SUBADDITIVE COCYCLES AND HOROFUNCTIONS

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The aim of this text is to present and put in perspective the results we have proved with Anders Karlsson in the article [GK15]. The topic of this article is the study, in an ergodic theoretic context, of some subadditivity properties, and their relationships with dynamical questions with a more geometric flavor, dealing with the asymptotic behavior of random semicontractions on general metric spaces. This text is translated from an article written in French on the occasion of the first congress of the French Mathematical Society [Gou17]. The proof of the main ergodic-theoretic result in [GK15] has been completely formalized and checked in the computer proof assistant Isabelle/HOL [Gou16].

1. Iteration of a semicontraction on Euclidean space

In order to explain the problems we want to consider, it is enlightening to start with a more elementary example, showing how subadditivity techniques can be useful to understand a deterministic semicontraction. In the next section, we will see how these results can be extended to random semicontractions.

Definition 1.1. *A transformation T on a metric space X is a semicontraction if it is* 1*-Lipschitz, i.e., if $d(T(x), T(y)) \leqslant d(x, y)$ for all $x, y \in X$.*

If T is a semicontraction, its iterates also are. Hence, for any points x and y, the distance between $T^n(x)$ and $T^n(y)$ remains uniformly bounded, by $d(x, y)$ (where we write $T^n = T \circ \cdots \circ T$). As a consequence, the asymptotic behavior of $T^n(x)$ (up to bounded error) is independent of x.

In the Euclidean space $\mathbb{R}^d$, the first examples of semicontractions are given by translations (where $T^n(x)$ tends to infinity as $nv + O(1)$, where v is the translation vector) and homotheties with ratio $\leqslant 1$ (for which $T^n(x)$ remains bounded). The following theorem, proved in 1981 in [KN81] under slightly stronger assumptions, shows that these examples are typical since there always exists an asymptotic translation vector. The proof we give is due to Karlsson [Kar01].

Theorem 1.2. *Consider a semicontraction $T : X \to X$ on a subset X of Euclidean space $\mathbb{R}^d$. Then there exists a vector v such that $T^n(x)/n$ converges to v for all $x \in X$.*

Note that the asymptotic behavior of $T^n(x)/n$ does not depend on x, therefore it suffices to prove the theorem for one single point x. Translating everything if necessary, we can assume $0 \in X$ and take $x = 0$ to simplify notations.

The proof relies crucially on the subadditivity properties of the sequence $u_n = d(0, T^n(0))$.

Definition 1.3. *A sequence $(u_n)_{n \in \mathbb{N}}$ of real numbers is* subadditive *if $u_{k+\ell} \leqslant u_k + u_\ell$ for all k, ℓ.*

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The main property of such a sequence is given in the next lemma, due to Fekete.

Lemma 1.4. *Let u_n be a subadditive sequence. Then u_n/n converges, to* $\operatorname{Inf}\{u_n/n, n > 0\} \in \mathbb{R} \cup \{-\infty\}$.

Proof. Fix a positive integer N. It follows from the subadditivity of u that $u_{kN+r} \leqslant ku_N + u_r$. Writing an arbitrary integer n as $kN + r$ with $r < N$, dividing by n and taking the limit, we get $\limsup u_n/n \leqslant u_N/N$. Hence, $\limsup u_n/n \leqslant \operatorname{Inf}\{u_N/N\}$. The result follows as $\liminf u_n/n \geqslant \operatorname{Inf}\{u_N/N\}$. □

Recalling the notation $u_n = d(0, T^n(0))$, we have

$$\begin{aligned} u_{k+\ell} &= d(0, T^{k+\ell}(0)) \leqslant d(0, T^k(0)) + d(T^k(0), T^k(T^\ell(0))) \\ &\leqslant d(0, T^k(0)) + d(0, T^\ell(0)) = u_k + u_\ell, \end{aligned} \tag{1.1}$$

where we used the semicontractivity of T^k. Hence, Fekete's Lemma shows that u_n/n converges to a limit $A \geqslant 0$. At time n, the point $T^n(0)$ is close to the sphere of radius An centered at 0. If $A = 0$, this proves Theorem 1.2. However, if $A > 0$, we should also prove the directional convergence of $T^n(0)$. For this, we will use times where the sequence u is almost additive, given by the following lemma.

Lemma 1.5. *Let $\varepsilon > 0$. Consider a subadditive sequence u_n such that $u_n/n \to A \in \mathbb{R}$. Then there exist arbitrarily large integers n such that, for all $1 \leqslant \ell \leqslant n$,*

$$u_n \geqslant u_{n-\ell} + (A - \varepsilon)\ell. \tag{1.2}$$

As u_ℓ is of magnitude $A\ell$, this inequality can informally be read as $u_n \geqslant u_{n-\ell} + u_\ell - \delta$, where δ is small. It entails additivity of the sequence at *all* intermediate times between 1 and n, up to a well controlled error.

