The anisotropic Calderón problem on 3-dimensional conformally Stäckel manifolds

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Abstract

Conformally Stäckel manifolds can be characterized as the class of n-dimensional pseudo-Riemannian manifolds (M, G) on which the Hamilton-Jacobi equation $G(\nabla u, \nabla u) = 0$ for null geodesics and the Laplace equation $-\Delta_G \psi = 0$ are solvable by R-separation of variables. In the particular case in which the metric has Riemannian signature, they provide explicit examples of metrics admitting a set of $n-1$ *commuting conformal symmetry operators* for the Laplace-Beltrami operator Δ_G. In this paper, we solve the anisotropic Calderón problem on compact 3-dimensional Riemannian manifolds with boundary which are conformally Stäckel, that is we show that the metric of such manifolds is uniquely determined by the Dirichlet-to-Neumann map measured on the boundary of the manifold, up to diffeomorphims that preserve the boundary.

Keywords. Inverse problem, anisotropic Calderon problem, conformally Stäckel manifolds, fixed energy R-separation, Weyl-Titchmarsh function, complex angular momentum.

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Contents

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1 Introduction

1.1 A prototype of an inverse problem: the anisotropic Calderón problem

The anisotropic Calderón problem, named after the seminal paper [10] by Alberto Pedro Calderón, addresses the question of determining the anisotropic conductivity of a body (*i.e.* a domain in $\mathbb{R}^n$) from current and voltage measurements made only *on the boundary* of the body, up to a change of coordinates fixing the boundary. It is well known (see [60]) that, in dimension three or higher, this problem can be reformulated in a geometric manner as the problem of determining the Riemannian metric of a compact connected Riemannian manifold with boundary from the Dirichlet-to-Neumann (DN) map (the quantities measured on the boundary of the manifold), up to diffeomorphisms fixing the boundary [1].

There has been an intense activity around the anisotropic Calderón problem in the last 30 years. In dimension 2, a complete positive answer was given in [60, 59] in the case of smooth metrics and in [1] for L^∞ metrics. In dimension 3 or higher, the anisotropic Calderón problem has been solved positively for *real-analytic metrics* in the serie of papers [60, 59, 58]. All these works use crucially the boundary determination results of [60] (see also [55] for a local version), that is the fact that the DN map determines uniquely the metric and all its derivatives (including the normal ones) on the boundary of the manifold. Then, the real-analyticity of the metric allows one to extend the boundary determination of the metric to the whole manifold, up to natural gauge invariances. We also refer to the recent paper [57] where a new proof of uniqueness in the Calderón problem for real-analytic metrics is given that uses a different approach. The anisotropic Calderón problem was also solved for Einstein metrics (which are real-analytic in the interior of the manifold) in [38].

[1] Note that in dimension 2, there exists another gauge invariance for this problem due to the covariance of the Laplace-Beltrami operator under conformal changes of the metric. The anisotropic Calderón problem amounts then to determining the Riemannian metric of a smooth compact connected Riemannian manifold with boundary from the DN map measured on the boundary of the manifold, up to diffeomorphisms fixing the boundary and/or a conformal change of the metric

The anisotropic Calderón problem for *smooth* metrics in dimension $n \geq 3$ remains however a major open problem even though some important results have been obtained in the last decade in [27, 28] for classes of smooth compact connected Riemannian manifolds with boundary that are *conformally transversally anisotropic* (CTA), meaning that

$$M \subset\subset \mathbb{R} \times M_0, \quad G = c(e \oplus g_0),$$

where (M_0, g_0) is a $n-1$ dimensional smooth compact connected Riemannian manifold with boundary, e is the Euclidean metric on the real line and c is a smooth strictly positive function in the cylinder $\mathbb{R} \times M_0$. It has been shown in [27], Theorem 1.2 and Lemma 2.9, that CTA manifolds are characterized by the existence of a limiting Carleman weight [2] or equivalently, by the existence of a nontrivial conformal Killing vector field. The existence of a limiting Carleman weight can then be used to construct complex geometrical optics (CGO) solutions on the manifolds. Under some additional conditions on the transversal part (M_0, g_0) of the cylinder (simplicity in [27] and injectivity of the geodesic ray-transform in [28]), CGO's technique can then be used to prove that the conformal factor c is uniquely determined from the knowledge of the DN map. We refer to [39, 56, 70, 76, 77] for surveys on the use of CGO's technique in the anisotropic Calderón problem. Finally, we mention the recent paper [29] that provides a positive solution to the linearized Calderón problem on CTA manifolds under less restrictive conditions on the transversal part (M_0, g_0).

Even though uniqueness is expected in the *global* Calderón problem on *smooth compact* connected Riemannian manifolds, some counterexamples to uniqueness are known in several cases where the above hypotheses fail. For instance, it is shown in [58] that there exists a pair of compact and complete non-compact 2-dimensional manifolds with boundary having the same DN map. This counterexample was obtained using a blow-up map. Some analogous non-uniqueness results for highly singular metrics on a compact manifold have been obtained in the study of invisibility phenomena in [37, 34, 35, 36, 26, 21]. The idea behind these cloaking devices is to hide an arbitrary object from measurements by coating it with a meta-material that corresponds to a *degenerate* Riemannian metric. Counterexamples to uniqueness in the case of *Hölder continuous* metrics for a *local* (meaning that the DN map is measured on a proper open subset of the boundary) Calderón problem has been obtained recently in [16] while several types of non-uniqueness results have been obtained in the Calderón problem with *disjoint Dirichlet and Neumann data* in [15, 17, 20].

As conveyed by the title of this section, the anisotropic Calderón problem is the prototype of an inverse problem that shares many similarities with other important inverse problems; for instance, inverse scattering experiments on some non-compact Riemannian manifolds having ends where the question amounts to determining the metric from the knowledge of the scattering matrix at a fixed energy, the scattering matrix being understood as a kind of DN map measured at infinity (*i.e.* the ends). Particularly close to the anisotropic Calderón problem on a compact manifold with boundary are the inverse scattering problems at fixed energy on *asymptotically*

[2]In fact, the precise result states that if an open manifold (M, G) admits a limiting Carleman weight, then it must be locally a CTA manifold.

hyperbolic manifolds, a class of manifolds in which the usual boundary is replaced by a conformal boundary of hyperbolic type. We refer to [43, 44, 45, 46] and especially the book [47], Section 5.2. for the link between inverse boundary value problems and inverse scattering on asymptotically hyperbolic manifolds. Though a complete answer of the inverse scattering problem at a fixed energy is not known in that setting, some interesting and important results in that direction have been proved in [49, 69, 47, 48, 40, 41] for some general asymptotically hyperbolic manifolds. However, a complete positive answer was found in [22, 24, 14, 19, 33] for some Riemannian asymptotically hyperbolic manifolds or de-Sitter like black holes spacetimes having a sufficient amount of (hidden) symmetries.

In this paper, we wish to continue the series of works [22, 24, 14, 19, 33] by studying the anisotropic Calderón problem on *conformally Stäckel manifolds*, a class of n-dimensional completely integrable Riemannian manifolds with the property that the Laplace-Beltrami operator possess $n-1$ commuting second order conformal symmetry operators that allows to solve the corresponding Laplace PDE by separation of variables. Important to say in this introduction are the facts that :

- This class of n-dimensional conformally Stäckel manifolds is rather large since we will show below that it depends locally on n^2 functions of one variable and a function of $n-1$ variables, precisely a function $\eta \in C^\infty(\partial M)$.
- Moreover, it does not belong to the class of CTA manifolds that are characterized by the existence of a conformal Killing vector field. Instead, the class of conformally Stäckel manifolds are "almost" characterized by the existence of $n-1$ independent *conformal Killing tensors of rank 2* [3].
- Since conformally Stäckel manifolds are not CTA manifolds, we stress the fact that our proof of uniqueness in the Calderón problem does not use the usual CGO techniques but relies rather a mix of boundary determination results and what we would like to call the multi-parameter complex angular momentum (CAM) method, which is made possible thanks to the (hidden) symmetries of the manifolds under consideration.

In order to keep the presentation of the model and of the ideas of the proof transparent, and to keep the notation as light as possible, we will now introduce conformally Stäckel manifolds and solve the corresponding anisotropic Calderón problems in dimension three. Except for additional notational complexity, the extension to higher dimensions is in every regard identical.

1.2 The model of conformally Stäckel manifolds

We follow the presentation of conformally Stäckel manifolds given in [11] (see also [2, 3, 4]) and also refer to [30, 50, 51, 52, 53, 63, 68, 71] for other classical references in the variable separation

[3] A symmetric contravariant two-tensor $K = (K^{ij})$ is a conformal Killing tensor for the contravariant metric $G = (G^{ij})$ if there exists a vector field $X = (X^i)$ such that $[G, K] = 2X \odot G$ where we denote by $[.,.]$ the Lie-Schouten bracket on contravariant symmetric tensors and by $\odot$ the symmetric tensor product.

theory. Even though the description of these models is only local in nature, we make it global by considering Ω to be a smooth compact connected three-dimensional manifold with smooth boundary having the global topology of a toric cylinder,

$$\Omega = [0, A] \times \mathbb{T}^2 .$$

Let us denote by (x^1, x^2, x^3) a global coordinate system on Ω and note that the boundary $\partial\Omega$ of Ω has two connected components given by

$$\partial\Omega = \Omega_0 \cup \Omega_1 , \quad \Omega_0 = \{0\} \times \mathbb{T}^2 , \quad \Omega_1 = \{A\} \times \mathbb{T}^2 .$$

We equip the manifold Ω with a smooth Riemannian metric G of the form

$$G = c^4 g = \sum_{i=1}^{3} H_i^2 (dx^i)^2 . \tag{1.1}$$

In the above expression, the Riemannian metric g is a Stäckel metric, that is

$$g = \sum_{i=1}^{3} h_i^2 (dx^i)^2 , \quad h_i^2 = \frac{\det S}{s^{i1}} , \tag{1.2}$$

with S being a Stäckel matrix, that is a non-singular matrix of the form

$$S = \begin{pmatrix} s_{11}(x^1) & s_{12}(x^1) & s_{13}(x^1) \\ s_{21}(x^2) & s_{22}(x^2) & s_{23}(x^2) \\ s_{31}(x^3) & s_{32}(x^3) & s_{33}(x^3) \end{pmatrix} , \tag{1.3}$$

and s^{ij} denotes the cofactor of the component s_{ij} of the matrix S. Observe that the diagonal components h_i^2 of the Stäckel metric are given by the entries of the first column of the inverse Stäckel matrix $A = S^{-1}$. Of course we demand the diagonal coefficients h_i^2 of the metric g to be positive.

Furthermore, the conformal factor c^4 is assumed to be a positive solution of the linear elliptic PDE on Ω given by:

$$-\Delta_g c - \sum_{i=1}^{3} h_i^2 \big(\phi_i + \frac{1}{4}\gamma_i^2 - \frac{1}{2}\partial_i \gamma_i\big) c = 0 . \tag{1.4}$$

where Δ_g denotes the Laplace-Beltrami operator associated to the Stäckel metric g on Ω, given in local coordinates by

$$-\Delta_g = -\frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{3} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j \right) ,$$

and

$$\gamma_i := -\partial_i \log \frac{h_1 h_2 h_3}{h_i^2} ,$$

are the contracted Christoffel symbols associated to g and $\phi_i = \phi_i(x^i)$ are arbitrary smooth functions of the indicated variable.

We shall review in Section 2.1 the theory of variable separation on conformally Stäckel manifolds for the Hamilton-Jacobi and Laplace equations, and recall some intrinsic characterizations of such manifolds in terms of the existence of conformal Killing tensors having certain properties. However, let us emphasize here the remarkable fact that all solutions of the Laplace equation

$$-\Delta_G \psi = 0, \quad \text{on } \Omega, \tag{1.5}$$

can be written as an infinite (countable) linear superposition of functions of the form

$$\psi = R(x^1, x^2, x^3)u, \quad u = u_1(x^1)u_2(x^2)u_3(x^3),$$

for a well chosen factor R. This will be crucial in the later analysis. More precisely, we will show that any solution ψ of (1.5) can be written as

$$\psi = R(x^1, x^2, x^3) \sum_{m=1}^{\infty} u_m(x^1) Y_m(x^2, x^3), \quad Y_m(x^2, x^3) = v_m(x^2) w_m(x^3), \tag{1.6}$$

such that, for a convenient choice of factor R, each u_m, v_m, w_m satisfies the coupled separated ODEs :

$$-u_m'' + [\,\mu_m^2 s_{12}(x^1) + \nu_m^2 s_{13}(x^1) - \phi_1(x^1)]\, u_m = 0, \tag{1.7}$$

$$-v_m'' + [\,\mu_m^2 s_{22}(x^2) + \nu_m^2 s_{23}(x^2) - \phi_2(x^2)]\, v_m = 0, \tag{1.8}$$

$$-w_m'' + [\,\mu_m^2 s_{32}(x^3) + \nu_m^2 s_{33}(x^3) - \phi_3(x^3)]\, w_m = 0. \tag{1.9}$$

Here the constants of separation (μ_m^2, ν_m^2) can be understood as the joint spectrum of the commuting elliptic selfadjoint operators (H, L) on $\mathbb{T}^2$ defined by :

$$\begin{pmatrix} H \\ L \end{pmatrix} = \frac{1}{s^{11}} \begin{pmatrix} -s_{33} & s_{23} \\ s_{32} & -s_{22} \end{pmatrix} \begin{pmatrix} A_2 \\ A_3 \end{pmatrix}, \tag{1.10}$$

where for all $j = 1, 2, 3$, we set :

$$A_j = -\partial_j^2 - \phi_j(x^j). \tag{1.11}$$

The common eigenfunctions of (H, L) take the form $Y_m = v_m(x^2) w_m(x^3)$ and satisfy (by definition) :

$$HY_m = \mu_m^2 Y_m, \quad LY_m = \nu_m^2 Y_m, \quad \forall m \geq 1. \tag{1.12}$$

Finally, the eigenfunctions Y_m form a Hilbert basis of $L^2(\mathbb{T}^2)$ in the following sense :

$$L^2(\mathbb{T}^2\,; s^{11} dx^2 dx^3) = \bigoplus_{m \geq 1} \langle Y_m \rangle. \tag{1.13}$$

The proof of the above statement will be given at the beginning of Section 2.2. As a consequence, we will be able to show that the DN map possesses a very special structure. Precisely, we will show that the DN map can be "almost" diagonalized onto the Hilbert basis $(Y_m)_{m\geq 1}$ as follows. First recall that the boundary of the cylinder Ω has two connected components Ω_0 and Ω_1 both isomorphic to $\mathbb{T}^2$. We thus identify the Sobolev spaces $H^s(\partial\Omega),\ s\in\mathbb{R}$, with

$$H^s(\partial\Omega) = H^s(\Omega_0)\oplus H^s(\Omega_1), \quad H^s(\Omega_j)\simeq H^s(\mathbb{T}^2),\ j=0,1,$$

and use a 2×2-matrix notation for the DN map $\Lambda_G : H^{\frac{1}{2}}(\partial\Omega)\longrightarrow H^{-\frac{1}{2}}(\partial\Omega)$, *i.e.*

$$\Lambda_G = \begin{pmatrix} \Lambda_{G,\Omega_0,\Omega_0} & \Lambda_{G,\Omega_0,\Omega_1} \\ \Lambda_{G,\Omega_1,\Omega_0} & \Lambda_{G,\Omega_1,\Omega_1} \end{pmatrix}.$$

Here the operators $\Lambda_{G,\Omega_i,\Omega_j} : H^{\frac{1}{2}}(\mathbb{T}^2)\longrightarrow H^{-\frac{1}{2}}(\mathbb{T}^2)$ correspond to the DN map when the Dirichlet data are imposed on Ω_i and the Neumann data are measured on Ω_j. From (1.13), we see that for $s\geq 0$, any element of the Sobolev spaces $H^s(\Omega_j),\ j=0,1$, can be decomposed onto the Hilbert basis $(Y_m)_{m\geq 1}$. With these notations, we will show that the DN map on a conformally Stäckel cylinder has the following structure :

$$\Lambda_G = \begin{pmatrix} \frac{-1}{H_1(0,x^2,x^3)} & 0 \\ 0 & \frac{1}{H_1(A,x^2,x^3)} \end{pmatrix} \Bigg[\begin{pmatrix} \frac{\Gamma_1(0,x^2,x^3)}{2} & 0 \\ 0 & \frac{\Gamma_1(A,x^2,x^3)}{2} \end{pmatrix} \tag{1.14}$$

$$+ \begin{pmatrix} R(0,x^2,x^3) & 0 \\ 0 & R(A,x^2,x^3) \end{pmatrix} A_G \begin{pmatrix} \frac{1}{R(0,x^2,x^3)} & 0 \\ 0 & \frac{1}{R(A,x^2,x^3)} \end{pmatrix} \Bigg]$$

where

$$\Gamma_i := -\partial_i \log \frac{H_1H_2H_3}{H_i^2}, \quad i=1,2,3,$$

are the contracted Christoffel symbols associated to the conformally Stäckel metric G and where the operator A_G is completely diagonalizable onto the Hilbert basis $(Y_m)_{m\geq 1}$, its restriction on $\langle Y_m\rangle$ being defined by :

$$(A_G)_{|\langle Y_m\rangle} := \begin{pmatrix} M(\mu_m^2,\nu_m^2) & \frac{1}{\Delta(\mu_m^2,\nu_m^2)} \\ \frac{1}{\Delta(\mu_m^2,\nu_m^2)} & N(\mu_m^2,\nu_m^2) \end{pmatrix}. \tag{1.15}$$

Finally, the function $\Delta(\mu^2,\nu^2)$ and the functions $M(\mu^2,\nu^2)$ and $N(\mu^2,\nu^2)$ are the (not so classical) *characteristic* and *Weyl-Titchmarsh* (WT) functions associated to the radial ODE (1.7) with Dirichlet boundary conditions. We emphasize the worlds *not so classical* since the characteristic and WT functions depend on the *two* spectral parameters μ^2,ν^2 which appear as the constants of separation in the variable separation procedure.

