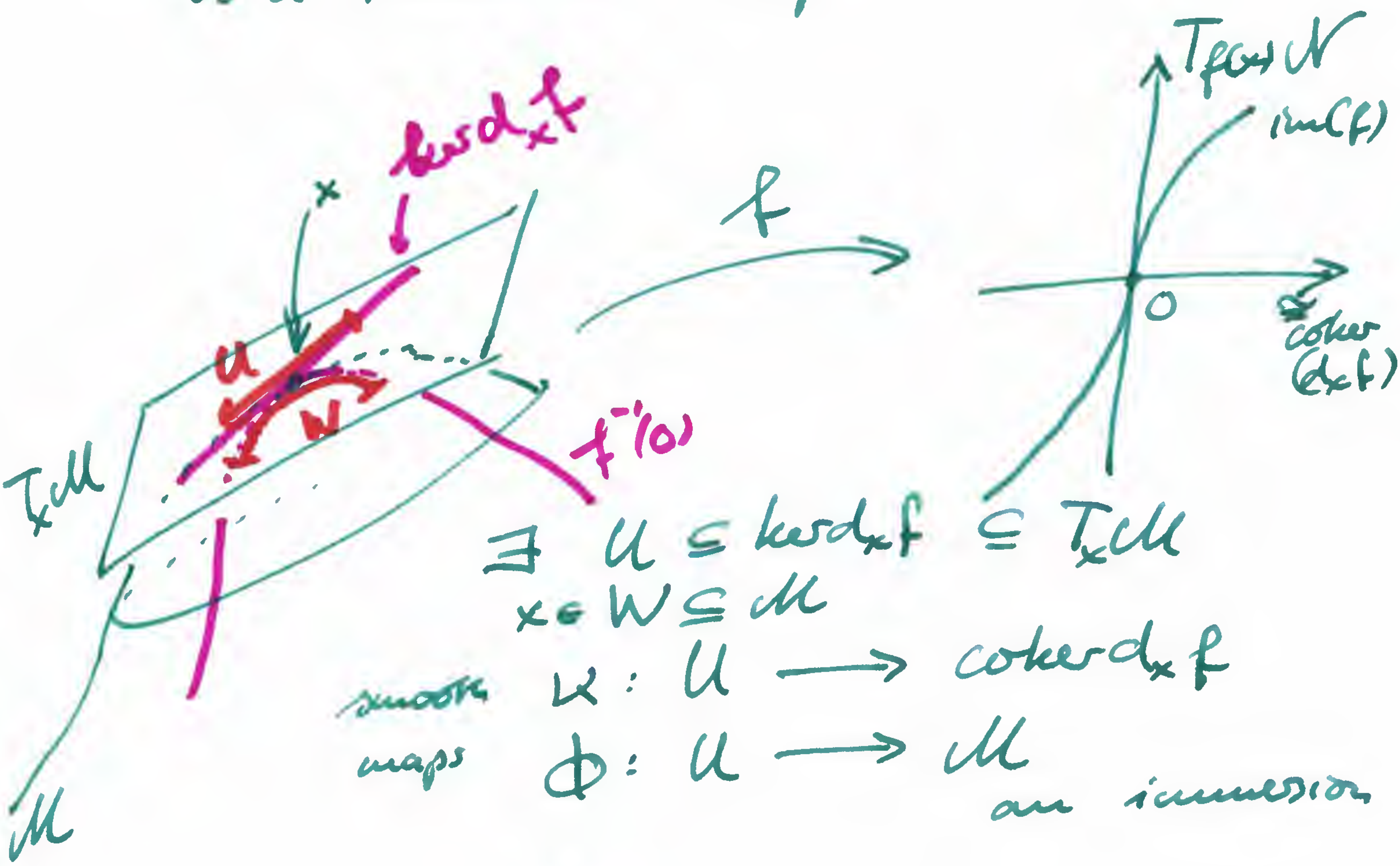
<u>Def:</u> A map between Hilbert mflds

$$f: \mathcal{M} \longrightarrow \mathcal{N}$$

is called <u>Fredholm</u> if $\forall x \in \mathcal{M}$

$$d_x f : T_x\mathcal{M} \longrightarrow T_{f(x)}\mathcal{N}$$

is a Fredholm map.



$\exists$ $U \subseteq \ker d_x f \subseteq T_x\mathcal{M}$

$x \in W \subseteq \mathcal{M}$

smooth maps

$\kappa : U \longrightarrow \operatorname{coker} d_x f$

$\phi : U \longrightarrow \mathcal{M}$ an immersion

s.th. $\phi(0) = x$, $d\kappa(0) = 0$,

$$id = d_0\phi : T_0 U \longrightarrow \ker d_x f$$

$$\phi(\kappa^{-1}(0)) = f^{-1}(0) \cap W$$

<u>Pf:</u> Explicit fct thm! (Exercise!)