Proof. The sequence $u_n - (A - \varepsilon)n$ is equivalent to εn, and tends therefore to infinity. In particular, there exist arbitrarily large times n which are records for this sequence, beating every previous value. For such an n, we have for $\ell \leqslant n$ the inequality $u_{n-\ell} - (A - \varepsilon)(n - \ell) \leqslant u_n - (A - \varepsilon)n$, which is equivalent to the result we claim. □

For $\varepsilon_i = 2^{-i}$, let us consider a corresponding sequence of times n_i given by Lemma 1.5, tending to infinity. Let h_i be a norm-1 linear form, equal to $-\|T^{n_i}(0)\|$ on $T^{n_i}(0)$. Then, for all $\ell \leqslant n_i$,

$$\begin{aligned} h_i(T^\ell(0)) = h_i(T^\ell(0) - T^{n_i}(0)) + h_i(T^{n_i}(0)) &\leqslant \|T^\ell(0) - T^{n_i}(0)\| - \|T^{n_i}(0)\| \\ &\leqslant \|T^{n_i-\ell}(0)\| - \|T^{n_i}(0)\| = u_{n_i-\ell} - u_{n_i} \leqslant -(A - \varepsilon_i)\ell, \end{aligned}$$

where the last inequality follows from (1.2). In the inequality $h_i(T^\ell(0)) \leqslant -(A - \varepsilon_i)\ell$ that we just obtained, it is remarkable that every mention of n_i has disappeared.

Let us now consider h a limit (weak or strong, as we are in finite dimension) of the sequence h_i, it is a norm-1 linear form. As ε_i tends to 0 with i, we deduce from the above the following inequality:

$$\text{for every integer } \ell, \quad h(T^\ell(0)) \leqslant -A\ell. \tag{1.3}$$

This inequality entails that $T^\ell(0)$ belongs to the half-space directed by h, at distance $A\ell$ from the origin. As it also has essentially norm $A\ell$, we deduce that it is essentially pointing in the direction of h (see Figure 1). This shows the convergence of $T^\ell(0)/\ell$. If one wants a more explicit argument, one can for instance consider a cluster value v of $T^\ell(0)/\ell$. It is a vector of

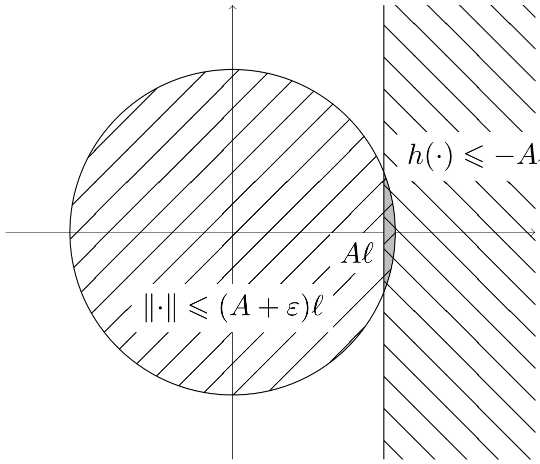

FIGURE 1. $T^\ell(0)$ belongs to the intersection of the dashed areas

norm A, satisfying $h(v) = -A$. As h has norm 1, this determines uniquely v thanks to the strict convexity of the Euclidean norm. Therefore, $T^\ell(0)/\ell$ has a unique cluster value, and it converges. This concludes the proof of Theorem 1.2. □

Remark 1.6. The proof has not used finite-dimensionality (if one replaces strong limits with weak limits). Therefore, the result is still true in Hilbert spaces, or more generally in uniformly convex Banach spaces.

Remark 1.7. Most of the proof is valid in a general Banach space: there always exists a linear form h with norm at most 1 such that $h(T^\ell(0)) \leqslant -A\ell$ for all ℓ (this already implies non-trivial results, for instance the sequence $(T^\ell(0))_{\ell \geqslant 0}$ is contained in a half-space if $A > 0$, as the linear form h is necessarily nonzero in this case). The only point where the proof breaks is the last argument, relying on strict convexity of the norm.

One may wonder if this is a limitation of the proof, or if the proof captures all the relevant information. In fact, the above theorem is wrong without convexity assumptions on the norm. Let us describe quickly a counter-example due to [KN81], in $\mathbb{R}^2$ with the sup norm. Let us fix the two vectors $v_+ = (1,1)$ and $v_- = (1,-1)$, both of norm 1. We define a continuous path $\gamma : \mathbb{R}_+ \to \mathbb{R}^2$ starting from 0, of the form $\gamma(t) = (t, \varphi(t))$, by following the direction v_+ during a time S_0, then the direction v_- during a time $S_1 \gg S_0$, then the direction v_+ during a time $S_2 \gg S_1$, and so on. One can ensure that the angle between $\gamma(t)$ and the horizontal line fluctuates between $-\pi/4$ et $\pi/4$. As the slopes of v_+ and v_- are 1, the path γ is an isometry from $\mathbb{R}_+$ onto its image. Let $h(x_1, x_2) = x_1$ be the first coordinate. Then the map $T : x \mapsto \gamma(|h(x)| + 1)$ is a semicontraction, as a composition of 1-Lipschitz functions. One checks easily that $T^n(0) = \gamma(n)$. Therefore, by construction, $T^n(0)/n$ does not converge.

2. Horofunctions

Many interesting geometric spaces, which are not vector spaces, have semicontractions. We would like to have a version of Theorem 1.2 for these spaces. The conclusion of the theorem can not be of the form "$T^n(x)/n$ is converging" as division by n makes no sense. It is always true that $d(T^n(x), x)/n$ converges to a limit $A \geqslant 0$, by subadditivity. However, the meaning to give to directional convergence is less obvious. Such theorems already exist in different contexts. Let us mention for instance the following Denjoy-Wolff Theorem [Den26, Wol26]:

Theorem 2.1. *Let T be a holomorphic map from the unit disk $\mathbb{D}$ in $\mathbb{C}$ into itself. Then, either T has a fixed point in the disk, or $T^n(0)$ converges to a point on the unit circle.*

This statement is indeed a particular case of the previous discussion, as a holomorphic map of the unit disk if a semicontraction for the hyperbolic distance.