The construction and the explanation of this special structure of the DN map as well as a review of the elementary properties of the characteristic and WT functions will be done in Section 2.2.

For the moment, since we are interested in the anisotropic Calderón problem on conformally Stäckel manifolds, let us simply (and formally) count the number of unknown functions defining them. A priori, a Stäckel metric g depends on nine functions $s_{ij}(x^i)$ of one variable while the conformal factor c depends on three additional unknown functions $\phi(x^i)$ through the Laplace type pde (1.4). Let us choose (this is always possible) the functions ϕ_i in such a way that the zero-order term be nonnegative, *i.e.*

$$-\sum_{i=1}^{3} h_i^2\big(\phi_i + \frac{1}{4}\Gamma_i^2 - \frac{1}{2}\partial_i\Gamma_i\big) \geq 0, \tag{1.16}$$

and solve the Dirichlet problem

$$\left\{ \begin{array}{rcl} -\Delta_g c - \sum_{i=1}^3 h_i^2\big(\phi_i + \frac{1}{4}\Gamma_i^2 - \frac{1}{2}\partial_i\Gamma_i\big)c & = & 0 \quad \text{on} \quad \Omega, \\ c & = & \eta, \quad \text{on} \quad \partial\Omega. \end{array} \right. \tag{1.17}$$

According to the maximum principle [31, 74], for any positive boundary function η on $\partial\Omega$, there exists a unique positive solution c of (1.17). We conclude that the conformal factor c depends roughly speaking on three unknown functions $\phi_i(x^i)$ of one variable (that satisfiy (1.16)) and a positive function $\eta \in C^\infty(\partial\Omega)$.

This makes twelve unknown functions of one variable and one unknown function of two variables for the metric g. Nevertheless, it is possible to remove three of the unknown functions of one variable by a simple change of coordinates that preserves the conformally Stäckel structure. Indeed, given positive functions $f_i(x^i)$, define the new variables y^i by :

$$y^i = \int_0^{x^i} \sqrt{f_i(s)}ds. \tag{1.18}$$

Then the new metric $\bar{G}$ is given

$$\bar{G} = \bar{c}^4\bar{g}, \tag{1.19}$$

where $\bar{g}$ is the metric

$$\bar{g} = \sum_{i=1}^{3} \bar{h}_i^2(dy^i)^2\,, \quad \bar{h}_i^2 = \frac{h_i^2(x^i(y^i))}{f_i(x^i(y^i))}, \tag{1.20}$$

which can be shown to be a Stäckel metric

$$\bar{g} = \sum_{i=1}^{3} \bar{h}_i^2(dy^i)^2\,, \quad \bar{h}_i^2 = \frac{\det \bar{S}}{\bar{s}^{i1}}\,, \tag{1.21}$$

associated to the new Stäckel matrix

$$\bar{S} = \big(\bar{s}_{ij}(y^i)\big)_{1\leq i,j\leq 3} := \left(\frac{s_{ij}(x^i(y^i))}{f_i(x^i(y^i))}\right)_{1\leq i,j\leq 3}. \tag{1.22}$$

In other words, the change of variables (1.18) amounts to dividing each line of the initial Stäckel matrix by the functions f_i, a step which allows us to remove one unknown function in each variable x^i. Note finally that the conformal factor $\bar{c}$ now satisfies :

$$-\Delta_{\bar{g}}\bar{c} - \sum_{i=1}^{3} \bar{h}_i^2\big(\bar{\phi}_i + \frac{1}{4}\bar{\gamma}_i^2 - \frac{1}{2}\partial_i\bar{\gamma}_i\big)\bar{c} = 0\,,$$

where the arbitrary functions $\bar{\phi}_i = \bar{\phi}_i(y^i)$ are given by

$$\bar{\phi}_i = \frac{\phi_i}{f_i} - \frac{(\log \dot{f_i})^2}{16} - \frac{(\log \ddot{f_i})}{4}, \tag{1.23}$$

(here the dot denotes the derivative with respect to y^i) and the $\bar{\gamma}_i$ are the contracted Christoffel symbols associated to the metric $\bar{g}$. In conclusion, we deduce that a conformally Stäckel metric effectively depends on $9 = 3^2$ unknown functions of one variable and one positive function $\eta \in C^\infty(\partial\Omega)$ of 2 variables.

1.3 The results

We will study the anisotropic Calderón problem in the class of smooth compact connected Riemannian manifolds with boundary (M, G) that are embedded in a conformally Stäckel cylinder Ω, *i.e.* we will consider (M, G) where

$$M \subset\subset \Omega = [0, A] \times \mathbb{T}^2, \tag{1.24}$$

and G is a Riemannian metric on M that possesses a smooth extension (still denoted by G) to the whole cylinder Ω given by (1.1) - (1.3).

Let us consider the corresponding Dirichlet problem

$$\left\{ \begin{array}{ll} -\Delta_G\, u = 0, & \text{on } M, \\ u = f, & \text{on } \partial M. \end{array} \right. \tag{1.25}$$

It is well known [70] that, for any $f \in H^{1/2}(\partial M)$, there exists a unique weak solution $u \in H^1(M)$ of the Dirichlet problem (1.25). So, we can define the Dirichlet-to-Neumann map as the operator Λ_G from $H^{1/2}(\partial M)$ to $H^{-1/2}(\partial M)$ given by

$$\Lambda_G(f) = (\partial_\nu u)_{|\partial M}\,. \tag{1.26}$$

Here, u is the unique solution of (1.25) and $(\partial_\nu u)_{|\partial M}$ is its normal derivative with respect to the unit outer normal ν on ∂M. Note that this normal derivative has to be understood in the weak sense as an element of $H^{-1/2}(\partial M)$ via the bilinear form

$$\langle \Lambda_G(f)\,, h\rangle = \int_M \langle du, dv\rangle_G \;\mathrm{dVol}_G,$$

where $f, h \in H^{1/2}(\partial M)$, u is the unique solution of the Dirichlet problem (1.25), and where v is any element of $H^1(M)$ such that $v_{|\partial M} = h$. Of course, when f is sufficiently smooth, this definition coincides with the usual one in local coordinates, that is

$$\partial_\nu u = \sum_i \nu^i \partial_i u. \tag{1.27}$$

Finally, we will use the notations $\Lambda_G = \Lambda_{G,M}$ and $\Lambda_{G,\Omega}$ to distinguish between the DN map measured on M and Ω respectively.

Let us now formulate our main result.

Theorem 1.1. *Let (M,G) and $(M,\tilde{G})$ be two conformally Stäckel manifolds satisfying (1.1) - (1.3) and (1.24). We will add a subscript $\tilde{}$ to all the quantities related to $(M,\tilde{G})$. Assume that*

$$\Lambda_G = \Lambda_{\tilde{G}}.$$

Then there exists a diffeomorphism $\varphi : M \longrightarrow M$ with $\varphi_{|\partial M} = Id$ whose pull-back satisfies

$$\tilde{G} = \varphi^* G,$$

Let us make some comments on this result.

1. The diffeomorphim φ appearing in the statement of the main Theorem is simply a change of variables of the special form (1.18) that preserves the structure of conformally Stäckel manifolds.

2. Theorem 1.1 solves positively the uniqueness issue in the Calderón problem on conformally Stäckel manifolds. It is an extension of the results in [33] where the inverse scattering problem at a fixed energy on Stäckel asymptotically hyperbolic manifolds was considered. One of the main differences between the model in [33] and our model is the fact that in [33] the conformal factor c is assumed to be identically 1 and the PDE (1.4) on the conformal factor c is then replaced by the so called Robertson conditions $\partial_j \gamma_i = 0, \ \forall 1 \leq i \neq j \leq 3$ (see [68]). Note that under these hypotheses, the PDE (1.4) is trivially satisfied. The Robertson conditions restrict the class of Stäckel metrics drastically and this is the sense in which our result extends [33]. We refer to [33], Example 1.2, for a list of examples of Stäckel metrics satisfying the Robertson conditions.

3. As already mentioned, conformally Stäckel manifolds aren't generically CTA manifolds. It would be the case however if one of the line in the Stäckel matrix S was a line of constant functions. Assume for instance that the s_{1j} are constants for all $j = 1,2,3$. Then ∂_{x^1} is a Killing vector field for the Stäckel metric g and thus a conformal Killing vector field for the metric G. Hence (M,G) would lie within the class of CTA manifolds. This is clear if we notice that the metric G can then be written as:

$$G = \left(c^4 \frac{\det(S)}{s^{11}} \right) \left[(dx^1)^2 + g_0 \right], \quad g_0 = \frac{s^{11}}{s^{21}} (dx^2)^2 + \frac{s^{11}}{s^{31}} (dx^3)^2.$$

Even in that case however, we could not apply directly the results of [27, 28] since the injectivity of the geodesic X-ray transform on the *closed* transversal manifold

$$(M_0, g_0) = (\mathbb{T}^2, g_0),$$

is not guaranteed in general.

The proof of Theorem 1.1 will be divided in four steps.

Step 1. Extension to the whole cylinder Ω. Note first that the Laplace equation $-\Delta_G \psi = 0$ on M is usually not separable since the boundary ∂M need not be compatible with variable separation, unlike the case on the whole cylinder (Ω, G). Hence we cannot use a priori the form (1.6) for the solutions of the Laplace equation as well as the structure (1.14) of the DN map. However we can reduce the Calderón problem on (M, G) to the Calderón problem on the extended cylinder (Ω, G) by the following result which is similar to the corresponding results on asymptotically hyperbolic manifolds from [47], chapter 5, Theorems 2.3 and 4.6.

Theorem 1.2. *Let $M_1 \subset\subset M_2$ be two smooth compact connected manifolds with boundary. Let G and $\tilde{G}$ be two Riemannian metrics on M_2 such that $G = \tilde{G}$ on $M_2 \setminus M_1$. Denote by $\Lambda_{G,j}$ the DN map associated to G on M_j for $j = 1, 2$. Then*

$$\Lambda_{G,1} = \Lambda_{\tilde{G},1} \implies \Lambda_{G,2} = \Lambda_{\tilde{G},2}.$$

Together with the well-known boundary determination results from [55, 60] or [27], Section 8, we will deduce from Theorem 1.2 :

Proposition 1.1. *Assume that the hypotheses of Theorem 1.2 hold. Then*

$$\Lambda_{G,M} = \Lambda_{\tilde{G},M} \implies \Lambda_{G,\Omega} = \Lambda_{\tilde{G},\Omega},$$

where the extended metrics G and $\tilde{G}$ on the whole cylinder Ω are conformally Stäckel metrics that can be chosen so as to satisfy $G = \tilde{G}$ on $\Omega \setminus M$ and the generic condition :

$$\begin{pmatrix} -s_{13}(0) \\ s_{12}(0) \end{pmatrix}, \begin{pmatrix} -s_{13}(A) \\ s_{12}(A) \end{pmatrix} \text{ are linearly independent.} \tag{1.28}$$

In conclusion, it will be enough to prove uniqueness in the Calderón problem on conformally Stäckel cylinders (Ω, G) for which we can use separation of variables. The proof of Theorem 1.2 and Proposition 1.1 will be given in Section 3.1.

Step 2. Boundary determination. After reducing the Calderón problem to the whole conformally Stäckel cylinders Ω satisfying the conclusions of Proposition 1.1, we use the standard boundary determination results [4] from [55, 60] and the particular structure of the metrics G and $\tilde{G}$ given by (1.1) - (1.3) to prove in a successive series of steps that first (from the equality of the metrics on the boundary)

$$\begin{pmatrix} s_{22} & s_{23} \\ s_{32} & s_{33} \end{pmatrix} = \begin{pmatrix} \tilde{s}_{22} & \tilde{s}_{23} \\ \tilde{s}_{32} & \tilde{s}_{33} \end{pmatrix}, \tag{1.29}$$

as functions of x^2, x^3 and

$$\left\{ \begin{array}{r} (c^4 \det S)(x^1, x^2, x^3) = (\tilde{c}^4 \det \tilde{S})(x^1, x^2, x^3), \\ R(x^1, x^2, x^3) = \tilde{R}(x^1, x^2, x^3), \\ H_1(x^1, x^2, x^3) = \tilde{H}_1(x^1, x^2, x^3), \end{array} \right. \quad x^1 = 0, A, \ \forall x^2, x^3. \tag{1.30}$$

[4]Precisely we use the fact $\Lambda_{G,\Omega} = \Lambda_{\tilde{G},\Omega}$ imply the equality of $G_{|\partial\Omega}$ and $\tilde{G}_{|\partial\Omega}$ as well as the equality between the normal derivatives $(\partial_\nu G)_{|\partial\Omega}$ and $(\partial_{\tilde{\nu}} \tilde{G})_{|\partial\Omega}$ on the boundary $\partial\Omega$.

and second (from the equality of the normal derivatives of the metrics on the boundary)

$$\begin{cases} (\partial_1 \log c^4 \ \det S)(x^1, x^2, x^3) = (\partial_1 \log \tilde{c}^4 \ \det \tilde{S})(x^1, x^2, x^3), \\ \Gamma_1(x^1, x^2, x^3) = \tilde{\Gamma}_1(x^1, x^2, x^3), \end{cases} \quad x^1 = 0, A, \ \forall x^2, x^3. \tag{1.31}$$

where we recall that

$$\Gamma_i := -\partial_i \log \frac{H_1 H_2 H_3}{H_i^2}, \quad i = 1, 2, 3.$$

Then, using the special structure (1.14) of the DN map, we will infer from (1.29) - (1.31) that

$$A_G = A_{\tilde{G}}, \tag{1.32}$$

where the operator A_G is defined in (1.15). From this and some additional work, we will be able to show the equality of the eigenfunctions Y_m

$$Y_m = \tilde{Y}_m, \ \forall m, \tag{1.33}$$

the equality of the joint spectra

$$(\mu_m^2, \nu_m^2) = (\tilde{\mu}_m^2, \tilde{\nu}_m^2), \quad \forall m, \tag{1.34}$$

and the equality between the ϕ_2 and ϕ_3

$$\phi_2 = \tilde{\phi}_2, \quad \phi_3 = \tilde{\phi}_3. \tag{1.35}$$

Hence at the end of the second step, we will have recovered most of the unknown functions of one variable depending on one of the angular variables x^2, x^3, and in fact all of them if we keep in mind the possibility of removing some of these unknown functions thanks to the change of variables (1.18).

Note finally that the above results clearly depend on the particular structure of conformally Stäckel metrics on the cylinder Ω but also on a clear understanding of the different invariances of the Stäckel metrics with respect to different but equivalent choices of the associated Stäckel matrices. These invariances will be explained in Section 2.1 whereas the boundary determination results and their consequences will be given in Section 3.2.

Step 3. The multi-parameter CAM method. At this stage, it remains essentially to determine the unknown functions depending on the radial variable x^1 and the conformal factor c. To determine the former, we start from the equality

$$M(\mu_m^2, \nu_m^2) = \tilde{M}(\mu_m^2, \nu_m^2), \quad \forall m, \tag{1.36}$$

which is a consequence of (1.32) and (1.34). Recall that the WT function M only depends on the radial ODE (1.7) and contains all the information on the functions s_{12}, s_{13}, ϕ_1 through the well known Borg-Marchenko Theorem [5, 8, 9, 32]. Our first task is thus to extend the equality (1.36) which is initially true on the joint spectrum $J = \{(\mu_m^2, \nu_m^2), \ m \geq 1\}$ to the whole plane $\mathbb{C}^2$, that is we aim to complexify the angular momenta as it was done for the first time by Regge in [67] and applied to solve some inverse problems in [25, 22, 23, 24, 14, 19, 20, 21, 15, 17, 33, 62, 64, 66] and references therein. For this, we use a multi-parameter CAM method as in [33] which allows us to prove that

$$M(\mu^2, \nu^2) = \tilde{M}(\mu^2, \nu^2), \quad \forall \mu, \nu \in \mathbb{C} \backslash \{poles\} \tag{1.37}$$

Once it is done, an application of Borg-Marchenko Theorem leads to

$$\phi_1 = \tilde{\phi}_1, \quad s_{12} = \tilde{s}_{12}, \quad s_{13} = \tilde{s}_{13}, \tag{1.38}$$

up to a change of x^1-variable of the type (1.18). We would like to emphasize that the multi-parameter CAM method that permits to infer (1.37) from (1.36) is far from being as simple as in the case of a single angular momentum. Indeed, the method lies within the realm of functions of several complex variables and not one complex variable. Moreover, a good understanding of the joint spectrum J is needed. We follow here the corresponding results obtained by Gobin [33], which should be useful in other contexts as well. The results on the CAM method will be given in Section 3.3.

