Inverse Spectral Analysis of Singular Radial AKNS Operators

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Abstract

We study an inverse spectral problem for singular AKNS operators based on spectral data associated with two distinct values of the effective angular momentum parameter κ. Our main focus is the local inverse problem near the zero potential. For the pairs $(\kappa_1, \kappa_2) = (0,1)$, $(1,2)$ and $(0,3)$, we establish local uniqueness. For $(0,2)$, we prove that the Fréchet differential of the spectral map at the origin is injective, while the question whether its range is closed remains open.

1 Introduction

Inverse spectral theory for one-dimensional singular differential operators arises naturally in the analysis of radially symmetric quantum systems. The classical example is the radial Schrödinger operator, obtained from the three-dimensional Schrödinger equation after separation of variables in spherical coordinates. For a real-valued square-integrable potential $q \in L^2(0,1)$, the radial equation takes the form

$$H_\ell(q)u := -\frac{d^2u}{dx^2} + \frac{\ell(\ell+1)}{x^2}u + q(x)u = \lambda u, \qquad x \in (0,1), \quad \ell \in \mathbb{N}. \tag{1.1}$$

The regularity condition is $u(x) = O(x^{\ell+1})$ as $x \to 0$, together with the Dirichlet boundary condition $u(1) = 0$. For each angular momentum ℓ, this defines a self-adjoint operator on $L^2(0,1)$ with discrete, simple, real spectrum.

The inverse spectral problem consists in determining the potential $q(x)$ from its spectral data. A classical result due to Pöschel–Trubowitz [28], Carlson [11], Guillot–Ralston [16] and Zhornitskaya–Serov [38] asserts that the Dirichlet spectrum together with suitable norming constants forms a real-analytic coordinate system on $L^2(0,1)$ for each fixed ℓ. Recently, in the context of radial Schrödinger operators with distinct angular momenta, we proved in [14] that the potential is uniquely determined by the Dirichlet spectra corresponding to infinitely many values of ℓ satisfying a Müntz-type condition, and that in a neighborhood of the zero potential, knowing two spectra (for the pairs $(\ell_1, \ell_2) = (0,1)$, $(1,2)$ or $(0,3)$) already implies uniqueness. These results rely on the explicit structure of the eigenfunctions in terms of Bessel functions, specifically of the form

$$u_{\ell,n}(x) = c_{\ell,n}\, x^{1/2} J_\nu(j_{\nu,n}x), \qquad \nu = \ell + \tfrac{1}{2}, \tag{1.2}$$

together with delicate completeness properties of the squared eigenfunctions, following earlier works of Rundell and Sacks [29].

The aim of the present work is to investigate the analog of this spectral problem for singular radial AKNS operators. The mathematical model considered here arises from certain physical models, whose derivation is briefly outlined in the appendix. After separation of variables, one is led to a family of singular radial AKNS operators parameterized by what we call an *effective angular momentum parameter* $\kappa \in \mathbb{Z}$. We

emphasize that the parameter κ does not, in general, correspond to a genuine angular momentum, in contrast with the Schrödinger case. In the 3D Dirac framework, $-\kappa$ arises as an eigenvalue of the spin–orbit operator K, whereas in the 2D model it can be interpreted as an effective angular momentum (see (7.97) in [36] and Appendix A of the present paper).

The associated singular AKNS operator is

$$H_\kappa(V)\,Z = H_\kappa(p,q)\,Z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Z' + \begin{pmatrix} 0 & -\dfrac{\kappa}{x} \\ -\dfrac{\kappa}{x} & 0 \end{pmatrix} Z + V(x)\,Z\,, \qquad Z = (Z_1, Z_2)^{\mathsf{T}}\,. \tag{1.3}$$

Here the potential matrix V is given by

$$V(x) = \begin{pmatrix} -q(x) & p(x) \\ p(x) & q(x) \end{pmatrix}\,, \qquad p, q \in L^2_{\mathbb{R}}(0,1)\,. \tag{1.4}$$

We impose the following boundary conditions. Let $(\theta_1, \theta_2) \in \mathbb{R}^2$.

- When $\kappa = 0$,
$$Z(0) \cdot u_{\theta_1} = 0\,, \qquad Z(1) \cdot u_{\theta_2} = 0\,, \qquad u_{\theta_j} = \begin{pmatrix} \sin \theta_j \\ \cos \theta_j \end{pmatrix}\,. \tag{1.5}$$

- When $\kappa \neq 0$,
$$Z(1) \cdot u_{\theta_2} = 0\,. \tag{1.6}$$

The AKNS system enjoys symmetries associated with the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{1.7}$$

A direct computation shows that

$$\sigma_1 H_\kappa(p,q)\sigma_1 = -H_{-\kappa}(-p,q), \tag{1.8}$$
$$\sigma_2 H_\kappa(p,q)\sigma_2 = H_{-\kappa}(-p,-q), \tag{1.9}$$
$$\sigma_3 H_\kappa(p,q)\sigma_3 = -H_\kappa(p,-q). \tag{1.10}$$

In particular, if Z solves $H_\kappa(p,q)Z = \lambda Z$, then $\sigma_1 Z$, $\sigma_2 Z$ and $\sigma_3 Z$ solve the corresponding transformed systems with the same or opposite spectral parameter according to (1.8)–(1.10).

In the case $p = 0$, the σ_1-symmetry shows that the knowledge of the spectra corresponding to κ and $-\kappa$ (with the same boundary condition $\theta_2 = 0$) is equivalent, up to a reindexation, to the knowledge of the two spectra associated with $\theta_2 = 0$ and $\theta_2 = \pi/2$ for H_κ. Therefore, provided that the resulting sequences satisfy the technical interlacing property required in Theorem 1.1 of [3], the potential q is uniquely determined.

For these reasons, and other technical difficulties[1], we restrict ourselves, in the present paper, to the case

$$\kappa \geq 0 \text{ and } \theta_2 = 0\,.$$

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[1] It would actually be interesting to treat the general case, but, except for the case $\theta_2 = \frac{\pi}{2}$, this would indeed be technically more involved.

In the case $\kappa = 0$, we also set $\theta_1 = 0$. As explained in the appendix, $\theta_2 = 0$ corresponds in the motivating Dirac radial model to the so-called Zig–Zag condition, while the MIT bag condition corresponds to $\theta_2 = \pm\frac{\pi}{4}$.

The domain of the operator is then defined as follows:

$$D(H_0) = \left\{ Z = (Z_1, Z_2) \in L^2(0,1)^2 \ : \ H_0 Z \in L^2(0,1)^2, \ Z_2(0) = 0, \ Z_2(1) = 0 \right\}. \tag{1.11}$$

For $\kappa > 0$, we set

$$D(H_\kappa) = \left\{ Z = (Z_1, Z_2) \in L^2(0,1)^2 \ : \ H_\kappa Z \in L^2(0,1)^2, \ Z_2(1) = 0 \right\}. \tag{1.12}$$

As shown in [32], this realizes a self-adjoint operator with purely discrete and *simple* spectrum which can be written as a doubly infinite sequence

$$\{\lambda_{\kappa,n}(p,q)\}_{n \in \mathbb{Z}}, \tag{1.13}$$

ordered as

$$\cdots < \lambda_{\kappa,-2}(p,q) < \lambda_{\kappa,-1}(p,q) < \lambda_{\kappa,0}(p,q) < \lambda_{\kappa,1}(p,q) < \lambda_{\kappa,2}(p,q) < \cdots. \tag{1.14}$$

The labelling is uniquely determined by the asymptotic behavior (see [32], Theorem 3.1):

$$\lambda_{\kappa,n}(p,q) = \left(n + \mathrm{sgn}(n)\, \tfrac{\kappa}{2} \right) \pi + \ell^2(n), \ \text{as } |n| \to \infty, \tag{1.15}$$

which governs both ends of the sequence. Here the notation $\alpha_n = \beta_n + \ell^2(n)$, $n \in \mathbb{Z}$, means that the sequence $(\alpha_n - \beta_n)_{n \in \mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z})$.

In addition to the eigenvalues, Serier introduced suitable norming constants which, together with the spectrum, form a complete system of spectral coordinates. More precisely, the combined data (eigenvalues and norming constants) provide a locally stable parameterization of the potential and yield a Borg–Levinson type uniqueness result (see [32]).

However, these classical results rely crucially on the availability of norming constants. From the physical and inverse point of view, such quantities are in general *not observable*. This naturally leads to a different and more challenging question: can one determine the potential uniquely from spectral data alone, *without* any norming constants?

As in the corresponding inverse problem for the radial Schrödinger operator, the spectral data associated with a single effective angular momentum are not sufficient to ensure uniqueness. This naturally leads to combining information from *at least two distinct effective angular momenta* κ. More precisely, we consider the spectra corresponding to two distinct effective angular momenta $\kappa_1 \neq \kappa_2$ and study whether this purely spectral information, without any norming constants, determines the potential.

We now state our first main result, which gives several cases where two spectra are sufficient to recover the potential locally near the trivial configuration.

Theorem 1.1 (Local uniqueness for the pairs $(0,1)$, $(1,2)$ and $(0,3)$)**.** *Let $(\kappa_1, \kappa_2) = (0,1)$, $(1,2)$ or $(0,3)$. Then the knowledge of the spectra associated with the effective angular momenta κ_1 and κ_2 uniquely determines the potential $V = (p,q) \in L^2(0,1) \times L^2(0,1)$ in a neighborhood of the zero potential $V_0 = 0$.*

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The proof of Theorem 1.1 relies on the analysis of the Fréchet differential of the associated spectral map at the zero potential. We show that this differential is injective with closed range, which yields the desired local uniqueness result.

We now briefly introduce this spectral map. Let $\widetilde{\lambda}_{\kappa,n}(p,q)$ denote the renormalized eigenvalues, implicitly defined by the asymptotic formula (1.15) and explicitly given by

$$\widetilde{\lambda}_{\kappa,n}(p,q) = \lambda_{\kappa,n}(p,q) - \left(n + \mathrm{sgn}(n)\,\tfrac{\kappa}{2} \right) \pi \,.$$

We choose two distinct effective angular momenta $\kappa_1 \neq \kappa_2$, which are fixed integers and we consider the associated spectral map

$$\mathcal{S}_{\kappa_1,\kappa_2} \,:\, L^2(0,1) \times L^2(0,1) \longrightarrow \ell^2_{\mathbb{R}}(\mathbb{Z}) \times \ell^2_{\mathbb{R}}(\mathbb{Z}) \,,$$

defined by

$$\mathcal{S}_{\kappa_1,\kappa_2}(p,q) = \left(\left(\widetilde{\lambda}_{\kappa_1,n}(p,q) \right)_{n \in \mathbb{Z}}, \, \left(\widetilde{\lambda}_{\kappa_2,n}(p,q) \right)_{n \in \mathbb{Z}} \right) \,. \tag{1.16}$$

We now state the second main result of this paper, which discusses the injectivity of the Fréchet differential of the spectral map at the zero potential for three pairs of effective angular momenta.

Theorem 1.2 (Behavior of the differential of the spectral map). *Let $\kappa_1 \neq \kappa_2$ be two distinct integers and consider the spectral map*

$$\mathcal{S}_{\kappa_1,\kappa_2} \,:\, L^2(0,1) \times L^2(0,1) \longrightarrow \ell^2_{\mathbb{R}}(\mathbb{Z}) \times \ell^2_{\mathbb{R}}(\mathbb{Z}) \,.$$

Then, at the zero potential $V = 0$, the Fréchet differential of $\mathcal{S}_{\kappa_1,\kappa_2}$ satisfies:

- *For $(\kappa_1,\kappa_2) = (0,1)$, $(1,2)$ or $(0,3)$, the differential is injective and has closed range.*

- *For $(\kappa_1,\kappa_2) = (0,2)$, the differential is injective.*

The proof of Theorem 1.2 is given in Sections 6–7. The case $(0,2)$ remains open, as we have not been able to prove that the differential has closed range. Theorem 1.1 then follows from the local injectivity result stated in Proposition 7.2.

In the appendix, we will describe how these questions arise in the analysis of inverse spectral problems for the radial Dirac operator (with possible addition of an Aharonov–Bohm potential) in dimension two and three and present some remaining open questions.

2 Eigenvalue analysis in the unperturbed case $V = 0$

In this section we analyze the spectral problem in the unperturbed case $V = 0$, which serves as the reference configuration for the perturbative and inverse analysis developed later.

When $V = 0$, the AKNS operator reduces to the first-order matrix equation

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} Z'(x) + \begin{pmatrix} 0 & -\dfrac{\kappa}{x} \\ -\dfrac{\kappa}{x} & 0 \end{pmatrix} Z(x) = \lambda Z(x), \qquad x \in (0,1), \tag{2.1}$$

with the boundary condition

$$Z_2(0) = 0 \,,\; Z_2(1) = 0 \quad \text{for } \kappa = 0 \,, \qquad Z_2(1) = 0 \,, \quad \text{for } \kappa \neq 0 \,. \tag{2.2}$$

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2.1 The case $\lambda = 0$

The case $\lambda = 0$ requires a specific discussion. Setting $\lambda = 0$ in the unperturbed Dirac equation yields

$$Z_1' = \frac{\kappa}{x}\, Z_1\,, \qquad Z_2' = -\frac{\kappa}{x}\, Z_2\,.$$

Hence the general solutions are

$$Z_1(x) = C_1\, x^\kappa\,, \qquad Z_2(x) = C_2\, x^{-\kappa}\,.$$

For $\kappa = 0$, the boundary condition $Z_2(0) = 0$ forces $C_2 = 0$. For $\kappa \geq 1$, the condition $Z_2 \in L^2(0,1)$ near $x = 0$ again forces $C_2 = 0$. so that

$$Z(x) = \begin{pmatrix} C_1\, x^\kappa \\ 0 \end{pmatrix}.$$

The boundary condition at $x = 1$ is satisfied, and therefore 0 is an eigenvalue of $H_\kappa(0)$. The associated eigenfunction is thus given by

$$Z_{\kappa,0}^{(0)}(x) = c_{\kappa,0} \begin{pmatrix} x^\kappa \\ 0 \end{pmatrix}, \qquad c_{\kappa,0} > 0\,,$$

and the normalization condition $\|Z_{\kappa,0}^{(0)}\|_{L^2(0,1)^2} = 1$, yields

$$c_{\kappa,0} = \sqrt{2\kappa + 1}\,, \qquad \text{so that} \qquad Z_{\kappa,0}^{(0)}(x) = \sqrt{2\kappa + 1} \begin{pmatrix} x^\kappa \\ 0 \end{pmatrix}.$$

2.2 The case $\lambda \neq 0$

We now consider the case $\lambda \neq 0$, for which the system (2.1) decouples into two scalar equations for Z_1 and Z_2:

$$\begin{cases} Z_1''(x) + \left(\lambda^2 - \dfrac{\kappa(\kappa - 1)}{x^2} \right) Z_1(x) = 0\,, \\[2mm] Z_2''(x) + \left(\lambda^2 - \dfrac{\kappa(\kappa + 1)}{x^2} \right) Z_2(x) = 0\,. \end{cases} \tag{2.3}$$

We set $\nu = \kappa + \frac{1}{2}$. After the standard substitution $Z_j(x) = \sqrt{x}\, u_j(\lambda x)$, the system reduces to Bessel equations of orders $\nu - 1$ and ν. Accordingly, a fundamental system of solutions is given by

$$Z^{(0)}(x,\lambda) = \sqrt{\frac{\pi \lambda x}{2}} \begin{pmatrix} J_{\nu-1}(\lambda x) \\ -J_\nu(\lambda x) \end{pmatrix}, \qquad W^{(0)}(x,\lambda) = \sqrt{\frac{\pi \lambda x}{2}} \begin{pmatrix} -Y_{\nu-1}(\lambda x) \\ Y_\nu(\lambda x) \end{pmatrix}. \tag{2.4}$$

where J_ν and Y_ν denote the Bessel functions of the first and second kinds (see [21]). We recall that for $z \in \mathbb{C} \setminus (-\infty, 0]$ and any real or complex parameter ν, one has

$$J_\nu(z) = \left(\frac{z}{2} \right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{z^2}{4} \right)^k}{k!\, \Gamma(\nu + k + 1)}\,. \tag{2.5}$$

For $\nu \notin \mathbb{Z}$, the function Y_ν is given by [2]

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[2] For an integer n, the function $Y_n(z)$ is defined by the limit $Y_n(z) = \lim_{\nu \to n} Y_\nu(z)$.

$$Y_\nu(z) = \frac{J_\nu(z) \, \cos(\nu\pi) \, - \, J_{-\nu}(z)}{\sin(\nu\pi)} \,. \tag{2.6}$$

We emphasize that for half-integer orders $\nu \in \frac{1}{2} + \mathbb{N}$, the function $\sqrt{z} \, J_\nu(z)$ is entire, whereas $\sqrt{z} \, Y_\nu(z)$ is generally meromorphic because of its singularity at $z = 0$.

As $x \to 0$, these functions satisfy the classical asymptotics :

$$J_\nu(x) \sim \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu, \qquad Y_\nu(x) \sim -\frac{\Gamma(\nu)}{\pi} \left(\frac{2}{x}\right)^\nu, \qquad \nu > 0\,. \tag{2.7}$$

Hence J_ν is regular at the origin, whereas Y_ν is singular. In particular, among the two fundamental solutions in (2.4), only $Z^{(0)}(x, \lambda)$ is square–integrable near $x = 0$ and satisfies the regularity condition. For each eigenvalue $\lambda = \lambda_{\kappa,n}(0,0)$ of the unperturbed operator, we define the associated eigenfunction by

$$Z_{\kappa,n}^{(0)}(x) = c_{\kappa,n} \begin{pmatrix} Z_{1,\kappa,n}^{(0)}(x) \\ Z_{2,\kappa,n}^{(0)}(x) \end{pmatrix} = c_{\kappa,n} \sqrt{\lambda_{\kappa,n}(0,0)x} \begin{pmatrix} J_{\nu-1}\big(\lambda_{\kappa,n}(0,0)\,x\big) \\ -\,J_\nu\big(\lambda_{\kappa,n}(0,0)\,x\big) \end{pmatrix}, \tag{2.8}$$

where $c_{\kappa,n} > 0$ is a normalization constant to be specified later. Using the boundary condition at $x = 1$, we obtain that the eigenvalues of $H_\kappa(0)$ are the simple zeros of J_ν, as will be detailed in the paragraph below.

2.3   Symmetries

In this paragraph we recall the symmetry properties of the unperturbed spectrum for $V = 0$.

For half–integer orders $\nu = \kappa + \frac{1}{2}$, the entire function $z \mapsto \sqrt{z} \, J_\nu(z)$ is an even function when κ is odd, and an odd function when κ is even. This parity property immediately yields the following symmetry for the nonzero eigenvalues:

$$\lambda_{\kappa,-n}(0,0) = -\,\lambda_{\kappa,n}(0,0)\,, \text{ for } n \geq 1\,. \tag{2.9}$$

We recall that the boundary condition leads to the characteristic equation $J_\nu(\lambda) = 0$. If $\{j_{\nu,n}\}_{n\geq 1}$ denotes the sequence of positive zeros of J_ν, then the nonzero eigenvalues of $H_\kappa(0)$ are

$$\lambda_{\kappa,\pm n}(0,0) = \pm j_{\nu,n}\,, \qquad n \geq 1\,. \tag{2.10}$$

Moreover, we have seen that $\lambda = 0$ is an eigenvalue. Thus the full spectrum is symmetric and ordered as

$$\cdots < -j_{\nu,2} < -j_{\nu,1} < 0 < j_{\nu,1} < j_{\nu,2} < \cdots, \tag{2.11}$$

and we assign $\lambda = 0$ to the index $n = 0$ in the bi-infinite enumeration $\{\lambda_{\kappa,n}(0)\}_{n\in\mathbb{Z}}$.

2.4   Summary and notation

The spectrum of the unperturbed operator $H_\kappa(0)$ consists of the simple eigenvalue $\lambda_{\kappa,0}(0) = 0$ and of the nonzero eigenvalues $\lambda_{\kappa,\pm n}(0) = \pm j_{\nu,n}$ for $n \geq 1$, forming a symmetric bi-infinite sequence indexed by $n \in \mathbb{Z}$ (see (2.11)). This enumeration is consistent with the asymptotic formula (1.15) (see [26], 10.21 (vi)).