In the general case, the counter-example from Remark 1.7 shows that one can not hope to have convergence at infinity in a strong sense without additional assumptions of geometric nature on the space. If we follow the proof of Theorem 1.2 in the context of a general metric space, we see that it is possible to make sense of all arguments up to the inequality (1.3), in terms of horofunctions.

Definition 2.2. *Let (X, d) be a metric space with a basepoint x_0. For $x \in X$, we say that the function $h_x : y \mapsto d(x, y) - d(x, x_0)$ is an* internal horofunction. *A* horofunction *is an element of the closure of the set of internal horofunctions, for the topology of pointwise convergence.*

For every $x \in X$, the internal horofunction h_x vanishes at x_0 and it is 1-Lipschitz. Therefore, $h_x(y)$ belongs to the compact interval $[-d(y, x_0), d(y, x_0)]$. As a product of compact spaces is compact for the topology of pointwise convergence (i.e., the product topology), we deduce that the set $\overline{X}^B$ of horofunctions, endowed with the topology of pointwise convergence, is a compact space in which the set X (seen as the set of internal horofunctions) is dense. A horofunction vanishes at x_0 and is 1-Lipschitz, as these properties are preserved by pointwise limits.

In the same way that we distinguish between a point $x \in X$ and the corresponding internal horofunction, we will distinguish by the notations between an abstract point $\xi \in \overline{X}^B$ and the corresponding horofunction h_ξ.

Remark 2.3. In general, X is *not* an open subset of $\overline{X}^B$, contrary to the usual requirements for compactifications. For instance, consider for X a countable number of rays $\mathbb{R}^+$, all coming from the same point x_0, with the graph distance. If a sequence converges to infinity along one of the rays (say the ray with index i), then the sequence of corresponding internal horofunctions converges to an (external) horofunction h_i. When i tends to infinity, one checks easily that h_i tends to h_{x_0}.

On the other hand, if the space is proper (i.e., every closed ball $\overline{B}(x, r)$ is compact) and geodesic (between any two points x and y, there is a geodesic, i.e., a path isometric to the segment $[0, d(x, y)]$), then $\overline{X}^B$ is a compactification of X in the usual sense.

One should think of external horofunctions as analogues of linear forms, but on general metric spaces. In the case of Euclidean space, the two notions coincide exactly. In geometric terms, what is interesting is not so much the horofunction h itself, than the sequence of horoballs $\{x : h(x) \leqslant c\}$ it defines, for $c \in \mathbb{R}$. This is a kind of family of half-spaces, increasing with c, defining a direction at infinity when $c \to -\infty$.

The notion of horofunction is exactly the one we need to extend the above proof of Theorem 1.2 to a general metric space:

Theorem 2.4 (Karlsson [Kar01])**.** *Let T be a semicontraction on a metric space (X, d) with a basepoint x_0. Then $d(T^n(x_0), x_0)/n$ converges to a limit $A \geqslant 0$. Moreover, there exists a horofunction h such that, for all $\ell \in \mathbb{N}$, we have $h(T^\ell(x_0)) \leqslant -A\ell$.*

Proof. The proof is exactly the same as the proof of the inequality (1.3), if one replaces the notion of linear form (which relied on the linearity of the underlying space) with the notion of horofunction. Indeed, let us define A as in the proof of this theorem, by subadditivity. Let $\varepsilon_i = 2^{-i}$, and consider an increasing sequence n_i such that, for all $\ell \leqslant n_i$, holds $d(x_0, T^{n_i}x_0) \geqslant d(x_0, T^{n_i-\ell}x_0) + (A - \varepsilon_i)\ell$, thanks to Lemma 1.5. Then, we use the internal horofunction based at $T^{n_i}x_0$. It satisfies, for $\ell \leqslant n_i$,

$$\begin{aligned} h_{T^{n_i}(x_0)}(T^\ell(x_0)) = d(T^{n_i}(x_0), T^\ell(x_0)) - d(T^{n_i}(x_0), x_0) \leqslant d(T^{n_i-\ell}(x_0), x_0) - d(T^{n_i}(x_0), x_0) \\ \leqslant -(A - \varepsilon_i)\ell. \end{aligned}$$

This shows that the set of horofunctions satisfying $h(T^\ell(x_0)) \leqslant -(A - \varepsilon_i)\ell$ for all $\ell \leqslant n_i$ is nonempty. Moreover, it is compact, and decreases with i. As the set of horofunctions is compact, the intersection of these sets is nonempty. Any element h of this intersection satisfies $h(T^\ell(x_0)) \leqslant -A\ell$ for all ℓ, as desired. (In the case where the space X is second-countable, the topology on $\overline{X}^B$ is metrizable, and one can just take for h any cluster value of the sequence $h_{T^{n_i}(x_0)}$.) □

This theorem entails that $T^\ell(x_0)$ is in the intersection of the ball of radius $(A + \varepsilon)\ell$ and of the half-space $\{h \leqslant -A\ell\}$ for ℓ large enough, as in Figure 1, with the difference that the shapes of the ball and the half-space depend on the geometry of (X, d). Deciding if one can deduce from this statement a stronger convergence at infinity will thus depend on X. For instance, this is true in a uniformly convex Banach space, thanks to Remark 1.6, but this is false in $\mathbb{R}^2$ with the sup norm, by Remark 1.7.