Step 4. A unique continuation argument for the conformal factor. We finish the proof of our main Theorem by remarking first that the metric G can be written as

$$G = \alpha\, g_0, \quad \alpha = c^4 \det S, \quad g_0 = \frac{1}{s^{11}}(dx^1)^2 + \frac{1}{s^{21}}(dx^2)^2 + \frac{1}{s^{31}}(dx^3)^2,$$

Note from the results of Steps 1 to 3 that we have

$$g_0 = \tilde{g}_0, \tag{1.39}$$

up to a change of coordinates (1.18). Thus it only remains to prove that $\alpha = \tilde{\alpha}$. The second crucial remark consists in using (1.4) to show that the conformal factor α satisfies the elliptic PDE

$$-\Delta_{g_0}\alpha - Q_{g_0,\phi_i}\alpha = 0, \tag{1.40}$$

where

$$Q_{g_0,\phi_i} = \sum_{i=1}^{3} g_0^{ii} \left[\frac{\partial_{ii}^2 \log\det g_0}{4} + \frac{\partial_i \log\det g_0}{8} + \frac{(\partial_i \log\det g_0)^2}{16} + \phi_i \right]. \tag{1.41}$$

Thanks to (1.35), (1.38) and (1.39), we thus observe one additional (and last) remarkable fact: the conformal factors α and $\tilde{\alpha}$ satisfy the *same* second order elliptic PDE (1.40). Finally, we use (1.30), (1.31) and a classical unique continuation principle [42, 72, 73] to prove $\alpha = \tilde{\alpha}$. As a consequence, we find that

$$G = \tilde{G},$$

up to some isometries of the type (1.18) that preserve the boundary. The derivation of the elliptic PDE (1.40) satisfied by α and the unique continuation argument will be given in Section 3.4.

2 The DN map on conformally Stäckel manifolds

2.1 A review of complete integrability and separability's properties on conformally Stäckel manifolds

The case of Stäckel manifolds. Stäckel manifolds (or systems) date back to the work by Stäckel [71], Robertson [68] and Eisenhart [30] on the theory of orthogonal variable separation for the Hamilton-Jacobi (HJ) equation $g(\nabla u, \nabla u) = E$ and the Helmholtz equation $-\Delta_g \psi = E\psi$ on a n-dimensional pseudo-Riemannian manifold (M, g). Here by *orthogonal separation*, we mean that we look for diagonal metrics g satisfying $g_{ij} = 0,\ i \neq j$ such that :

- the HJ equation possesses locally a solution $u(x, c)$ parametrized by n constants $c = (c_1, \dots, c_n)$ of the form

$$u(x, c) = \sum_{i=1}^{n} u_i(x^i, c), \quad x = (x^1, \dots, x^n),$$

  satisfying the completeness condition

$$\det \left[\frac{\partial^2 u}{\partial x^i \partial c_j} \right] \neq 0.$$

- the Helmholtz equation possesses locally a solution $\psi(x, c)$ parametrized by $2n$ constants $c = (c_1, \dots, c_{2n})$ of the form

$$\psi(x, c) = \prod_{i=1}^{n} \psi_i(x^i, c), \quad x = (x^1, \dots, x^n),$$

  satisfying the completeness condition

$$\det \begin{bmatrix} \frac{\partial u_i}{\partial c_J} \\ \frac{\partial v_i}{\partial c_J} \end{bmatrix} \neq 0, \quad u_i = \frac{\psi_i'}{\psi_i}, \quad v_i = \frac{\psi_i''}{\psi_i}.$$

The three classical theorems of Stäckel, Robertson and Eisenhart are as follows

Theorem 2.1 (Stäckel, 1893). *The HJ equation is separable in orthogonal coordinates $x = (x^1, \dots, x^n)$ for all values of the energy E if and only if the metric is of the form*

$$g = \sum_{i=1}^{n} h_i^2 (dx^i)^2 , \quad h_i^2 = \frac{\det S}{s^{i1}} ,$$

with $S = \big(s_{ij}(x^i)\big)$ being a Stäckel matrix, that is a non-singular matrix such that each entry s_{ij} depends only the coordinate x^i, and s^{ij} denotes the cofactor of the component s_{ij} of the matrix S.

Theorem 2.2 (Robertson, 1928). *The Helmholtz equation is separable in orthogonal coordinates $x = (x^i)$ for all values of the energy E if and only if in these coordinates the metric g is Stäckel and moreover the following Robertson condition is satisfied*

$$\partial_j \gamma_i = 0, \quad i \neq j,$$

with

$$\gamma_i := -\partial_i \log \frac{h_1 h_2 h_3}{h_i^2}\,.$$

Theorem 2.3 (Eisenhart, 1934). *The Robertson condition is satisfied if and only if the Ricci tensor is diagonal, i.e. $R_{ij} = 0$ for $i \neq j$.*

More intrinsic characterizations of the separability properties of the HJ and Helmholtz equations have been obtained later by Kalnins and Miller (see for instance [50] and the survey [63]) and Benenti (see for instance the surveys [2, 3]). In order to state them, let us recall some standard definitions. Let $K = (K^{ij})$ be a symmetric contravariant two-tensor. We denote by P_K the fiber-wise homogeneous polynomial function on the cotangent bundle T^*M given by $P_K = K^{ij} p_i p_j$. We say that two such symmetric contravariant two-tensors K and K' are in involution if the corresponding polynomial functions are in involution, *i.e.* if their Poisson bracket relative to the canonical symplectic structure of T^*M vanishes identically

$$\{P_K, P_{K'}\} = 0.$$

We say that a symmetric contravariant two-tensor K is a Killing tensor on (M, g) if and only if P_K is a first integral of the geodesic flow, *i.e.* it is in involution with the geodesic Hamiltonian $H = g^{ij} p_i p_j$, *i.e.*

$$\{P_K, H\} = 0, \tag{2.1}$$

or equivalently

$$\nabla^{(h} K^{ij)} = 0,$$

where ∇ denotes the covariant derivative with respect to the Levi-Civita connection and the parentheses $(\dots)$ denotes the symmetrization of the indices. Finally, to a symmetric contravariant two-tensor $K = (K^{ij})$, we can associate by means of the pseudo-Riemannian metric g a linear operator $\mathbf{K}$ acting on vector fields $X = (X^i)$ by means of

$$(\mathbf{K}X)^i = K^{ij} g_{jh} X^h = K^i_j X^j.$$

Hence we can talk about eigenvalues, eigenvectors, etc... of a symmetric contravariant two-tensor K throught this identification.

Let us state now the intrinsic characterizations of Stäckel metrics in the formulation given in [2].

Theorem 2.4 (Intrinsic characterizations for HJ and Helmholtz equations)**.** *1) The HJ equation on (M,g) is orthogonally separable if and only if there exists a Killing tensor K with pointwise simple eigenvalues and normal (i.e. orthogonally integrable or surface forming) eigenvectors.*
2) The HJ equation on (M,g) is orthogonally separable if and only if there exist n pointwise independent Killing tensors $K_a,\ a=1,\dots,n$ commuting as linear operators and in involution. Moreover, the contravariant metric $g=g^{ij}$ belongs to the algebra generated by the tensors K_a and can be chosen equal to K_1.
3) The Helmholtz equation is orthogonally separable on (M,g) if and only of there exists a Killing tensor K with simple eigenvalues and normal eigenvectors that commutes with the Ricci tensor, i.e. $K^{ij}R_{jk}-R_{ij}K^{jk}=0$.

Finally, we make the link between the above intrinsic characterization of the separability of the Helmholtz equation with the existence of second order symmetry operators for the Laplace-Beltrami operator $-\Delta_g$. First let us associate to the n Killing tensors $K_a,\ a=1,\dots,n$ from Theorem 2.4, the pseudo-Laplacian Δ_{K_a} by

$$\Delta_{K_a}\psi=\nabla_i(K_a^{ij}\nabla_j)\psi,$$

where ∇_i denotes the covariant derivative with respect to the Levi-Civita connection. Then, we have the following result (see Theorems 6.2 and 6.3 in [3])

Theorem 2.5. *All pseudo-Laplacian $\Delta_{K_a},\ a=1,\dots,n$ pairwise commute and thus (since $\Delta_{K_1}=\Delta_g$) commute with the Laplace-Beltrami operator Δ_g.*

The case of conformally Stäckel manifolds. The above separability results in the HJ and Helmholtz equations are valid *at all energies E*. What happens if the energy E is fixed and furthermore set equal to 0? Note that in the Hamilton-Jacobi case, the corresponding orthogonal variable separation theory would only apply to the case of null geodesics and would therefore require that the metric have indefinite signature. Even though we shall eventually only be concerned with the case of Riemannian signature for the purposes of the Calderón problem studied in this paper, we shall for now recall the definitions and characterizations of separability for the null HJ and Laplace-Beltrami equations in general pseudo-Riemannian signature, following the classical results of Kalnins and Miller [51, 52, 53] and Benenti, Chanu and Rastelli [4, 11]. In the Riemannian case, this will give rise to the class of conformally Stäckel manifolds studied in this paper.

First, the definitions of separated solutions of the null HJ and Laplace equations slightly differ from the previous ones since we now allow R-separability. Precisely

- the null HJ equation is said to be separable if it possesses locally a solution $u(x,c)$ parametrized by $n-1$ constants $c=(c_1,\dots,c_{n-1})$ of the form

$$u(x,c)=\sum_{i=1}^{n}u_i(x^i,c),\quad x=(x^1,\dots,x^n),$$

satisfying the rank condition

$$\operatorname{rank}\left[\frac{\partial^2 u}{\partial x^i \partial c_j}\right] = n-1.$$

- the Laplace equation is said to be R-separable if there exists a function R such that the Laplace equation possesses locally a solution $\psi(x,c)$ parametrized by $2n-1$ constants $c=(c_1,\ldots,c_{2n-1})$ of the form

$$\psi(x,c) = R\prod_{i=1}^{n}\psi_i(x^i,c), \quad x=(x^1,\ldots,x^n),$$

satisfying the rank condition

$$\operatorname{rank}\begin{bmatrix}\frac{\partial u_i}{\partial c_J}\\ \frac{\partial v_i}{\partial c_J}\end{bmatrix} = 2n-1, \quad u_i = \frac{\psi_i'}{\psi_i}, \quad v_i = \frac{\psi_i''}{\psi_i}.$$

Second, we say that orthogonal coordinates $x=(x^i)$ are *conformally separable* on a Riemannian manifold (M,G) if there exists a smooth positive function c (playing the role of a conformal factor) and a Stäckel metric g such that

$$G = c^4\, g = \sum_{i=1}^{n} H_i^2 (dx^i)^2.$$

Then we have the following characterizations :

Theorem 2.6 (Kalnins-Miller [52], Benenti-Chanu-Rastelli [4])**.** *The null HJ equation is separable in orthogonal coordinates $x=(x^i)$ if and only if these coordinates are conformally separable.*

Theorem 2.7 (Kalnins-Miller [53], Chanu-Rastelli [11])**.** *The Laplace equation is separable in orthogonal coordinates $x=(x^i)$ if and only if these coordinates are conformally separable and there exist functions $\phi_i=\phi_i(x^i),\ i=1,\ldots,n$ such that the generalized Robertson condition is satisfied*

$$\frac{1}{4}\sum_{i=1}^{n} G^{ii}(2\partial_i\Gamma_i - \Gamma_i^2) = \sum_{i=1}^{n} G^{ii}\phi_i, \tag{2.2}$$

with

$$\Gamma_i := -\partial_i \log\frac{H_1H_2H_3}{H_i^2}\,.$$

In this case, the function R is any solution of

$$2\partial_i \ln R = \Gamma_i - \xi_i(x^i), \tag{2.3}$$

for arbitrary functions $\xi_i=\xi_i(x^i)$.

In analogy with (2.1), a symmetric contravariant two-tensor $K = (K^{ij})$ is said to be a conformal Killing tensor for the contravariant metric $G = (G^{ij})$ if there exists a vector field C such that

$$\{P_G, P_K\} = 2P_C P_G,$$

or equivalently

$$\nabla^{(i}K^{jk)} = C^{(i}G^{jk)}.$$

Then we have the following intrinsic characterization of the separability of the null HJ and Laplace equations.

Theorem 2.8 (Kalnins-Miller [52], Benenti-Chanu-Rastelli [4], Chanu-Rastelli [11]). *1) The null HJ equation is separable in orthogonal coordinates if and only if there exists a conformal Killing tensor with simple eigenvalues and normal eigenvectors.*
2) The null HJ equation is separable in orthogonal coordinates if and only there exist n conformal Killing tensors $K_a,\ a = 1, \dots, n$ pointwise independent, with common eigenvectors and in involution.
3) The Laplace equation is separable in orthogonal coordinates if and only if there exist n conformal Killing tensors $K_a,\ a = 1, \dots, n$ pointwise independent, with common eigenvectors, in involution and such that the generalized Robertson condition (2.2) is satisfied.

Remark 2.1. *Observe that there does not seem to exist in the literature an intrinsic characterization for the generalized Robertson condition as is the case for Stäckel metrics in terms of commutation property with the Ricci tensor, see Theorem 2.3 or Theorem 2.4, 3)).*

Finally, we can prove that the Laplace-Beltrami operator Δ_G possesses $n-1$ conformal symmetry operators. For all $a = 1, \dots, n$, let us associate to the Killing tensors K_a corresponding to the Stäckel metric $g = c^{-4}G$, the second-order operators

$$H_a := \Delta_{K_a} - \frac{1}{R}\Delta_{K_a} R,$$

where the pseudo-Laplacian are defined by

$$\Delta_{K_a} := \nabla_i (K_a^{ii} \nabla_i) = \sum_{i=1}^{n} K_a^{ii} (\partial_{ii}^2 - \Gamma_i \partial_i).$$

Notice that $H_1 = \Delta_g$. Moreover, we say that an operator H is a *conformal symmetry operator* for Δ_G if $H\psi = 0$ if

$$[H, \Delta_G] = L\Delta_G,$$

for some first-order operator L. Then we have

Theorem 2.9 (Chanu-Rastelli [11]). *The operators $H_a,\ a = 1, \dots, n$ pairwise commute, i.e. for all a, b*

$$[H_a, H_b] = 0.$$

Moreover they are conformal symmetry operators for the Laplace-Beltrami operator Δ_G.

2.2 3D conformally Stäckel manifolds and the structure of the DN map

After this quick review on the separability properties of the HJ and Helmholtz equations on conformally Stäckel manifolds, we would like to specialize the procedure to the three-dimensional case for the Laplace equation that appears in the construction of the DN map and make explicit the *completeness* of the set of separated solutions.

Three-dimensional conformally Stäckel cylinders. Assume that

$$\Omega = [0, A] \times \mathbb{T}^2,$$

is a toric cylinder and denote by $x = (x^1, x^2, x^3)$ a global coordinate ssystem on Ω. We consider a Riemannian metric G on Ω given by

$$G = c^4\, g = \sum_{i=1}^{3} H_i^2 (dx^i)^2\,, \tag{2.4}$$

with g a Stäckel metric

$$g = \sum_{i=1}^{3} h_i^2 (dx^i)^2\,, \quad h_i^2 = \frac{\det S}{s^{i1}}\,,$$

with $S = (s_{ij}(x^i))$ being a non-singular Stäckel matrix. In order to insure R-separability of the Laplace equation, we assume that the generalized Robertson condition (2.2) holds. Writing this equation in coordinates and remarking that $\Gamma_i = \gamma_i - 2\partial_i \ln c$, the generalized Robertson condition is seen to be equivalent to the PDE (1.4) on the conformal factor c, *i.e.*

$$-\Delta_g c - \sum_{i=1}^{3} h_i^2 \big(\phi_i + \frac{1}{4}\gamma_i^2 - \frac{1}{2}\partial_i \gamma_i\big) c = 0\,.$$

Let us calculate now the Laplace equation in this coordinate system and see how variables separation naturally appears in the calculations. We start with

$$-\Delta_G\, \psi = 0,$$

and we look for a solution ψ under the form $\psi = Ru$. Then u must satisfy

$$-\Delta_G\, u - \frac{2}{R} G^{-1}(dR, du) - \frac{\Delta_G\, R}{R} u = 0,$$

or equivalently in the conformally separable Stäckel coordinates $x = (x^i)$

$$\sum_{i=1}^{3} \left[H_i^{-2}(-\partial_{ii}^2 u + \Gamma_i \partial_i u) - 2H_i^{-2} \partial_i \ln R\, \partial_i u \right] - \frac{\Delta_G\, R}{R} u = 0. \tag{2.5}$$

Choose the R factor so as to satisfy (2.3), *i.e.* $2\partial_i \ln R = \Gamma_i$. (Since the functions ξ_i appearing in (2.3) are arbitrary in (2.3), we choose them to be zero for convenience.)