Hence the moduli space $f^{-1}(0)$ is locally equivalent (via Φ) to a smooth map

$$K : \underbrace{\ker d_x f}_{\text{finite dim'l}} \longrightarrow \underbrace{\operatorname{coker}(d_x f)}_{\text{finite dim'l}}$$

"formal dimension of moduli space"

$$\operatorname{ind}(d_x f) = \dim(\ker d_x f) - \dim \operatorname{coker}(d_x f)$$

Observe: If $\operatorname{coker}(d_x f)$, then $f^{-1}(0)$ has the structure of a smooth manifold in a neighbourhood of x, of $\dim = \operatorname{ind}(d_x f)$

Remember that

$$S_{A,3/2}(\varepsilon)\big/\mathrm{Stab}(A) \xrightarrow[\text{nbhd of } [A]]{\text{homeo onto some}} \mathcal{A}/\mathcal{G}$$

And via this map $\overbrace{}^{\text{(on } S_{A,3/2}(\varepsilon)\text{)}}$ the ASD-equ for $A+a$ is (if A is ASD)

$$\begin{cases} d_A^* a = 0 \\ F^+_{A+a} = 0 \iff \cancel{F_A^+}^{\,0} + d_A^+ a + \tfrac{1}{2}[a\wedge a]^+ = 0 \end{cases}$$

$$\iff \begin{cases} d_A^* a = 0 \\ d_A^+ a = -\tfrac{1}{2}[a\wedge a]^+ \end{cases}$$

This is a <u>Fredholm map</u>!
with linearisation at $a=0$:

$$L = d_A^* \oplus d_A^+$$

Hence

$$S_{A,3/2} \longrightarrow L^2_{1/2}(X;\Lambda^2_+ \otimes \mathrm{ad}(P))$$

$$a \longmapsto F^+_{A+a}$$

$$\boxed{\begin{array}{l}\text{is Fredholm of} \\ \text{index} = \mathrm{ind}(d_A^* \oplus d_A^+) + \dim(\mathrm{Stab}(A))\end{array}}$$

See it as two set of

a map

$$L^2_{3/2}(X;\Lambda^1\otimes ad(P)) \to L^2_{1/2}(X; ad(P)) \oplus L^2_{1/2}(X;\Lambda^2_+\otimes ad(P))$$

<u>Thm</u> (Freed–Uhlenbeck) ~~If $b_2^+(X)\geq 1$~~,
~~thus~~ For generic metric the moduli space

$$M_P^* = \{A\in \mathcal{A}_P \mid F_A^+=0\}_{A \text{ irreducible}}/\mathcal{G}$$

is <u>cut out transversally</u> by the equns

($\operatorname{coker}(d_A^+)=0$ $\forall$ instantons A),

hence is a smooth mfld

<u>Thm</u> (Freed–Uhlenbeck?)

If $b_2^+(X)\geq 1$ then

$$M_P = M_P^*$$

for generic metric (can avoid reducibles)

If $b_2^+(X)\geq 2$ then any two generic

$M_P^{g_1}$, $M_P^{g_2}$ ~~can be~~ are cobordant.

Discussion of reducibles

Def$^{\underline{n}}$ A is called reducible, if

$$\mathrm{Stab}(A) = \{u \in \mathcal{G} \mid u(A) = A\}$$
$$\neq Z(G)$$

$\mathrm{Hol}_\gamma(A) \in \mathrm{Ad}(P)_{x_0}$ if γ is a loop based at x_0

If $u \in \mathrm{Stab}(A)$, then $d_A u \equiv 0$, and u is determined by parallel transport from $u(x_0)$.

$\Rightarrow \mathrm{Stab}(A) \subseteq \mathrm{Ad}(P)_{x_0}$

Observe: $\mathrm{Stab}(A) = \mathrm{Commutant}\left(\{\mathrm{Hol}_\gamma(A) \mid \gamma \in \text{loop at } x_0\}\right)$

$u \in \mathrm{Commutant}(\mathrm{Hol}_{x_0}(A))$

$\Leftrightarrow$ u commutes with parallel transport

$\Leftrightarrow$ $d_A u \equiv 0$

(infinitesimal version of parallel transport)

So $Z(G)$ always lies in $\mathrm{Stab}(A)$ ~~and acts trivially~~

$G = SU(2)$

Consequence of observation:

$$\mathrm{Stab}(A) = \mathbb{Z}/2 \ , \ U(1) \ , \text{ or } SU(2)$$

centralises $\underline{SU(2)}$, $\underline{U(1)}$, id

closed connected subgroups are

$\mathbb{Z}/2$, $U(1)$, $SU(2)$

$\left(\overline{\{ \mathrm{Hol}_\gamma(A) \mid \gamma \in \text{Loop at } x \}} \text{ is connected} \right)$

generic case

A preserves a sum of line bundles

$E = L_1 \oplus L_2$

A is a trivial connection (on a trivial bundle)