The associated eigenfunctions are given by the regular solutions $Z_{\kappa,n}^{(0)}$. For the zero eigenvalue, one has

$$Z_{\kappa,0}^{(0)}(x) = \sqrt{2\kappa + 1} \begin{pmatrix} x^\kappa \\ 0 \end{pmatrix},$$

while for $n \neq 0$ they are expressed in terms of Bessel functions.

By symmetry, the eigenfunctions corresponding to $\lambda_{\kappa,n}(0)$ and $\lambda_{\kappa,-n}(0)$ differ only by a sign in their oscillatory components. In particular, the normalization constants depend only on $|n|$, and

$$c_{\kappa,-n} = c_{\kappa,n}, \text{ for } n \geq 1.$$

Accordingly, we index the spectrum by $n \in \mathbb{Z}$, with $n = 0$ corresponding to the zero eigenvalue.

Remark 2.1 (Normalization constants). *We will need the asymptotic behavior of these normalization constants as $n \to \infty$ (see Section 3.3). In fact, the constants $c_{\kappa,n}$ can be computed explicitly using the following standard finite-interval identity (often referred to as a* Lommel-type *formula, see ([26], 10.22.5)): for any α, $\mu \in \mathbb{R}^+$,*

$$\int_0^1 x\, J_\mu(\alpha x)^2\, dx = \frac{1}{2}\Big(J_\mu(\alpha)^2 - J_{\mu-1}(\alpha)\, J_{\mu+1}(\alpha)\Big). \tag{2.12}$$

Applying (2.12) with $\alpha = j_{\nu,|n|}$ and $\mu = \nu$, and using $J_\nu(j_{\nu,|n|}) = 0$ together with the recurrence

$$J_{\nu-1}(\alpha) + J_{\nu+1}(\alpha) = \frac{2\nu}{\alpha}J_\nu(\alpha),$$

we obtain

$$J_{\nu-1}(j_{\nu,|n|}) = -J_{\nu+1}(j_{\nu,|n|})$$

and hence

$$\int_0^1 x\, J_\nu\big(j_{\nu,|n|}x\big)^2\, dx = \frac{1}{2}\, J_{\nu+1}\big(j_{\nu,|n|}\big)^2.$$

Similarly, applying (2.12) with $\mu = \nu - 1$ yields

$$\int_0^1 x\, J_{\nu-1}\big(j_{\nu,|n|}x\big)^2\, dx = \frac{1}{2}\, J_{\nu-1}\big(j_{\nu,|n|}\big)^2 = \frac{1}{2}\, J_{\nu+1}\big(j_{\nu,|n|}\big)^2.$$

Substituting these identities into the normalization condition $\|Z_{\kappa,n}^{(0)}\|_{L^2(0,1)^2} = 1$ (see (2.8)) gives

$$c_{\kappa,n} = \frac{1}{\sqrt{j_{\nu,|n|}}\,\big|J_{\nu+1}(j_{\nu,|n|})\big|}, \text{ as } n \neq 0. \tag{2.13}$$

The classical asymptotic expansions for Bessel functions and for their positive zeros $j_{\nu,n}$ (see, e.g., Watson [37]) yield

$$c_{\kappa,n}^2 = \frac{\pi}{2} + \frac{4\nu^2 - 1}{16\pi\left(n + \nu/2 - 1/4\right)^2} + O\big(n^{-4}\big), \qquad n \to \infty. \tag{2.14}$$

Finally, in the zero-eigenvalue case, we recall that the normalization condition yields

$$c_{\kappa,0} = \sqrt{2\kappa + 1}.$$

We conclude this section with the explicit analysis of the special case $\kappa = 0$. In this case, the computations are particularly simple and allow us to illustrate the preceding constructions in a fully explicit manner.

2.5 The case $\kappa = 0$

We now consider the case $\kappa = 0$, for which $\nu = \frac{1}{2}$ and the singular term κ/x disappears from the system. The analysis of this particular case is especially interesting (see Section 3.4). In this setting, the Dirac system (2.1) reduces to

$$Z_1' = \lambda Z_2 \,, \qquad Z_2' = -\lambda Z_1 \,,$$

and the regular solution, characterized by the condition $Y_2(0) = 0$, is given by

$$Z^{(0)}(x, \lambda) = \begin{pmatrix} \cos(\lambda x) \\ -\sin(\lambda x) \end{pmatrix} .$$

Imposing the boundary condition at $x = 1$ yields the characteristic equation $\sin(\lambda) = 0$, so that the nonzero eigenvalues are explicitly given by

$$\lambda_{0,n}(0,0) = n\pi \,, \qquad n \in \mathbb{Z} \setminus \{0\} \,. \tag{2.15}$$

For each $n \in \mathbb{Z}$, the associated eigenfunction is

$$Z_{0,n}^{(0)}(x) = \begin{pmatrix} \cos(n\pi x) \\ -\sin(n\pi x) \end{pmatrix} , \tag{2.16}$$

since the $L^2(0,1)^2$-norm of this vector-valued function equals 1.

Remark 2.2. *Recall that the Bessel functions of order $\frac{1}{2}$ admit the elementary representations*

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z \,, \qquad J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos z \,. \tag{2.17}$$

3 Spectral map and the linearized problem at $V = 0$.

In this section we introduce the spectral map and analyze its linearization at the unperturbed configuration $V = 0$. This linearized analysis provides the key tool for the local inverse results established later.

3.1 Differential of $\lambda_{\kappa,n}(p,q)$ and the spectral map

In this subsection we recall the analytic dependence of the eigenvalues on the AKNS potential $V = (p, q)$ and describe their Fréchet differential.

Following Serier [32, Prop. 3.1], for each fixed pair $(\kappa, n) \in \mathbb{N} \times \mathbb{Z}$ the map

$$(p, q) \in L^2(0,1) \times L^2(0,1) \longmapsto \lambda_{\kappa,n}(p,q)$$

is real-analytic. Moreover, if $\lambda_{\kappa,n}(p,q)$ is a simple eigenvalue of $H_\kappa(V)$ with normalized eigenfunction

$$Z_{\kappa,n}(x; p, q) = \begin{pmatrix} Z_{1,\kappa,n}(x; p, q) \\ Z_{2,\kappa,n}(x; p, q) \end{pmatrix} , \qquad \|Z_{\kappa,n}(\cdot; p, q)\|_{L^2(0,1)^2} = 1 \,,$$

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then the Fréchet differential at (p, q) in the direction $(v_1, v_2) \in L^2(0,1) \times L^2(0,1)$ is given by

$$
\begin{aligned}
D_{(p,q)}\lambda_{\kappa,n}(p,q) \cdot (v_1, v_2) = \int_0^1 \Big(& 2\, Z_{1,\kappa,n}(x; p, q)\, Z_{2,\kappa,n}(x; p, q)\, v_1(x) \\
& + \big(Z_{2,\kappa,n}(x; p, q)^2 - Z_{1,\kappa,n}(x; p, q)^2 \big)\, v_2(x) \Big)\, dx\,.
\end{aligned}
\tag{3.1}
$$

The relation (3.1) therefore allows us to compute the differential of the spectral map introduced in (1.16).

3.2 The linearized problem at $V = 0$

We now investigate the Fréchet derivative of the spectral map at the unperturbed potential $V = 0$, which leads to the formulation of the linearized inverse problem. Our objective is to determine whether, for two distinct effective angular momenta $\kappa_1 \neq \kappa_2$, the associated map

$$
V = (p, q) \longmapsto \big(\lambda_{\kappa_1,n}(p, q)\,,\, \lambda_{\kappa_2,n}(p, q) \big)_{n \in \mathbb{Z}}
$$

is locally injective at $V = 0$. Equivalently, the same question can be formulated in terms of the renormalized eigenvalues $\widetilde{\lambda}_{\kappa,n}$, since the renormalization consists in subtracting an explicit function of n which is independent of (p, q) and therefore does not affect the Fréchet differential at $V = 0$.

From the general variation formula (3.1), one obtains

$$
D_{(p,q)}\lambda_{\kappa,n}(0,0) \cdot (v_1, v_2) = \int_0^1 \Big(2\, Z^{(0)}_{1,\kappa,n}(x)\, Z^{(0)}_{2,\kappa,n}(x)\, v_1(x) + \big[\big(Z^{(0)}_{2,\kappa,n}(x) \big)^2 - \big(Z^{(0)}_{1,\kappa,n}(x) \big)^2 \big]\, v_2(x) \Big)\, dx\,,
\tag{3.2}
$$

where

$$
Z^{(0)}_{\kappa,n}(x) = \begin{pmatrix} Z^{(0)}_{1,\kappa,n}(x) \\ Z^{(0)}_{2,\kappa,n}(x) \end{pmatrix}\,,
$$

is the normalized eigenfunction of $H_\kappa(0)$ associated with the eigenvalue $\lambda_{\kappa,n}(0,0)$.

We begin with the case of nonzero eigenvalues. Using the results of the previous sections and the symmetry properties of the unperturbed spectrum, we obtain for every $n \in \mathbb{Z}^\pm$

$$
\begin{aligned}
D_{(p,q)}\lambda_{\kappa,n}(0,0) \cdot (v_1, v_2) = c_{\kappa,|n|}^2 \int_0^1 j_{\nu,|n|}\, x\, \Big(& \mp 2\, J_{\nu-1}\big(j_{\nu,|n|} x \big)\, J_\nu\big(j_{\nu,|n|} x \big)\, v_1(x) \\
& + \big[J_\nu\big(j_{\nu,|n|} x \big)^2 - J_{\nu-1}\big(j_{\nu,|n|} x \big)^2 \big]\, v_2(x) \Big)\, dx\,.
\end{aligned}
\tag{3.3}
$$

We next consider the zero eigenvalue. Recalling that

$$
Z^{(0)}_{\kappa,0}(x) = c_{\kappa,0} \begin{pmatrix} x^\kappa \\ 0 \end{pmatrix}\,, \qquad c_{\kappa,0} = \sqrt{2\kappa + 1}\,,
$$

substituting this expression into (3.1) yields

$$
D_{(p,q)}\lambda_{\kappa,0}(0,0) \cdot (v_1, v_2) = -(2\kappa + 1) \int_0^1 x^{2\kappa}\, v_2(x)\, dx\,.
\tag{3.4}
$$

The structure of system (3.3) suggests that the contributions of v_1 and v_2 can be separated. We now formalize this observation by introducing a bounded isomorphism on the target space which exactly decouples the differential of the spectral map.

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3.3 Decoupling of the differential via a continuous isomorphism

We show that, up to a bounded isomorphism on the target space, the differential of the spectral map can be reduced to a fully decoupled system. This allows us to study independently the contributions of v_1 and v_2. We denote this differential by

$$S := D_{(p,q)}\mathcal{S}_{\kappa_1,\kappa_2}(0,0)\,.$$

Notation. For each $\kappa \in \{\kappa_1, \kappa_2\}$ and each $n \geq 1$ (with $\nu = \kappa + \frac{1}{2}$), we introduce the bounded linear functionals on $L^2(0,1)$

$$A_{\kappa,n}(v_1) := \int_0^1 2j_{\nu,n}x\, J_{\nu-1}\big(j_{\nu,n}x\big)\, J_\nu\big(j_{\nu,n}x\big)\, v_1(x)\,dx\,,$$

$$B_{\kappa,n}(v_2) := \int_0^1 j_{\nu,n}x\Big(J_\nu\big(j_{\nu,n}x\big)^2 - J_{\nu-1}\big(j_{\nu,n}x\big)^2\Big)\, v_2(x)\,dx\,.$$

In this notation, for $n \geq 1$,

$$D_{(p,q)}\lambda_{\kappa,\pm n}(0,0)\cdot(v_1,v_2) = c_{\kappa,n}^2\big(\mp A_{\kappa,n}(v_1) + B_{\kappa,n}(v_2)\big)\,,$$

while for the zero mode one has

$$D_{(p,q)}\lambda_{\kappa,0}(0,0)\cdot(v_1,v_2) = -c_{\kappa,0}^2\int_0^1 x^{2\kappa}\, v_2(x)\,dx\,, \qquad c_{\kappa,0} = \sqrt{2\kappa+1}\,. \tag{3.5}$$

For $\kappa \in \{\kappa_1, \kappa_2\}$, define

$$d_\kappa(v) := \big(d_{\kappa,n}(v)\big)_{n\in\mathbb{Z}} := \big(D_{(p,q)}\lambda_{\kappa,n}(0,0)\cdot v\big)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\,, \qquad v = (v_1,v_2)\,,$$

so that

$$S(v) = (d_{\kappa_1}(v),\, d_{\kappa_2}(v)) \in \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})\,.$$

Now, for fixed κ, define the linear map

$$\mathcal{U}_\kappa : \ell^2(\mathbb{Z}) \longrightarrow \mathbb{R} \times \ell^2(\mathbb{N}^*) \times \ell^2(\mathbb{N}^*)$$

by

$$\mathcal{U}_\kappa(a) = \left(\frac{a_0}{c_{\kappa,0}^2}\,,\, \left(\frac{a_{-n} - a_n}{2\,c_{\kappa,n}^2}\right)_{n\geq 1}\,,\, \left(\frac{a_{-n} + a_n}{2\,c_{\kappa,n}^2}\right)_{n\geq 1}\right)\,, \tag{3.6}$$

where $\mathbb{N}^* := \{1,2,3,\dots\}$ denotes the set of positive integers. Using (2.14), we see that $(c_{\kappa,n})_{n\in\mathbb{Z}}$ is uniformly bounded above and below, so $\mathcal{U}_\kappa$ is a bounded isomorphism. Applying $\mathcal{U}_\kappa$ to $d_\kappa(v)$ and using the formulas above yields the exact decoupling

$$\mathcal{U}_\kappa\big(d_\kappa(v)\big) = \left(-\int_0^1 x^{2\kappa}v_2\,,\, \big(A_{\kappa,n}(v_1)\big)_{n\geq 1}\,,\, \big(B_{\kappa,n}(v_2)\big)_{n\geq 1}\right)\,. \tag{3.7}$$

Finally, set $\mathcal{U} := \mathcal{U}_{\kappa_1} \times \mathcal{U}_{\kappa_2}$, which is a bounded isomorphism on $\ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})$. Then

$$(\mathcal{U} \circ S)(v_1, v_2) = \Big(\mathcal{M}(v_2), \, \mathcal{A}_{\kappa_1, \kappa_2}(v_1), \, \mathcal{B}_{\kappa_1, \kappa_2}(v_2) \Big), \tag{3.8}$$

where

$$\mathcal{M}(v_2) = \left(-\int_0^1 x^{2\kappa_1} v_2 \, , \, -\int_0^1 x^{2\kappa_2} v_2 \right) \in \mathbb{R}^2 \, ,$$

and

$$\mathcal{A}_{\kappa_1, \kappa_2}(v_1) = \Big((A_{\kappa_1, n}(v_1))_{n \geq 1} \, , \, (A_{\kappa_2, n}(v_1))_{n \geq 1} \Big), \qquad \mathcal{B}_{\kappa_1, \kappa_2}(v_2) = \Big((B_{\kappa_1, n}(v_2))_{n \geq 1} \, , \, (B_{\kappa_2, n}(v_2))_{n \geq 1} \Big).$$

In particular, $\mathcal{U} \circ S$ is block diagonal: v_1 only enters $\mathcal{A}_{\kappa_1, \kappa_2}$, whereas v_2 only enters $(\mathcal{M}, \mathcal{B}_{\kappa_1, \kappa_2})$.

3.4   Reformulation of the injectivity problem

We now reformulate the injectivity of the Fréchet differential of the spectral map $\mathcal{S}_{\kappa_1, \kappa_2}$ at $(p, q) = (0, 0)$. By definition, injectivity amounts to characterizing all perturbations (v_1, v_2) such that

$$S(v_1, v_2) = 0 \, , \qquad S = D_{(p,q)} \mathcal{S}_{\kappa_1, \kappa_2}(0, 0) \, .$$

Since $\mathcal{U}$ is a bounded isomorphism on the target space, this condition is equivalent to

$$(\mathcal{U} \circ S)(v_1, v_2) = 0 \, .$$

Using the block diagonal structure (3.8), the kernel condition reduces to the decoupled system

$$\mathcal{A}_{\kappa_1, \kappa_2}(v_1) = 0 \, , \qquad \mathcal{M}(v_2) = 0 \, , \qquad \mathcal{B}_{\kappa_1, \kappa_2}(v_2) = 0 \, . \tag{3.9}$$

This decoupling also reflects the choice of boundary conditions, which is encoded in the structure of the eigenfunctions. Thus, the study of the kernel separates into two independent problems: one involving only the component v_1, governed by $\mathcal{A}_{\kappa_1, \kappa_2}$, and one involving only the component v_2, governed by $(\mathcal{M}, \mathcal{B}_{\kappa_1, \kappa_2})$.

For a fixed effective angular momentum κ, the above conditions reduce to

$$A_{\kappa, n}(v_1) = 0, \quad n \geq 1 \, , \qquad B_{\kappa, n}(v_2) = 0, \quad n \geq 1 \, , \qquad \int_0^1 x^{2\kappa} v_2(x) \, dx = 0 \, . \tag{3.10}$$

As an illustration, consider the simple case $\kappa = 0$. Using the explicit expressions of the Bessel functions of order $\pm \frac{1}{2}$, one obtains

$$x \, J_{-1/2}(n\pi x) \, J_{1/2}(n\pi x) = \tfrac{1}{2} \sin(2n\pi x) \, , \qquad x \big(J_{1/2}(n\pi x)^2 - J_{-1/2}(n\pi x)^2 \big) = -\cos(2n\pi x) \, .$$

Hence (3.10) becomes

$$\int_0^1 \sin(2n\pi x) \, v_1(x) \, dx = 0 \, , \qquad n \geq 1,$$

and

$$\int_0^1 \cos(2n\pi x) \, v_2(x) \, dx = 0 \, , \qquad n \geq 1, \qquad \int_0^1 v_2(x) \, dx = 0 \, .$$

These relations show that v_1 is even and v_2 is odd with respect to $x = \frac{1}{2}$. Conversely, if v_1 is even and v_2 is odd with respect to $x = \frac{1}{2}$, then all the above integrals vanish. Hence these parity conditions are necessary and sufficient for $S(v_1, v_2) = 0$ in the case $\kappa = 0$.

4   Kneser–Sommerfeld–Type Expansions

The classical Kneser–Sommerfeld identity provides a series expansion over the zeros $j_{\nu,n}$ of J_ν. Its correct form, first given by Buchholz [9] and later clarified by Hayashi [17] and Martin [23], differs from the formula stated by Watson, which omits an essential integral term. The valid expansion (4.1) played a central role in our previous analysis of the radial Schrödinger operator:

$$\sum_{n\geq 1} \frac{J_\nu(j_{\nu,n}x)\,J_\nu(j_{\nu,n}X)}{(z^2 - j_{\nu,n}^2)\,[J_\nu'(j_{\nu,n})]^2} = \frac{\pi}{4\,J_\nu(z)}\,J_\nu(xz)\left[J_\nu(z)\,Y_\nu(Xz) - Y_\nu(z)\,J_\nu(Xz)\right], \tag{4.1}$$

for $0 < x \leq X \leq 1$.

In the present AKNS setting, the linearized system (3.3) involves both squared and mixed Bessel products. The relevant combinations are, with $\nu = \kappa + \frac{1}{2}$,

$$J_\nu(j_{\nu,n}x)^2, \quad J_{\nu-1}(j_{\nu,n}x)^2 \quad \text{and} \quad J_{\nu-1}(j_{\nu,n}x)\,J_\nu(j_{\nu,n}x)\,.$$

However, the last two expressions fall outside the framework of the classical Kneser–Sommerfeld expansion, which treats only diagonal products of the form $J_\nu(xj_{\nu,n})J_\nu(Xj_{\nu,n})$.