One can therefore say that Theorem 2.4 decouples the dynamics from the geometry, capturing all the information about iterations of semicontractions on metric spaces, and reducing the question of convergence at infinity to a purely geometric question on the geometric shape of horofunctions.

Example 2.5. An important class of metric spaces is the CAT(0) spaces, i.e., metric spaces (they do not have to be manifolds) which have non-positive curvature in an extended sense, see [BH99]. In such a space, there is a natural geometric notion of boundary at infinity, which turns out to be in bijection with external horofunctions. Moreover, the horofunctions can be described with sufficient precision to extend the argument given above in Euclidean space: If a sequence satisfies $d(x_n, x_0)/n \to A > 0$ and $h(x_n)/n \to -A$ where h is a horofunction, then x_n converges to the point at infinity corresponding to h. This applies to $x_n = T^n(x_0)$ when T is a semicontraction. We obtain a generalization of Theorem 1.2 to a much broader class of metric spaces.

A weakness of the previous result is that it does not give much when $A = 0$. For instance, it does not seem to reprove Theorem 2.1 of Denjoy and Wolff when $A = 0$ (while the convergence to a point on the boundary follows directly when $A > 0$, as the disk with the hyperbolic distance is CAT(0) – and even CAT(−1)). In fact, one can fully recover Theorem 2.1 from Theorem 2.4 thanks to the following lemma due to Całka [Cał84], for which we give a direct proof.

Lemma 2.6. *Let T be a semicontraction of a proper metric space. Let $x_0 \in X$. If there exists a subsequence n_i along which $d(x_0, T^{n_i}x_0)$ stays bounded, then the whole sequence $d(x_0, T^n x_0)$ is bounded.*

Proof. Let $\mathcal{O}$ be the orbit of x_0. It has a cluster point x_1 by assumption. Let $B = \overline{\mathcal{O}} \cap \overline{B}(x_1, 1)$. By properness, B is covered by a finite number of balls $B_i = \overline{\mathcal{O}} \cap \overline{B}(x_i, 1/2)$, with $x_i \in \mathcal{O}$. For each i, choose $k_i > 0$ such that $T^{k_i}(x_i) \in \overline{B}(x_1, 1/2)$, this is possible as x_1 is a cluster point of $\mathcal{O}$. Then $T^{k_i}(B_i) \subseteq B$ as T is a semicontraction.

Consider now $n > \max k_i$. Then

$$T^n(B) \subseteq \bigcup_i T^n(B_i) = \bigcup_i T^{n-k_i}(T^{k_i} B_i) \subseteq \bigcup_i T^{n-k_i} B \subseteq \bigcup_{m<n} T^m(B).$$

We deduce by induction that $T^n(B) \subseteq \bigcup_{m \leqslant \max k_i} T^m(B)$. Hence, $\bigcup_n T^n(B)$ is within bounded distance of x_0. Finally, x_0 has an iterate that enters B. All its subsequent iterates remain in the above set. □

Proof of Theorem 2.1 of Denjoy-Wolff. We endow the unit disk with the hyperbolic distance, for which any holomorphic map is a semicontraction.

Assume first that $T^n(0)$ stays bounded for this distance. Then $K = \bigcap_n \overline{\bigcup_{m \geqslant n} \{T^m(0)\}}$ is a nonempty compact set, satisfying $T(K) = K$. The set K is contained in a unique ball of minimal radius (this is a general property of nonpositive curvature, see [BH99, Proposition 2.7]) that we denote by $\overline{B}(x, r)$. Then $K = T(K)$ is included in $\overline{B}(T(x), r)$ as T is a semicontraction. By uniqueness, $x = T(x)$, and T has a fixed point.

Assume now that $T^n(0)$ is unbounded. By Lemma 2.6, it tends to infinity in the hyperbolic disk, i.e., to the unit circle S^1 in $\mathbb{C}$. Moreover, Theorem 2.4 shows that the sequence $T^n(0)$ stays in a horoball $\{x : h(x) \leqslant 0\}$ for some horofunction h. In this setting, horoballs are Euclidean disks with 0 in their boundary and tangent to the unit circle. In particular, the closure of such a horoball meets S^1 at a unique point, to which $T^n(0)$ must converge. □

3. Iteration of random semicontractions

The problem of interest to us is the composition of random semicontractions. Let us describe it in the simplest case. Fix a metric space (X, d) with a basepoint x_0, consider a finite number of semicontractions $T_1, \dots, T_I$ on X, and fix a probability measure $\mathbb{P}_0$ on $\{1, \dots, I\}$, i.e., a sequence of positive real numbers $p_i > 0$ with $\sum p_i = 1$. Then we can describe a left random walk L_n on X as follows. At time 0, let $L_0 = x_0$. Then, choose randomly a semicontraction $T^{(1)}$ among $T_1, \dots, T_I$, taking T_i with probability p_i, and jump to $L_1 = T^{(1)}(x_0)$. Then, choose $T^{(2)}$ like $T^{(1)}$, independently of the choices already made, and jump to $L_2 = T^{(2)}(L_1)$. And so on. Formally,

$$L_n = T^{(n)} \circ \dots \circ T^{(1)}(x_0),$$

where the $T^{(k)}$ are random semicontractions, chosen independently according to the distribution $\mathbb{P}_0$. We should write $T^{(k)} = T^{(k)}(\omega)$ and $L_n = L_n(\omega)$ where ω is a random parameter, living in a probability space which parameterizes all objects we use (here, we can take $\Omega = \{1, \dots, I\}^{\mathbb{N}}$ with the probability measure $\mathbb{P} = \mathbb{P}_0^{\otimes \mathbb{N}}$). As usual in probability theory, we will not write explicitly the parameter ω to get simpler formulas (but it will reappear in the more general context we will describe later on).