Remark 2.2. *Recall that*

$$\begin{aligned}\Gamma_i &:= -\partial_i \ln \frac{H_1 H_2 H_3}{H_i^2} = -\partial_i \ln \frac{c^2 h_1 h_2 h_3}{h_i^2} = -\partial_i \ln \frac{c^2 \sqrt{\det S} s^{i1}}{\sqrt{s^{11} s^{12} s^{13}}}, \\ &= -\partial_i \ln \frac{c^2 \sqrt{\det S}}{\sqrt{s^{11} s^{12} s^{13}}}, \quad \text{since} \ \ \partial_i \, s^{i1} = 0.\end{aligned}$$

Comparing with (2.3), we see that R may be written as

$$R = \left(\frac{s^{11} s^{21} s^{31}}{c^4 \ \det S} \right)^{\frac{1}{4}}. \tag{2.6}$$

Under the assumption (2.3) or equivalenty (2.6), we calculate

$$\frac{\Delta_G R}{R} = \frac{1}{4} H_i^{-2} \left(2 \partial_i \Gamma_i - \Gamma_i^2 \right). \tag{2.7}$$

Putting together (2.3), (2.5) and (2.7), we get the following expression for the Laplace equation

$$\sum_{i=1}^{3} H_i^{-2} \left[-\partial_{ii}^2 u - \frac{1}{4} H_i^{-2} \left(2 \partial_i \Gamma_i - \Gamma_i^2 \right) \right] u = 0.$$

Finally, using (2.2), we obtain

$$\sum_{i=1}^{3} H_i^{-2} \left[-\partial_{ii}^2 u - \phi_i(x^i) \right] u = 0. \tag{2.8}$$

Let us introduce the ordinary differential operators $A_i = -\partial_{ii}^2 - \phi_i(x^i)$ from (1.11). Then we continue the separation of variables procedure as follows

$$\begin{aligned} -\Delta_G \psi \quad &\Longleftrightarrow \quad \sum_{i=1}^{3} H_i^{-2} A_i u = 0, \\ &\Longleftrightarrow \quad A_1 u + \frac{H_1^2}{H_2^2} A_2 u + \frac{H_1^2}{H_3^2} A_3 u = 0, \\ &\Longleftrightarrow \quad A_1 u + s_{12}(x^1) H u + s_{13}(x^1) L u = 0, \end{aligned} \tag{2.9}$$

where the operators (H, L) are given by (1.10), *i.e.*

$$\begin{pmatrix} H \\ L \end{pmatrix} = \frac{1}{s^{11}} \begin{pmatrix} -s_{33} & s_{23} \\ s_{32} & -s_{22} \end{pmatrix} \begin{pmatrix} A_2 \\ A_3 \end{pmatrix},$$

and are computed thanks to the Stäckel structure of g. We recall the following Lemma from Gobin [33] (Lemma 2.5, Remarks 2.6 and 2.7).

Lemma 2.1. *The operators H and L are elliptic selfadjoint operators on $L^2(\mathbb{T}^2;\, s^{11}dx^1dx^2)$ that commute, i.e. $[H, L] = 0$. The basis of common eigenfunctions $(Y_m)_{m\geq 1}$, with joint spectrum denoted by (μ_m^2, ν_m^2), i.e.*

$$HY_m = \mu_m^2 Y_m, \quad LY_m = \nu_m^2 Y_m,$$

can be written as $Y_m = v_m(x^2)w_m(x^3)$ and satisfy

$$L^2(\mathbb{T}^2;\, s^{11}\, dx^2dx^3) = \bigoplus_m \langle Y_m \rangle.$$

Remark 2.3. *Note that since $s^{11} \neq 0$, we have*

$$\begin{pmatrix} A_2 \\ A_3 \end{pmatrix} = -\begin{pmatrix} s_{22} & s_{23} \\ s_{32} & s_{33} \end{pmatrix} \begin{pmatrix} H \\ L \end{pmatrix}. \tag{2.10}$$

We now finish the separation of variables procedure by looking for the solutions ψ of $-\Delta_G\, \psi = 0$ under the form

$$\psi = R \sum_{m=1}^{\infty} u_m(x^1) Y_m, \quad Y_m = v_m(x^2)w_m(x^3). \tag{2.11}$$

Putting (2.11) into (2.9) and (2.10), we deduce that u_m, v_m, w_m satisfy the three separated ODEs (1.7) - (1.9), *i.e.*

$$\begin{aligned}
-u_m'' + [\, \mu_m^2 s_{12}(x^1) + \nu_m^2 s_{13}(x^1) - \phi_1(x^1)]\, u_m &= 0, \\
-v_m'' + [\, \mu_m^2 s_{22}(x^2) + \nu_m^2 s_{23}(x^2) - \phi_2(x^2)]\, v_m &= 0, \\
-w_m'' + [\, \mu_m^2 s_{32}(x^3) + \nu_m^2 s_{33}(x^3) - \phi_3(x^3)]\, w_m &= 0.
\end{aligned}$$

This finishes the procedure of variables separation for the Laplace equation on a conformally Stäckel manifold.

Some hidden invariances. When solving the inverse Calderón problem on conformally Stäckel manifolds in Section 3, we will need to understand some underlying invariances in the definition of our metrics and in the procedure of variables separation. The first and main invariance comes from the fact that a Stäckel metric g as in (1.2) is *not* determined by a unique Stäckel matrix S. Precisely, we quote the following Proposition from Gobin [33]

Proposition 2.1. *Let S be a Stäckel matrix and g_S the corresponding Stäckel metric.*
1. Let $G \in GL_2(\mathbb{R})$ a constant matrix and define the new Stäckel matrix

$$\hat{S} = \begin{pmatrix} s_{11}(x^1) & \hat{s}_{12}(x^1) & \hat{s}_{13}(x^1) \\ s_{21}(x^2) & \hat{s}_{22}(x^2) & \hat{s}_{23}(x^2) \\ s_{31}(x^3) & \hat{s}_{32}(x^3) & \hat{s}_{33}(x^3) \end{pmatrix},$$

satisfying

$$\begin{pmatrix} s_{i2} & s_{i3} \end{pmatrix} = \begin{pmatrix} \hat{s}_{i2} & \hat{s}_{i3} \end{pmatrix} G, \quad \forall i \in \{1, 2, 3\}.$$

Then $g_{\hat{S}} = g_S$.
2) Define the Stäckel matrix

$$\hat{S} = \begin{pmatrix} \hat{s}_{11}(x^1) & s_{12}(x^1) & s_{13}(x^1) \\ \hat{s}_{21}(x^2) & s_{22}(x^2) & s_{23}(x^2) \\ \hat{s}_{31}(x^3) & s_{32}(x^3) & s_{33}(x^3) \end{pmatrix},$$

where,

$$\begin{cases} \hat{s}_{11}(x^1) = s_{11}(x^1) + C_1 s_{12}(x^1) + C_2 s_{13}(x^1) \\ \hat{s}_{21}(x^2) = s_{21}(x^2) + C_1 s_{22}(x^2) + C_2 s_{23}(x^2) \\ \hat{s}_{31}(x^3) = s_{31}(x^3) + C_1 s_{32}(x^3) + C_2 s_{33}(x^3) \end{cases},$$

where C_1 *and* C_2 *are real constants. Then* $g_{\hat{S}} = g_S$.

These invariances are important and will naturally appear at several stages in our proof of uniqueness in the inverse problem. Note moreover that these invariances allow us to assume from the very beginning and without loss of generality certain properties for the Stäckel matrix S used to represent a given Stäckel metric g_S. Precisely, we have

Proposition 2.2 (Gobin [33], Prop. 1.17 and Remark 1.18)**.** *Using the above invariances, we can always choose a Stäckel matrix* S *associated to a Riemannian Stäckel metric* g_S *such that*

$$\begin{cases} \hat{s}_{12}(x^1) > 0 \quad \textit{and} \quad \hat{s}_{13}(x^1) > 0, \quad \forall x^1 \\ \hat{s}_{22}(x^2) < 0 \quad \textit{and} \quad \hat{s}_{23}(x^2) > 0, \quad \forall x^2 \\ \hat{s}_{32}(x^3) > 0 \quad \textit{and} \quad \hat{s}_{33}(x^3) < 0, \quad \forall x^3 \end{cases}.$$

As a consequence, we can always assume from the very beginning

$$s^{11},\ s^{21},\ s^{31},\ \det S > 0.$$

Finally, note that

$$s^{11} > 0 \quad \Longleftrightarrow \quad \frac{s_{22}}{s_{23}} < \frac{s_{32}}{s_{33}}, \quad \forall x^2, x^3.$$

There is a second and last invariance that we need to understand before attacking the inverse problem. This invariance appears in the procedure of variables separation when we look for solutions of the Laplace equation decomposed onto the Hilbert basis of angular harmonics Y_m which are common eigenfunctions of the operators (H, L). This decomposition is not unique since we could for example have decomposed the solutions onto the Hilbert basis of common eigenfunctions of the operators

$$\hat{H} = H + B_1, \quad \hat{L} = L + B_2,$$

where B_1, B_2 are two constants. The common eigenfunctions, which are still the Y_m, would be then associated to the joint spectrum

$$\hat{\mu}_m^2 = \mu_m^2 + B_1, \quad \hat{\nu}_m^2 = \nu_m^2 + B_2. \tag{2.12}$$

Then the separability procedure would remain unaffected and would still lead to the separated ODEs (1.7) - (1.9) with the only modification (2.12). We thus have the freedom to choose the constants B_1, B_2 as we wish in the separated equations. This invariance will be important at one point in Section 3.2.

The construction of the DN map and its structure. We construct here the DN map associated to a conformally Stäckel manifold (M, G). Consider the Dirichlet problem

$$\left\{ \begin{array}{rcl} -\Delta_G \psi & = & 0, \text{ on } \Omega, \\ \psi & = & f, \text{ on } \partial\Omega. \end{array} \right. \tag{2.13}$$

Recall that $\partial\Omega = \Omega_0 \cup \Omega_1$ where $\Omega_j \simeq \mathbb{T}^2$. Hence we can identify the Dirichlet data $f \in H^{\frac{1}{2}}(\partial\Omega)$ with the two-component vector

$$f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \in H^{\frac{1}{2}}(\Omega_0) \oplus H^{\frac{1}{2}}(\Omega_1).$$

By definition, the DN map is given by

$$\Lambda_G f := (\partial_\nu \psi)_{|\partial\Omega} = \begin{pmatrix} (\partial_{\nu_0} \psi)_{|\Omega_0} \\ (\partial_{\nu_1} \psi)_{|\Omega_1} \end{pmatrix}$$

where ψ is the unique solution of (2.13) and ν_0 and ν_1 are the outgoing unit normal vectors on Ω_0 and Ω_1 respectively. A short calculation using the form (2.4) of the metric G leads to

$$\Lambda_G f = \begin{pmatrix} \left(-\frac{1}{H_1} \partial_{x^1} \psi\right)_{|x^1=0} \\ \left(\frac{1}{H_1} \partial_{x^1} \psi\right)_{|x^1=A} \end{pmatrix}.$$

Recalling that $\psi = R\,u$ with the R-factor given by (2.3) or explicitly by (2.6), we get

$$\Lambda_G f = \begin{pmatrix} -\frac{R}{H_1} \left[(\partial_1 \ln R) u + \partial_1 u\right]_{|x^1=0} \\ \frac{R}{H_1} \left[(\partial_1 \ln R) u + \partial_1 u\right]_{|x^1=A} \end{pmatrix}.$$

But we know from (2.3) that $\partial_1 \ln R = \frac{1}{2}\Gamma_1$. Hence

$$\Lambda_G f = \begin{pmatrix} -\frac{R}{H_1} \left[\frac{1}{2}\Gamma_1 u + \partial_1 u\right]_{|x^1=0} \\ \frac{R}{H_1} \left[\frac{1}{2}\Gamma_1 u + \partial_1 u\right]_{|x^1=A} \end{pmatrix}.$$

Let us use at this point the separated form (2.11) of the solution ψ, *i.e.*

$$\psi = R\,u, \quad u = \sum_{m \geq 1}^{\infty} u_m(x^1)\, Y_m,$$

and the corresponding Fourier decomposition of the Dirichlet data f on $\partial\Omega$:

$$f = R\,\varphi, \quad \varphi = \begin{pmatrix} \varphi^0 \\ \varphi^1 \end{pmatrix}, \quad \varphi^j = \sum_{m\geq 1}^{\infty} \varphi^j_m\, Y_m, \quad j=0,1.$$

We observe that the functions $u_m(x^1)$ satisfy the one-dimensional Dirichlet problem on $[0,A]$:

$$\begin{cases} -u_m'' + [\mu_m^2 s_{12}(x^1) + \nu_m^2 s_{13}(x^1) - \phi_1(x^1)]u_m = 0, \\ u_m(0) = \varphi^0_m, \ u_m(A) = \varphi^1_m. \end{cases}$$

We thus obtain the following decomposition for the DN map Λ_G

$$\Lambda_G\, f = \sum_{m\geq 1}^{\infty} \begin{pmatrix} -\frac{R(0,x^2,x^3)}{H_1(0,x^2,x^3)} \left[\frac{1}{2}\Gamma_1(0,x^2,x^3)\varphi^0_m + u_m'(0)\right] \\ \frac{R(A,x^2,x^3)}{H_1(A,x^2,x^3)} \left[\frac{1}{2}\Gamma_1(A,x^2,x^3)\varphi^1_m + u_m'(A)\right] \end{pmatrix} Y_m. \tag{2.14}$$

It remains essentially to express the derivatives $u_m'(0)$ and $u_m'(A)$ in terms of the Dirichlet data φ^0_m and φ^1_m. This can be done as follows.

Denote by $\{c_0, s_0\}$ and $\{c_1, s_1\}$ the fundamental systems of solutions (FSS) of the separated ODE

$$-u'' + [\mu^2 s_{12}(x^1) + \nu^2 s_{13}(x^1) - \phi_1(x^1)]u = 0, \tag{2.15}$$

(where μ^2 and ν^2 are here any constants) which satisfy the Cauchy conditions of sine and cosine type at $x^1 = 0$ and $x^1 = A$, *i.e.*

$$c_0(0) = 1, \ c_0'(0) = 0, \ s_0(0) = 0, \ s_0'(0) = 1,$$
$$c_1(A) = 1, \ c_1'(A) = 0, \ s_1(A) = 0, \ s_1'(A) = 1.$$

Clearly, the functions $c_j, \ s_j, \ j = 0,1$ are analytic separately in the parameters $\mu, \nu \in \mathbb{C}$ and their Wronskians satisfy

$$W(c_j, s_j) = 1, \ j = 0,1,$$

where $W(f,g) := fg' - f'g$.