To handle the AKNS structure, we therefore require *modified* Kneser–Sommerfeld–type identities. In what follows, we now state three additional identities of the same type. For simplicity, we assume that $\nu \in \mathbb{R}^+\backslash\mathbb{N}$, although the formulas extend to arbitrary complex values of ν.[3]

Proposition 4.1. *Let $\nu \in \mathbb{R}_+ \setminus \mathbb{N}$, $z \neq 0$, $z \neq j_{\nu,n}$ and $0 < x \leq X \leq 1$. The following Kneser–Sommerfeld–type identities hold:*

$$\sum_{n\geq 1} \frac{J_{\nu-1}(j_{\nu,n}x)\,J_{\nu-1}(j_{\nu,n}X)}{(z^2 - j_{\nu,n}^2)\left[J_\nu'(j_{\nu,n})\right]^2} = -\frac{\nu}{z^2}\,(xX)^{\nu-1}$$
$$+ \frac{\pi}{4J_\nu(z)}\,J_{\nu-1}(xz)\left(J_\nu(z)\,Y_{\nu-1}(Xz) - Y_\nu(z)\,J_{\nu-1}(Xz)\right), \tag{4.2}$$

$$\sum_{n\geq 1} \frac{J_{\nu-1}(j_{\nu,n}x)\,J_\nu(j_{\nu,n}X)}{(z^2 - j_{\nu,n}^2)\,j_{\nu,n}\left[J_\nu'(j_{\nu,n})\right]^2} = \frac{1}{2z^2}\,x^{\nu-1}(X^{-\nu} - X^\nu)$$
$$+ \frac{\pi}{4z\,J_\nu(z)}\,J_{\nu-1}(xz)\left(J_\nu(z)\,Y_\nu(Xz) - Y_\nu(z)\,J_\nu(Xz)\right), \tag{4.3}$$

$$\sum_{n\geq 1} \frac{J_{\nu-1}(j_{\nu,n}X)\,J_\nu(j_{\nu,n}x)}{(z^2 - j_{\nu,n}^2)\,j_{\nu,n}\left[J_\nu'(j_{\nu,n})\right]^2} = -\frac{1}{2z^2}\,x^\nu X^{\nu-1}$$
$$+ \frac{\pi}{4z\,J_\nu(z)}\,J_\nu(xz)\left(J_\nu(z)\,Y_{\nu-1}(Xz) - Y_\nu(z)\,J_{\nu-1}(Xz)\right)\,. \tag{4.4}$$

Proof. We follow, step by step, the contour-integral argument of Watson for the Kneser-Sommerfeld formula (see [37]), in the corrected form later clarified by Buchholz, Hayashi and Martin. We first give the proof of (4.2).

---

[3]Here we use the definition of $Y_n(z)$ recalled above.

Let $z \in \mathbb{C}$ with, for instance, $\Re z > 0$ and $z \neq j_{\nu,n}$ for all $n \geq 1$. Following Watson's approach, we introduce, for $w \in \mathbb{C}$ and fixed $0 < X \leq 1$, the auxiliary function

$$W_\nu(w, X) := J_\nu(w)\, Y_{\nu-1}(Xw) \ - \ Y_\nu(w)\, J_{\nu-1}(Xw)\,. \tag{4.5}$$

Using the small-w asymptotics of J_ν and Y_ν (see (2.7)), one readily obtains

$$W_\nu(w, X) = \frac{2X^{\nu-1}}{\pi w} \ + \ O(1)\,, \qquad w \to 0\,. \tag{4.6}$$

Recall the Hankel functions

$$H_\nu^{(1)}(w) = J_\nu(w) + i\, Y_\nu(w)\,, \qquad H_\nu^{(2)}(w) = J_\nu(w) - i\, Y_\nu(w)\,. \tag{4.7}$$

A short computation gives the equivalent representation

$$W_\nu(w, X) = -\frac{1}{2i}\left[H_\nu^{(1)}(w)\, H_{\nu-1}^{(2)}(Xw) - H_\nu^{(2)}(w)\, H_{\nu-1}^{(1)}(Xw) \right]\,. \tag{4.8}$$

Finally, from the large-w asymptotics ([21], (5.11.4)-(5.11.5)), valid for $|\arg w| \leq \pi - \delta$, $\delta \in (0, \pi)$,

$$H_\nu^{(1)}(w) \sim \sqrt{\frac{2}{\pi w}}\, e^{i(w - \frac{\nu\pi}{2} - \frac{\pi}{4})}\,, \qquad H_\nu^{(2)}(w) \sim \sqrt{\frac{2}{\pi w}}\, e^{-i(w - \frac{\nu\pi}{2} - \frac{\pi}{4})}\,, \tag{4.9}$$

we obtain, for $|w|$ large with $\Re w > 0$,

$$|W_\nu(w, X)| \lesssim \frac{e^{(1-X)|\Im w|}}{|w|}\,, \qquad 0 < X \leq 1\,. \tag{4.10}$$
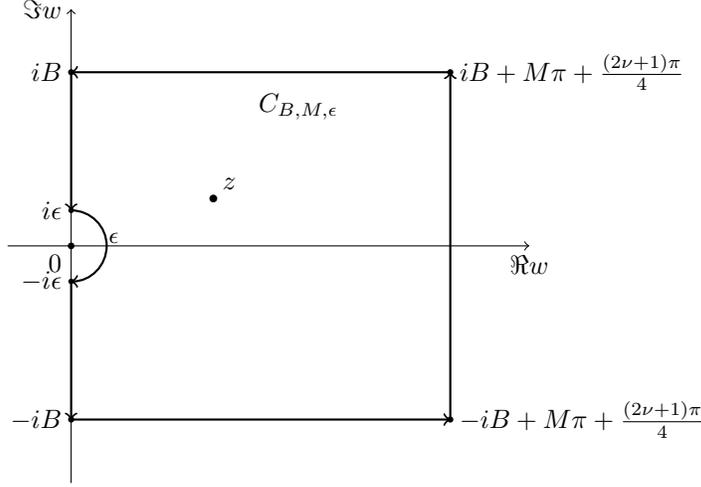
We consider the contour integral

$$I_{B,M,\epsilon} = \oint_{C_{B,M,\epsilon}} \frac{W_\nu(w, X)}{w^2 - z^2}\, \frac{w\, J_{\nu-1}(xw)}{J_\nu(w)}\, dw\,, \tag{4.11}$$

where $C_{B,M,\epsilon}$ is the rectangle in the half-plane $\Re w \geq 0$, with vertices

$$\pm iB, \qquad \pm iB + M\pi + \frac{(2\nu+1)\pi}{4}\,,$$

for $B > 0$ and $M \in \mathbb{N}$ large enough, indented at the origin with a half-circle of radius $\epsilon > 0$ in the half-plane $\Re w > 0$:

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Using again the small-w asymptotics of J_ν and Y_ν (see (2.7)), one readily obtains

$$\frac{W_\nu(w,X)}{w^2-z^2}\,\frac{w\,J_{\nu-1}(xw)}{J_\nu(w)} = -\frac{4\nu}{\pi z^2 w}(xX)^{\nu-1} + O(1)\,,\quad w\to 0\,. \tag{4.12}$$

Using the parity identities

$$H_\nu^{(1)}(-w) = e^{-i\pi\nu}H_\nu^{(1)}(w)\,,\quad H_\nu^{(2)}(-w) = e^{i\pi\nu}H_\nu^{(2)}(w)\,,$$

which hold for all w on the imaginary axis, we observe that $W_\nu(w,X)$ is an odd function of w on this axis. Furthermore, the map

$$w \mapsto \frac{w\,J_{\nu-1}(xw)}{J_\nu(w)}$$

is even in w. Thus, the contribution from the vertical union of the intervals $[-iB,-i\epsilon]\cup[i\epsilon,iB]$ cancels out.

Let us set

$$f(w) = \frac{W_\nu(w,X)}{w^2-z^2}\,\frac{w\,J_{\nu-1}(xw)}{J_\nu(w)}\,. \tag{4.13}$$

Using (4.12), the contribution of the integral over the small circle centered at the origin and of radius ϵ (traversed in the counterclockwise direction) converges, in the limit $\epsilon \to 0$, to

$$\frac{4i\nu}{z^2}\,(xX)^{\nu-1}\,. \tag{4.14}$$

Using Bessel asymptotics ([21], (5.11.6))

$$J_\nu(w) = \sqrt{\frac{2}{\pi w}}\left[\cos\!\left(w - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O\!\left(\frac{e^{|\Im w|}}{|w|}\right)\right]\,,\quad |\arg w|\le \pi-\delta\,,\ |w|\to+\infty\,, \tag{4.15}$$

with $\delta \in (0,\pi)$, we deduce that no Bessel zero $j_{\nu,n}$ lies on the vertical segments and that on the three other sides,

$$\left|\frac{H_\nu^{(1)}(w)\,H_{\nu-1}^{(2)}(Xw) - H_\nu^{(2)}(w)\,H_{\nu-1}^{(1)}(Xw)}{w^2-z^2}\,\frac{w\,J_{\nu-1}(xw)}{J_\nu(w)}\right| \lesssim \frac{e^{(x-X)|\Im w|}}{|w|^2}\,. \tag{4.16}$$

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Since by the assumption in the proposition $x \leq X$, the integrand decays like $|w|^{-2}$ on these sides. Letting $B, M \to \infty$, all these contributions vanish. Therefore, by the residue theorem, letting $B, M \to \infty$ in (4.11) and $\epsilon \to 0$, we obtain

$$\frac{4i\nu}{z^2} (xX)^{\nu-1} = 2\pi i \sum \mathrm{Res}(f) , \tag{4.17}$$

where the sum runs over the poles inside the contour, namely

$$w = z \quad \text{and} \quad w = j_{\nu,n} , \qquad n \geq 1 .$$

a. Residue at $w = j_{\nu,n}$: the integrand has a simple pole, giving

$$\mathop{\mathrm{Res}}_{w=j_{\nu,n}} (f) = \frac{J_{\nu-1}(X j_{\nu,n}) \, j_{\nu,n} \, J_{\nu-1}(x j_{\nu,n}) \, Y_\nu(j_{\nu,n})}{(z^2 - j_{\nu,n}^2) \, J_\nu'(j_{\nu,n})} . \tag{4.18}$$

We use the identity

$$Y_\nu(j_{\nu,n}) = \frac{Y_\nu(j_{\nu,n}) J_\nu'(j_{\nu,n}) - J_\nu(j_{\nu,n}) Y_\nu'(j_{\nu,n})}{J_\nu'(j_{\nu,n})} ,$$

and the Wronskian formula ([21], (5.9.2))

$$J_\nu(z) Y_\nu'(z) - J_\nu'(z) Y_\nu(z) = \frac{2}{\pi z} ,$$

to obtain

$$Y_\nu(j_{\nu,n}) = -\frac{2}{\pi \, j_{\nu,n} \, J_\nu'(j_{\nu,n})} .$$

Substituting this into (4.18) gives

$$\mathop{\mathrm{Res}}_{w=j_{\nu,n}} (f) = -\frac{2}{\pi} \frac{J_{\nu-1}(X j_{\nu,n}) \, J_{\nu-1}(x j_{\nu,n})}{(z^2 - j_{\nu,n}^2) \, [J_\nu'(j_{\nu,n})]^2} . \tag{4.19}$$

b. Residue at $w = z$: a direct computation shows:

$$\mathop{\mathrm{Res}}_{w=z}(f) = \frac{1}{2 J_\nu(z)} J_{\nu-1}(xz) \left[J_\nu(z) Y_{\nu-1}(Xz) - Y_\nu(z) J_{\nu-1}(Xz) \right] . \tag{4.20}$$

Summing the residues (4.19) and (4.20) and invoking (4.17) yields precisely the identity (4.2).

To prove the identity (4.3), we introduce a new auxiliary function

$$\tilde{W}_\nu(w, X) := J_\nu(w) \, Y_\nu(Xw) - Y_\nu(w) \, J_\nu(Xw) , \tag{4.21}$$

and we consider the contour integral

$$I_{B,M,\epsilon} = \oint_{C_{B,M,\epsilon}} \frac{\tilde{W}_\nu(w, X)}{w^2 - z^2} \frac{J_{\nu-1}(xw)}{J_\nu(w)} \, dw , \tag{4.22}$$

where $C_{B,M,\epsilon}$ is the same rectangle as above, indented at the origin with a half-circle of radius $\epsilon > 0$. We conclude in exactly the same way.

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Similarly, to prove the identity (4.4), we use again the auxiliary function (4.5) and consider the contour integral

$$I_{B,M,\epsilon} = \oint_{C_{B,M,\epsilon}} \frac{W_\nu(w,X)}{w^2 - z^2} \frac{J_\nu(xw)}{J_\nu(w)}\, dw\,,\tag{4.23}$$

where $C_{B,M,\epsilon}$ is the same contour as before. The proof then follows analogously. □

The following result is readily obtained:

Corollary 4.2. *Let $\nu \in \mathbb{R}^+\backslash\mathbb{N}$, and $x \in (0,1]$. Then, for all $z \notin \{0, j_{\nu,n}\}_{n\geq 1}$,*

$$\begin{aligned}
\sum_{n\geq 1} \frac{x\, J_{\nu-1}(j_{\nu,n}x)\, J_\nu(j_{\nu,n}x)}{(z^2 - j_{\nu,n}^2)\, j_{\nu,n}\, \left[J_\nu'(j_{\nu,n})\right]^2} &= \frac{1}{4z^2}\left(1 - 2x^{2\nu}\right)\\
&\quad + \frac{\pi\, x}{8z\, J_\nu(z)}\, \big[J_\nu(z)\left(J_{\nu-1}(zx)\, Y_\nu(zx) + Y_{\nu-1}(zx)\, J_\nu(zx)\right)\\
&\quad -2Y_\nu(z)\, J_{\nu-1}(zx)\, J_\nu(zx)\big]\,.
\end{aligned}\tag{4.24}$$

Proof. The result follows by adding (4.3) and (4.4) with $x = X$ and multiplying by $\frac{x}{2}$. □

5 Application of the Kneser–Sommerfeld representation

5.1 Preliminaries

We have seen that for a fixed effective angular momentum κ the linearized condition

$$D_{(p,q)}\lambda_{\kappa,n}(0,0)\cdot(v_1,v_2) = 0\,, \qquad \text{for all } n \in \mathbb{Z}\,,$$

is equivalent to the relations (see (3.10))

$$\int_0^1 x\, J_{\nu-1}(j_{\nu,n}x)\, J_\nu(j_{\nu,n}x)\, v_1(x)\, dx = 0\,, \qquad n \geq 1\,,\tag{5.1}$$

and

$$\int_0^1 x\left[J_\nu(j_{\nu,n}x)^2 - J_{\nu-1}(j_{\nu,n}x)^2\right] v_2(x)\, dx = 0\,, \qquad n \geq 1\,,\tag{5.2}$$

with the additional constraint

$$\int_0^1 x^{2\kappa}\, v_2(x)\, dx = 0\,.\tag{5.3}$$

Let us first examine the conditions (5.2)-(5.3). We use the classical Kneser-Sommerfeld expansion (4.1) and the relation (4.2) with $x = X$. We multiply (5.2) by

$$\frac{1}{(z^2 - j_{\nu,n}^2)\left[J_\nu'(j_{\nu,n})\right]^2}$$

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with $z \notin \{0, j_{\nu,n}\}_{n \geq 1}$. Summing over n, and using $2\nu - 1 = 2\kappa$, we obtain for such z,

$$-\frac{\nu}{z^2} \int_0^1 x^{2\nu-1} v_2(x)\, dx + \frac{\pi}{4 J_\nu(z)} \int_0^1 x \Big(J_\nu(zx) \left[J_\nu(z) Y_\nu(zx) - Y_\nu(z) J_\nu(zx) \right]$$
$$- J_{\nu-1}(zx) \left[J_\nu(z) Y_{\nu-1}(zx) - Y_\nu(z) J_{\nu-1}(zx) \right] \Big) v_2(x)\, dx = 0\,. \quad (5.4)$$

Since $2\nu - 1 = 2\kappa$, the first integral in (5.4) vanishes thanks to the constraint (5.3). Hence we obtain the simplified identity:

$$\int_0^1 x \Big(J_\nu(zx) \left[J_\nu(z) Y_\nu(zx) - Y_\nu(z) J_\nu(zx) \right]$$
$$- J_{\nu-1}(zx) \left[J_\nu(z) Y_{\nu-1}(zx) - Y_\nu(z) J_{\nu-1}(zx) \right] \Big) v_2(x)\, dx = 0\,, \quad (5.5)$$

which can be rewritten for $z \notin \{0, j_{\nu,n}\}_{n \geq 1}$ as

$$J_\nu(z) \int_0^1 x \Big(J_\nu(zx) Y_\nu(zx) - J_{\nu-1}(zx) Y_{\nu-1}(zx) \Big) v_2(x)\, dx$$
$$- Y_\nu(z) \int_0^1 x \Big(J_\nu(zx)^2 - J_{\nu-1}(zx)^2 \Big) v_2(x)\, dx = 0\,. \quad (5.6)$$

By continuity with respect to z, the identity (5.6) extends to $z \in \mathbb{C}^*$.

In the same way, using Corollary 4.2, we obtain from (5.1), for $z \notin \{0, j_{\nu,n}\}_{n \geq 1}$

$$\int_0^1 (1 - 2x^{2\nu})\, v_1(x)\, dx + \frac{\pi}{2 J_\nu(z)} \int_0^1 (zx) \Big(J_\nu(z) \left[J_{\nu-1}(zx) Y_\nu(zx) + Y_{\nu-1}(zx) J_\nu(zx) \right]$$
$$- 2 Y_\nu(z) J_{\nu-1}(zx) J_\nu(zx) \Big) v_1(x)\, dx = 0\,. \quad (5.7)$$

Thus, using the large-w asymptotics for Bessel functions (4.15) together with the asymptotics for $Y_\nu(w)$ (see [21], (5.11.7)),

$$Y_\nu(w) = \sqrt{\frac{2}{\pi w}} \left[\sin\Big(w - \frac{\nu\pi}{2} - \frac{\pi}{4}\Big) + O\Big(\frac{e^{|\Im w|}}{|w|}\Big) \right]\,, \qquad |\arg w| \leq \pi - \delta\,, \quad |w| \to +\infty\,, \quad (5.8)$$

we deduce using the Riemann-Lebesgue lemma, that the integral in (5.7) is $o(1)$ as $z \to +\infty$ away from the points $j_{\nu,n}$. Consequently,

$$\int_0^1 \big(1 - 2x^{2\nu}\big)\, v_1(x)\, dx = 0\,. \quad (5.9)$$

As a consequence, for all $z \notin \{0, j_{\nu,n}\}_{n \geq 1}$, we obtain

$$J_\nu(z) \int_0^1 x \big(J_{\nu-1}(zx) Y_\nu(zx) + Y_{\nu-1}(zx) J_\nu(zx) \big) v_1(x)\, dx$$
$$+ Y_\nu(z) \int_0^1 x \big(-2 J_{\nu-1}(zx) J_\nu(zx) \big) v_1(x)\, dx = 0\,, \quad (5.10)$$

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and this identity extends to all $z \in \mathbb{C}^*$.

Now, we use the vector functions introduced by Serier:

$$\Phi_\kappa(x) = \begin{pmatrix} \Phi_{\kappa,1}(x) \\ \Phi_{\kappa,2}(x) \end{pmatrix} = \frac{\pi x}{2} \begin{pmatrix} -2\, J_{\nu-1}(x)\, J_\nu(x) \\ J_\nu(x)^2 - J_{\nu-1}(x)^2 \end{pmatrix},$$

$$\Psi_\kappa(x) = \begin{pmatrix} \Psi_{\kappa,1}(x) \\ \Psi_{\kappa,2}(x) \end{pmatrix} = \frac{\pi x}{2} \begin{pmatrix} J_{\nu-1}(x)Y_\nu(x) + J_\nu(x)Y_{\nu-1}(x) \\ J_{\nu-1}(x)Y_{\nu-1}(x) - J_\nu(x)Y_\nu(x) \end{pmatrix},$$

with $\nu = \kappa + \frac{1}{2}$. With this notation, the previous computations can be summarized in the following proposition.