A a reducible ASD connection $\Rightarrow$ E splits as a parallel of line bundles each carrying an abelian connection

$E = L_1 \oplus L_2$

$$\Rightarrow \quad F_A = \begin{pmatrix} F_{A_1} & 0 \\ 0 & F_{A_2} \end{pmatrix}$$

with respect to that splitting

$$0 = F_A^+ = \begin{pmatrix} F_{A_1}^+ & 0 \\ 0 & F_{A_2}^+ \end{pmatrix}$$

Bianchi identity becomes

$$dF_{A_1} = 0 \quad , \quad dF_{A_2} = 0$$

Otoh $\quad F_{A_i}^+ = 0 \Rightarrow d^* F_{A_i} = 0$

Therefore F_{A_i} is a <u>harmonic self-dual</u> form for both i

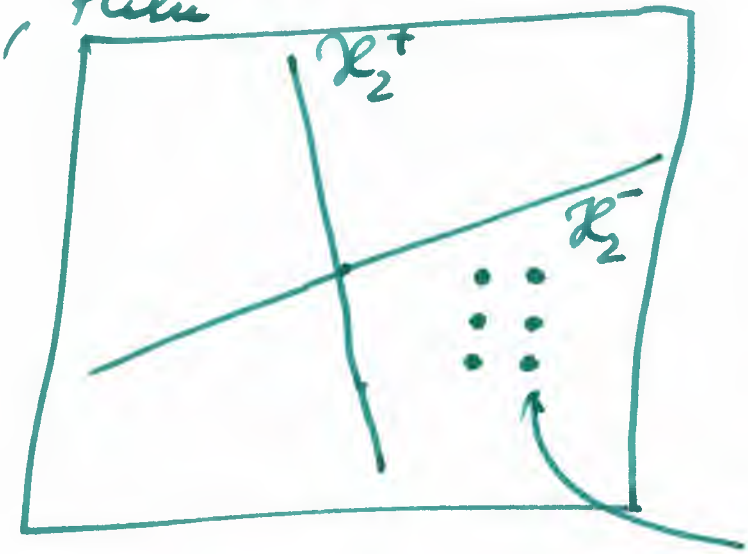
E top'ly non-triv $\Rightarrow$ One of L_1 or L_2 is non-triv.

$$\mathcal{H}^2 = \mathcal{H}_2^+ \oplus \mathcal{H}_2^-$$
$$\cong H^2_{dR}$$

If $b_2^+ = 0$, then $\mathcal{H}_2^- = \mathcal{H}_2$

If $b_2^+ > 0$, then

$H^2_{dR} =$



integer classes.

$\mathcal{H}_2^+$ and $\mathcal{H}_2^-$ <u>move</u> inside H^2_{dR} as one changes the metric

$\mathcal{H}_2^-$ has codim b_2^+.

<u>Fact</u> If $b_2^+ > 0$, then for "generic" metrics $\mathcal{H}_2^-$ will <u>miss</u> non-zero integer classes such as

$$\left[\frac{-i}{2\pi} F_{A_1}\right] = c_1(L_1)$$

Donaldson's thm: $\quad X$ neg. definite

Need

(1.) The reducibles ASD mod gauge are $\cong \{a, -a\} \subseteq H^2(X;\mathbb{Z})$ with
$$q_X(a,a) = -1$$
(gauge theory)

(2.) A nbhd of a reducible is homeom. to $\mathrm{Cone}(\mathbb{C}P^2)$ (no claim on orientation)

(3.) q_X is diagonalizable iff $\exists\ r = \mathrm{rk}(q_X)$ pairs $\{a,-a\}$ with $q_X(a,a) = -1$

Proof of 3.

($\Leftarrow$) Suppose $\{b, -b\} \neq \{a, -a\}$ is a different pair. Then

Suppose $q_X(a, b) \neq 0$. Write $b = b^{\parallel} + b^{\perp}$, where

$$b^{\parallel} \in \langle a \rangle_{\mathbb{Z}}$$
$$b^{\perp} \in \langle a \rangle_{\mathbb{Z}}^{\perp}.$$

Then

$$q_X(a, b) = q_X(a, b^{\parallel}) + q_X(a, b^{\perp})$$

Suppose $q_X(a, b) \neq 0$. Then $b^{\parallel} \neq 0$.