Associated to the ODE (2.15) with Dirichlet boundary conditions, we introduce first the characteristic function

$$\Delta(\mu^2, \nu^2) = W(s_0, s_1). \tag{2.16}$$

Second, introduce the Weyl solutions of (2.15) given by the particular linear combinations

$$\Psi = c_0 + M(\mu^2,\nu^2)s_0, \quad \Phi = c_1 - N(\mu^2,\nu^2)s_1,$$

by demanding that they satisfy the Dirichlet boundary condition at $x = A$ and $x = 0$ respectively. The coefficients M, N are the Weyl-Titchmarsh (WT) functions and one easily verifies that they can be expressed as follows in terms of the Wronskians and characteristic functions,

$$M(\mu^2,\nu^2) = -\frac{W(c_0,s_1)}{\Delta(\mu^2,\nu^2)} = -\frac{D(\mu^2,\nu^2)}{\Delta(\mu^2,\nu^2)}, \qquad N(\mu^2,\nu^2) = \frac{W(s_0,c_1)}{\Delta(\mu^2,\nu^2)} = \frac{E(\mu^2,\nu^2)}{\Delta(\mu^2,\nu^2)}. \tag{2.17}$$

Finally it is an easy calculation to show that

$$\begin{array}{l} u_m'(0) = M(\mu_m^2, \nu_m^2)\, \varphi_m^0 + \frac{1}{\Delta(\mu_m^2,\nu_m^2)}\, \varphi_m^1, \\ u_m'(A) = \frac{1}{\Delta(\mu_m^2,\nu_m^2)}\, \varphi_m^0 + N(\mu_m^2, \nu_m^2)\, \varphi_m^1. \end{array} \tag{2.18}$$

Coming back to the expression of the DN map, we obtain from (2.14) and (2.18) the following expression

$$\Lambda_G\, f = \sum_{m\geq 1}^{\infty} \begin{pmatrix} -\frac{R(0)}{H_1(0)} \left[\frac{1}{2}\Gamma_1(0) + M(\mu_m^2, \nu_m^2)\right] & -\frac{R(0)}{H_1(0)} \frac{1}{\Delta(\mu_m^2,\nu_m^2)} \\ \frac{R(A)}{H_1(A)} \frac{1}{\Delta(\mu_m^2,\nu_m^2)} & \frac{R(A)}{H_1(A)} \left[\frac{1}{2}\Gamma_1(A) + N(\mu_m^2, \nu_m^2)\right] \end{pmatrix} \begin{pmatrix} \varphi_m^0 \\ \varphi_m^1 \end{pmatrix} Y_m,$$

where we used the notations

$$\begin{array}{c} R(0) = R(0, x^2, x^3),\ H_1(0) = H_1(0, x^2, x^3),\ \Gamma_1(0) = \Gamma_1(0, x^2, x^3), \\ R(A) = R(A, x^2, x^3),\ H_1(A) = H_1(A, x^2, x^3),\ \Gamma_1(A) = \Gamma_1(A, x^2, x^3), \end{array}$$

as well as a 2×2-matrix valued notation for the DN map on each harmonic Y_m. A last manipulation of the above expression of the DN map leads to the form (1.14) - (1.15) announced in the Introduction, *i.e.*

$$\Lambda_G = \begin{pmatrix} \frac{-1}{H_1(0)} & 0 \\ 0 & \frac{1}{H_1(A)} \end{pmatrix} \left[\begin{pmatrix} \frac{\Gamma_1(0)}{2} & 0 \\ 0 & \frac{\Gamma_1(A)}{2} \end{pmatrix} + \begin{pmatrix} R(0) & 0 \\ 0 & R(A) \end{pmatrix} A_G \begin{pmatrix} \frac{1}{R(0)} & 0 \\ 0 & \frac{1}{R(A)} \end{pmatrix} \right]$$

where

$$A_G = \bigoplus_{m\geq 1} A_G^m, \qquad A_G^m := (A_G)_{|\langle Y_m \rangle} := \begin{pmatrix} M(\mu_m^2, \nu_m^2) & \frac{1}{\Delta(\mu_m^2,\nu_m^2)} \\ \frac{1}{\Delta(\mu_m^2,\nu_m^2)} & N(\mu_m^2, \nu_m^2) \end{pmatrix}.$$

From the above structure of the DN map, we make two comments :

1) The full DN map Λ_G is essentially encoded in the operator A_G which is diagonalizable on the Hilbert basis $(Y_m)_{m\geq 1}$. The radial part of the metric (*i.e.* the functions depending on x^1) appears there in the definition of the characteristic and Weyl-Titchmarsh functions (2.16) and (2.17). On the contrary, the angular part of the metric (*i.e.* the functions depending on (x^2, x^3)) appears in the joint spectrum $J = \{(\mu_m^2, \nu_m^2),\ m \geq 1\}$ at which the characteristic and WT functions are evaluated. The inverse problem for the operator A_G will be studied thanks to the multi-parameter CAM method in Section 3.3.

2) The full DN map differs from the operator A_G by explicit boundary values of the essential functions H_1, R and Γ_1 which are a priori unknown in the inverse problem. These boundary values will be uniquely determined however thanks to usual boundary determination results in Section 3.2.

3 The Calderón inverse problem

3.1 Reduction to an inverse problem on the whole cylinder Ω

Recall that we consider smooth compact connected Riemannian manifolds (M, G) and $(M, \tilde{G})$ such that :

1) $M \subset\subset \Omega = [0, A] \times \mathbb{T}^2$.

2) G and $\tilde{G}$ are Riemannian metrics on M having the form (1.1) - (1.3).

3) $\Lambda_G = \Lambda_{\tilde{G}}$.

We aim to show in this section that it is enough to prove uniqueness in the inverse Calderón problem for conformally Stäckel metrics G and $\tilde{G}$ *on the whole cylinder* Ω on which we will be able to use separation of variables. This will be done using the extension procedure explained in the Introduction, that is by proving Theorem 1.2 and its consequence Proposition 1.1.

Proof of Theorem 1.2. We assume that

$$\Lambda_{G,1} = \Lambda_{\tilde{G},1}.$$

From the boundary determination results in [60, 55], recall that the metrics G and $\tilde{G}$ coincide on ∂M_1 as well as all their normal derivatives. Hence we can identify the normal derivative operators ∂_ν and $\partial_{\tilde{\nu}}$ on ∂M_1.

Let $f \in H^{\frac{1}{2}}(M_2)$ and consider the unique solutions u, $\tilde{u}$ of the Dirichlet problems

$$\left\{ \begin{array}{rcll} -\Delta_G u &=& 0, & \text{on } M_2, \\ u &=& f, & \text{on } \partial M_2, \end{array} \right. \qquad \left\{ \begin{array}{rcll} -\Delta_{\tilde{G}} \tilde{u} &=& 0, & \text{on } M_2, \\ \tilde{u} &=& f, & \text{on } \partial M_2. \end{array} \right.$$

Since $G = \tilde{G}$ on $M_2 \setminus M_1$, we aim to show that $\partial_\nu u = \partial_\nu \tilde{u}$ on ∂M_2.

For this, introduce the solution u_{in} of the Dirichlet problem

$$\left\{ \begin{array}{rcll} -\Delta_{\tilde{G}} u_{in} &=& 0, & \text{on } M_1, \\ u_{in} &=& \psi, & \text{on } \partial M_1, \end{array} \right.$$

where $\psi = u_{|\partial M_1}$ and define the function

$$v := \left\{ \begin{array}{l} u_{in} \text{ on } M_1, \\ u \text{ on } M_2 \setminus M_1. \end{array} \right.$$

Since $G = \tilde{G}$ on $M_2 \setminus M_1$, we clearly have

$$\Delta_{\tilde{G}} v = 0, \quad \text{on } M_1 \cup (M_2 \setminus M_1),$$

and $v = u$ on ∂M_1 by definition of u_{in}. Let us study the traces of the normal derivatives of v at the interface ∂M_1 when the normal derivatives are taken from the exterior and the interior. We have for the former

$$\partial_\nu v_{|\partial M_1^+} = \partial_\nu u_{|\partial M_1^+} = \Lambda_{G,1}\psi,$$

and for the latter

$$\partial_\nu v_{|\partial M_1^-} = \partial_\nu (u_{in})_{|\partial M_1^-} = \Lambda_{\tilde{G},1}\psi = \Lambda_{G,1}\psi,$$

thanks to our main hypothesis. We deduce that v and $\partial_\nu v$ are continuous on ∂M_1 and thus

$$\begin{cases} -\Delta_{\tilde{G}}\, v &= 0, \quad \text{on } M_2, \\ \qquad v &= f, \quad \text{on } \partial M_2. \end{cases}$$

By uniqueness of the Dirichlet problem, we infer that $v = \tilde{u}$ on M_2. This implies that $u = \tilde{u}$ on $M_2 \setminus M_1$ and therefore $\partial_\nu u = \partial_\nu \tilde{u}$ on ∂M_2, whence

$$\Lambda_{G,2} = \Lambda_{\tilde{G},2}.$$

□

Let us apply now this extension result to the inverse Calderón problem on conformally Stäckel manifolds. From the boundary determination results of [60, 55], we know that there exists a neighbourhood U of ∂M and a diffeomorphism $\phi \in$Diff(U) such that

$$\sum_{|\alpha|=0}^{N} \sup_{x\in\partial M} |\partial^\alpha(\tilde{G}(x) - \phi^* G(x))| = 0, \quad \forall N \geq 0.$$

In particular, the metrics G and $\tilde{G}$ coincide on ∂M as well as all their tangential and normal derivatives at any order N. We finish the extension procedure in the the following way:

Proof of Proposition 1.1. We first extend the metric G on M to a (still) conformally Stäckel metric to the whole cylinder Ω and we demand that G satisfies the generic assumption (1.28). Then we extend the metric $\tilde{G}$ on M to a conformally Stäckel metric $\hat{G}$ on Ω by defining

$$\hat{G} = \begin{cases} \tilde{G} = \tilde{G}, & \text{on } M, \\ \tilde{G} = G, & \text{on } \Omega \setminus M. \end{cases}$$

The new metric $\hat{G}$ on Ω is smooth thanks to the above boundary determination results, is clearly conformally Stäckel since G and $\tilde{G}$ are, and satisfies

$$\hat{G} = G, \quad \text{on } \Omega \setminus M.$$

Hence Theorem 1.2 implies that

$$\Lambda_{G,\Omega} = \Lambda_{\hat{G},\Omega}.$$

□

3.2 Boundary determination results

Using the results of Section 3.1, we are led to study the Calderón problem on conformally Stäckel *cylinders*. Precisely, we consider two smooth compact connected Riemannian manifolds (Ω, G) and $(\tilde{\Omega}, \tilde{G})$ such that :

1) $\Omega = [0, A] \times \mathbb{T}^2$ and $\tilde{\Omega} = [0, \tilde{A}] \times \mathbb{T}^2$ are toric cylinders.

2) The Riemannian metrics G and $\tilde{G}$ on Ω have the conformally Stäckel form (1.1) - (1.3) and satisfy the generic assumption (1.28).

3) Their DN maps coincide, that is $\Lambda_G = \Lambda_{\tilde{G}}$ [5].

In this Section, we use the boundary determination results of [60, 55] to obtain the maximum of informations on the metrics G and $\tilde{G}$. Precisely we will use the facts that the metrics G and $\tilde{G}$ and their normal derivatives coincide on $\partial\Omega = \partial\tilde{\Omega}$. We divide our boundary determination results into three steps.

Step 1. Assume first that

$$G = \tilde{G}, \quad \text{on } \partial\Omega. \tag{3.1}$$

Recall that $\Omega = \Omega_0 \cup \Omega_1$ and observe that

$$\begin{cases} G_{|\Omega_0} = H_2^2(0, x^2, x^3)(dx^2)^2 + H_3^2(0, x^2, x^3)(dx^3)^2, \\ G_{|\Omega_1} = H_2^2(A, x^2, x^3)(dx^2)^2 + H_3^2(A, x^2, x^3)(dx^3)^2. \end{cases} \tag{3.2}$$

Hence we get from (3.1) and (3.2) that

$$H_j(0) = \tilde{H}_j(0), \quad H_j(A) = \tilde{H}_j(\tilde{A}), \quad j = 2, 3, \tag{3.3}$$

where we used the shorthand notations [6]

$$H_j(0) = H_j(0, x^2, x^3), \quad H_j(A) = H_j(A, x^2, x^3), \quad j = 2, 3.$$

In particular, using the the definition of the diagonal coefficients H_j given by (1.1) and (1.2), we have at $x^1 = 0$

$$\begin{aligned} \frac{H_2^2(0)}{H_3^2(0)} = \frac{\tilde{H}_2^2(0)}{\tilde{H}_3^2(0)} \quad &\Longleftrightarrow \quad \frac{h_2^2(0)}{h_3^2(0)} = \frac{\tilde{h}_2^2(0)}{\tilde{h}_3^2(0)}, \\ &\Longleftrightarrow \quad \frac{s^{31}(0)}{s^{21}(0)} = \frac{\tilde{s}^{31}(0)}{\tilde{s}^{21}(0)}, \\ &\Longleftrightarrow \quad \frac{s^{31}(0)}{\tilde{s}^{31}(0)} = \frac{s^{21}(0)}{\tilde{s}^{21}(0)}. \end{aligned} \tag{3.4}$$

[5] Note that we identify the boundaries $\partial\Omega = \partial\tilde{\Omega}$ when stating this equality.

[6] More generally, in this Section, given a function $f = f(x^1, x^2, x^3)$, we use the notations $f(0)$ and $f(A)$ for $f(0, x^2, x^3)$ and $f(A, x^2, x^3)$ respectively.

The remarkable properties of Stäckel metrics manifest themselves here since the functions $\frac{s^{31}(0)}{\tilde{s}^{31}(0)}$ and $\frac{s^{21}(0)}{\tilde{s}^{21}(0)}$ appearing in (3.4) only depend on x^2 and x^3 respectively. We thus infer from (3.4) that there exists a constant C_0 such that

$$s^{31}(0) = C_0 \tilde{s}^{31}(0), \quad s^{21}(0) = C_0 \tilde{s}^{21}(0). \tag{3.5}$$

Similarly, working at $x^1 = A$, we see that there exists a constant C_1 such that

$$s^{31}(A) = C_1 \tilde{s}^{31}(\tilde{A}), \quad s^{21}(A) = C_1 \tilde{s}^{21}(\tilde{A}). \tag{3.6}$$

Recalling that

$$s^{21} = s_{13}s_{32} - s_{12}s_{33}, \quad s^{31} = s_{12}s_{23} - s_{13}s_{22},$$

we get from (3.5)

$$\begin{pmatrix} s_{22} & s_{23} \\ s_{32} & s_{33} \end{pmatrix} \begin{pmatrix} -s_{13}(0) \\ s_{12}(0) \end{pmatrix} = C_0 \begin{pmatrix} \tilde{s}_{22} & \tilde{s}_{23} \\ \tilde{s}_{32} & \tilde{s}_{33} \end{pmatrix} \begin{pmatrix} -\tilde{s}_{13}(0) \\ \tilde{s}_{12}(0) \end{pmatrix}. \tag{3.7}$$

Let us introduce the notation

$$T = \begin{pmatrix} s_{22} & s_{23} \\ s_{32} & s_{33} \end{pmatrix}, \tag{3.8}$$

and observe that $\det T = s^{11} \neq 0$ since G is a Riemannian metric. Hence (3.7) can be rewritten as

$$\frac{1}{C_0} \tilde{T}^{-1} T \begin{pmatrix} -s_{13}(0) \\ s_{12}(0) \end{pmatrix} = \begin{pmatrix} -\tilde{s}_{13}(0) \\ \tilde{s}_{12}(0) \end{pmatrix}. \tag{3.9}$$

In this equality, only the 2×2-matrix $\tilde{T}^{-1}T$ depends on the variables x^2, x^3. Hence differentiating (3.9) with respect to x^2 and x^3, we obtain

$$\left(\partial_j \tilde{T}^{-1} T\right) \begin{pmatrix} -s_{13}(0) \\ s_{12}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad j = 2, 3.$$

We deduce from this that

$$\begin{pmatrix} -s_{13}(0) \\ s_{12}(0) \end{pmatrix} \in \ker \left(\partial_j \tilde{T}^{-1} T\right), \quad j = 2, 3. \tag{3.10}$$

Since a similar analysis can be done at $x^1 = A$, we also have

$$\begin{pmatrix} -s_{13}(A) \\ s_{12}(A) \end{pmatrix} \in \ker \left(\partial_j \tilde{T}^{-1} T\right), \quad j = 2, 3. \tag{3.11}$$

Thanks to our generic hypothesis (1.28), we infer from (3.10) and (3.11) that $\dim \ker \left(\partial_j \tilde{T}^{-1} T\right) = 2$ and thus

$$\partial_j \tilde{T}^{-1} T = 0, \quad j = 2, 3.$$

We deduce from this that there exists a constant invertible matrix $G \in GL_2(\mathbb{R})$ such that

$$T = \tilde{T} G. \tag{3.12}$$

Finally, we recall from the hidden invariances stated in Proposition 2.1 that we do not alter the Stäckel metrics g and $\tilde{g}$ by multiplying from the right the last two columns of their Stäckel matrices by an invertible matrix G. Hence we can use this invariance and (3.12) to assume from now on that

$$T = \tilde{T}, \quad \text{and thus} \quad s^{11} = \tilde{s}^{11}. \tag{3.13}$$

Let us come back to the equality

$$H_2^2(0) = \tilde{H}_2^2(0) \quad \Longleftrightarrow \quad \frac{(c^4 \det S)(0)}{s^{21}(0)} = \frac{(\tilde{c}^4 \det \tilde{S})(0)}{\tilde{s}^{21}(0)}. \tag{3.14}$$

From (3.5) we thus obtain

$$(c^4 \det S)(0) = C_0\, (\tilde{c}^4 \det \tilde{S})(0). \tag{3.15}$$

Likewise by working at $x^1 = A$, we obtain similarly

$$(c^4 \det S)(A) = C_1\, (\tilde{c}^4 \det \tilde{S})(\tilde{A}). \tag{3.16}$$

Remark 3.1. *Witout loss of generality, we can assume that the constants C_0 and C_1 appearing in (3.5) and (3.6) are equal to 1 in the following way. Recall that the DN maps are invariant by pullback by a diffeomorphim ϕ that is the identity on the boundary. Among such diffeomorphisms, we can use a change of coordinates of the form (1.18)*

$$y^1 = \int_0^{x^1} \sqrt{f_1}(s) ds \ \in [0, A_1],$$

which leads to a new but equivalent expression of the metric G given by (1.19) - (1.22) without changing the DN map Λ_G. In particular, in this new coordinate system, we can replace the initial first line of the Stäckel matrix

$$(s_{11}, s_{12}, s_{13}),$$

by the new first line

$$(\frac{s_{11}}{f_1}, \frac{s_{12}}{f_1}, \frac{s_{13}}{f_1}).$$

We see that we can always choose f_1 such that

$$f_1(0) = C_0, \quad f_1(A) = C_1. \tag{3.17}$$

Putting this into (3.5) and (3.6), we immediately see that, in this new coordinate system, the constants C_0 and C_1 disappear from (3.5) and (3.6). Equivalently, this means that we can always assume from the very beginning that

$$C_0 = C_1 = 1. \tag{3.18}$$

Note that the equalities (3.6) and (3.16) become in this new coordinate system

$$s^{31}(A_1) = \tilde{s}^{31}(\tilde{A}), \quad s^{21}(A_1) = \tilde{s}^{21}(\tilde{A}), \quad (c^4 \det S)(A_1) = (\tilde{c}^4 \det \tilde{S})(\tilde{A}), \tag{3.19}$$

since the boundary $\{x^1 = A\}$ becomes $\{y^1 = A_1\}$. To avoid confusion, we identify A with A_1 in what follows, that is we assume from the very beginning that we work with the variable $x^1 = y^1$.