Proposition 5.1. *Let (v_1, v_2) satisfy the linearized spectral conditions* (5.1)–(5.2) *together with the constraint* (5.3). *Then the following statements hold.*

1. *The function v_1 satisfies*

$$\int_0^1 (1 - 2x^{2\nu})\, v_1(x)\, dx = 0. \tag{5.11}$$

   Moreover, for all $z \in \mathbb{C}^*$,

$$\int_0^1 \left[J_\nu(z)\, \Psi_{\kappa,1}(zx) + Y_\nu(z)\, \Phi_{\kappa,1}(zx) \right] v_1(x)\, dx = 0. \tag{5.12}$$

2. *For all $z \in \mathbb{C}^*$, the function v_2 satisfies*

$$\int_0^1 \left[J_\nu(z)\, \Psi_{\kappa,2}(zx) + Y_\nu(z)\, \Phi_{\kappa,2}(zx) \right] v_2(x)\, dx = 0. \tag{5.13}$$

5.2   A first injectivity result

We have seen in the previous subsection that the linearization condition implies the integral constraint

$$\int_0^1 \left(1 - 2x^{2\nu}\right) v_1(x)\, dx = 0, \qquad \nu = \kappa + \frac{1}{2}, \tag{5.14}$$

while the presence of the zero eigenvalue imposes (see (3.10))

$$\int_0^1 x^{2\nu-1}\, v_2(x)\, dx = 0. \tag{5.15}$$

Our first injectivity result for the Fréchet differential in the AKNS setting is based on the classical Müntz–Szász theorem [24, 25, 34].

Theorem 5.2. *Let $(v_1, v_2) \in L^2(0,1)^2$ be a real-valued vector function satisfying the AKNS linearized constraints* (5.14)–(5.15) *for an infinite increasing sequence $\{\kappa_k\}_{k \geq 1} \subset \mathbb{N}^*$, with $\nu_k := \kappa_k + \frac{1}{2}$. Assume moreover that*

$$\sum_{k=1}^\infty \frac{1}{\kappa_k} = +\infty.$$

Then $(v_1, v_2) = (0, 0)$ almost everywhere in $(0, 1)$.

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Proof. From the identity

$$\int_0^1 \left(1 - 2x^{2\nu_k}\right) v_1(x)\, dx = 0\,, \tag{5.16}$$

we let $k \to \infty$ and we get

$$\int_0^1 v_1(x)\, dx = 0\,.$$

Substituting this back into (5.16), we obtain the moment identities

$$\int_0^1 x^{2\nu_k}\, v_1(x)\, dx = 0\,, \qquad \text{for all } k \geq 1\,.$$

Since

$$\sum_{k=1}^{\infty} \frac{1}{\kappa_k} = +\infty\,,$$

the classical Müntz-Szász theorem applies to the family $\{x^{2\nu_k}\}_{k \geq 1}$ on $(0, 1)$ and implies that

$$v_1 = 0 \quad \text{almost everywhere on } (0, 1)\,.$$

The argument for v_2 is identical, using the constraint (5.15). Hence $(v_1, v_2) = (0, 0)$ a.e., which completes the proof. $\qquad \square$

5.3   Transformation operators and Green's identity

We recall the definition of the transformation operators introduced in the work of Serier. Such operators first appeared in the seminal paper of Guillot and Ralston [16] in connection with the inverse spectral problem for the radial Schrödinger operator (the case $\kappa = 1$). They were later extended to general integer κ by Rundell and Sacks [29], and subsequently refined in [31].

In the AKNS setting, Serier constructed similar operators adapted to the first-order matrix structure. A key difference with the Schrödinger case is that the inverse operators have a more favorable structure.

Throughout this subsection we use the vector-valued functions Φ_κ and Ψ_κ introduced in the previous section, and we keep the notation

$$\Phi_\kappa(x) = \begin{pmatrix} \Phi_{\kappa,1}(x) \\ \Phi_{\kappa,2}(x) \end{pmatrix}\,, \qquad \Psi_\kappa(x) = \begin{pmatrix} \Psi_{\kappa,1}(x) \\ \Psi_{\kappa,2}(x) \end{pmatrix}\,.$$

The next lemma is taken from [32] and will be essential for analyzing the inverse problem in the AKNS setting. First, let us give some notation[4].

Notation 5.1. *For all $n \in \mathbb{N}$, let U_n and V_n be defined by*

$$U_n(x) = \begin{bmatrix} 0 \\ x^n \end{bmatrix} \quad \text{and} \quad V_n(x) = \begin{bmatrix} x^n \\ 0 \end{bmatrix} \quad x \in [0, 1]\,.$$

---

[4]We adopt the same notation as that introduced by Serier [32].

Lemma 5.3. *For each $\kappa \in \mathbb{N}$, define the operator*

$$S_{\kappa+1} : \ L^2(0,1)^2 \longrightarrow L^2(0,1)^2 \,, \qquad S_{\kappa+1}[p,q] := \big(S_{\kappa,1}[p], \ S_{\kappa,2}[q]\big)\,,$$

where

$$S_{\kappa,1}[p](x) = p(x) - 2(2\kappa+1)x^{2\kappa}\int_x^1 \frac{p(t)}{t^{2\kappa+1}}dt\,, \qquad S_{\kappa,2}[q](x) = q(x) - 2(2\kappa+1)x^{2\kappa+1}\int_x^1 \frac{q(t)}{t^{2\kappa+2}}dt\,.$$

We also set $S_0 := \mathrm{Id}$. The operators $\{S_\kappa\}_{\kappa \geq 0}$ satisfy:

(i) *The adjoint is given by*

$$S_{\kappa+1}^*[f,g] = \big(S_{\kappa,1}^*[f], \ S_{\kappa,2}^*[g]\big)\,,$$

with

$$S_{\kappa,1}^*[f](x) = f(x) - \frac{2(2\kappa+1)}{x^{2\kappa+1}}\int_0^x t^{2\kappa}f(t)\,dt\,, \qquad S_{\kappa,2}^*[g](x) = g(x) - \frac{2(2\kappa+1)}{x^{2\kappa+2}}\int_0^x t^{2\kappa+1}g(t)\,dt\,.$$

(ii) *The family $\{S_\kappa\}$ is commuting:*

$$S_\kappa S_m = S_m S_\kappa \qquad \forall\, \kappa, m \in \mathbb{N}\,.$$

(iii) *Each S_κ is bounded on $L^2(0,1)^2$.*

(iv) *With $N_{\kappa+1} := \ker S_{\kappa+1}^*$, one has*

$$N_{\kappa+1} = \mathrm{Vect}(U_{2\kappa}, V_{2\kappa+1})\,.$$

(v) *$S_{\kappa+1}$ is an isomorphism from $L^2(0,1)^2$ onto $N_{\kappa+1}^\perp$, with inverse*

$$A_{\kappa+1}[f,g] := \big(S_{\kappa,2}^*[f], \ S_{\kappa,1}^*[g]\big)\,.$$

(vi) *The functions Φ_κ and Ψ_κ satisfy the reduction relations*

$$\Phi_{\kappa+1} = -S_{\kappa+1}^*[\Phi_\kappa]\,, \qquad \Psi_{\kappa+1} = -S_{\kappa+1}^*[\Psi_\kappa]\,.$$

We will also need the following complementary result which is analogous to Lemma 3.4 in [29].

Lemma 5.4. *Let $\kappa \geq 0$ and let $f, g \in L^2(0,1)$. Then:*

1. *If $g = S_{\kappa,1}[f]$, then in the sense of distributions on $(0,1)$,*

$$g^{(2\kappa+1)}(x) = \frac{4\kappa+2}{x}\, f^{(2\kappa)}(x) + f^{(2\kappa+1)}(x)\,. \tag{5.17}$$

2. *If $g = S_{\kappa,2}[f]$, then in the sense of distributions on $(0,1)$,*

$$g^{(2\kappa+2)}(x) = \frac{4\kappa+2}{x}\, f^{(2\kappa+1)}(x) + f^{(2\kappa+2)}(x)\,. \tag{5.18}$$

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Proof. We adapt the argument of [29] for the first identity (5.17). Starting from

$$g(x) = f(x) - 2(2\kappa + 1) x^{2\kappa} \int_x^1 s^{-2\kappa-1} f(s) \, ds \,, \tag{5.19}$$

a single differentiation yields

$$g'(x) = f'(x) - 4\kappa(2\kappa + 1) x^{2\kappa-1} \int_x^1 s^{-2\kappa-1} f(s) \, ds + \frac{2(2\kappa + 1)}{x} f(x) \,. \tag{5.20}$$

To eliminate the integral term, consider

$$2\kappa \, (5.19) \; - \; x(5.20) \,,$$

which gives

$$2\kappa g(x) - xg'(x) = -(2\kappa + 2)f(x) - xf'(x) \,.$$

Differentiating once more,

$$(2\kappa - 1)g'(x) - xg''(x) = -(2\kappa + 3)f'(x) - xf''(x) \,,$$

and by iterating this procedure k times one obtains

$$(2\kappa - k) \, g^{(k)}(x) - xg^{(k+1)}(x) = -(2\kappa + k + 2) \, f^{(k)}(x) - xf^{(k+1)}(x) \,.$$

Setting $k = 2\kappa$ and dividing by x yields exactly (5.17). The proof of (5.18) is entirely analogous. $\qquad\square$

We now consider the composite operator T_κ, obtained by composing the index-reduction operators $S_1, \ldots, S_\kappa$, which carries Bessel kernels to trigonometric ones.

Lemma 5.5. *For every $\kappa \in \mathbb{N}$, define*

$$T_\kappa = (-1)^{\kappa+1} S_\kappa S_{\kappa-1} \cdots S_1 \,, \qquad T_0 := -S_0 \,.$$

Write $T_\kappa[p, q] = (T_\kappa^1[p], T_\kappa^2[q])$. Then:

(i) T_κ is bounded and injective, and for all p, q and all $\lambda \in \mathbb{C}$,

$$\int_0^1 \Phi_\kappa(\lambda t) \cdot \begin{pmatrix} p(t) \\ q(t) \end{pmatrix} dt = \int_0^1 \begin{pmatrix} \sin(2\lambda t) \\ \cos(2\lambda t) \end{pmatrix} \cdot T_\kappa[p, q](t) \, dt \,,$$

$$\int_0^1 \Psi_\kappa(\lambda t) \cdot \begin{pmatrix} p(t) \\ q(t) \end{pmatrix} dt = \int_0^1 \begin{pmatrix} \cos(2\lambda t) \\ -\sin(2\lambda t) \end{pmatrix} \cdot T_\kappa[p, q](t) \, dt \,.$$

(ii) The adjoint T_κ^* satisfies

$$\Phi_\kappa(\lambda x) = T_\kappa^* \begin{pmatrix} \sin(2\lambda\cdot) \\ \cos(2\lambda\cdot) \end{pmatrix}(x), \qquad \Psi_\kappa(\lambda x) = T_\kappa^* \begin{pmatrix} \cos(2\lambda\cdot) \\ -\sin(2\lambda\cdot) \end{pmatrix}(x).$$

and

$$\ker T_\kappa^* = \bigoplus_{k=1}^\kappa N_k \,.$$

(iii) T_κ *defines an isomorphism from* $L^2(0,1)^2$ *onto* $\left(\bigoplus_{k=1}^{\kappa} N_k\right)^{\perp}$, *with inverse*

$$B_\kappa[f,g] = \left((T_\kappa^2)^*[f],\, (T_\kappa^1)^*[g]\right).$$

Remark 5.6. *Taking, for instance,* $(p,q) = (p,0)$ *in Lemma 5.5(i), we obtain that, for every* p *and every* $\lambda \in \mathbb{C}$,

$$\int_0^1 \Phi_{\kappa,1}(\lambda t)\, p(t)\, dt = \int_0^1 \sin(2\lambda t)\, T_\kappa^1(p)(t)\, dt\,.$$

We now apply Lemma 5.5(i). Using the classical identity

$$Y_\nu(x) = (-1)^{\kappa+1}\, J_{-\nu}(x)\,,$$

Proposition 5.1 can be rewritten in the following equivalent form: for all $z \in \mathbb{C}^*$,

$$\begin{cases} \displaystyle \int_0^1 \left[J_\nu(z)\cos(2zx)\, T_\kappa^1[v_1](x) + (-1)^{\kappa+1} J_{-\nu}(z)\sin(2zx)\, T_\kappa^1[v_1](x) \right] dx = 0\,, \\[2mm] \displaystyle \int_0^1 \left[-J_\nu(z)\sin(2zx)\, T_\kappa^2[v_2](x) + (-1)^{\kappa+1} J_{-\nu}(z)\cos(2zx)\, T_\kappa^2[v_2](x) \right] dx = 0\,. \end{cases} \tag{5.21}$$

For later use, we recall the explicit formulas for Bessel functions of half-integer order together with the associated polynomials introduced in [6, 10.1.19–20]. When $\kappa = 0, 1, 2, \ldots$ and $z \in \mathbb{C}$, one has the classical representations

$$J_{\kappa+\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left(P_\kappa\left(\tfrac{1}{z}\right) \sin z \;-\; Q_{\kappa-1}\left(\tfrac{1}{z}\right) \cos z \right), \tag{5.22}$$

$$J_{-\kappa-\frac{1}{2}}(z) = (-1)^\kappa \sqrt{\frac{2}{\pi z}} \left(P_\kappa\left(\tfrac{1}{z}\right) \cos z \;+\; Q_{\kappa-1}\left(\tfrac{1}{z}\right) \sin z \right). \tag{5.23}$$

The polynomials P_κ and Q_κ, each of degree κ, are generated by the three-term recurrences

$$P_{\kappa+1}(t) = (2\kappa+1)\, t\, P_\kappa(t) - P_{\kappa-1}(t)\,, \qquad \kappa \geq 1\,, \tag{5.24}$$
$$Q_{\kappa+1}(t) = (2\kappa+3)\, t\, Q_\kappa(t) - Q_{\kappa-1}(t)\,, \qquad \kappa \geq 0\,, \tag{5.25}$$

with initial values

$$P_0(t) = 1\,, \qquad P_1(t) = t\,, \qquad Q_{-1}(t) = 0\,, \qquad Q_0(t) = 1\,.$$

Observe that P_κ and Q_κ inherit the parity of κ: they are even functions when κ is even and odd functions when κ is odd.

For illustration, the lowest half-integer orders give

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\, \sin z\,, \qquad J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\, \cos z\,. \tag{5.26}$$

The next pair is

$$J_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left(\frac{\sin z}{z} - \cos z \right), \qquad J_{-\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left(-\frac{\cos z}{z} - \sin z \right). \tag{5.27}$$

Using the recurrence relation, the first few polynomials are

$$
\begin{array}{ll}
P_0(t) = 1\,, & Q_{-1}(t) = 0\,, \\
P_1(t) = t\,, & Q_0(t) = 1\,, \\
P_2(t) = 3t^2 - 1\,, & Q_1(t) = 3t\,, \\
P_3(t) = 15t^3 - 6t\,, & Q_2(t) = 15t^2 - 1\,.
\end{array}
\tag{5.28}
$$

Gathering the previous identities, we arrive at the following statement.

Proposition 5.7. *Assume that for $\kappa \in \mathbb{N}$,*

$$
D_{(p,q)}\lambda_{\kappa,n}(0,0) \cdot (v_1, v_2) = 0\,, \qquad \text{for all } n \in \mathbb{Z}\,.
$$

Then, for every $z \in \mathbb{C}$ and every integer $\kappa \geq 0$, one obtains the following identity: for all $z \in \mathbb{C}^*$,

$$
\begin{cases}
\displaystyle \int_0^1 \left[\left(P_\kappa\!\left(\tfrac{1}{z}\right) \sin\!\big(z(2x-1)\big) + Q_{\kappa-1}\!\left(\tfrac{1}{z}\right) \cos\!\big(z(2x-1)\big) \right) T_\kappa^1[v_1](x) \right] dx = 0, \\[4mm]
\displaystyle \int_0^1 \left[\left(P_\kappa\!\left(\tfrac{1}{z}\right) \cos\!\big(z(2x-1)\big) - Q_{\kappa-1}\!\left(\tfrac{1}{z}\right) \sin\!\big(z(2x-1)\big) \right) T_\kappa^2[v_2](x) \right] dx = 0.
\end{cases}
\tag{5.29}
$$

Proof. The identity follows directly from (5.21) together with the half-integer representations (5.22)-(5.23), after rewriting the products of Bessel functions using elementary trigonometric relations. □

We now introduce the sequence of polynomials $\{A_\kappa(t)\}_{\kappa \in \mathbb{N}}$, defined recursively by

$$
A_0(t) = 1, \qquad A_1(t) = 1 - \frac{t}{2}\,,
$$

and, for all $\kappa \geq 1$,

$$
A_{\kappa+1}(t) = (2\kappa + 1)\, A_\kappa(t) + \frac{t^2}{4}\, A_{\kappa-1}(t)\,.
$$

Remark 5.8. *The first polynomials of the sequence beyond A_1 are explicitly given by*

$$
A_2(t) = \tfrac{1}{4}t^2 - \tfrac{3}{2}t + 3\,, \qquad A_3(t) = -\tfrac{1}{8}t^3 + \tfrac{3}{2}t^2 - \tfrac{15}{2}t + 15\,.
$$

The second equation of the system (5.29) coincides with the equation already studied in [14, Proposition 5.1][5]. We may therefore directly invoke [14, Theorem 6.6]. The first equation of the system (5.29) can be handled in the same way, by closely following the proof of [14, Theorem 6.6]. We therefore obtain the following result, where $D = \frac{d}{dx}$.

Theorem 5.9. *Let $(v_1, v_2) \in L^2(0,1)^2$. Assume that, for some $\kappa \in \mathbb{N}$,*

$$
D_{(p,q)}\lambda_{\kappa,n}(0,0) \cdot (v_1, v_2) = 0, \qquad \text{for all } n \in \mathbb{Z}.
$$

Then, in the sense of distributions, the functions

$$
A_\kappa(D)\big[T_\kappa^j[v_j]\big], \qquad j \in \{1, 2\},
$$

are even for $j = 1$ and odd for $j = 2$ with respect to the midpoint $x = \frac{1}{2}$.

---

[5]In the case $q = -m$, where m is a constant interpreted as a mass, the AKNS system is closely related to a scalar Schrödinger equation (see [3, Eq. (1.4)] and Appendix A (*Open problems*) of the present paper). Consequently, the analysis reduces to a second-order Schrödinger-type problem already studied in [14].

6 Kernel of the Fréchet differential

6.1 Injectivity of the differential for the pair $(\kappa_1, \kappa_2) = (0, 1)$

In this subsection, we assume that the perturbation (v_1, v_2) satisfies the linearized spectral condition

$$D_{(p,q)}\lambda_{\kappa,n}(0,0) \cdot (v_1, v_2) = 0, \qquad n \in \mathbb{Z},$$

for both effective angular momenta $\kappa = 0$ and $\kappa = 1$.

For $\kappa = 0$, we already know that v_1 is even and v_2 is odd about $x = \frac{1}{2}$. We now apply Theorem 5.9 with $\kappa = 1$, which yields that $A_1(D)\big[T_1^j[v_j]\big]$ is even for $j = 1$ and odd for $j = 2$, with respect to the same midpoint.