One can also consider a right random walk R_n given by

$$R_n = T^{(1)} \circ \dots \circ T^{(n)}(x_0).$$

Its geometric meaning is less clear at first sight, but its convergence behavior is much better as we will explain now. In general, L_n can be very far away from L_{n-1}, while

$$d(R_n, R_{n-1}) = d(T^{(1)} \circ \dots \circ T^{(n)}(x_0), T^{(1)} \circ \dots \circ T^{(n-1)}(x_0)) \leqslant d(T^{(n)}(x_0), x_0),$$

where the last inequality follows from the fact that $T^{(1)} \circ \cdots \circ T^{(n-1)}$ is a semicontraction. Therefore, R_n is within bounded distance of R_{n-1}. The random walk R_n makes bounded jumps, contrary to L_n.

Example 3.1. The isometries of the hyperbolic disk are of three type: elliptic (with a fixed point inside the disk), parabolic (with a unique fixed point on the boundary, the dynamics is a rotation on horospheres centered at this point) and loxodromic (with two fixed points on the boundary, one attractive and one repulsive). Assume that all T_i are loxodromic isometries, with attractive fixed point ξ_i. We expect that, independently of the position of L_{n-1}, the map $T^{(n)} = T_{i_n}$ sends it close to its attractive point ξ_{i_n}. Hence, the sequence L_n should tend towards the circle at infinity, but alternate between different possible limit points since, almost surely, i_n will take every value in $\{1, \dots, I\}$ infinitely often when n tends to infinity. In particular, we should not expect L_n to converge typically. On the other hand, in R_n, the map that is applied last is always $T^{(1)}$, so that R_n should be close to ξ_{i_1}, up to an error depending on the next terms in the sequence. As the maps we are composing are contractions on the boundary (away from their repulsive fixed point), the influence of the n-th map should be exponentially small. Therefore, R_n should typically be a Cauchy sequence in $\overline{\mathbb{D}}$, and it should converge (to a random limit, that depends on the random parameter ω). In this geometric context, this heuristic description is correct (the almost sure convergence of R_n to a limit point is due to Furstenberg, in a broader context).

We will consider a more general setting, encompassing the previous one, in which the semicontractions we compose are not any more independent from each other.

Let us consider a space Ω with a probability measure $\mathbb{P}$ and a measurable map U which preserves the measure (i.e., for every measurable subset B, we have $\mathbb{P}(U^{-1}B) = \mathbb{P}(B)$). We will moreover assume that U is ergodic: any measurable set B with $U^{-1}(B) = B$ has measure 0 or 1. Finally, let us fix a map $\omega \mapsto T(\omega)$ associating to $\omega \in \Omega$ a semicontraction $T(\omega)$ on the space (X, d), in a measurable way. We also require an integrability assumption: we will always assume $\int d(x_0, T(\omega)x_0)\,\mathrm{d}\mathbb{P}(\omega) < \infty$. Then we can define "random walks" on (X, d) as follows. Writing x_0 for a basepoint in X, let $L_n(\omega) = T(U^{n-1}(\omega)) \circ \cdots \circ T(\omega)(x_0)$ and $R_n(\omega) = T(\omega) \circ \cdots \circ T(U^{n-1}\omega)(x_0)$. We will mainly be interested in $R_n(\omega)$, since this is the walk for which one can expect convergence results, as explained in Example 3.1. Therefore, let us write $T^n(\omega) = T(\omega) \circ \cdots \circ T(U^{n-1}\omega)$.

This setting is a generalization of the case of random compositions: It is recovered by taking $\Omega = \{1, \cdots I\}^{\mathbb{N}}$ and $\mathbb{P} = \mathbb{P}_0^{\mathbb{N}}$ and U the left shift (given by $U((\omega_k)_{k\in\mathbb{N}}) = (\omega_{k+1})_{k\in\mathbb{N}}$) and $T((\omega_k)_{k\in\mathbb{N}}) = T_{\omega_0}$. The non-independent case is in general more delicate to study since several probabilistic tools do not apply any more (for instance, Furstenberg's proof in Example 3.1 relies on the martingale convergence theorem, which does not hold in this broader context).

The setting we have studied in Sections 1 and 2, of a single semicontraction, is also a particular case of the general setting, taking Ω reduced to a point. We can ask how much of the results proved in this particular case extend to the general situation.