Let us finish with the boundary determination results coming from (3.1). Recalling that the R-factor is given by (2.6), *i.e.*

$$R = \left(\frac{s^{11} s^{21} s^{31}}{c^4 \det S} \right)^{\frac{1}{4}},$$

we see from (3.5), (3.13), (3.15), (3.18) and (3.19) that

$$R(0) = \tilde{R}(0), \quad R(A) = \tilde{R}(\tilde{A}). \tag{3.20}$$

Finally, having in mind the definition of the diagonal component H_1^2 of the metric G

$$H_1^2 = \frac{c^4 \det S}{s^{11}},$$

we deduce from (3.13), (3.15), (3.18) and (3.19) that

$$H_1(0) = \tilde{H}_1(0), \quad H_1(A) = \tilde{H}_1(\tilde{A}). \tag{3.21}$$

Step 2. Assume next that the normal derivatives of the metrics G and $\tilde{G}$ coincide on the boundary $\partial\Omega$, *i.e.*

$$\partial_\nu G = \partial_{\tilde{\nu}} \tilde{G}, \quad \text{on } \partial\Omega. \tag{3.22}$$

At the boundary $\Omega_0 = \{x^1 = 0\}$, we observe that

$$\partial_\nu G_{|\Omega_0} = -\frac{1}{H_1(0)} \left((\partial_1 H_2^2)(0)(dx^2)^2 + (\partial_1 H_3^2)(0)(dx^3)^2 \right).$$

Hence from (3.22) we get

$$\frac{(\partial_1 H_j^2)(0)}{H_1(0)} = \frac{(\partial_1 \tilde{H}_j^2)(0)}{\tilde{H}_1(0)}, \quad j = 2, 3,$$

which can be rewritten using (3.3) and (3.21) as

$$(\partial_1 \log H_j^2)(0) = (\partial_1 \log \tilde{H}_j^2)(0), \quad j = 2, 3. \tag{3.23}$$

Recalling that

$$H_j^2 = \frac{c^4 \det S}{s^{j1}}, \quad j = 2, 3,$$

we infer from (3.23) that

$$\left\{\begin{array}{lcl}(\partial_1 \log c^4 \det S)(0) - (\partial_1 \log s^{21})(0) & = & (\partial_1 \log \tilde{c}^4 \det \tilde{S})(0) - (\partial_1 \log \tilde{s}^{21})(0), \\ (\partial_1 \log c^4 \det S)(0) - (\partial_1 \log s^{31})(0) & = & (\partial_1 \log \tilde{c}^4 \det \tilde{S})(0) - (\partial_1 \log \tilde{s}^{31})(0).\end{array}\right. \tag{3.24}$$

Note here that we can use the same change of variables of the form (1.18) as in Remark 3.1 to assume from the very beginning that

$$(\partial_1 \log s^{21})(0) = (\partial_1 \log \tilde{s}^{21})(0). \tag{3.25}$$

Indeed, it suffices to choose f_1 such that (3.17) and the additional condition

$$(\partial_1 \log s^{21})(0) - (\partial_1 \log f_1)(0) = (\partial_1 \log \tilde{s}^{21})(0), \tag{3.26}$$

hold. Hence we obtain from (3.24) and (3.25)

$$\left\{\begin{array}{rcl}(\partial_1 \log s^{21})(0) & = & (\partial_1 \log \tilde{s}^{21})(0), \\ (\partial_1 \log s^{31})(0) & = & (\partial_1 \log \tilde{s}^{31})(0), \\ (\partial_1 \log c^4 \det S)(0) & = & (\partial_1 \log \tilde{c}^4 \det \tilde{S})(0).\end{array}\right. \tag{3.27}$$

Of course, a similar analysis can be performed at the boundary component given by $\partial\Omega_1 = \{x^1 = A\}$. Thus we also obtain

$$\left\{\begin{array}{rcl}(\partial_1 \log s^{21})(A) & = & (\partial_1 \log \tilde{s}^{21})(\tilde{A}), \\ (\partial_1 \log s^{31})(A) & = & (\partial_1 \log \tilde{s}^{31})(\tilde{A}), \\ (\partial_1 \log c^4 \det S)(A) & = & (\partial_1 \log \tilde{c}^4 \det \tilde{S})(\tilde{A}).\end{array}\right. \tag{3.28}$$

Finally, recalling the definition of the contracted Christoffel symbol

$$\Gamma_1 = -\partial_1 \log \frac{H_2 H_3}{H_1} = -\frac{1}{2}\partial_1 \log \left(\frac{c^4 \det S\, s^{11}}{s^{21} s^{31}} \right) = -\frac{1}{2}\partial_1 \log \left(\frac{c^4 \det S}{s^{21} s^{31}} \right),$$

we immediately obtain from (3.27) and (3.28)

$$\Gamma_1(0) = \tilde{\Gamma}_1(0), \quad \Gamma_1(A) = \tilde{\Gamma}_1(\tilde{A}). \tag{3.29}$$

Remark 3.2. *The previous boundary determination results have been obtained using the variable y^1 defined by (1.18) such that (3.17) and (3.26) hold. Observe that, putting (3.18) into (3.7) and using (3.13), the use of the variable y^1 with the identification $A_1 = A$ implies the following equalities*

$$s_{12}(0) = \tilde{s}_{12}(0), \quad s_{13}(0) = \tilde{s}_{13}(0), \quad s_{12}(A) = \tilde{s}_{12}(\tilde{A}), \quad s_{13}(A) = \tilde{s}_{13}(\tilde{A}). \tag{3.30}$$

Similarly, the two first lines of (3.27) together with (3.5) can be rewritten as

$$T \begin{pmatrix} -s'_{13}(0) \\ s'_{12}(0) \end{pmatrix} = \tilde{T} \begin{pmatrix} -\tilde{s}'_{13}(0) \\ \tilde{s}'_{12}(0) \end{pmatrix}.$$

Hence, using (3.13) once more and a similar analysis at $y^1 = A$, we obtain the equalities

$$s_{12}'(0) = \tilde{s}_{12}'(0), \quad s_{13}'(0) = \tilde{s}_{13}'(0), \quad s_{12}'(A) = \tilde{s}_{12}'(\tilde{A}), \quad s_{13}'(A) = \tilde{s}_{13}'(\tilde{A}). \tag{3.31}$$

Notice that the equalities (3.30) and (3.31) could be used to define the change of variable y^1 from the outset.

Step 3. The previous results obtained in Steps 1 and 2 exhaust all the information that we can extract from boundary determination arguments. To go further, we need to exploit the particular structure (1.14) - (1.15) of the DN map.

Recall first that

$$\Lambda_G = \begin{pmatrix} \frac{-1}{H_1(0)} & 0 \\ 0 & \frac{1}{H_1(A)} \end{pmatrix} \left[\begin{pmatrix} \frac{\Gamma_1(0)}{2} & 0 \\ 0 & \frac{\Gamma_1(A)}{2} \end{pmatrix} + \begin{pmatrix} R(0) & 0 \\ 0 & R(A) \end{pmatrix} A_G \begin{pmatrix} \frac{1}{R(0)} & 0 \\ 0 & \frac{1}{R(A)} \end{pmatrix} \right]$$

where

$$A_G = \bigoplus_{m=1}^{\infty} A_G^m, \quad A_G^m := (A_G)_{|\langle Y_m \rangle} = \begin{pmatrix} M(\mu_m^2, \nu_m^2) & \frac{1}{\Delta(\mu_m^2, \nu_m^2)} \\ \frac{1}{\Delta(\mu_m^2, \nu_m^2)} & N(\mu_m^2, \nu_m^2) \end{pmatrix}.$$

From $\Lambda_G = \Lambda_{\tilde{G}}$ and from the boundary determination results (3.20), (3.21) and (3.29), we obtain immediately the equality between operators

$$A_G = A_{\tilde{G}}, \quad \text{on} \ \ L^2(\mathbb{T}^2;\, s^{11} dx^2 dx^3) \otimes L^2(\mathbb{T}^2;\, s^{11} dx^2 dx^3). \tag{3.32}$$

Denote by $(\omega_m, \varphi_m)_{m \in \mathbb{Z}^*}$ and $(\tilde{\omega}_m, \tilde{\varphi}_m)_{m \in \mathbb{Z}^*}$ the eigenvalues and eigenfunctions of A_G and $A_{\tilde{G}}$ respectively. As a consequence of (3.32) we have [7]

$$\omega_m = \tilde{\omega}_m, \quad \varphi_m = \tilde{\varphi}_m, \quad \forall m \in \mathbb{Z}^*. \tag{3.33}$$

Let us make explicit the eigenvalues $(\omega_m)_{m \in \mathbb{Z}^*}$ and their corresponding eigenfunctions $(\varphi_m)_{m \in \mathbb{Z}^*}$. Recall that on each harmonic $\langle Y_m \rangle$, $m \geq 1$, the operator A_G simplifies in the 2×2-matrix given by (1.15), *i.e.*

$$A_G^m = \begin{pmatrix} M(\mu_m^2, \nu_m^2) & \frac{1}{\Delta(\mu_m^2, \nu_m^2)} \\ \frac{1}{\Delta(\mu_m^2, \nu_m^2)} & N(\mu_m^2, \nu_m^2) \end{pmatrix},$$

Diagonalizing, we obtain two eigenvalues attached to each index $m \geq 1$

$$\omega_{\pm m} := \frac{M_m + N_m \pm \sqrt{(M_m - N_m)^2 + \frac{4}{\Delta_m^2}}}{2}, \quad m \geq 1, \tag{3.34}$$

[7] In fact, the common eigenvalues and corresponding eigenspaces (ω_m, E_m) of A_G and $A_{\tilde{G}}$ coincide. But remember that the operator A_G does not depend on the choice of a Hilbert basis within an eigenspace E_m so that without loss of generality we can also identify the eigenfunctions.

associated to the eigenfunctions

$$\varphi_m^+ = \begin{pmatrix} 1 \\ x_m^+ \end{pmatrix} \otimes Y_m, \quad \varphi_m^- = \begin{pmatrix} 1 \\ x_m^- \end{pmatrix} \otimes Y_m, \quad m \geq 1, \tag{3.35}$$

where we used the shorthand notation

$$\Delta_m = \Delta(\mu_m^2, \nu_m^2), \quad M_m = M(\mu_m^2, \nu_m^2), \quad N_m = N(\mu_m^2, \nu_m^2),$$

and

$$x_m^\pm = \frac{\Delta_m}{2}\left(N_m - M_m \pm \sqrt{(M_m + N_m)^2 + \frac{4}{\Delta_m^2}}\right).$$

In particular, using (3.33), (3.34) and (3.35), we see that there exists a bijection $\theta : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ such that

$$Y_m = \tilde{Y}_{\theta(m)}, \quad \forall m \geq 1. \tag{3.36}$$

It is not clear at this stage why the bijection θ should be the identity. However, we can proceed in the following way. Recall from (1.10) and (1.11) that

$$\begin{pmatrix} -\partial_2^2 \\ -\partial_3^2 \end{pmatrix} = -T \begin{pmatrix} H \\ L \end{pmatrix} + \begin{pmatrix} \phi_2 \\ \phi_3 \end{pmatrix} = -\tilde{T} \begin{pmatrix} \tilde{H} \\ \tilde{L} \end{pmatrix} + \begin{pmatrix} \tilde{\phi}_2 \\ \tilde{\phi}_3 \end{pmatrix}, \tag{3.37}$$

where T denotes the 2×2-matrix (3.8) and $T = \tilde{T}$ thanks to (3.13). If we apply the operators in the equality (3.37) to $Y_m = \tilde{Y}_{\theta(m)}$, we thus obtain

$$\left[-T \begin{pmatrix} \mu_m^2 \\ \nu_m^2 \end{pmatrix} + \begin{pmatrix} \phi_2 \\ \phi_3 \end{pmatrix}\right] \otimes Y_m = \left[-T \begin{pmatrix} \tilde{\mu}_{\theta(m)}^2 \\ \tilde{\nu}_{\theta(m)}^2 \end{pmatrix} + \begin{pmatrix} \tilde{\phi}_2 \\ \tilde{\phi}_3 \end{pmatrix}\right] \otimes Y_m. \tag{3.38}$$

If we divide (3.38) by Y_m (outside the nodal sets), we get using the invertibility of T that

$$T^{-1} \begin{pmatrix} \tilde{\phi}_2 - \phi_2 \\ \tilde{\phi}_3 - \phi_3 \end{pmatrix} = \begin{pmatrix} \tilde{\mu}_{\theta(m)}^2 - \mu_m^2 \\ \tilde{\nu}_{\theta(m)}^2 - \nu_m^2 \end{pmatrix}. \tag{3.39}$$

The remarkable properties of Stäckel metrics manifest themselves again here since the left-hand side of (3.39) depends on the variables (x^2, x^3) while the right-hand side of (3.39) is given by constants. Hence we infer that there exist two constants B_1 and B_2 such that

$$T^{-1} \begin{pmatrix} \tilde{\phi}_2 - \phi_2 \\ \tilde{\phi}_3 - \phi_3 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} \tilde{\mu}_{\theta(m)}^2 - \mu_m^2 \\ \tilde{\nu}_{\theta(m)}^2 - \nu_m^2 \end{pmatrix}. \tag{3.40}$$

Putting the first equality of (3.40) into (3.37), we get easily

$$\begin{pmatrix} \tilde{H} \\ \tilde{L} \end{pmatrix} = \begin{pmatrix} H \\ L \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \tag{3.41}$$

that is the angular operators (H, L) differ from $(\tilde{H}, \tilde{L})$ by mere constants B_1 and B_2. In particular, their common eigenfunctions are the same

$$Y_m = \tilde{Y}_m, \quad \forall m \geq 1, \tag{3.42}$$

and their corresponding eigenvalues differ by the same constants

$$\tilde{\mu}_m^2 = \mu_m^2 + B_1, \quad \tilde{\nu}_m^2 = \nu_m^2 + B_2, \quad \forall m \geq 1.$$

Finally, as pointed out in Section 2.2, after Proposition 2.2, we can set the constants B_1 and B_2 equal to zero since this corresponds to an intrinsic invariance of the separation of variables' procedure. We thus assume from now on that

$$\mu_m^2 = \tilde{\mu}_m^2, \quad \nu_m^2 = \tilde{\nu}_m^2, \quad \forall m \geq 1, \tag{3.43}$$

and (from (3.40))

$$\phi_2 = \tilde{\phi}_2, \quad \phi_3 = \tilde{\phi}_3. \tag{3.44}$$

Summary. At this stage of the proof, under our main assumption $\Lambda_G = \Lambda_{\tilde{G}}$, we have shown

$$\begin{pmatrix} s_{22} & s_{23} \\ s_{32} & s_{33} \end{pmatrix} = \begin{pmatrix} \tilde{s}_{22} & \tilde{s}_{23} \\ \tilde{s}_{32} & \tilde{s}_{33} \end{pmatrix}, \quad \phi_2 = \tilde{\phi}_2, \quad \phi_3 = \tilde{\phi}_3,$$

and

$$\mu_m^2 = \tilde{\mu}_m^2, \quad \nu_m^2 = \tilde{\nu}_m^2, \quad Y_m = \tilde{Y}_m, \quad \forall m \geq 1.$$

3.3 The multi-parameter CAM method

We continue extracting more informations on the metrics G and $\tilde{G}$ from the equality between operators

$$A_G = A_{\tilde{G}},$$

and the coincidence between the angular parts of the metrics G and $\tilde{G}$ that leads to the equalities

$$\mu_m^2 = \tilde{\mu}_m^2, \quad \nu_m^2 = \tilde{\nu}_m^2, \quad Y_m = \tilde{Y}_m, \quad \forall m \geq 1,$$

proved in Section 3.2. Using the definition of A_G, this implies in particular that

$$M(\mu_m^2, \nu_m^2) = \tilde{M}(\mu_m^2, \nu_m^2), \quad \forall m \geq 1. \tag{3.45}$$

Hence the two WT functions associated to the separated ODE

$$-u_m'' + [\,\mu_m^2 s_{12}(x^1) + \nu_m^2 s_{13}(x^1) - \phi_1(x^1)]\, u_m \;=\; 0, \qquad x^1 \in (0, A), \tag{3.46}$$

$$-\tilde{u}_m'' + [\,\mu_m^2 \tilde{s}_{12}(x^1) + \nu_m^2 \tilde{s}_{13}(x^1) - \tilde{\phi}_1(x^1)]\, \tilde{u}_m \;=\; 0, \qquad x^1 \in (0, \tilde{A}), \tag{3.47}$$

coincide when evaluated on the joint spectrum $J := \{(\mu_m^2, \nu_m^2),\ m \geq 1\}$.