We begin with the simpler case $j = 2$. A straightforward computation yields

$$2A_1(D)\big[T_1^2[v_2]\big](x) = 2A_1(D)\big[S_{0,2}[v_2]\big](x) = -v_2'(x) + \Big(2 - \frac{2}{x}\Big)v_2(x) - (4x - 2)\int_x^1 \frac{v_2(t)}{t^2}\,dt. \quad (6.1)$$

Setting $y(x) := v_2'(x)$ and evaluating (6.1) at $x = \frac{1}{2}$, we obtain $y(\frac{1}{2}) = 0$, since v_2 is odd. We further compute

$$\begin{aligned}
G(x) &:= D^2 A_1(D)\big[T_1^2[v_2]\big](x) \\
&= A_1(D)\big[D^2(S_{0,2}[v_2])\big](x) \\
&= A_1(D)\big[\tfrac{2}{x}y + y'\big](x) \\
&= -\tfrac{1}{2}y''(x) + \Big(1 - \frac{1}{x}\Big)y'(x) + \Big(\frac{2}{x} + \frac{1}{x^2}\Big)y(x).
\end{aligned} \quad (6.2)$$

where we have used Lemma 5.4 (2) in the third line. Recalling that G is odd, the identity

$$G(x) + G(1 - x) = 0,$$

holds for all $x \in (0, 1)$. Since y is even, this identity implies that y satisfies a linear second-order differential equation on $(0, 1)$, together with the conditions

$$y\big(\tfrac{1}{2}\big) = 0, \qquad y'\big(\tfrac{1}{2}\big) = 0.$$

By the Cauchy–Lipschitz theorem, we conclude that $y \equiv 0$. Therefore $v_2' = y = 0$, so v_2 is constant. Since v_2 is odd with respect to $x = \frac{1}{2}$, this constant must vanish, and thus $v_2 \equiv 0$.

We now examine the case $j = 1$. Using Lemma 5.4 (1), a straightforward computation yields

$$G(x) := DA_1(D)\big[T_1^1[v_1]\big](x) = -\tfrac{1}{2}v_1''(x) + \Big(1 - \frac{1}{x}\Big)v_1'(x) + \Big(\frac{2}{x} + \frac{1}{x^2}\Big)v_1(x). \quad (6.3)$$

The function G is odd. Writing $G(x) + G(1 - x) = 0$ and using the fact that v_1 is even, we infer that v_1 satisfies the following second–order linear ordinary differential equation on $(0, 1)$:

$$v_1''(x) + \Big(\frac{1}{x} - \frac{1}{1 - x}\Big)v_1'(x) - \Big(\frac{2}{x} + \frac{2}{1 - x} + \frac{1}{x^2} + \frac{1}{(1 - x)^2}\Big)v_1(x) = 0, \qquad x \in (0, 1). \quad (6.4)$$

We now assume that there exists a solution v_1 of (6.4) which is even with respect to the midpoint $x = \frac{1}{2}$, and we impose the normalization conditions

$$v_1\big(\tfrac{1}{2}\big) = 1, \qquad v_1'\big(\tfrac{1}{2}\big) = 0.$$

Using MATHEMATICA, we obtain the explicit closed-form expression

$$v_1(x) = \frac{2x^2 - 2x + 1}{2x(1-x)} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{1-x}\right) - 1, \qquad x \in (0,1). \tag{6.5}$$

In particular, v_1 blows up like $\frac{1}{2x}$ as $x \to 0^+$ and therefore does not belong to $L^2(0,1)$. It follows that one must have $v_1\left(\frac{1}{2}\right) = 0$, and the Cauchy–Lipschitz theorem then implies that $v_1 \equiv 0$ on $(0,1)$.

Combining the conclusions of the two cases $j = 1$ and $j = 2$, we obtain that

$$(v_1, v_2) = (0,0).$$

Hence the kernel of the Fréchet differential of the spectral map at the zero potential is trivial for the pair of effective angular momenta $(\kappa_1, \kappa_2) = (0,1)$. We have therefore proved the following result.

Theorem 6.1 (Injectivity for the pair $(0,1)$). *For $(\kappa_1, \kappa_2) = (0,1)$, the Fréchet differential of the spectral map*

$$D_{(p,q)}\mathcal{S}_{0,1}(0,0) : L^2(0,1) \times L^2(0,1) \longrightarrow \ell^2_{\mathbb{R}}(\mathbb{Z}) \times \ell^2_{\mathbb{R}}(\mathbb{Z})$$

is one to one.

6.2 Injectivity of the differential for the pair $(\kappa_1, \kappa_2) = (0,2)$

Throughout this subsection, we assume that the perturbation (v_1, v_2) fulfills the linearized spectral constraint

$$D_{(p,q)}\lambda_{\kappa,n}(0,0) \cdot (v_1, v_2) = 0, \qquad n \in \mathbb{Z},$$

simultaneously for the effective angular momenta $\kappa = 0$ and $\kappa = 2$.

For $\kappa = 0$, as before, v_1 is even and v_2 is odd with respect to the midpoint $x = \frac{1}{2}$.

Let us now examine the case $\kappa = 2$. According to Theorem 5.9, and setting $w_j := T_2^j[v_j]$, $j = 1, 2$, we have

$$A_2(D)\left[w_j\right] \quad \text{is even if } j = 1, \text{ and odd if } j = 2. \tag{6.6}$$

Here $A_2(t) = \frac{1}{4}t^2 - \frac{3}{2}t + 3$ and $D = \frac{d}{dx}$.

We begin by studying the case $j = 2$. We introduce the following notation. Set $f = S_{0,2}[v_2]$, so that $w_2 = -S_{1,2}[f]$. Differentiating (6.6) four times and applying Lemma 5.4 (2) with $\kappa = 1$, we obtain

$$A_2(D)\left(f^{(4)}(x) + \frac{6}{x}f^{(3)}(x)\right) \quad \text{is odd.} \tag{6.7}$$

On the other hand, since $f = S_{0,2}[v_2]$, a second application of Lemma 5.4 (2), now with $\kappa = 0$, yields

$$f''(x) = \frac{2}{x}v_2'(x) + v_2''(x). \tag{6.8}$$

Setting $y(x) := v_2'(x)$ (which is even), we obtain after simplification

$$
\begin{aligned}
G(x) := 4A_2(D)&\left(f^{(4)}(x) + \frac{6}{x} f^{(3)}(x) \right) \\
= y^{(5)}(x) &+ \left(\frac{8}{x} - 6 \right) y^{(4)}(x) + \left(12 - \frac{48}{x} - \frac{8}{x^2} \right) y^{(3)}(x) \\
&+ \left(\frac{96}{x} - \frac{24}{x^3} \right) y''(x) + \left(\frac{96}{x^2} + \frac{144}{x^3} + \frac{96}{x^4} \right) y'(x) \\
&+ \left(-\frac{96}{x^3} - \frac{144}{x^4} - \frac{96}{x^5} \right) y(x) .
\end{aligned}
\tag{6.9}
$$

Because (6.6) asserts that $A_2(D)[w_2]$ is odd, and differentiation four times does not alter odd parity, we conclude that $G(x)$ is itself odd. Writing $G(x) + G(1-x) = 0$ and using the fact that y is even, we see that y satisfies a linear differential equation of order 4. We denote by

$$
v_2(x) := \int_{1/2}^{x} y(t)\, dt
\tag{6.10}
$$

the unique odd primitive of $y(x)$. An immediate computation gives the following expression:

$$
w_2(x) = T_2^2[v_2](x) = -S_{1,2}\big[S_{0,2}[v_2]\big](x) = -v_2(x) - 4x \int_x^1 \frac{v_2(t)}{t^2}\, dt + 12x^3 \int_x^1 \frac{v_2(t)}{t^4}\, dt .
\tag{6.11}
$$

Applying the differential operator $4A_2(D) = D^2 - 6D + 12$ to $w_2(x)$, we obtain

$$
\begin{aligned}
4A_2(D)\big[w_2\big](x) = \ &- v_2''(x) + \left(6 - \frac{8}{x} \right) v_2'(x) + \left(-12 + \frac{48}{x} - \frac{24}{x^2} \right) v_2(x) \\
&+ \big(24 - 48x \big) \int_x^1 \frac{v_2(t)}{t^2}\, dt \\
&+ \big(72x - 216x^2 + 144x^3 \big) \int_x^1 \frac{v_2(t)}{t^4}\, dt .
\end{aligned}
\tag{6.12}
$$

Evaluating this expression at $x = \frac{1}{2}$ and using that v_2 is odd, we obtain

$$
4A_2(D)\big[w_2\big]\big(\tfrac{1}{2}\big) = -v_2''\big(\tfrac{1}{2}\big) - 10\, v_2'\big(\tfrac{1}{2}\big) - 12\, v_2\big(\tfrac{1}{2}\big) = -10\, y\big(\tfrac{1}{2}\big) .
\tag{6.13}
$$

Since $4A_2(D)\big[w_2\big](x)$ is odd, we therefore conclude that $y\big(\frac{1}{2}\big) = 0$.

Proceeding in the same way, we compute

$$
\begin{aligned}
4\, D^2 A_2(D)\big[w_2\big](x) = \ &- v_2^{(4)}(x) + \left(6 - \frac{8}{x} \right) v_2^{(3)}(x) + \left(-12 + \frac{48}{x} - \frac{8}{x^2} \right) v_2''(x) \\
&+ \left(-\frac{96}{x} + \frac{96}{x^2} + \frac{8}{x^3} \right) v_2'(x) + \left(-\frac{288}{x^2} + \frac{144}{x^3} \right) v_2(x) \\
&+ \big(-432 + 864x \big) \int_x^1 \frac{v_2(t)}{t^4}\, dt .
\end{aligned}
\tag{6.14}
$$

Recalling that $y(x) = v_2'(x)$ is even and $y\left(\frac{1}{2}\right) = 0$, we evaluate at $x = \frac{1}{2}$ and obtain

$$4\,D^2 A_2(D)\big[w_2\big]\left(\tfrac{1}{2}\right) = -y^{(3)}\left(\tfrac{1}{2}\right) - 10\,y''\left(\tfrac{1}{2}\right) + 52\,y'\left(\tfrac{1}{2}\right) + 256\,y\left(\tfrac{1}{2}\right) = -10\,y''\left(\tfrac{1}{2}\right)\,.$$

Since $4\,D^2 A_2(D)\big[w_2\big](x)$ is also an odd function, we conclude, as above, that $y''\left(\frac{1}{2}\right) = 0$.

In conclusion, y satisfies a fourth–order linear differential equation with the initial conditions

$$y\left(\tfrac{1}{2}\right) = 0\,, \qquad y'\left(\tfrac{1}{2}\right) = 0\,, \qquad y''\left(\tfrac{1}{2}\right) = 0\,, \qquad y^{(3)}\left(\tfrac{1}{2}\right) = 0\,. \tag{6.15}$$

The Cauchy–Lipschitz theorem then implies that $y \equiv 0$. Since $y = v_2'$ and v_2 is odd, this in turn forces $v_2 \equiv 0$.

We now turn to the analysis in the case $j = 1$. In this case, $A_2(D)\,[w_1]$ is an even function. We introduce

$$w_1(x) := -S_{1,1}\big[S_{0,1}[v_1]\big](x) = -v_1(x) - 4\int_x^1 \frac{v_1(t)}{t}\,dt + 12x^2 \int_x^1 \frac{v_1(t)}{t^3}\,dt\,.$$

A direct computation yields

$$\begin{aligned}
DA_2(D)[w_1](x) = {} & -\frac{1}{4}\,v_1^{(3)}(x) + \left(\frac{3}{2} - \frac{2}{x}\right)v_1''(x) + \left(\frac{12}{x} - \frac{2}{x^2} - 3\right)v_1'(x) \\
& + \left(\frac{2}{x^3} + \frac{24}{x^2} - \frac{24}{x}\right)v_1(x) + 36(2x - 1)\int_x^1 \frac{v_1(t)}{t^3}\,dt\,.
\end{aligned} \tag{6.16}$$

The function $DA_2(D)[w_1]$ is odd with respect to $x = \frac{1}{2}$. Moreover, in the case $\kappa = 0$, we recall that v_1 is even. Evaluating (6.16) at $x = \frac{1}{2}$, we have $v_1'\left(\frac{1}{2}\right) = v_1^{(3)}\left(\frac{1}{2}\right) = 0$, and the integral term vanishes since $2x - 1 = 0$. Therefore,

$$64\,v_1\left(\tfrac{1}{2}\right) - \frac{5}{2}\,v_1''\left(\tfrac{1}{2}\right) = 0\,. \tag{6.17}$$

Now, following the usual convention, we introduce

$$f = S_{0,1}[v_1]\,, \qquad \text{so that} \qquad w_1 = -S_{1,1}[f]\,.$$

After differentiating three times $A_2(D)\,[w_1]$ and invoking Lemma 5.4 (1) with $\kappa = 1$, we arrive at

$$A_2(D)\left(f^{(3)}(x) + \frac{6}{x}f^{(2)}(x)\right) \quad \text{is odd.} \tag{6.18}$$

On the other hand, because $f = S_{0,1}[v_1]$, a second application of Lemma 5.4 (1), now with $\kappa = 0$, yields

$$f'(x) = \frac{2}{x}\,v_1(x) + v_1'(x)\,. \tag{6.19}$$

Setting

$$G(x) := 4A_2(D)\left(f^{(3)}(x) + \frac{6}{x}f^{(2)}(x)\right)\,,$$

a straightforward simplification yields the following differential expression:

$$G(x) = v_1^{(5)}(x) + \left(\frac{8}{x} - 6\right)v_1^{(4)}(x) + \left(-\frac{8}{x^2} - \frac{48}{x} + 12\right)v_1^{(3)}(x)$$

$$+ \left(-\frac{24}{x^3} + \frac{96}{x}\right)v_1''(x) + \left(\frac{96}{x^4} + \frac{144}{x^3} + \frac{96}{x^2}\right)v_1'(x)$$

$$+ \left(-\frac{96}{x^5} - \frac{144}{x^4} - \frac{96}{x^3}\right)v_1(x).$$

We therefore recover exactly the same odd function $G(x)$ as in the previous case with v_1 even.

Writing $G(x) + G(1-x) = 0$, we get :

$$\left(\frac{8}{x} + \frac{8}{1-x} - 12\right)v_1^{(4)}(x)$$

$$+ \left(-\frac{8}{x^2} + \frac{8}{(1-x)^2} - \frac{48}{x} + \frac{48}{1-x}\right)v_1^{(3)}(x)$$

$$+ \left(-\frac{24}{x^3} - \frac{24}{(1-x)^3} + \frac{96}{x} + \frac{96}{1-x}\right)v_1''(x) \tag{6.20}$$

$$+ \left(\frac{96}{x^4} - \frac{96}{(1-x)^4} + \frac{144}{x^3} - \frac{144}{(1-x)^3} + \frac{96}{x^2} - \frac{96}{(1-x)^2}\right)v_1'(x)$$

$$+ \left(-\frac{96}{x^5} - \frac{96}{(1-x)^5} - \frac{144}{x^4} - \frac{144}{(1-x)^4} - \frac{96}{x^3} - \frac{96}{(1-x)^3}\right)v_1(x) = 0.$$

Indicial roots and determination of the solution. Using MATHEMATICA, we compute the indicial equation of (6.20) at the singular point $x = 0$. This yields

$$8(\rho - 4)(\rho - 3)(\rho - 1)(\rho + 1) = 0,$$

so that the indicial roots are

$$\rho \in \{-1,\, 1,\, 3,\, 4\}.$$

We now look for the solution of (6.20) satisfying the normalization condition

$$v_1\left(\tfrac{1}{2}\right) = 1.$$

Since v_1 is even with respect to $x = \tfrac{1}{2}$, we have

$$v_1'\left(\tfrac{1}{2}\right) = v_1^{(3)}\left(\tfrac{1}{2}\right) = 0,$$

and, from (6.17), it follows that

$$v_1''\left(\tfrac{1}{2}\right) = \frac{128}{5}.$$

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By the Cauchy–Lipschitz theorem, these conditions uniquely determine a solution on $(0, 1)$. Using Math-ematica, we obtain the following explicit expression:

$$v_1(x) = \frac{5}{3} - \frac{5}{8x} - \frac{5}{8(1-x)} - \frac{15}{8\left(2 - 3x + 3x^2\right)} + \frac{25}{12\left(2 - 3x + 3x^2\right)^2} \, . \tag{6.21}$$

This solution exhibits a non-integrable blow–up at the boundary. In particular,

$$v_1 \notin L^2(0, 1) \, .$$

We deduce that one must impose

$$v_1\left(\tfrac{1}{2}\right) = 0 \, .$$

It then follows that all derivatives of v_1 at $x = \frac{1}{2}$ up to order three vanish. By uniqueness of the Cauchy problem, this implies that

$$v_1 \equiv 0 \qquad \text{on } (0, 1) \, .$$

Thus, we have established the following injectivity result in the case $(0, 2)$:

Theorem 6.2 (Injectivity for the pair $(0, 2)$). *For $(\kappa_1, \kappa_2) = (0, 2)$, the Fréchet differential of the spectral map*

$$D_{(p,q)}\mathcal{S}_{0,2}(0,0) \, : \, L^2(0,1) \times L^2(0,1) \longrightarrow \ell_{\mathbb{R}}^2(\mathbb{Z}) \times \ell_{\mathbb{R}}^2(\mathbb{Z})$$

is injective.

6.3 Injectivity of the differential for the pair $(\kappa_1, \kappa_2) = (1, 2)$

Throughout this subsection, we assume that $(v_1, v_2) \in L^2(0,1)^2$ satisfies the linearized spectral condition

$$D_{(p,q)}\lambda_{\kappa,n}(0,0) \cdot (v_1, v_2) = 0 \, , \qquad n \in \mathbb{Z},$$

for both effective angular momenta $\kappa = 1$ and $\kappa = 2$.

Applying Theorem 5.9 with $\kappa = 1$ and $\kappa = 2$, we obtain that $A_\kappa(D)\big[T_\kappa^j[v_j]\big]$ is even for $j = 1$ and odd for $j = 2$, with respect to $x = \frac{1}{2}$.