The first result (asymptotic behavior of the distance to the origin) follows directly from an ergodic theorem, Kingman's Theorem, which is the analogue of Fekete's Lemma in an ergodic context. We will come back later to this statement, given below as Theorem 4.2. This theorem readily implies the following:

Proposition 3.2. *There exists $A \geqslant 0$ such that $d(x_0, T^n(\omega)x_0)/n \to A$ for almost every ω.*

To go further and obtain a directional convergence, we would like the analogue of Theorem 2.4, i.e., obtain for almost every ω a horofunction h^ω describing the asymptotic behavior of the walk. This result is considerably more delicate. We have proved it in full generality with Karlsson in [GK15], after several partial results:

Theorem 3.3 (Karlsson-Margulis [KM99]). *Let $\varepsilon > 0$. For almost every ω, there exists a horofunction h^ω such that all cluster values of the sequence $h^\omega(T^n(\omega)x_0)/n$ belong to the interval $[-A, -A+\varepsilon]$.*

Theorem 3.4 (Karlsson-Ledrappier [KL06]). *Assume moreover that all the maps $T(\omega)$ are isometries of X. For almost every ω, there exists a horofunction h^ω such that $h^\omega(T^n(\omega)x_0)/n \to -A$.*

Theorem 3.5 (Gouëzel-Karlsson [GK15]). *Without further assumptions, for almost every ω, there exists a horofunction h^ω such that $h^\omega(T^n(\omega)x_0)/n \to -A$.*

The last theorem realizes the full decoupling between dynamics and geometry that we already explained in Section 2 for the dynamics of a single semicontraction: Assume that, for $A > 0$, a sequence satisfying $d(x_n, x_0) \sim An$ and $h(x_n) \sim -An$ converges necessarily towards a point in a given geometric compactification of X (this is purely a geometric property of X and its compactification). Then we deduce that, for almost every ω, the sequence $T^n(\omega)x_0$ converges in the compactification. This is for instance the case when X is CAT(0), as explained in Example 2.5. However, we note that, in the CAT(0) case, Theorem 3.3 is sufficient to obtain this convergence (see [KM99]), thanks to additional geometric arguments (that can be completely avoided if one uses Theorem 3.5).

These theorems have many applications in different contexts. For instance, if one applies them to isometries of the symmetric space associated to $\mathrm{GL}(d, \mathbb{R})$ (which is CAT(0), so that any of the above theorems would suffice), one can recover Oseledets' Theorem on random products of matrices. One can also obtain a random version of the theorem of Denjoy and Wolff (Theorem 2.1), or applications to operator theory, to Teichmüller theory. This note is not devoted to applications, we refer the reader to the articles cited above. We rather want to explore a little bit the proofs of these statements: contrary to the intuition, they have nothing geometric, they rely exclusively on subadditivity arguments (just like the proofs in Sections 1 and 2).

This is not completely true for the proof given by Karlsson and Ledrappier of Theorem 3.4: they take advantage of the fact that the maps are isometries by arguing that isometries act on the set of horofunctions. One can then use a cocycle on this space, which is geometric in spirit. However, this is true for the proofs of Theorems 3.3 et 3.5, that we will sketch in the next section.

Remark 3.6. Theorem 3.5 constructs a horofunction that satisfies $h^\omega(T^n(\omega)x_0) \leqslant -An + o(n)$, which is weaker than the conclusion of Theorem 2.4 giving $h(T^n x_0) \leqslant -An$ in the case of a single semicontraction. It is easy to see that it is impossible to get such a strong conclusion in the random case: it would for instance imply that the points $T^n(\omega)x_0$ would almost surely stay in a horoball. This is not the case if the $T(\omega)$ can go in every direction, for instance if one chooses on $\mathbb{R}$ uniformly between the translation of 2 and the translation of -1 (we have chosen two vectors with different norms to ensure that A is nonzero).

4. Ergodic theory and subadditivity

The analogue of subadditive sequences in a dynamical setting is given by the notion of subadditive cocycle (this terminology is very bad, as a subadditive cocycle is not a cocycle, the word subcocycle would certainly be better, but it is too late to change).

Definition 4.1. *Let $(\Omega, \mathbb{P})$ be a probability space and $U : \Omega \to \Omega$ an ergodic map preserving the measure. A measurable function $u : \mathbb{N} \times \Omega \to \mathbb{R}$ is a* subadditive cocycle *if, for all k, ℓ and for almost every ω,*

$$u(k+\ell, \omega) \leqslant u(k, \omega) + u(\ell, U^k \omega).$$

A subadditive cocycle is integrable *if $\int u^+(1, \omega) \,\mathrm{d}\mathbb{P}(\omega) < \infty$, where u^+ is the positive part of u.*

Let us consider for instance a family of semicontractions $T(\omega)$ depending measurably on $\omega \in \Omega$. Let $u(n, \omega) = d(x_0, T^n(\omega)(x_0))$. This is a subadditive cocycle: for all k, ℓ and ω, we have

$$\begin{aligned} u(k+\ell, \omega) &= d(x_0, T^{k+\ell}(\omega)x_0) = d(x_0, T^k(\omega)(T^\ell(U^k\omega)(x_0))) \\ &\leqslant d(x_0, T^k(\omega)(x_0)) + d(T^k(\omega)(x_0), T^k(\omega)(T^\ell(U^k\omega)(x_0))) \\ &\leqslant d(x_0, T^k(\omega)(x_0)) + d(x_0, T^\ell(U^k\omega)(x_0)) = u(k, \omega) + u(\ell, U^k\omega), \end{aligned}$$

where we used the triangular inequality to go from the first to the second line, and the fact that $T^k(\omega)$ is a semicontraction to go from the second to the third line. This is precisely the same computation as for one single semicontraction in (1.1), with an additional dependency on ω that has to be written correctly.