Our first task will be to show that the equality (3.45) on the discrete subset J can be extended to the whole plane $\mathbb{C}^2$, *i.e.*

$$M(\mu^2, \nu^2) = \tilde{M}(\mu^2, \nu^2), \quad \forall (\mu, \nu) \in \mathbb{C}^2 \backslash \{poles\} \tag{3.48}$$

The passage from (3.45) to (3.48) is what we call the multi-parameter CAM method and turns out to be the central technical tool from which we will able to solve the inverse problem. To do this, note first that (3.45) can be rewritten using (2.17) as

$$D(\mu_m^2, \nu_m^2)\tilde{\Delta}(\mu_m^2, \nu_m^2) - \tilde{D}(\mu_m^2, \nu_m^2)\Delta(\mu_m^2, \nu_m^2) = 0, \quad \forall m \geq 1. \tag{3.49}$$

Define now the function

$$F(\mu, \nu) := D(\mu^2, \nu^2)\tilde{\Delta}(\mu^2, \nu^2) - \tilde{D}(\mu^2, \nu^2)\Delta(\mu^2, \nu^2). \tag{3.50}$$

Then F is clearly analytic [8] on $\mathbb{C}^2$ and vanishes on the "square-root" of the joint spectrum J thanks to (3.49). Hence, in order to prove (3.48), it will be enough to prove that F vanishes identically.

To go further, we will use the following result of Berndtsson [6] (that we took from Bloom [7]) which provides a sufficient condition for a discrete set to be a *uniqueness set* of a bounded analytic function of several variables.

Theorem 3.1 (Berndtsson, 1978). *Let K be an open cone in $\mathbb{R}^n$ with vertex at the origin and $T(K) = \{z \in \mathbb{C}^n \ / \ \Re(z) \in K\}$. Suppose f is bounded and analytic on $T(K)$. Let E be a discrete subset of K such that for some constant $h > 0$, $e_1, e_2 \in E$ implies that $|e_1 - e_2| \geq h$. Let $n(r) = \# E \cap B(0, r)$. Assume that f vanishes on E. Then f is identically 0 if*

$$\overline{\lim_{r \to \infty}} \frac{n(r)}{r^n} > 0.$$

In order to apply Theorem 3.1, we need to define an analytic function that is bounded on a conic set of the form $T(K)$ and that satisfies the above properties. The natural candidate - the function F - is not bounded and we need to rescale it in a convenient way. Hence let us first prove some universal estimates for F.

Proposition 3.1. *There exist positive constants $\bar{A}, \bar{B}, C > 0$ such that for all $(\mu, \nu) \in \mathbb{C}^2$*

$$|D(\mu^2, \nu^2)|,\ |\Delta(\mu^2, \nu^2)| \ \leq \ C\, e^{\frac{\bar{A}}{2}|\Re(\mu)| + \frac{\bar{B}}{2}|\Re(\nu)|}.$$

As a consequence,

$$|F(\mu, \nu)| \ \leq \ C\, e^{\bar{A}|\Re(\mu)| + \bar{B}|\Re(\nu)|}.$$

[8] The functions c_j, s_j, $j = 0, 1$ and thus the functions Δ, D and F are analytic in the variables μ and ν independently thanks to standard theorems on ODE depending analytically on parameters. Hence the function F is analytic on $\mathbb{C}^2$ due to the Hartogs Theorem.

The proof of this Proposition requires some preliminary steps.

Step 1. We first prove some estimates in μ and ν independently.

Lemma 3.1. *1. For each $\nu \in \mathbb{C}$ fixed, there exists positive constants $\bar{A}$, $C(\nu) > 0$ such that*

$$|D(\mu^2,\nu^2)|,\ |\Delta(\mu^2,\nu^2)|,\ |F(\mu,\nu)| \ \leq \ C(\nu)\, e^{\bar{A}|\Re(\mu)|}.$$

2. For each $\mu \in \mathbb{C}$ fixed, there exists positive constants $\bar{B}$, $C(\nu) > 0$ such that

$$|D(\mu^2,\nu^2)|,\ |\Delta(\mu^2,\nu^2)|,\ |F(\mu,\nu)| \ \leq \ C(\mu)\, e^{\bar{B}|\Re(\nu)|}.$$

In order to prove this Lemma, we need to recast the separated ODE (3.46) and (3.47) into *normal forms*, that is Schrödinger equations with spectral parameters $-\mu^2$ or $-\nu^2$. For instance, if we choose $-\mu^2$ as spectral parameter, we introduce the new radial coordinate

$$u^1 = \int_0^{x^1} \sqrt{s_{12}(t)}dt \quad \in \quad [0,\bar{A}], \tag{3.51}$$

and remark that, if $u(x^1,\mu^2,\nu^2)$ is a solution of the separated ODE

$$-u'' + [\,\mu^2 s_{12}(x^1) + \nu^2 s_{13}(x^1) - \phi_1(x^1)]\, u = 0,$$

then the function $U(u^1,\mu^2,\nu^2) := (s_{12}(x^1(u^1)))^{\frac{1}{4}} u(x^1(u^1),\mu^2,\nu^2)$ is a solution of the ODE

$$-\ddot{U} + q_\nu(u^1)U = -\mu^2 U, \tag{3.52}$$

where

$$q_\nu = \nu^2 \bar{s}_{13} - \bar{\phi}_1, \quad \bar{s}_{13} = \frac{s_{13}}{s_{12}}, \quad \bar{\phi}_1 = \frac{\phi_1}{s_{12}} - \frac{(\log\dot{} s_{12})^2}{16} + \frac{(\log\ddot{} s_{12})}{4}. \tag{3.53}$$

Here, the notation $\dot{}$ denotes the derivative with respect to u^1. For fixed $\nu \in \mathbb{C}$, (3.52) is now a classical Schrödinger equation with $-\mu^2$ as spectral parameter.

Introduce the FSS $\{U_0, V_0\}$ and $\{U_1, V_1\}$ of (3.52) defined by the Cauchy conditions

$$\begin{aligned} U_0(0) = 1,\ \dot{U}_0(0) = 0, \quad V_0(0) = 0,\ \dot{V}_0(0) = 1, \\ U_1(\bar{A}) = 1,\ \dot{U}_1(\bar{A}) = 0, \quad V_1(\bar{A}) = 0,\ \dot{V}_1(\bar{A}) = 1, \end{aligned}$$

as well as the characteristic and WT functions

$$\Delta_{q_\nu}(\mu^2) = W(V_0,V_1), \quad D_{q_\nu}(\mu^2) = W(U_0,V_1), \quad M_{q_\nu}(\mu^2) = -\frac{D_{q_\nu}(\mu^2)}{\Delta_{q_\nu}(\mu^2)}, \tag{3.54}$$

where $W(f,g) := f\dot{g} - \dot{f}g$ is the Wronskian. Introduce also the FSS $\{C_0, S_0\}$ and $\{C_1, C_1\}$ of (3.52) defined by

$$\begin{aligned} C_j(u^1,\mu^2,\nu^2) := (s_{12}(x^1(u^1)))^{\frac{1}{4}} c_j(x^1(u^1),\mu^2,\nu^2), \quad j = 0,1, \\ S_j(u^1,\mu^2,\nu^2) := (s_{12}(x^1(u^1)))^{\frac{1}{4}} s_j(x^1(u^1),\mu^2,\nu^2), \quad j = 0,1. \end{aligned} \tag{3.55}$$

Then a straighforward though tedious calculation shows that

$$C_0 = (s_{12}(0))^{\frac{1}{4}} U_0 + \left(\frac{s_{12}'(0)}{4(s_{12}(0))^{\frac{5}{4}}} \right) V_0, \quad S_0 = \frac{V_0}{(s_{12}(0))^{\frac{1}{4}}}, \tag{3.56}$$

and

$$C_1 = (s_{12}(A))^{\frac{1}{4}} U_1 + \left(\frac{s_{12}'(A)}{4(s_{12}(A))^{\frac{5}{4}}} \right) V_1, \quad S_1 = \frac{V_A}{(s_{12}(A))^{\frac{1}{4}}}. \tag{3.57}$$

From (3.56) and (3.57), we also obtain

$$\Delta(\mu^2, \nu^2) = \frac{1}{(s_{12}(0)\, s_{12}(A))^{\frac{1}{4}}} \Delta_{q_\nu}(\mu^2), \tag{3.58}$$

$$D(\mu^2, \nu^2) = \left(\frac{s_{12}(0)}{s_{12}(A)} \right)^{\frac{1}{4}} D_{q_\nu}(\mu^2) + \frac{s_{12}'(0)}{4(s_{12}(0))^{\frac{5}{4}} (s_{12}(A))^{\frac{1}{4}}} \Delta_{q_\nu}(\mu^2), \tag{3.59}$$

and

$$M(\mu^2, \nu^2) = \frac{1}{4} (\log s_{12})'(0) + \sqrt{s_{12}(0)}\, M_{q_\nu}(\mu^2). \tag{3.60}$$

The interest in introducing such a normal form comes from the existence of universal asymptotics as $|\mu| \to \infty$ for the functions U_j, V_j, $j = 0, 1$ (see [65], Theorem 3, p.13). Precisely we have for fixed $\nu \in \mathbb{C}$

$$\left\{ \begin{array}{lcl} U_0(u^1, \mu^2, \nu^2) & = & \cosh(\mu u^1) + O\left(\frac{1}{|\mu|} e^{|\Re(\mu)| u^1 + \|q_\nu\| \sqrt{u^1}} \right), \\ \dot{U}_0(u^1, \mu^2, \nu^2) & = & -\mu \sinh(\mu u^1) + O\left(\|q_\nu\| e^{|\Re(\mu)| u^1 + \|q_\nu\| \sqrt{u^1}} \right), \\ V_0(u^1, \mu^2, \nu^2) & = & \frac{\sinh(\mu u^1)}{\mu} + O\left(\frac{1}{|\mu|^2} e^{|\Re(\mu)| u^1 + \|q_\nu\| \sqrt{u^1}} \right), \\ \dot{V}_0(u^1, \mu^2, \nu^2) & = & \cosh(\mu u^1) + O\left(\frac{\|q_\nu\|}{|\mu|} e^{|\Re(\mu)| u^1 + \|q_\nu\| \sqrt{u^1}} \right), \end{array} \right. \quad |\mu| \to \infty, \tag{3.61}$$

and

$$\left\{ \begin{array}{lcl} U_1(u^1, \mu^2, \nu^2) & = & \cosh(\mu(\bar{A} - u^1)) + O\left(\frac{1}{|\mu|} e^{|\Re(\mu)|(\bar{A} - u^1) + \|q_\nu\| \sqrt{\bar{A} - u^1}} \right), \\ \dot{U}_1(u^1, \mu^2, \nu^2) & = & -\mu \sinh(\mu(\bar{A} - u^1)) + O\left(\|q_\nu\| e^{|\Re(\mu)|(\bar{A} - u^1) + \|q_\nu\| \sqrt{\bar{A} - u^1}} \right), \\ V_1(u^1, \mu^2, \nu^2) & = & -\frac{\sinh(\mu(\bar{A} - u^1))}{\mu} + O\left(\frac{1}{|\mu|^2} e^{|\Re(\mu)|(\bar{A} - u^1) + \|q_\nu\| \sqrt{\bar{A} - u^1}} \right), \\ \dot{V}_1(u^1, \mu^2, \nu^2) & = & \cosh(\mu(\bar{A} - u^1)) + O\left(\frac{\|q_\nu\|}{|\mu|} e^{|\Re(\mu)|(\bar{A} - u^1) + \|q_\nu\| \sqrt{\bar{A} - u^1}} \right), \end{array} \right. \quad |\mu| \to \infty. \tag{3.62}$$

From these universal estimates and the definition of $\Delta_{q_\nu}(\mu^2)$, $D_{q_\nu}(\mu^2)$ given by (3.54), we obtain that for each fixed $\nu \in \mathbb{C}$, there exists a constant $C(\nu)$ such that

$$\Delta_{q_\nu}(\mu^2) = \frac{\sinh(\bar{A}\mu)}{\mu} + O\left(C(\nu) \frac{e^{|\Re(\mu)| \bar{A}}}{|\mu|^2} \right), \quad |\mu| \to \infty, \tag{3.63}$$

and

$$D_{q_\nu}(\mu^2) = \cosh(\bar{A}\mu) + O\left(C(\nu) \frac{e^{|\Re(\mu)|\bar{A}}}{|\mu|} \right), \quad |\mu| \to \infty. \tag{3.64}$$

We can now give the proof of Lemma 3.1.

Proof of Lemma 3.1. The proof of 1. follows directly from the estimates (3.63) and (3.64) together with (3.50), (3.58) and (3.59). The proof of 2. is similar to 1. inverting the role [9] of the spectral parameters μ^2 and ν^2. We leave the details to the readers. □

Step 2. Second we need a uniform estimate when $(\mu, \nu) = (iy, iy') \in (i\mathbb{R})^2$.

Lemma 3.2. *There exists a constant $C > 0$ such that for all $(y, y') \in \mathbb{R}^2$*

$$|D(-y^2, -y'^2)|,\ |\Delta(-y^2, -y'^2)|,\ |F(iy, iy')| \ \leq \ C.$$

Proof. When $(\mu, \nu) = (iy, iy') \in (i\mathbb{R})^2$, the separated ODE (3.46) takes the form

$$-u'' - \phi_1(x^1) u = \omega^2 r_{yy'}(x^1) u, \tag{3.65}$$

where

$$\omega^2 = y^2 + y'^2, \quad r_{yy'}(x^1) = \frac{y^2 s_{12}(x^1) + y'^2 s_{13}(x^1)}{y^2 + y'^2}. \tag{3.66}$$

We introduce the change of variable (which is dependent on y, y'!)

$$w^1 = w^1_{yy'} = \int_0^{x^1} \sqrt{r_{yy'}(t)} dt \quad \in \quad [0, \bar{C}_{yy'}],$$

and remark that, if $u(x^1, \mu^2, \nu^2)$ is a solution of the separated ODE

$$-u'' + [\,\mu^2 s_{12}(x^1) + \nu^2 s_{13}(x^1) - \phi_1(x^1)]\, u = 0,$$

then the function $W(w^1, -y^2, -y'^2) := (r_{yy'}(x^1(w^1)))^{\frac{1}{4}} W(x^1(w^1), -y^2, -y'^2)$ is a solution of the ODE

$$-\ddot{W} + q_{yy'}(w^1) W = \omega^2 W, \tag{3.67}$$

[9] For this, it suffices to use the new coordinate

$$v^1 = \int_0^{x^1} \sqrt{s_{13}(t)} dt \quad \in \quad [0, \bar{B}],$$

and remark that, if $u(x^1, \mu^2, \nu^2)$ is a solution of the separated ODE $-u'' + [\,\mu^2 s_{12}(x^1) + \nu^2 s_{13}(x^1) - \phi_1(x^1)]\, u = 0$, then the function $V(v^1, \mu^2, \nu^2) := (s_{13}(x^1(v^1)))^{\frac{1}{4}} u(x^1(v^1), \mu^2, \nu^2)$ is a solution of the ODE

$$-\ddot{V} + q_\mu(v^1) V = -\nu^2 V, \quad q_\mu = \mu^2 \frac{s_{12}}{s_{13}} - \frac{\phi_1}{s_{13}} + \frac{(\log \dot{s}_{13})^2}{16} - \frac{(\log \ddot{s}_{13})}{4}.$$

where

$$q_{yy'} = -\frac{\phi_1}{r_{yy'}} + \frac{(\log \dot{r}_{yy'})^2}{16} - \frac{(\log \ddot{r}_{yy'})}{4}. \tag{3.68}$$