We first consider the case $j = 2$ with $\kappa = 1$. Set $f = S_{0,2}[v_2]$. A direct computation gives

$$2A_1(D)\big[T_1^2[v_2]\big](x) = 2A_1(D)\big[S_{0,2}[v_2]\big](x) = 2A_1(D)[f](x) = -f'(x) + 2f(x) \, . \tag{6.22}$$

Decomposing f into its even and odd parts,

$$f = f_e + f_o \, , \tag{6.23}$$

where f_e is even and f_o is odd with respect to $x = \frac{1}{2}$, we immediately obtain

$$f_e = \frac{1}{2}\, f_o' \, , \tag{6.24}$$

since the function $-f_o' + 2f_e$ is even and odd and therefore

$$f = \frac{1}{2}\, f_o' + f_o \, . \tag{6.25}$$

We now exploit the case $\kappa = 2$. We define

$$g := S_{1,2}[f] = f(x) - 6x^3 \int_x^1 \frac{f(t)}{t^4} \, dt,$$

so that $g = -T_2^2[v_2]$. By assumption, $A_2(D)\big[T_2^2[v_2]\big]$ is odd in the sense of distributions, where

$$A_2(t) = \tfrac{1}{4}t^2 - \tfrac{3}{2}t + 3.$$

A straightforward computation yields

$$4A_2(D)[g](x) = f''(x) + 6\Big(\frac{1}{x} - 1\Big)f'(x) + \Big(\frac{12}{x^2} - \frac{36}{x} + 12\Big)f(x) - 36x(1-x)(1-2x)\int_x^1 \frac{f(t)}{t^4}\, dt. \quad (6.26)$$

Since $4A_2(D)[g]$ is odd with respect to $x = \tfrac{1}{2}$, evaluating (6.26) at $x = \tfrac{1}{2}$ yields

$$f''\big(\tfrac{1}{2}\big) + 6f'\big(\tfrac{1}{2}\big) - 12f\big(\tfrac{1}{2}\big) = 0.$$

Using the decomposition $f = \tfrac{1}{2}f_o' + f_o$, where f_o is odd, we obtain

$$f_o^{(3)}\big(\tfrac{1}{2}\big) = 0. \quad (6.27)$$

Similarly, differentiating (6.26) twice yields

$$\begin{aligned}
D^2\big(4A_2(D)[g]\big)(x) = {} & f^{(4)}(x) + 6\Big(\frac{1}{x} - 1\Big)f^{(3)}(x) + \Big(12 - \frac{36}{x}\Big)f''(x) \\
& + \Big(\frac{72}{x} - \frac{36}{x^2}\Big)f'(x) + \Big(\frac{144}{x^2} - \frac{72}{x^3}\Big)f(x) \\
& + 216(1-2x)\int_x^1 \frac{f(t)}{t^4}\, dt.
\end{aligned} \quad (6.28)$$

Evaluating the identity (6.28) at $x = \tfrac{1}{2}$ yields

$$f_o^{(5)}\big(\tfrac{1}{2}\big) = 48\, f_o^{(3)}\big(\tfrac{1}{2}\big) = 0. \quad (6.29)$$

Finally, a similar computation yields

$$\begin{aligned}
G(x) := D^4\big(4A_2(D)[g]\big)(x) = {} & f^{(6)}(x) + 6\Big(\frac{1}{x} - 1\Big)f^{(5)}(x) \\
& + \Big(12 - \frac{36}{x} - \frac{12}{x^2}\Big)f^{(4)}(x) + \Big(\frac{72}{x} + \frac{36}{x^2} + \frac{12}{x^3}\Big)f^{(3)}(x).
\end{aligned} \quad (6.30)$$

Replacing $f = \tfrac{1}{2}f_o' + f_o$, we get

$$\begin{aligned}
G(x) = {} & \frac{1}{2}\, f_o^{(7)}(x) + \Big(\frac{3}{x} - 2\Big)f_o^{(6)}(x) \\
& - \Big(\frac{12}{x} + \frac{6}{x^2}\Big)f_o^{(5)}(x) + \Big(12 + \frac{6}{x^2} + \frac{6}{x^3}\Big)f_o^{(4)}(x) \\
& + \Big(\frac{72}{x} + \frac{36}{x^2} + \frac{12}{x^3}\Big)f_o^{(3)}(x).
\end{aligned} \quad (6.31)$$

We now use the symmetry condition $G(x) + G(1-x) = 0$. This yields the following differential equation:

$$
\begin{aligned}
0 = {} & f_o^{(7)}(x) + 3\Big(\frac{1}{x} - \frac{1}{1-x}\Big) f_o^{(6)}(x) \\
& - \Big(\frac{12}{x} + \frac{6}{x^2} + \frac{12}{1-x} + \frac{6}{(1-x)^2}\Big) f_o^{(5)}(x) \\
& + 6\Big(\frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{(1-x)^2} - \frac{1}{(1-x)^3}\Big) f_o^{(4)}(x) \\
& + \Big(\frac{72}{x} + \frac{36}{x^2} + \frac{12}{x^3} + \frac{72}{1-x} + \frac{36}{(1-x)^2} + \frac{12}{(1-x)^3}\Big) f_o^{(3)}(x) \, .
\end{aligned}
\tag{6.32}
$$

Finally, introducing

$$
y(x) = f_o^{(3)}(x) \, ,
$$

we are led to a fourth–order differential equation satisfied by y. The function y is even with respect to $x = \frac{1}{2}$, and the previous identities imply

$$
y^{(k)}\big(\tfrac{1}{2}\big) = 0, \qquad k = 0, 1, 2, 3 \, .
$$

The Cauchy–Lipschitz theorem then yields that $y \equiv 0$. Consequently, f_o must be an odd polynomial of degree at most two, hence it necessarily takes the form

$$
f_o(x) = a \, (x - \tfrac{1}{2}) \, .
\tag{6.33}
$$

Using once more the relation $f = \frac{1}{2} f_o' + f_o$, we deduce

$$
f(x) = \frac{a}{2} + a \, (x - \tfrac{1}{2}) = a \, x \, .
\tag{6.34}
$$

Since $f = S_{0,2}[v_2]$, we recover v_2 by applying the left inverse $S_{0,1}^*$ given in Lemma 5.3(v), namely

$$
v_2(x) = S_{0,1}^*[f](x) = f(x) - \frac{2}{x} \int_0^x f(t) \, dt \, .
$$

A direct computation yields

$$
v_2(x) = 0 \, .
\tag{6.35}
$$

Let us now examine the case $j = 1$. By Theorem 5.9 applied with $\kappa = 1$ and $\kappa = 2$, we know that $A_\kappa(D)\big[T_\kappa^1[v_1]\big]$ is even. Set $f = S_{0,1}[v_1]$. As in the case $j = 2$, the analysis of the case $\kappa = 1$ yields

$$
f = f_e + \frac{1}{2} \, f_e' \, .
\tag{6.36}
$$

(As before, f_o denotes the odd part of f with respect to $x = \frac{1}{2}$, and f_e its even part.) We now exploit the case $\kappa = 2$. We define

$$
g := S_{1,1}[f] = f(x) - 6x^2 \int_x^1 \frac{f(t)}{t^3} \, dt \, ,
$$

so that $g = -T_2^1[v_1]$. By assumption, $DA_2(D)\big[T_2^1[v_1]\big]$ is odd in the sense of distributions. A straightforward computation yields

$$4DA_2(D)[g](x) = f^{(3)}(x) + 6\Big(\frac{1}{x} - 1\Big)f''(x) + \Big(12 - \frac{36}{x}\Big)f'(x) + \Big(\frac{72}{x} - \frac{36}{x^2}\Big)f(x)$$
$$+ 72(1 - 2x)\int_x^1 \frac{f(t)}{t^3}\,dt\,.$$
$$(6.37)$$

Since $4DA_2(D)[g]$ is odd with respect to $x = \frac{1}{2}$, evaluating (6.37) at $x = \frac{1}{2}$ gives

$$f_o^{(3)}\big(\tfrac{1}{2}\big) = 48 f_o'\big(\tfrac{1}{2}\big)\,. \qquad (6.38)$$

Similarly, differentiating (6.37) twice yields

$$H(x) := 4D^3 A_2(D)[g](x) = f^{(5)}(x) + 6\Big(\frac{1}{x} - 1\Big)f^{(4)}(x)$$
$$+ \Big(12 - \frac{36}{x} - \frac{12}{x^2}\Big)f^{(3)}(x)$$
$$+ \Big(\frac{72}{x} + \frac{36}{x^2} + \frac{12}{x^3}\Big)f''(x)\,.$$
$$(6.39)$$

Replacing $f = f_e + \frac{1}{2}f_e'$ in $H(x)$ and using the relation $f_o = \frac{1}{2}f_e'$, we can express H entirely in terms of f_o. A straightforward computation yields

$$H(x) = f_o^{(5)}(x) + \Big(\frac{6}{x} - 4\Big)f_o^{(4)}(x) - \Big(\frac{24}{x} + \frac{12}{x^2}\Big)f_o^{(3)}(x)$$
$$+ \Big(24 + \frac{12}{x^2} + \frac{12}{x^3}\Big)f_o''(x) + \Big(\frac{144}{x} + \frac{72}{x^2} + \frac{24}{x^3}\Big)f_o'(x)\,.$$
$$(6.40)$$

We now use the symmetry condition $H(x) + H(1 - x) = 0$. This yields the following differential equation:

$$0 = 2f_o^{(5)}(x) + \Big(\frac{6}{x} - \frac{6}{1 - x}\Big)f_o^{(4)}(x)$$
$$- \Big(\frac{24}{x} + \frac{24}{1 - x} + \frac{12}{x^2} + \frac{12}{(1 - x)^2}\Big)f_o^{(3)}(x)$$
$$+ \Big(\frac{12}{x^2} + \frac{12}{x^3} - \frac{12}{(1 - x)^2} - \frac{12}{(1 - x)^3}\Big)f_o''(x)$$
$$+ \Big(\frac{144}{x} + \frac{144}{1 - x} + \frac{72}{x^2} + \frac{72}{(1 - x)^2} + \frac{24}{x^3} + \frac{24}{(1 - x)^3}\Big)f_o'(x)\,.$$
$$(6.41)$$

Finally, introducing $y(x) = f_o'(x)$, we are led to a fourth–order differential equation satisfied by y:

$$0 = 2y^{(4)}(x) + \Big(\frac{6}{x} - \frac{6}{1 - x}\Big)y^{(3)}(x)$$
$$- \Big(\frac{24}{x} + \frac{24}{1 - x} + \frac{12}{x^2} + \frac{12}{(1 - x)^2}\Big)y''(x)$$
$$+ \Big(\frac{12}{x^2} + \frac{12}{x^3} - \frac{12}{(1 - x)^2} - \frac{12}{(1 - x)^3}\Big)y'(x)$$
$$+ \Big(\frac{144}{x} + \frac{144}{1 - x} + \frac{72}{x^2} + \frac{72}{(1 - x)^2} + \frac{24}{x^3} + \frac{24}{(1 - x)^3}\Big)y(x)\,.$$
$$(6.42)$$

A direct computation shows that the roots of the indicial equation are -2, 0, 2, and 3.

The function y is even with respect to $x = \frac{1}{2}$. From the previous computations, if we normalize by choosing $f_o'\left(\frac{1}{2}\right) = 1$, then

$$y\left(\tfrac{1}{2}\right) = 1, \qquad y'\left(\tfrac{1}{2}\right) = 0, \qquad y''\left(\tfrac{1}{2}\right) = 48. \qquad y^{(3)}\left(\tfrac{1}{2}\right) = 0.$$

Using MATHEMATICA, the unique solution is given by

$$y(x) = -1 + \frac{1}{4(x-1)^2} + \frac{1}{4x^2}. \tag{6.43}$$

We recall that $y = f_0'$. Since f_0 is odd, we obtain

$$f_0(x) = -x + \frac{1}{4(1-x)} - \frac{1}{4x} + \frac{1}{2},$$

and, since $f_e' = 2f_0$, there exists a real constant C such that

$$f_e(x) = 2\left(-\frac{x^2}{2} - \frac{1}{4}\ln(1-x) - \frac{1}{4}\ln(x) + \frac{x}{2}\right) + C.$$

We thus obtain,

$$f(x) = f_e(x) + f_0(x) = -x^2 - \frac{1}{2}\ln(1-x) - \frac{1}{2}\ln(x) + \frac{1}{4(1-x)} - \frac{1}{4x} + C. \tag{6.44}$$

This leads to a contradiction, since $f = S_{0,1}[v_1]$ must belong to $L^2(0,1)$, whereas the function (6.44) is not square integrable near $x = 0$ and $x = 1$. Consequently, the initial condition must satisfy

$$f_o'\left(\tfrac{1}{2}\right) = 0.$$

By the Cauchy–Lipschitz Theorem, the corresponding solution of the differential equation then satisfies $y \equiv 0$. Hence f_o is a constant, which must be zero since f_o is odd. Therefore f itself is a constant function.

Since $f = S_{0,1}[v_1]$, we recover v_1 by applying the left inverse $S_{0,2}^*$ given in Lemma 5.3(v), namely

$$v_1(x) = S_{0,2}^*[f](x) = f(x) - \frac{2}{x^2}\int_0^x t\,f(t)\,dt.$$

A direct computation yields

$$v_1(x) = 0. \tag{6.45}$$

Thus, we have established the following injectivity result in the case $(1,2)$:

Theorem 6.3 (Injectivity for the pair $(1,2)$). *For $(\kappa_1, \kappa_2) = (1,2)$, the Fréchet differential of the spectral map*

$$D_{(p,q)}\mathcal{S}_{1,2}(0,0) : L^2(0,1) \times L^2(0,1) \longrightarrow \ell^2_{\mathbb{R}}(\mathbb{Z}) \times \ell^2_{\mathbb{R}}(\mathbb{Z})$$

is one to one.

6.4 Injectivity of the differential for the pair $(\kappa_1, \kappa_2) = (0, 3)$

Throughout this subsection, we consider $(v_1, v_2) \in L^2(0,1)^2$ satisfying the linearized spectral constraint

$$D_{(p,q)}\lambda_{\kappa,n}(0,0) \cdot (v_1, v_2) = 0, \qquad n \in \mathbb{Z},$$

simultaneously for the two effective angular momenta $\kappa = 0$ and $\kappa = 3$.

In the case $\kappa = 0$, one has, as in the previous section, that v_1 is even and v_2 is odd with respect to the midpoint $x = \frac{1}{2}$.

We now turn to the case $\kappa = 3$. According to Theorem 5.9, and setting $w_j := T_3^j[v_j]$, $j = 1, 2$, we have

$$A_3(D)\left[w_j\right] \quad \text{is even if } j = 1, \text{ and odd if } j = 2, \tag{6.46}$$

where

$$A_3(t) = -\tfrac{1}{8}t^3 + \tfrac{3}{2}t^2 - \tfrac{15}{2}t + 15, \qquad D = \tfrac{d}{dx}.$$

We first analyze the case $j = 2$. The relevant transformation operator can be written explicitly. For $x \in (0, 1)$, one has

$$T_3^2[v_2](x) = v_2(x) - 6x \int_x^1 \frac{v_2(t)}{t^2}\, dt + 48x^3 \int_x^1 \frac{v_2(t)}{t^4}\, dt - 60x^5 \int_x^1 \frac{v_2(t)}{t^6}\, dt. \tag{6.47}$$

Moreover, introducing the differential operator $D = \frac{d}{dx}$ and the polynomial

$$A_3(t) = -\tfrac{1}{8}t^3 + \tfrac{3}{2}t^2 - \tfrac{15}{2}t + 15, \qquad \text{so that} \qquad 8A_3(D) = -D^3 + 12D^2 - 60D + 120,$$

we obtain the following explicit identity

$$
\begin{aligned}
F(x) &= 8A_3(D)\left[T_3^2[v_2]\right](x) \\
&= -v_2^{(3)}(x) + \left(12 - \frac{18}{x}\right)v_2''(x) + \left(-60 + \frac{216}{x} - \frac{126}{x^2}\right)v_2'(x) \\
&\quad + \left(120 - \frac{1080}{x} + \frac{1728}{x^2} - \frac{624}{x^3}\right)v_2(x) + 360(1 - 2x)\int_x^1 \frac{v_2(t)}{t^2}\, dt \\
&\quad + 288\left(20x^3 - 30x^2 + 12x - 1\right)\int_x^1 \frac{v_2(t)}{t^4}\, dt \\
&\quad - 3600x^2\left(2x^3 - 5x^2 + 4x - 1\right)\int_x^1 \frac{v_2(t)}{t^6}\, dt.
\end{aligned}
\tag{6.48}
$$

Evaluating (6.48) at $x = \frac{1}{2}$, we use that the left-hand side is odd with respect to $\frac{1}{2}$, hence it vanishes at $x = \frac{1}{2}$. Since v_2 is also odd, one has

$$v_2\left(\tfrac{1}{2}\right) = 0, \qquad v_2''\left(\tfrac{1}{2}\right) = 0,$$

Thus,

$$v_2^{(3)}\left(\tfrac{1}{2}\right) = -132\, v_2'\left(\tfrac{1}{2}\right).$$

We now compute the second derivative of $F(x) = 8A_3(D)\big[T_3^2[v_2]\big](x)$. Differentiating (6.48) twice, we obtain

$$
\begin{aligned}
F''(x) = &-v_2^{(5)}(x) + \Big(12 - \frac{18}{x}\Big)v_2^{(4)}(x) + \Big(-60 + \frac{216}{x} - \frac{90}{x^2}\Big)v_2^{(3)}(x) \\
&+ \Big(120 - \frac{1080}{x} + \frac{1296}{x^2} - \frac{156}{x^3}\Big)v_2''(x) \\
&+ \frac{36\big(60x^3 - 210x^2 + 124x - 9\big)}{x^4}\,v_2'(x) + \frac{1440\big(12x^3 - 26x^2 + 12x - 1\big)}{x^5}\,v_2(x) \\
&+ 17280(2x-1)\int_x^1 \frac{v_2(t)}{t^4}\,dt - 7200\big(20x^3 - 30x^2 + 12x - 1\big)\int_x^1 \frac{v_2(t)}{t^6}\,dt\,.
\end{aligned}
\tag{6.49}
$$

Evaluating (6.49) at $x = \frac{1}{2}$ and using the oddness of F and v_2, all nonlocal terms vanish and we obtain

$$
0 = 4608\,v_2'\big(\tfrac{1}{2}\big) + 12\,v_2^{(3)}\big(\tfrac{1}{2}\big) - v_2^{(5)}\big(\tfrac{1}{2}\big)\,.
$$

We now differentiate (6.49) twice. This yields

$$
\begin{aligned}
F^{(4)}(x) = &-v_2^{(7)}(x) + \Big(12 - \frac{18}{x}\Big)v_2^{(6)}(x) + \Big(-60 + \frac{216}{x} - \frac{54}{x^2}\Big)v_2^{(5)}(x) \\
&+ \Big(120 - \frac{1080}{x} + \frac{864}{x^2} + \frac{168}{x^3}\Big)v_2^{(4)}(x) \\
&+ \Big(\frac{2160}{x} - \frac{5400}{x^2} - \frac{288}{x^3} + \frac{72}{x^4}\Big)v_2^{(3)}(x) \\
&+ \Big(\frac{12960}{x^2} - \frac{9360}{x^3} - \frac{1728}{x^4} - \frac{720}{x^5}\Big)v_2''(x) \\
&+ \Big(\frac{44640}{x^3} - \frac{19440}{x^4} + \frac{1728}{x^5} + \frac{720}{x^6}\Big)v_2'(x) + \Big(\frac{172800}{x^4} - \frac{86400}{x^5}\Big)v_2(x) \\
&+ 432000\,(1 - 2x)\int_x^1 \frac{v_2(t)}{t^6}\,dt\,.
\end{aligned}
\tag{6.50}
$$

Evaluating at $x = \frac{1}{2}$, since F is odd with respect to $x = \frac{1}{2}$ one has $F^{(4)}\big(\frac{1}{2}\big) = 0$, and the nonlocal term vanishes. Moreover, since v_2 is also odd with respect to $x = \frac{1}{2}$, we have $v_2\big(\frac{1}{2}\big) = v_2''\big(\frac{1}{2}\big) = v_2^{(4)}\big(\frac{1}{2}\big) = v_2^{(6)}\big(\frac{1}{2}\big) = 0$. Therefore,

$$
0 = -v_2^{(7)}\big(\tfrac{1}{2}\big) + 156\,v_2^{(5)}\big(\tfrac{1}{2}\big) - 18432\,v_2^{(3)}\big(\tfrac{1}{2}\big) + 147456\,v_2'\big(\tfrac{1}{2}\big)\,.
$$

Differentiating twice (6.50) and collecting terms, we obtain

$$F^{(6)}(x) = - v_2^{(9)}(x) + \left(12 - \frac{18}{x}\right) v_2^{(8)}(x) + \left(-60 + \frac{216}{x} - \frac{18}{x^2}\right) v_2^{(7)}(x)$$
$$+ \left(120 - \frac{1080}{x} + \frac{432}{x^2} + \frac{348}{x^3}\right) v_2^{(6)}(x)$$
$$+ \left(\frac{2160}{x} - \frac{3240}{x^2} - \frac{3312}{x^3} - \frac{1260}{x^4}\right) v_2^{(5)}(x)$$
$$+ \left(\frac{8640}{x^2} + \frac{10080}{x^3} + \frac{5184}{x^4} + \frac{720}{x^5}\right) v_2^{(4)}(x) \qquad (6.51)$$
$$+ \left(-\frac{2880}{x^3} + \frac{4320}{x^4} + \frac{12096}{x^5} + \frac{9360}{x^6}\right) v_2^{(3)}(x)$$
$$+ \left(-\frac{17280}{x^4} - \frac{43200}{x^5} - \frac{51840}{x^6} - \frac{30240}{x^7}\right) v_2''(x)$$
$$+ \left(\frac{17280}{x^5} + \frac{43200}{x^6} + \frac{51840}{x^7} + \frac{30240}{x^8}\right) v_2'(x).$$

Let $G(x) = F^{(6)}(x)$. Since F is odd with respect to $x = \frac{1}{2}$, the same holds for G, and therefore

$$G(x) + G(1 - x) = 0, \qquad x \in (0, 1).$$

Using that v_2 is also odd and applying the reduction obtained above, we can rewrite the symmetry identity $G(x) + G(1 - x) = 0$ as a linear ODE for

$$w := v_2'.$$

This yields the following eighth–order symmetry equation:

$$-2\,w^{(8)}(x)$$

$$-\left(\frac{18}{x} - \frac{18}{1-x}\right)w^{(7)}(x)$$

$$+\left(-120 + 216\left(\frac{1}{x} + \frac{1}{1-x}\right) - 18\left(\frac{1}{x^2} + \frac{1}{(1-x)^2}\right)\right)w^{(6)}(x)$$

$$-\left(\frac{1080}{x} - \frac{1080}{1-x} - \frac{432}{x^2} + \frac{432}{(1-x)^2} - \frac{348}{x^3} + \frac{348}{(1-x)^3}\right)w^{(5)}(x)$$

$$+\left(\frac{2160}{x} + \frac{2160}{1-x} - \frac{3240}{x^2} - \frac{3240}{(1-x)^2} - \frac{3312}{x^3} - \frac{3312}{(1-x)^3} - \frac{1260}{x^4} - \frac{1260}{(1-x)^4}\right)w^{(4)}(x)$$

$$+\left(\frac{8640}{x^2} - \frac{8640}{(1-x)^2} + \frac{10080}{x^3} - \frac{10080}{(1-x)^3} + \frac{5184}{x^4} - \frac{5184}{(1-x)^4} + \frac{720}{x^5} - \frac{720}{(1-x)^5}\right)w^{(3)}(x)$$

$$+\left(-\frac{2880}{x^3} - \frac{2880}{(1-x)^3} + \frac{4320}{x^4} + \frac{4320}{(1-x)^4} + \frac{12096}{x^5} + \frac{12096}{(1-x)^5} + \frac{9360}{x^6} + \frac{9360}{(1-x)^6}\right)w''(x)$$

$$+\left(-\frac{17280}{x^4} + \frac{17280}{(1-x)^4} - \frac{43200}{x^5} + \frac{43200}{(1-x)^5} - \frac{51840}{x^6} + \frac{51840}{(1-x)^6} - \frac{30240}{x^7} + \frac{30240}{(1-x)^7}\right)w'(x)$$

$$+\left(\frac{17280}{x^5} + \frac{17280}{(1-x)^5} + \frac{43200}{x^6} + \frac{43200}{(1-x)^6} + \frac{51840}{x^7} + \frac{51840}{(1-x)^7} + \frac{30240}{x^8} + \frac{30240}{(1-x)^8}\right)w(x) = 0\,.$$

(6.52)

The indicial roots at $x = 0$ are

$$r \in \{7, 6, 5, 3, 1, -1\} \ \cup \ \{-1 \pm i\sqrt{23}\}\,.$$

From the previous analysis, solutions of this ODE are uniquely determined by the single parameter $v_2'\left(\frac{1}{2}\right)$. For instance, imposing the normalization

$$w\left(\tfrac{1}{2}\right) = v_2'\left(\tfrac{1}{2}\right) = 1$$

gives us a solution $w = v_2'$ of (6.52) satisfying the differential constraints at $x = \frac{1}{2}$,

$$v_2^{(3)}\left(\tfrac{1}{2}\right) = -132\,, \qquad v_2^{(5)}\left(\tfrac{1}{2}\right) = 3024\,, \qquad v_2^{(7)}\left(\tfrac{1}{2}\right) = 3052224\,,$$

which follow from the symmetry relations derived above.