In the same way that results on subadditive sequences (Lemmas 1.4 and 1.5) were instrumental in the proofs of Sections 1 and 2 on the behavior of one semicontraction, we will be able to analyze the behavior of random semicontractions if we have sufficiently precise tools on subadditive cocycles.

The first central result in this direction is Kingman's Theorem, replacing in this context Fekete's Lemma 1.4.

Theorem 4.2 (Kingman [Kin68])**.** *Let u be an integrable subadditive cocycle. There exists $A \in [-\infty, \infty)$ such that, almost surely, $u(n, \omega)/n \to A$. Moreover, if $A > -\infty$, the convergence also holds in L^1. Finally, A is the limit of the sequence $(\int u(n, \omega) \,\mathrm{d}\mathbb{P}(\omega))/n$, which is convergent by subadditivity.*

Since $d(x_0, T^n(\omega)(x_0))$ is a subadditive cocycle when the $T(\omega)$ are semicontractions, this result implies Proposition 3.2, i.e., the almost sure convergence of $d(x_0, T^n(\omega)(x_0))/n$.

There are many proofs of Kingman's Theorem in the literature. The simplest one is probably the proof of Steele [Ste89], that we will sketch now.

Proof sketch. Consider the measurable function $f(\omega) = \liminf u(n, \omega)/n$. The subadditivity of u implies that $f(\omega) \leqslant f(U\omega)$ almost surely. We deduce thanks to Poincaré recurrence theorem that $f(\omega) = f(U\omega)$ almost everywhere. Indeed, by this theorem, almost every point in $V_a = f^{-1}([-\infty, a])$ comes back infinitely often to V_a under the iteration of U. A point with $f(\omega) < f(U\omega)$ would belong to V_a for each rational a in $(f(\omega), f(U\omega))$ but can only come back to it if it belongs to a 0 measure set.

The function f, which is almost everywhere invariant, is almost everywhere constant by ergodicity, equal to some $A \in [-\infty, +\infty)$. Assume for instance $A > -\infty$, and take $\varepsilon > 0$. Fix also $N > 0$. For almost every ω, there exists an integer $n(\omega)$ with $u(n(\omega), \omega) \leqslant n(\omega)(A + \varepsilon)$, by definition of the inferior limit. Given a point ω, we define a sequence of times as follows. Start from $n_0 = 0$. If $n(U^{n_0}\omega) > N$ (i.e., we have to wait too long to see the almost realization of the liminf), we are not patient enough and we let $n_1 = n_0 + 1$. Otherwise, set $n_1 = n_0 + n(U^{n_0}\omega)$, so that $u(n_1 - n_0, U^{n_0}\omega) \leqslant (n_1 - n_0)(A + \varepsilon)$. We continue this construction by induction, therefore partitioning the integers into intervals $[n_i, n_{i+1} - 1]$. On most of them, the value of u is bounded by $(n_{i+1} - n_i) \cdot (A + \varepsilon)$ by construction. On the other ones, we do not have a good control, but their frequency is very small if N is large.

Combining these two estimates and using the subadditivity of u to bound $u(n_i, \omega)$ by the sum of the contributions of each individual interval, we obtain $u(n_i, \omega) \leqslant n_i \cdot (A+\varepsilon) + o_N(1)n_i$. This is bounded by $n_i \cdot (A + 2\varepsilon)$ if N is large enough. Finally, we obtain $\limsup u(n, \omega)/n \leqslant A + 2\varepsilon$ (first along the subsequence n_i, then for any integer as two consecutive terms of this sequence are separated at most by N). Letting ε tend to 0, we finally get $\limsup u(n, \omega)/n \leqslant A = \liminf u(n, \omega)/n$. This concludes the proof of the almost sure convergence. □

One can note that this proof looks very closely like the proof of Fekete's Lemma 1.4. The difference is that, instead of using subadditivity always with respect to the same time N (which almost realizes the liminf), one has to use a time which depends on the point we are currently at. Apart from this, the two proofs can be written completely in parallel.

To prove Theorem 3.5, we need a substitute for Lemma 1.5 if we want to use the proof strategy of Section 1. The direct analogue of this lemma in our context would be the following statement:

Let $\varepsilon > 0$. Let u be an integrable subadditive cocycle, such that $u(n, \omega)/n \to A > -\infty$ almost everywhere. For almost every ω, there exist arbitrarily large integers n such that, for all $1 \leqslant \ell \leqslant n$, we have $u(n, \omega) \geqslant u(n - \ell, U^\ell(\omega)) + (A - \varepsilon)\ell$.

However, this statement is wrong. Take for instance $u(n, \omega) = \sum_{k=0}^{n-1} v(U^k\omega)$ for some function v (this is an additive cocycle, whose limit A is equal to $\int v$). If the above statement holds, then taking $\ell = 1$ we get $v(\omega) \geqslant A - \varepsilon$. Letting ε tend to 0, we get $v(\omega) \geqslant A = \int v$ almost everywhere, which is wrong if v is not almost surely constant.

This argument shows that any valid statement has to allow some fluctuations for each ℓ. At the same time, it is crucial for the application to semicontractions to have a statement which controls all intermediate times between 1 and n. The main result of [GK15] is the following theorem, compatible with these two constraints.