$\Rightarrow$ $b^{\parallel} = n \cdot a$ for some $n \in \mathbb{Z}$, $n \neq 0$.

But now

- $b^{\perp} \neq 0 \Rightarrow -1 = q_X(b, b)$

$$= -n^2 + q_X(b^{\perp}, b^{\perp})$$

Shorter:

$$q_X(a, b) = q_X(a, b^{\parallel}) + \underbrace{q_X(a, b^{\perp})}_{=0}$$

with respect to q_X: $\langle a \rangle \oplus \langle a \rangle^{\perp}$.

If this is $\neq 0$, then

$$-1 = q_X(b, b) = \underbrace{q_X(b^{\parallel}, b^{\parallel})}_{<0} + q_X(b^{\perp}, b^{\perp})$$

Therefore $q_X(b^\perp, b^\perp) = 0 \Rightarrow b^\perp = 0$

$\Rightarrow b = \pm a$.

Hence any pair splits off an orthog. $\mathbb{Z}$-subspace. ∎

Proof of 1.

Suppose $\mathrm{Stab}(A) \neq \mathbb{Z}/2$.

$k=1$-bundle $\Rightarrow \mathrm{Stab}(A) = U(1)$.

$\Rightarrow \exists$ decomp.

$$E = P \times_{SU(2)} \mathbb{C}^2 = L \oplus L^{-1}$$

ass $SU(2)$-bundle

A-parallel (L^{-1} because $c_1(E)=0$)

Then $F_A = \begin{pmatrix} F_{A_1} & 0 \\ 0 & -F_{A_1} \end{pmatrix}$

curvature of the conn. in the line bundle

$$\left.\begin{array}{l} dF_{A_1} = 0 \\ F^+_{A_1} = 0 \\ \Leftrightarrow d^* F_{A_1} = 0 \end{array}\right\} \Rightarrow F_{A_1} \text{ is harmonic asd}$$

(Bianchi)

and $\underbrace{\left[\frac{i}{2\pi} F_{A_1}\right]}_{=:\omega} = c_1(L)$

-1

$\|$

$\langle c_2(E), [X] \rangle \overset{\text{shorter}}{=} \langle c_1(L) \smile c_1(L), [X] \rangle$

$= \int_X \omega \wedge \omega = -\frac{1}{4\pi^2} \int F_{A_1} \wedge F_{A_1} = -\frac{1}{8\pi^2} \int \mathrm{tr}(F_A \wedge F_A)$ $= -1$

<u>Also:</u> To any such pair

$\{a, -a\}$ take the decomp

$E = L \oplus L^{-1}$ where $c_1(L) = a$

($\Rightarrow c_2(L \oplus L^{-1}) = -a^2 = 1$)

Define parallel connections

$\nabla_{A} = \nabla_{A_1} \oplus \nabla_{A_2}$

$dF_{A_1} = 0$

Want $d^{+}F_{A_1} = 0$ also.

Can achieve with a gauge $u(1)$ transformation:

<u>False:</u> $F_{g(A_1)} = F_{A_1} + d(g^{-1}dg)$

(Want $F_{g(A_1)}$ to be the unique (asd) harmonic representative of $[F_{A_1}]$.)

If $H_1 = 0$ all $u(1)$-connections on a line bundles ought to be gauge equivalent

$\nabla_{g(a)} = \nabla_A + g\,dg^{-1}$,

$g = e^{i\xi}$ $\quad \xi$ a fctn

$g^{-1}dg = e^{-i\xi}\, i\,d\xi\, e^{i\xi} = i\,d\xi$

Neat statement:

If $H^1(X;\mathbb{R})=0$ and $L \to X$ is a line bundle, then for any ω rep. $c_1(L)$ there is a unique gauge equiv. class of connections with curvature $-2\pi i\omega$.

Pf: f: $\mathcal{A} \longrightarrow \mathcal{C}_L = \{\eta \in \Omega^2(X) \mid [\eta] = c_1(L)\}$

$A \mapsto F_A \cdot \frac{-i}{2\pi}$

This is surjective because

$$\frac{-i}{2\pi} F_{A+a} = \frac{-i}{2\pi} F_A \frac{-i}{2\pi} da$$

Thus got all

Suppose $F_{A_1} = F_{A_2} \Rightarrow d(A_1 - A_2) = 0$.

$ib = A_1 - A_2$

$H^1 = 0 \Rightarrow b$ is exact

Show: There is a gauge transfo

$u(A_1) = A_1 + i(du)u^{-1}$

If $u = e^{i\xi}$, then

$$(du)u^{-1} = i e^{i\xi} d\xi \, e^{-i\xi} = i\, d\xi$$

So $A_1 - A_2 = ib$ has a solution $b = d\xi$, hence A_1, A_2 are gauge equiv.