To gain further insight into the behaviour of solutions, we performed a numerical integration of equation (6.52) using MATHEMATICA. Starting from the normalization $v_2'\left(\frac{1}{2}\right) = 1$ together with the differential constraints above, the resulting functions $w(x) = v_2'(x)$ and $v_2(x)$ are displayed in Figure 1.
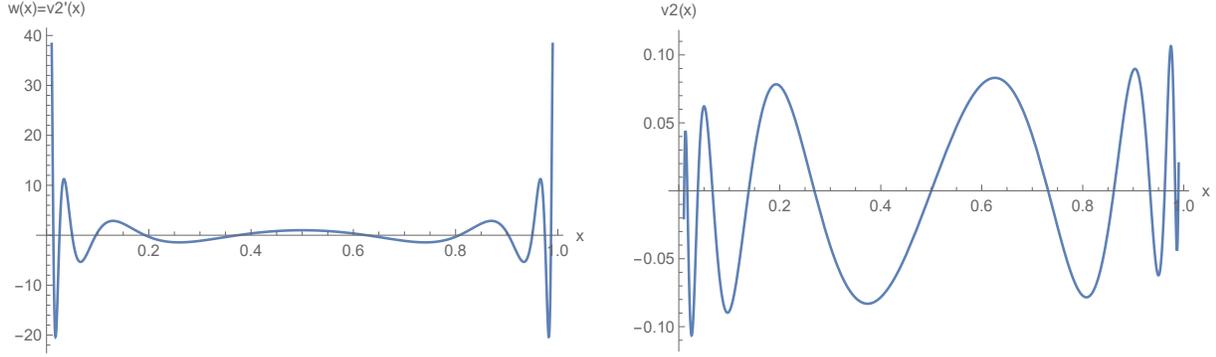
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Figure 1: Numerical solutions $w(x) = v_2'(x)$ (left) and $v_2(x)$ (right), computed with *Mathematica*.

The qualitative behaviour of w is consistent with the structure of the indicial roots. In particular, the complex pair $-1 \pm i\sqrt{23}$ produces oscillatory components in the local behaviour near the singular endpoints.

On the other hand, when $\kappa = 3$, the previous analysis (see (3.10)) yields the additional constraint

$$\int_0^1 v_2(x)\, x^6\, dx = 0\,.$$

However, using the numerical solution corresponding to the normalization $v_2'\left(\frac{1}{2}\right) = 1$, we obtain

$$\int_0^1 v_2(x)\, x^6\, dx \approx 1.6 \times 10^{-1}\,,$$

which is clearly non–zero. This leads to a numerical contradiction. Consequently, one must have

$$v_2'\left(\tfrac{1}{2}\right) = 0\,.$$

By the Cauchy–Lipschitz theorem applied to equation (6.52), this implies that $w \equiv 0$. Since v_2 is odd with respect to $x = \frac{1}{2}$, it follows that $v_2 \equiv 0$.

We now turn to the case $j = 1$. The analysis is completely analogous to the case $j = 2$. Since v_1 and $A_3(D)\left[T_3^1[v_1]\right]$ are even with respect to $x = \frac{1}{2}$, the successive symmetry identities at $x = \frac{1}{2}$ determine all higher even derivatives of v_1 from the two parameters $v_1\left(\frac{1}{2}\right)$ and $v_1''\left(\frac{1}{2}\right)$, while all odd derivatives vanish at $x = \frac{1}{2}$. Namely,

$$v_1^{(4)}\left(\tfrac{1}{2}\right) = 12\, v_1''\left(\tfrac{1}{2}\right) + 4608\, v_1\left(\tfrac{1}{2}\right)\,.$$

$$v_1^{(6)}\left(\tfrac{1}{2}\right) = 156\, v_1^{(4)}\left(\tfrac{1}{2}\right) - 18432\, v_1''\left(\tfrac{1}{2}\right) + 147456\, v_1\left(\tfrac{1}{2}\right)\,.$$

Then, proceeding exactly as in the case $j = 2$, one obtains the same eighth–order symmetry equation as above, with w replaced by v_1. Hence, once the two parameters $v_1\left(\frac{1}{2}\right)$ and $v_1''\left(\frac{1}{2}\right)$ are fixed, all higher derivatives at $x = \frac{1}{2}$ are uniquely determined, and the Cauchy–Lipschitz theorem yields a unique local even solution.

We therefore introduce the two fundamental even solutions corresponding to the initial data

$$u: \quad \left(v_1(\tfrac{1}{2}), v_1''(\tfrac{1}{2})\right) = (1, 0)\,, \qquad v: \quad \left(v_1(\tfrac{1}{2}), v_1''(\tfrac{1}{2})\right) = (0, 1)\,.$$

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Any even solution of the symmetry equation is then a linear combination

$$v_1 = \alpha \, u + \beta \, v \, .$$

To understand the behaviour of these solutions near $x = 0$, we performed a numerical Frobenius analysis using MATHEMATICA. Starting from the two independent even solutions (u, v), we integrate the eighth–order equation numerically on $(0, 1)$.

We have previously seen that any local solution near $x = 0$ is expected to have an expansion of the form

$$v_1(x) = x^{-1}\Big(A + B \cos(\sqrt{23}\log x) + C \sin(\sqrt{23}\log x) \Big) + O(x) \, .$$

For the pair (u, v), MATHEMATICA produces the numerical triples

$$(A_u, B_u, C_u) = (0.700136, -0.0937512, -0.0416037) \, ,$$

$$(A_v, B_v, C_v) = (0.00529329, 0.00201729, -0.00117865) \, .$$

To test whether a linear combination of these solutions could cancel the leading singular behaviour, we solve numerically

$$\alpha(A_u, B_u, C_u) + \beta(A_v, B_v, C_v) = (0, 0, 0) \, .$$

The computation yields only the trivial solution $\alpha = \beta = 0$. Consequently, the only solution which is L^2 at both endpoints $x = 0$ and $x = 1$ is the trivial one. This numerical analysis leads to the following theorem.

Theorem 6.4 (Injectivity for the pair $(0,3)$). *For $(\kappa_1, \kappa_2) = (0, 3)$, the Fréchet differential of the spectral map*

$$D_{(p,q)}\mathcal{S}_{0,3}(0, 0) \, : \, L^2(0, 1) \times L^2(0, 1) \longrightarrow \ell^2_{\mathbb{R}}(\mathbb{Z}) \times \ell^2_{\mathbb{R}}(\mathbb{Z})$$

is one to one.

7 Closed range of the linearized spectral map

In this section, we study the Fréchet differential of the spectral map at the zero potential and prove that its range is closed when $\kappa_2 - \kappa_1$ is odd. For the corresponding radial Schrödinger problem, the closed–range property was established by Carlson-Shubin and Shubin-Christ see, e.g., [10, 33].

7.1 Preliminaries

Let

$$S := D_{(p,q)}\mathcal{S}_{\kappa_1,\kappa_2}(0, 0).$$

We briefly recall the notation introduced earlier. For each $\kappa \in \{\kappa_1, \kappa_2\}$ and $n \geq 1$ (with $\nu = \kappa + \frac{1}{2}$), we introduce the linear functionals

$$A_{\kappa,n}(v_1) := \int_0^1 2j_{\nu,n}x \, J_{\nu-1}\big(j_{\nu,n}x\big) \, J_\nu\big(j_{\nu,n}x\big) \, v_1(x) \, dx \, ,$$

$$B_{\kappa,n}(v_2) := \int_0^1 j_{\nu,n}x \Big(J_\nu\big(j_{\nu,n}x\big)^2 - J_{\nu-1}\big(j_{\nu,n}x\big)^2 \Big) v_2(x) \, dx \, ,$$

which define bounded linear functionals on $L^2(0,1)$.

Using Lemma 5.5 and Remark 5.6, these linear forms admit the following transformation–operator representations: for $n \geq 1$,

$$A_{\kappa,n}(v_1) = -\frac{2}{\pi} \int_0^1 \Phi_{\kappa,1}(j_{\nu,n}x)\, v_1(x)\, dx \tag{7.1}$$

$$= -\frac{2}{\pi} \int_0^1 \sin(2j_{\nu,n}x)\, T_\kappa^1(v_1)(x)\, dx\,, \tag{7.2}$$

$$B_{\kappa,n}(v_2) = \frac{2}{\pi} \int_0^1 \Phi_{\kappa,2}(j_{\nu,n}x)\, v_2(x)\, dx \tag{7.3}$$

$$= \frac{2}{\pi} \int_0^1 \cos(2j_{\nu,n}x)\, T_\kappa^2(v_2)(x)\, dx\,. \tag{7.4}$$

As shown in Section 3.3, there exists a bounded isomorphism

$$\mathcal{U} : \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \longrightarrow \mathbb{R}^2 \times \ell^2(\mathbb{N}^*)^2 \times \ell^2(\mathbb{N}^*)^2$$

such that $\mathcal{U} \circ S$ is block diagonal:

$$(\mathcal{U} \circ S)(v_1, v_2) = \big(\mathcal{M}(v_2),\ \mathcal{A}_{\kappa_1,\kappa_2}(v_1),\ \mathcal{B}_{\kappa_1,\kappa_2}(v_2)\big)\,,$$

where

$$\mathcal{A}_{\kappa_1,\kappa_2}(v_1) = \big((A_{\kappa_1,n}(v_1))_{n\geq 1},\ (A_{\kappa_2,n}(v_1))_{n\geq 1}\big)\,,$$
$$\mathcal{B}_{\kappa_1,\kappa_2}(v_2) = \big((B_{\kappa_1,n}(v_2))_{n\geq 1},\ (B_{\kappa_2,n}(v_2))_{n\geq 1}\big)\,,$$
$$\mathcal{M}(v_2) = \left(-\int_0^1 x^{2\kappa_1} v_2,\ -\int_0^1 x^{2\kappa_2} v_2\right)\,.$$

Since $\mathcal{U}$ is an isomorphism, the range of S is closed if and only if the range of $\mathcal{U} \circ S$ is closed. The closed–range property thus reduces to the independent analysis of $\mathcal{M}$, $\mathcal{A}_{\kappa_1,\kappa_2}$ and $\mathcal{B}_{\kappa_1,\kappa_2}$.

7.2 Strategy of the proof

We now outline the strategy used to prove the closed–range property. For simplicity, we present the argument in the model case $(\kappa_1, \kappa_2) = (0,1)$, the general case is identical.

We approximate the operator $\mathcal{A}_{\kappa_1,\kappa_2}$ by replacing the Bessel zeros $j_{\nu,n}$ with their leading asymptotics, namely

$$j_{\frac{1}{2},n} \sim n\pi \quad (\kappa = 0)\,, \qquad j_{\frac{3}{2},n} \sim (n + \tfrac{1}{2})\pi \quad (\kappa = 1)\,.$$

This leads to a Fourier–type model operator whose kernels involve pure sine functions with frequencies $2n\pi$ and $2(n+\frac{1}{2})\pi$. We show that this model operator is injective with closed range, hence semi–Fredholm.

The difference between the original operator and the Fourier model operator is compact. The proof of this fact is identical to the one given in Appendix B of [14], and we therefore omit the details. Since the semi–Fredholm property is stable under compact perturbations, it follows that $\mathcal{A}_{\kappa_1,\kappa_2}$ has closed range.

Applying the same argument to the family $(B_{\kappa,n})_{n\geq 1}$ shows that the block operator $(\mathcal{M}, \mathcal{B}_{\kappa_1,\kappa_2})$ also has closed range, and this proves that the differential S has closed range.

7.3   Trigonometric model

Using the sine representation (7.2), we introduce a Fourier–type model operator corresponding to $(\kappa_1, \kappa_2) = (0, 1)$:

$$\mathcal{A}_{0,1}^{(0)}(v_1) = \left((\widetilde{A}_{0,n}(v_1))_{n \geq 1}, (\widetilde{A}_{1,n}(v_1))_{n \geq 1} \right),$$

where

$$\widetilde{A}_{0,n}(v_1) = \frac{2}{\pi} \int_0^1 \sin(2n\pi x)\, v_1(x)\, dx\,,$$

$$\widetilde{A}_{1,n}(v_1) = -\frac{2}{\pi} \int_0^1 \sin\big((2n+1)\pi x\big)\, \big(S_{0,1}[v_1]\big)(x)\, dx\,.$$

Similarly, using the cosine representation (7.4), we define

$$\mathcal{B}_{0,1}^{(0)}(v_2) = \left((\widetilde{B}_{0,n}(v_2))_{n \geq 1}, (\widetilde{B}_{1,n}(v_2))_{n \geq 1} \right),$$

with

$$\widetilde{B}_{0,n}(v_2) = \frac{2}{\pi} \int_0^1 \cos(2n\pi x)\, v_2(x)\, dx\,,$$

$$\widetilde{B}_{1,n}(v_2) = \frac{2}{\pi} \int_0^1 \cos\big((2n+1)\pi x\big)\, \big(S_{0,1}[v_2]\big)(x)\, dx\,.$$

From now on, we focus on $\mathcal{A}_{0,1}^{(0)}$ and, for simplicity, we write v instead of v_1. Using the trigonometric form above, we consider

$$\|\mathcal{A}_{0,1}^{(0)}v\|_{\ell^2(\mathbb{N}^*) \times \ell^2(\mathbb{N}^*)}^2 = \sum_{n \geq 1} |\widetilde{A}_{0,n}(v)|^2 + \sum_{n \geq 1} |\widetilde{A}_{1,n}(v)|^2\,.$$

The first term corresponds to the classical sine Fourier coefficients

$$\widetilde{A}_{0,n}(v) = \frac{2}{\pi} \int_0^1 \sin(2n\pi x)\, v(x)\, dx\,.$$

Since $\sin(2n\pi x)$ is odd with respect to $x = \frac{1}{2}$, only the odd part of v contributes. By Parseval's identity,

$$\sum_{n \geq 1} |\widetilde{A}_{0,n}(v)|^2 = \sum_{n \geq 1} |\widetilde{A}_{0,n}(v_{\mathrm{odd}})|^2 = \frac{2}{\pi^2} \|v_{\mathrm{odd}}\|_{L^2(0,1)}^2\,,$$

where $v_{\mathrm{odd}}(x) := \frac{1}{2}\big(v(x) - v(1-x)\big)$ denotes the odd part of v with respect to $x = \frac{1}{2}$.

The second term involves the shifted sine basis and the transform $S_{0,1}$:

$$\widetilde{A}_{1,n}(v) = -\frac{2}{\pi} \int_0^1 \sin\big((2n+1)\pi x\big)\, (S_{0,1}[v])(x)\, dx\,.$$

Since $\sin((2n+1)\pi x)$ is even with respect to $x = \frac{1}{2}$, only the even part of $S_{0,1}[v]$ contributes. Including the mode $n = 0$, Parseval's identity yields

$$\sum_{n \geq 0} |\widetilde{A}_{1,n}(v)|^2 = \frac{2}{\pi^2} \|(S_{0,1}[v])_{\mathrm{even}}\|_{L^2(0,1)}^2 \,,$$

where

$$(S_{0,1}[v])_{\mathrm{even}}(x) := \tfrac{1}{2} \big((S_{0,1}[v])(x) + (S_{0,1}[v])(1-x) \big) \,.$$

If we start the sum at $n = 1$, we subtract the contribution of the mode $\sin(\pi x)$, namely

$$|\widetilde{A}_{1,0}(v)|^2 = \frac{2}{\pi^2} \big| \langle (S_{0,1}[v])_{\mathrm{even}}, \, \sin(\pi x) \rangle_{L^2(0,1)} \big|^2 \,.$$

Let P denote the orthogonal projection onto the subspace $L^2_{\mathrm{even}}(0,1)$ (with respect to $x = \frac{1}{2}$). Then

$$(S_{0,1}[v])_{\mathrm{even}} = P S_{0,1}[v] \,.$$

Therefore, starting the sum at $n = 1$, Parseval's identity yields

$$\sum_{n \geq 1} |\widetilde{A}_{1,n}(v)|^2 = \frac{2}{\pi^2} \|(S_{0,1}[v])_{\mathrm{even}}\|_{L^2(0,1)}^2 - \frac{2}{\pi^2} \big| \langle P S_{0,1}[v], \, \sin(\pi x) \rangle_{L^2(0,1)} \big|^2 \,.$$

We may rewrite the scalar product using the adjoint of $P S_{0,1}$:

$$\langle P S_{0,1}[v], \, \sin(\pi x) \rangle_{L^2(0,1)} = \langle v, \, (P S_{0,1})^* \sin(\pi x) \rangle_{L^2(0,1)} \,.$$

Hence

$$\sum_{n \geq 1} |\widetilde{A}_{1,n}(v)|^2 = \frac{2}{\pi^2} \|(S_{0,1}[v])_{\mathrm{even}}\|_{L^2(0,1)}^2 - \frac{2}{\pi^2} \big| \langle v, \, (P S_{0,1})^* \sin(\pi x) \rangle_{L^2(0,1)} \big|^2 \,.$$

Combining both contributions, we obtain

$$\|\mathcal{A}_{0,1}^{(0)} v\|_{\ell^2(\mathbb{N}^*) \times \ell^2(\mathbb{N}^*)}^2 = \frac{2}{\pi^2} \Big(\|v_{\mathrm{odd}}\|_{L^2(0,1)}^2 + \|(S_{0,1}[v])_{\mathrm{even}}\|_{L^2(0,1)}^2 - \big| \langle v, \, (P S_{0,1})^* \sin(\pi x) \rangle_{L^2(0,1)} \big|^2 \Big) \,.$$

In order to exploit this identity, we focus on the even component of $S_{0,1}[v]$ and introduce the corresponding projected transform.