Theorem 4.3. *Let u be an integrable subadditive cocycle, such that $u(n, \omega)/n \to A > -\infty$ almost everywhere. For almost every ω, there exists a sequence $\delta_\ell \to 0$ and arbitrarily large integers n such that, for all $1 \leqslant \ell \leqslant n$, we have $u(n, \omega) \geqslant u(n - \ell, U^\ell(\omega)) + (A - \delta_\ell)\ell$.*

In a setting of random semicontractions, applying this theorem to the subadditive cocycle $u(n, \omega) = d(x_0, T^n(\omega)(x_0))$ and following the arguments of Section 2, we obtain readily Theorem 3.5.

Note that the subadditivity of u ensures that $u(n, \omega) \leqslant u(n - \ell, U^\ell(\omega)) + u(\ell, \omega)$. As $u(\ell, \omega) \sim A\ell$ by Kingman's theorem, an upper bound $u(n, \omega) \leqslant u(n - \ell, U^\ell(\omega)) + (A + \delta_\ell)\ell$ is automatic. The difficulty in Theorem 4.3 is that, instead, we are looking after a *lower* bound, ensuring that the subadditive cocycle u is in fact almost additive at all intermediate times between 1 and n, for some good times n.

To prove this theorem, a first idea is to try to use the concept of records, at the heart of the proof of Lemma 1.5. It would work very well to prove the existence of infinitely many times n for which $u(n, \omega) \geqslant u(n - \ell, \omega) + (A - \delta_\ell)\ell$ for all intermediate time ℓ. Unfortunately, this is not the statement we are interested in: we do not want a statement involving $u(n-\ell,\omega)$, but rather $u(n-\ell,U^\ell(\omega))$ since this is the quantity that is relevant for the application to semicontractions. We need a different argument.

The proof of Theorem 3.3 by Karlsson and Margulis in [KM99] relied on a statement which is slightly weaker than Theorem 4.3. In the same context, they show that, given $\varepsilon > 0$, there exist almost surely a time $k(\omega)$ and arbitrarily large integers n such that, for all $k(\omega) \leqslant \ell \leqslant n$, we have $u(n,\omega) \geqslant u(n-\ell,U^\ell(\omega)) + (A-\varepsilon)\ell$. This is enough to prove Theorem 3.3 by following the proof in Section 2. At first, one could think that this statement is very close to Theorem 4.3: a strategy to prove this theorem could be to start from the statement of Karlsson and Margulis for $\varepsilon_i = 2^{-i}$, and then apply some kind of diagonal argument to obtain times n that work simultaneously for $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_N$ (with N arbitrarily large). The problem with this approach is that the theorem of Karlsson and Margulis is not quantitative: it does not guarantee that there are many good times (and, in fact, their proof gives a very small set of good times). Typically, there is no integer which is good both for ε_0 and ε_1, ruining the diagonal argument!

If we want to use this kind of approach, we need large sets of good times, when ε is fixed. This is what we will do to prove Theorem 4.3. The notion of largeness we will use is the (lower) asymptotic density

$$\underline{\mathrm{Dens}}\, B = \liminf_{N\to\infty} \frac{\mathrm{Card}(B \cap \{1,\dots,N\})}{N}.$$

The main steps in the proof are the following. Going to the natural extension if necessary, we can assume that U is invertible. Then we define a new subadditive cocycle $\tilde{u}(n,\omega') = u(n,U^{-n}\omega')$ (it is subadditive for U^{-1}). Its interest is that, writing $\omega' = U^n\omega$, then

$$u(n,\omega) - u(n-\ell,U^\ell\omega) = \tilde{u}(n,\omega') - \tilde{u}(n-\ell,\omega').$$

In the right hand side term, the same point ω' appears in both instances of $\tilde{u}$. This will make it possible to use some combinatorial arguments that do not work directly for u. The price to pay is that good times for $\tilde{u}$ are not good times for u: there is an additional change of variables, which spoils the argument if the information on the set of good times is only qualitative, but which works if we have quantitative estimates in terms of asymptotic density for the set of good times.

Then, we show that $\tilde{u}$ has many good times, with the following lemmas:

Lemma 4.4. *Let $\delta > 0$. Then there exists $C > 0$ such that, for almost every ω,*

$$\underline{\mathrm{Dens}}\{n \in \mathbb{N} \,:\, \forall \ell \in [1,n], \tilde{u}(n,\omega) - \tilde{u}(n-\ell,\omega) \geqslant (A-C)\ell\} \geqslant 1-\delta.$$

Lemma 4.5. *Let $\delta > 0$ and $\varepsilon > 0$. Then there exists an integer k such that, for almost every ω,*

$$\underline{\mathrm{Dens}}\{n \in \mathbb{N} \,:\, \forall \ell \in [k,n], \tilde{u}(n,\omega) - \tilde{u}(n-\ell,\omega) \geqslant (A-\varepsilon)\ell\} \geqslant 1-\delta.$$

The second lemma is essentially a more precise variant of the first one. Their proofs are essentially combinatorial, and borrow some ideas to the proof by Steele of Kingman's Theorem that we have described above.

As the intersection of two sets with asymptotic density close to 1 still has an asymptotic density close to 1, we will then be able to intersect the sets of good times produced by these lemmas (and, in the case of Lemma 4.5, for different values of ε), while keeping sets with large density. This makes it possible to implement the diagonal argument alluded to earlier. After a final change of variables to go back to u, we finally obtain Theorem 4.3. The details are rather delicate and technical, the interested reader is referred to [GK15] for a full proof and to [Gou16] for a computer-checked formalization of the proof.

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