We recall that the integral operator $S_{0,1}$ is defined by

$$(S_{0,1}[v])(x) := v(x) - 2 \int_x^1 \frac{v(t)}{t} \, dt \,, \qquad x \in (0,1) \,, \tag{7.5}$$

and we define

$$T := P \circ S_{0,1} : \; L^2_{\mathrm{even}}(0,1) \longrightarrow L^2_{\mathrm{even}}(0,1) \,.$$

Lemma 7.1 (A bounded left inverse for T)**.** *Define the operator* $L : L^2_{\mathrm{even}}(0,1) \to L^2_{\mathrm{even}}(0,1)$ *by*

$$(Lg)(x) = g(x) - \frac{1}{x(1-x)} \int_0^x (1 - 2t) \, g(t) \, dt, \qquad 0 < x < 1. \tag{7.6}$$

Then:

1. *L is a left inverse of $T := P \circ S_{0,1}$ on $L^2_{\text{even}}(0,1)$, i.e.*

$$LTv = v, \qquad \forall v \in L^2_{\text{even}}(0,1). \tag{7.7}$$

2. *The operator L is bounded on $L^2_{\text{even}}(0,1)$. More precisely,*

$$\|Lg\|_{L^2(0,1)} \le 5 \|g\|_{L^2(0,1)}, \qquad \forall g \in L^2_{\text{even}}(0,1).$$

Proof. Step 1: derivation of the formula. By density, we may assume without loss of generality that $v \in C^1([0,1])$. We recall that v is even and set $g := Tv = PS_{0,1}[v]$. Using the definition of $S_{0,1}$ together with the symmetry $v(1-x) = v(x)$, one obtains for $x \in (0,1)$

$$g(x) = v(x) - \int_x^1 \frac{v(t)}{t}\, dt - \int_0^x \frac{v(t)}{1-t}\, dt.$$

Differentiating gives

$$g'(x) = v'(x) + \frac{v(x)}{x} - \frac{v(x)}{1-x} = v'(x) + \frac{1-2x}{x(1-x)}\, v(x),$$

hence

$$x(1-x)v'(x) + (1-2x)v(x) = x(1-x)g'(x), \qquad \text{i.e.} \qquad \big(x(1-x)v(x)\big)' = x(1-x)g'(x).$$

Integrating from 0 to x and performing one integration by parts yields

$$x(1-x)v(x) = x(1-x)g(x) - \int_0^x (1-2t)\, g(t)\, dt,$$

that is,

$$v(x) = g(x) - \frac{1}{x(1-x)} \int_0^x (1-2t)\, g(t)\, dt, \qquad 0 < x < 1.$$

Since g is even with respect to $\frac{1}{2}$, the right-hand side is also even, hence v is even as well. This is precisely the formula defining the left inverse L.

Step 2: boundedness on $L^2_{\text{even}}(0,1)$. Since Lg is even with respect to $\frac{1}{2}$, it suffices to work on $(0, \frac{1}{2})$:

$$\|Lg\|^2_{L^2(0,1)} = 2\|Lg\|^2_{L^2(0,1/2)}.$$

For $0 < x \le \frac{1}{2}$,

$$(Lg)(x) = g(x) - \frac{1}{x(1-x)} \int_0^x (1-2t)\, g(t)\, dt.$$

Using $|1-2t| \le 1$ and $1-x \ge \frac{1}{2}$, we obtain

$$|(Lg)(x)| \le |g(x)| + \frac{2}{x} \int_0^x |g(t)|\, dt.$$

By Hardy's inequality on $(0, \frac{1}{2})$,

$$\int_0^{1/2} \left(\frac{1}{x} \int_0^x |g(t)|\, dt \right)^2 dx \le 4 \int_0^{1/2} |g(x)|^2 dx,$$

hence

$$\|Lg\|_{L^2(0,1/2)} \le \|g\|_{L^2(0,1/2)} + 4\|g\|_{L^2(0,1/2)} = 5\|g\|_{L^2(0,1/2)}.$$

Therefore $\|Lg\|_{L^2(0,1)} \le 5\|g\|_{L^2(0,1)}$, and L is bounded on $L^2_{\text{even}}(0,1)$. $\qquad \square$

We now show that the two nonnegative contributions in the right–hand side of the following identity already control the full L^2–norm of v:

$$\|\mathcal{A}_{0,1}^{(0)}v\|_{\ell^2(\mathbb{N}^*)\times\ell^2(\mathbb{N}^*)}^2 = \frac{2}{\pi^2}\left(\|v_{\text{odd}}\|_{L^2(0,1)}^2 + \|(S_{0,1}[v])_{\text{even}}\|_{L^2(0,1)}^2 - |\langle v, w\rangle_{L^2(0,1)}|^2\right),\qquad (7.8)$$

where

$$w := (PS_{0,1})^* \sin(\pi x) \in L^2(0,1)\,.$$

We first prove that

$$\|v\|_{L^2(0,1)}^2 \lesssim \|v_{\text{odd}}\|_{L^2(0,1)}^2 + \|(S_{0,1}[v])_{\text{even}}\|_{L^2(0,1)}^2\,.\qquad (7.9)$$

Since $v = v_{\text{odd}} + v_{\text{even}}$, we have

$$(S_{0,1}[v])_{\text{even}} = (P \circ S_{0,1})v = (P \circ S_{0,1})v_{\text{even}} + (P \circ S_{0,1})v_{\text{odd}}\,.$$

We recall that $T = P \circ S_{0,1}$ on $L_{\text{even}}^2(0,1)$. Then

$$Tv_{\text{even}} = (P \circ S_{0,1})v_{\text{even}} = (S_{0,1}[v])_{\text{even}} - (P \circ S_{0,1})v_{\text{odd}}\,.$$

Since L is a bounded left inverse of T on $L_{\text{even}}^2(0,1)$, we obtain

$$v_{\text{even}} = L\Big((S_{0,1}[v])_{\text{even}} - (P \circ S_{0,1})v_{\text{odd}}\Big)\,,$$

and therefore, since $P \circ S_{0,1}$ is bounded on $L^2(0,1)$,

$$\|v_{\text{even}}\|_{L^2(0,1)} \leq C\Big(\|(S_{0,1}[v])_{\text{even}}\|_{L^2(0,1)} + \|v_{\text{odd}}\|_{L^2(0,1)}\Big)\,.$$

Squaring and adding the odd part yields (7.9). Combining (7.8) and (7.9) yields

$$\|v\|_{L^2(0,1)}^2 \lesssim \|\mathcal{A}_{0,1}^{(0)}v\|_{\ell^2(\mathbb{N}^*)\times\ell^2(\mathbb{N}^*)}^2 + |\langle v, w\rangle_{L^2(0,1)}|^2\,.$$

Equivalently,

$$\|v\|_{L^2(0,1)} \lesssim \left\|\big(\mathcal{A}_{0,1}^{(0)}v,\ \langle v, w\rangle_{L^2(0,1)}\big)\right\|\,.$$

In particular, the augmented operator

$$\widetilde{\mathcal{A}} := \big(\mathcal{A}_{0,1}^{(0)},\ \langle \cdot, w\rangle_{L^2(0,1)}\big) :\ L^2(0,1) \to (\ell^2(\mathbb{N}^*) \times \ell^2(\mathbb{N}^*)) \times \mathbb{R}$$

is injective and has closed range. Since $\langle \cdot, w\rangle$ is a rank–one operator, $\widetilde{\mathcal{A}}$ is a finite–rank extension of $\mathcal{A}_{0,1}^{(0)}$. Hence $\mathcal{A}_{0,1}^{(0)}$ is semi–Fredholm: its kernel is at most one–dimensional and its range is closed. This follows, for instance, from [8, Proposition 11.4].

Now, we recall below the following local injectivity result, which is a direct consequence of the mean value theorem and the open mapping theorem (see, for instance, [1], Theorem 2.5.10).

Proposition 7.2 (Local injectivity). *Let X and Y be Banach spaces, and let*

$$\mathcal{S} : U \subset X \longrightarrow Y$$

be a $\mathcal{C}^1$ map defined on an open neighborhood U of a point $x_0 \in X$. Assume that the Fréchet differential $d_{x_0}\mathcal{S} : X \to Y$ is injective and has closed range. Then there exists a neighborhood $V \subset U$ of x_0 such that $\mathcal{S}$ is injective on V.

Theorem 1.1 is a direct consequence of Proposition 7.2, Theorems 6.1, 6.3 and 6.4, and the closedness of the range of S.

Remark 7.3 (Other effective angular momenta). *The case $(\kappa_1, \kappa_2) = (0, 2)$ is more delicate. Indeed, the asymptotics of the corresponding Bessel zeros do not produce the half–integer phase shift that yields the interlaced frequencies appearing in the case $(0, 1)$. As a consequence, the associated trigonometric system is no longer complete: one only obtains a partial family (either sine or cosine), rather than a full sine–cosine system. In particular, the argument based on the coercive identity for the trigonometric model cannot be applied directly, since the missing family prevents a direct control of the whole L^2–norm. A refined analysis is then required to recover closed range in this case.*

A   Physical interpretation of the model: from radial Dirac operators to AKNS systems

The AKNS system appears in many models in Physics. We have selected below two models where the results established in the main text are relevant. This also suggests the consideration of many other questions.

A.1   Dirac in 3D

Following [36] (see also [2] and [32]), we recall that the MIT realization of the Dirac operator on $L^2(\mathcal{B}, \mathbb{C}^4)$ ($\mathcal{B}$ is the unit ball of $\mathbb{R}^3$) with a radial matrix potential

$$V(x) := \phi_{el}(r)I_4 + \phi_{sc}(r)\beta + i\beta\alpha \cdot e_r \phi_{am}(r) \, , \tag{A.1a}$$

where

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} , \, \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} , \, \alpha = (\alpha_1, \alpha_2, \alpha_3) \, , \tag{A.1b}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{A.1c}$$

the σ_i are the Pauli matrices,

$$e_r := \mathbf{x}/r \, , \tag{A.1d}$$

and ϕ_{el}, ϕ_{sc} and ϕ_{am} are radial potentials with a physical interpretation.

Although the case ϕ_{el} is interesting (one can find in [36] the analysis of the Coulomb case), we are concerned in this article with the case when $\phi_{el} = 0$, and use in the main text the notation $\phi_{sc} = p$ and $\phi_{am} = q$. Notice that, when ϕ_{el} is not 0, it is known from [22] (see also the discussion in the introduction in [3]) that the inverse problem is ill posed for the AKNS system already when $\kappa = 0$. Theorem 4.14 in [36] states that the Dirac operator

$$\mathbb{D}_V := \mathbb{D}_0 + V \, , \tag{A.2a}$$

with (m being the mass)

$$\mathbb{D}_0 = \sum_i \alpha_i D_{x_i} + \beta m \tag{A.2b}$$

is unitary equivalent to the direct sum of the so-called "partial wave" Dirac operators h_{m_j, κ_j}

$$\bigoplus_{j=\frac{1}{2}, \frac{3}{2}, \cdots}^{+\infty} \quad \bigoplus_{m_j=-j}^{j} \quad \bigoplus_{\kappa_j = \pm(j+\frac{1}{2})} h_{m_j, \kappa_j}$$

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where, in the basis $\{\Phi^+_{m_j,\kappa_j}, \Phi^-_{m_j,\kappa_j}\}$ (see (4.110)-(4.116) in [36]), h_{m_j,κ_j} is the operator H_{κ_j} with a suitable boundary condition at $r=1$.

Notice that in this decomposition we only meet (up to unitary equivalence) the Dirac operators H_κ on $L^2(0,1)$ for $\kappa \in \mathbb{Z} \setminus \{0\}$. Here $\mathbb{Z} \setminus \{0\}$ is interpreted as the eigenvalues of some selfadjoint operator K on $L^2(S^2, \mathbb{C}^4)$, where S^2 is the two dimensional unit sphere in $\mathbb{R}^3$. We emphasize that κ is not the angular momentum as sometimes wrongly written (for example in [3]).

Notice also that in the Subsection 4.6.6 in [36] only the case in $(0,+\infty)$ is considered but this does not change the "tangential" decomposition of $L^2(S^2, \mathbb{C}^4)$.

Hence we have to analyze more carefully the possible boundary conditions by coming back to the problem for the unit ball in $\mathbb{R}^3$. According to [4], the generalized MIT condition in a domain Ω is given by

$$\varphi = \frac{i}{2}(\lambda_e - \lambda_s \beta)(\alpha \cdot \nu)\varphi \text{ on } \partial\Omega,$$

with

$$\lambda_e^2 - \lambda_s^2 = -4.$$

Notice that the standard MIT model corresponds with $\lambda_e = 0$, $\lambda_s = \pm 2$.

In the case of the ball and for the standard case, we get

$$\varphi = -i\beta(\alpha \cdot e_r)\varphi \text{ on } S^2.$$

Using Lemma 4.13 in [36], the operators β and $\alpha \cdot e_r$ respect the decomposition and, with respect to the basis $\{\Phi^+_{m_j,\kappa_j}, \Phi^-_{m_j,\kappa_j}\}$, are represented by the 2×2 matrices

$$\beta_{m_j,\kappa_j} = \sigma_3 \quad \text{and} \quad (-i\alpha \cdot e_r) = -i\sigma_2.$$

The boundary condition consequently reads

$$(f^+, f^-)^T = \sigma_1 (f^+, f^-)^T, \text{ for } r=1,$$

or

$$f^+(1) + f^-(1) = 0.$$

This corresponds in the AKNS notation to $\beta = \frac{\pi}{4}$.

Let us consider now the general MIT condition. We get

$$(f^+, f^-)^T = \frac{1}{2} \begin{pmatrix} 0 & \lambda_e - \lambda_s \\ \lambda_s + \lambda_e & 0 \end{pmatrix} (f^+, f^-)^T, \text{ for } r=1,$$

which reads

$$f^+(1) = \frac{1}{2}(\lambda_e - \lambda_s)f^-(1).$$

If we take λ_s and λ_e of opposite sign and take $\lambda_e \to +\infty$, we get at the limit

$$f^-(1) = 0,$$

which corresponds in the AKNS formalism to $\theta_2 = 0$. This limit is analyzed in [4] and this justifies to consider this limiting case also called Zig-Zag model.

More directly, this model is analyzed in [18] who refers to [30]. Other properties for the radial Dirac operator are considered in [5, 15].

A.2 Dirac in 2D with Aharonov-Bohm potential

It is natural to consider the same problem in dimension 2. Here we refer to another section in [36] or to [4]. Here we naturally get an AKNS family with $\kappa \in \frac{1}{2} + \mathbb{Z}$. In this case, the Dirac operator is a 2×2 system. The free Dirac operator reads

$$\mathbb{D}_0 = \sigma_1 D_{x_1} + \sigma_2 D_{x_2} \,,$$

and we can add a potential in the form

$$V = \begin{pmatrix} -q & p \\ p & q \end{pmatrix} \,,$$

as it appears in the AKNS system.

The description of the decomposition in the radial case is simpler than in the $3D$ case and we have just to consider the polar coordinates. This is precisely described in Thaller's book ([36], Subsection 7.3.3) but we have to explain two points which are not present there. First, since we are interested in the case of the disk, we have to describe what would be the boundary condition. This is for example discussed for general domains with C^2 boundary in [7] (see also references therein), the simplest conditions becoming simply (Zig-Zag model):

$$(\gamma v_1)_{|\partial\Omega} = 0 \,,$$

or

$$(\gamma v_2)_{|\partial\Omega} = 0 \,,$$

where γ denotes the trace operator.

In the reduction using the decomposition in [36] we get the boundary condition $Y_2(0) = 0$. Other conditions could be discussed. According to Lemma 2.3 in [7], the general condition reads

$$(\gamma v_2)_{|\partial\Omega} = \frac{1 - \sin\eta}{\cos\eta} t(s)(\gamma v_2)_{|\partial\Omega} \,.$$

(where, for $s \in \partial\Omega$, $t(s) := t_1(s) + it_2(s)$, $(t_1(s), t_2(s))$ is the tangent vector to $\partial\Omega$ at s, see p.2, line -3 in [7]). In Theorem 1.1 in [7], it is assumed that (see Remark 2) $\cos\eta \neq 0$ for having a regular self-adjoint problem with compact resolvent. Nevertheless in the Zig-Zag case, one can also define a natural selfadjoint extension. 0 seems to belong to the essential spectrum. The results are described in the recent paper [13] which refers to a paper by K. Schmidt [30]. The corresponding family of the AKNS operators is indexed by $\kappa = \pm(1/2, 3/2, \cdots,)$ with boundary condition at $r = 1$ given by $\theta_2 = 0$.

Unfortunately, we do not know how to treat this problem when the κ are not in $\mathbb{Z}$.

As already observed in [36], one can perform the same decomposition in the case when the magnetic potential $A_\phi(r)e_\phi$ (with $e_\phi = \frac{1}{r}(-x_2, x_1)$) corresponds to a radial magnetic field $B(r)$. The decomposition leads simply to replace in the definition of the AKNS system $\frac{d}{dr} - \frac{\kappa}{r}$ by $\frac{d}{dr} - \frac{\kappa}{r} + A_\phi(r)$ (see Formula (7.103) in [36]).

We want to consider $A_\phi(r) = \frac{\alpha}{r}$. The formal part of the decomposition still works but the regularity assumption done in [36] is not satisfied since the corresponding magnetic field is $2\pi\alpha\delta_0$ where δ_0 denotes the Dirac measure at the origin. As usual we can reduce the analysis to $\alpha \in [0, 1)$. The case $\alpha = 0$ being the previously discussed case without magnetic potential, it remains to consider $\alpha \in (0, 1)$. Hence we have to define the domain of this magnetic Dirac operator in this so called Aharonov-Bohm situation. This is fortunately discussed in the literature ([27, 35]). The authors classify in the case of $\mathbb{R}^2$ all the possible selfadjoint extensions of the minimal realization starting from $C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2)$. As described

in [35], we choose the condition corresponding to the parameter $\zeta = 0$ and (taking also account of the boundary condition, which is not present in Tamura's paper [35]) the domain is

$$D(\mathbb{D}_{\alpha,V}) := \left\{ u = (u_1, u_2) \in L^2(\Omega)^2, D_\alpha u \in L^2(\Omega)^2 \,,\, \lim_{|x| \to 0} |x|^{1-\alpha} e^{-i\theta} u_2(x) = 0 \,,\, (\gamma u_2)_{\partial\Omega} = 0 \right\}.$$

In the case of the unit disk $\Omega = B^1$, we get the AKNS system in $(0,1)$ with κ replaced by $\kappa_\alpha = \kappa + \alpha$. When $\alpha = \frac{1}{2}$ we get a sequence of integers in $\mathbb{Z}$ for which the analysis of the main text is relevant.

A.3 Open problems

Notice that more generally, it is interesting to consider the AKNS systems without to assume that κ is an integer and with any boundary condition at the origin (for the relevant κ) and at $r = 1$.

In view of the application to the two-dimensional Dirac operator, we note in particular that Theorem 5.2 remains valid even when the parameter κ_k is not assumed to be an integer. It could also be interesting to look at the case with a mass $m \neq 0$. At the level of the AKNS system this seems to correspond to the study of a model where the variation of ϕ_{el} is considered and the other potentials are 0. In this direction, we refer to [20], where an Ambarzumian-type theorem is established for Dirac operators. This result provides a uniqueness statement at the unperturbed point, showing that the vanishing of the potential ϕ_{el} is uniquely determined by the corresponding spectral data.

Finally, in light of [3], it is natural to investigate the corresponding Schrödinger problems with Robin boundary conditions. This stems from the structural link between Dirac and Schrödinger frameworks: in the Dirac setting introduced in [3], when the scalar potential $\phi_{sc} = 0$, the system reduces to a second-order Schrödinger (Bessel-type) equation, and the boundary conditions naturally translate into Robin-type conditions for the associated Schrödinger operator.

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