We also observe the following properties related to this change of variable. Since s_{12}, s_{13} are positive and C^∞ on $[0, A]$, we deduce first that there exist positive constants $c, C > 0$ such that

$$c \leq r_{yy'} \leq C,$$

and second that the potential $q_{yy'}$ is uniformly bounded with respect to $(y, y') \in \mathbb{R}^2$. Finally, the variable w^1 lives on the (y, y')-dependent interval $[0, \bar{C} = \bar{C}_{yy'}]$ whose length satisfies the uniform estimate

$$\sqrt{c}A \leq \bar{C}_{yy'} = \int_0^A \sqrt{r_{yy'}(t)}dt \leq \sqrt{C}A.$$

Similarly to what we did previously, we introduce the FSS $\{W_0, X_0\}$ and $\{W_1, X_1\}$ of (3.67) defined by the Cauchy conditions

$$\begin{array}{rr} W_0(0) = 1,\ \dot{W}_0(0) = 0, & X_0(0) = 0,\ \dot{X}_0(0) = 1, \\ W_1(\bar{C}) = 1,\ \dot{W}_1(\bar{C}) = 0, & X_1(\bar{C}) = 0,\ \dot{X}_1(\bar{C}) = 1, \end{array}$$

as well as the characteristic functions

$$\Delta_{q_{yy'}}(\omega^2) = W(X_0, X_1), \quad D_{q_{yy'}}(\omega^2) = W(W_0, X_1). \tag{3.69}$$

These new characteristic functions are related to the initial ones by the formulas

$$\Delta(-y^2, -y'^2) = \frac{1}{(r_{yy'}(0)\, r_{yy'}(A))^{\frac{1}{4}}} \Delta_{q_{yy'}}(\omega^2), \tag{3.70}$$

$$D(-y^2, -y'^2) = \left(\frac{r_{yy'}(0)}{r_{yy'}(A)}\right)^{\frac{1}{4}} D_{q_{yy'}}(\omega^2) + \frac{r'_{yy'}(0)}{4(r_{yy'}(0))^{\frac{5}{4}}(r_{yy'}(A))^{\frac{1}{4}}} \Delta_{q_{yy'}}(\omega^2). \tag{3.71}$$

It is thus enough to show that the new characteristic functions are uniformly bounded on $(y, y') \in \mathbb{R}^2$. This can be done once again using the universal estimates from [65]. Precisely, since $\Im(\omega) = 0$, we have

$$\left\{ \begin{array}{rcl} W_0(w^1, -y^2, -y'^2) & = & \cos(\omega w^1) + O\left(\frac{1}{|\omega|} e^{\|q_{yy'}\| \sqrt{w^1}}\right), \\ \dot{W}_0(w^1, -y^2, -y'^2) & = & -\omega \sin(\omega w^1) + O\left(\|q_{yy'}\| e^{\|q_{yy'}\| \sqrt{w^1}}\right), \\ X_0(w^1, -y^2, -y'^2) & = & \frac{\sin(\omega w^1)}{\omega} + O\left(\frac{1}{|\omega|^2} e^{\|q_{yy'}\| \sqrt{w^1}}\right), \\ \dot{X}_0(w^1, -y^2, -y'^2) & = & \cos(\omega w^1) + O\left(\frac{\|q_{yy'}\|}{|\omega|} e^{\|q_{yy'}\| \sqrt{w^1}}\right), \end{array} \right. \quad |\omega| \to \infty, \tag{3.72}$$

and

$$
\begin{cases}
W_1(w^1,-y^2,-y'^2) &=& \cos(\omega(\bar{C}-w^1)) + O\left(\frac{1}{|\omega|}e^{\|q_{yy'}\|\sqrt{\bar{C}-w^1}}\right), \\
\dot{W}_1(w^1,-y^2,-y'^2) &=& -\omega\sin(\omega(\bar{C}-w^1)) + O\left(\|q_{yy'}\|e^{\|q_{yy'}\|\sqrt{\bar{C}-w^1}}\right), \\
X_1(w^1,-y^2,-y'^2) &=& -\frac{\sin(\omega(\bar{C}-w^1))}{\omega} + O\left(\frac{1}{|\omega|^2}e^{\|q_{yy'}\|\sqrt{\bar{C}-w^1}}\right), \\
\dot{X}_1(w^1,-y^2,-y'^2) &=& \cos(\omega(\bar{C}-w^1)) + O\left(\frac{\|q_{yy'}\|}{|\omega|}e^{\|q_{yy'}\|\sqrt{\bar{C}-w^1}}\right),
\end{cases}
\quad |\omega| \to \infty. \tag{3.73}
$$

Therefore we obtain (since the potentials $q_{yy'}$ and $\bar{C}_{yy'}$ are uniformly bounded with respect to $(y,y') \in \mathbb{R}^2$)

$$
\Delta_{q_{yy'}}(\omega^2) = \frac{\sin(\bar{C}_{yy'}\omega)}{\omega} + O\left(\frac{1}{|\omega|^2}\right), \quad |\omega| \to \infty, \tag{3.74}
$$

and

$$
D_{q_{yy'}}(\omega^2) = \cos(\bar{C}_{yy'}\omega) + O\left(\frac{1}{|\omega|}\right), \quad |\omega| \to \infty. \tag{3.75}
$$

Finally, we deduce from (3.50), (3.70), (3.71), (3.74) and (3.75) that there exists a constant $C > 0$ such that

$$
|F(iy,iy')| \leq \frac{C}{|\omega|}, \quad |\omega| \to \infty. \tag{3.76}
$$

The claim of the Lemma follows from (3.76).

□

We can now finish the proof of Proposition 3.1 by applying twice the Phragmen-Lindelöf principle.

Proof of Proposition 3.1. First we fix $\nu \in i\mathbb{R}$. According to Lemmas 3.1 and 3.2, the analytic function $\mu \longrightarrow F(\mu,\nu)$ satisfies

$$
\begin{cases}
|F(\mu,\nu)| \leq C(\nu)e^{\bar{A}|\Re(\mu)|}, & \forall \mu \in \mathbb{C}, \\
|F(\mu,\nu)| \leq C, & \forall \mu \in i\mathbb{R}.
\end{cases}
$$

Hence the Phragmen-Lindelöf principle (see for instance [61], Lecture 6., Theorem 3) yields

$$
|F(\mu,\nu)| \leq Ce^{\bar{A}|\Re(\mu)|}, \quad \forall (\mu,\nu) \in (\mathbb{C}, i\mathbb{R}). \tag{3.77}
$$

Second we fix $\mu \in \mathbb{C}$. Then, according to Lemma 3.1 and (3.77), the analytic function $\nu \longrightarrow F(\mu,\nu)$ satisfies

$$
\begin{cases}
|F(\mu,\nu)| \leq C(\mu)e^{\bar{B}|\Re(\nu)|}, & \forall \nu \in \mathbb{C}, \\
|F(\mu,\nu)| \leq Ce^{\bar{A}|\Re(\mu)|}, & \forall \nu \in i\mathbb{R}.
\end{cases}
$$

Applying once again the Phragmen-Lindelöf principle, we obtain

$$|F(\mu,\nu)| \leq Ce^{\bar{A}|\Re(\mu)|+\bar{B}|\Re(\nu)|}, \quad \forall(\mu,\nu)\in\mathbb{C}^2,$$

which proves the Proposition. □

We can now apply Theorem 3.1. First define the analytic function

$$f(\mu,\nu) := F(\mu,\nu)e^{-\bar{A}\mu-\bar{B}\nu},$$

where $\bar{A}$ and $\bar{B}$ are the positive constants appearing in Proposition 3.1. Then it is clear from Proposition 3.1 that f is bounded and analytic on the set

$$T((\mathbb{R}^+)^2) = \{(\mu,\nu)\in\mathbb{C}^2 \ | \ \Re(\mu,\nu)\in(\mathbb{R}^+)^2\}.$$

Second define the cone

$$\mathcal{C}_\epsilon = \{(\mu,\theta\mu)\in(\mathbb{R}^+)^2 \ / \ \mu\in\mathbb{R}^+, \ \sqrt{c_1+\epsilon}\leq\theta\leq\sqrt{c_2-\epsilon}\}, \quad 0<\epsilon<<1, \tag{3.78}$$

where

$$c_1 = \max\left(-\frac{s_{32}}{s_{33}}\right), \quad c_2 = \min\left(-\frac{s_{22}}{s_{23}}\right). \tag{3.79}$$

Remark 3.3. *The fact that $c_1 < c_2$ is ensured by Proposition 2.2.*

Define also the discrete set

$$E_M = \{(\mu_m,\nu_m)\in(\mathbb{R}^+)^2 \ / \ m\geq M\}, \tag{3.80}$$

where M is chosen large enough to ensure [10] that for all $m\geq M$, the joint spectrum (μ_m^2,ν_m^2) of the angular operators (H,L) belongs to $(\mathbb{R}^+)^2$. In that case, (μ_m,ν_m) simply denotes the positive square root of (μ_m^2,ν_m^2).

We now recall the following results shown by Gobin [33], Appendices B and C.

Lemma 3.3. *1. There exists $h>0$ such that*

$$|e_1-e_2|\geq h, \quad \forall(e_1,e_2)\in(E_m\cap\mathcal{C}_\epsilon)^2.$$

2. Set $N(r) = \#(E_m\cap\mathcal{C}_e)\cap B(0,r)$. Then

$$\overline{\lim_{r\to\infty}}\frac{N(r)}{r^2}>0.$$

[10] It has been shown by Gobin [33], Lemma 2.9., that there exist constants C_1, C_2, D_1, D_2 such that for all $m\geq 1$

$$C_1\mu_m^2 + D_1 \leq \nu_m^2 \leq C_2\mu_m^2 + D_2,$$

where

$$C_1 = \min\left(-\frac{s_{32}}{s_{33}}\right) > 0, \quad C_2 = \max\left(-\frac{s_{22}}{s_{23}}\right) > 0.$$

This result implies the asserted claim.

Remarks 3.1. ***1.*** *The proof of the second assertion in Lemma 3.3 is an application of the papers [12, 13] by Colin de Verdière in which the author studies basic properties of the joint spectrum of commuting pseudo-differential operators on manifolds (such as the density of the joint spectrum in certain cones which is needed here, or Bohr-Sommerfeld quantization formulas).*

2. *In [12], the joint spectrum J is counted with multiplicity and thus the density $N(r)$ estimated in Lemma 3.3 also counts this multiplicity. However, the discrete subset E considered by Berndtsson in Theorem 3.1 does not count multiplicity and so for the density $n(r)$. Fortunately, we can show easily that each (μ_m^2, ν_m^2) in the joint spectrum J has at most multiplicity 4 (see Gobin [33], Remark 2.7). Hence the density $N(r)$ calculated with Colin de Verdière's work differs at most by a factor 4 to the density $n(r)$ needed in Theorem 3.1. In all cases, the density $n(r)$ remains of order r^2 and we can apply Berndtsson's Theorem 3.1.*

Hence applying Theorem 3.1 to the bounded and analytic function $f(\mu,\nu)$ on $T(\mathcal{C}_\epsilon)$, we see that f vanishes identically on $T(\mathcal{C}_\epsilon)$ and thus on $\mathbb{C}^2$ by analytic continuation. Using the definition of f, we infer that the function $F(\mu,\nu)$ vanishes identically on $\mathbb{C}^2$ and by definition (3.50) of F, this means that

$$M(\mu^2,\nu^2) = \tilde{M}(\mu^2,\nu^2), \quad \forall (\mu,\nu) \in \mathbb{C}^2 \backslash \{poles\} \tag{3.81}$$

It remains now to extract the new information given by (3.81). Instead of working with the WT function $M(\mu^2,\nu^2)$ that depends on two variables, we prefer to work with the more classical WT function $M_{q_\nu}(\mu^2)$ defined in (3.54) that depends on one variable. Recalling the link between the two different WT functions given by (3.60) and using (3.30) - (3.31), we infer from (3.81) that

$$M_{q_\nu}(\mu^2) = \tilde{M}_{\tilde{q}_\nu}(\mu^2), \quad \forall (\mu,\nu) \in \mathbb{C}^2. \tag{3.82}$$

For each fixed $\nu \in \mathbb{C}$, we can apply the Borg-Marchenko Theorem (see for instance [5, 8, 9, 32, 75]) and obtain from (3.82)

$$\left\{ \begin{array}{rcl} \bar{A} & = & \bar{\tilde{A}}, \\ q_\nu(u^1) & = & \tilde{q}_\nu(u^1), \quad \forall u^1 \in [0,\bar{A}], \ \forall \nu \in \mathbb{C}. \end{array} \right. \tag{3.83}$$

Observe that working in the variable u^1, the cylinders Ω and $\tilde{\Omega}$ are exactly the same. Moreover, thanks to the definition (3.53) of the potential q_ν and playing with two different values of $\nu \in \mathbb{C}$, we finally obtain

$$\left\{ \begin{array}{rcl} \bar{s}_{13}(u^1) & = & \bar{\tilde{s}}_{13}(u^1), \quad \forall u^1 \in [0,\bar{A}], \\ \bar{\phi}_1(u^1) & = & \bar{\tilde{\phi}}_1(u^1), \quad \forall u^1 \in [0,\bar{A}], \end{array} \right. \tag{3.84}$$

where we recall that

$$\bar{s}_{13}(u^1) = \frac{s_{13}(x^1(u^1))}{s_{12}(x^1(u^1))}, \quad \bar{\phi}_1 = \frac{\phi_1}{s_{12}} - \frac{(\log \dot{s}_{12})^2}{16} + \frac{(\log \ddot{s}_{12})}{4}.$$

Let us pause a moment and see what information we have obtained exactly. It is easier to describe now our conformally Stäckel manifolds (M,G) and $(\tilde{M},\tilde{G})$ in the unique coordinate system (u^1, u^2, u^3) given by

$$u^1 = (3.51), \quad u^2 = x^2, \quad u^3 = x^3, \tag{3.85}$$

and remember that the change of variable u^1 does not break the conformally Stäckel structure of the metrics. In that case, we have shown (see (1.19) - (1.23)) that

$$\Omega = \tilde{\Omega} = [0, \bar{A}] \times \mathbb{T}^2,$$

and

$$\bar{G} = \bar{c}^4 \bar{g}, \quad \bar{\tilde{G}} = \bar{\tilde{c}}^4 \bar{\tilde{g}}. \tag{3.86}$$

Here $\bar{g}$, $\bar{\tilde{g}}$ are Stäckel metrics associated to the Stäckel matrices

$$\bar{S} = \begin{pmatrix} \bar{s}_{11}(u^1) & 1 & \bar{s}_{13}(u^1) \\ s_{21}(u^2) & s_{22}(u^2) & s_{23}(u^2) \\ s_{31}(u^3) & s_{32}(u^3) & s_{33}(u^3) \end{pmatrix}, \quad \bar{\tilde{S}} = \begin{pmatrix} \bar{\tilde{s}}_{11}(u^1) & 1 & \bar{s}_{13}(u^1) \\ \tilde{s}_{21}(u^2) & s_{22}(u^2) & s_{23}(u^2) \\ \tilde{s}_{31}(u^3) & s_{32}(u^3) & s_{33}(u^3) \end{pmatrix}, \tag{3.87}$$

thanks to (3.13) and (3.84). Moreover, the Stäckel metric $\bar{g}$ has the local expression

$$g = \sum_{i=1}^{3} \bar{h}_i^2 (du^i)^2, \quad \bar{h}_i^2 = \frac{\det \bar{S}}{s_{12}}, \; i = 1, 2, 3, \tag{3.88}$$

whereas the conformal factors $\bar{c}$, still satisfies the PDE (1.4) which has now the local expression

$$-\Delta_{\bar{g}} \bar{c} - \sum_{i=1}^{3} \bar{h}_i^2 \big(\bar{\phi}_i + \frac{1}{4}\bar{\gamma}_i^2 - \frac{1}{2}\partial_i \bar{\gamma}_i\big) c = 0\,, \tag{3.89}$$

with

$$\bar{\phi}_1 = \frac{\phi_1}{s_{12}} - \frac{(\log \dot{s}_{12})^2}{16} + \frac{(\log \ddot{s}_{12})}{4}, \quad \bar{\phi}_2 = \phi_2, \quad \bar{\phi}_3 = \phi_3. \tag{3.90}$$

The crucial remark is that, *in this coordinate system, the last two columns of $\bar{S}$ and $\bar{\tilde{S}}$ coincide.* We will use this observation in the next section to finish the proof of our uniqueness result for the inverse problem.

3.4 The elliptic PDE on the conformal factor

Working in the coordinate system (u^1, u^2, u^3) given by (3.85), we see that the metrics $\bar{G}$ can be written as

$$\bar{G} = \alpha\, g_0, \quad \alpha = \bar{c}^4 \det \bar{S}, \quad g_0 = \frac{1}{\bar{s}^{11}}(du^1)^2 + \frac{1}{\bar{s}^{21}}(du^2)^2 + \frac{1}{\bar{s}^{31}}(du^3)^2,$$

and a corresponding expression for $\bar{\tilde{G}}$ holds. Notice from (3.87) and (3.89) that

$$g_0 = \tilde{g}_0. \tag{3.91}$$

Thus it only remains to prove that $\alpha = \tilde{\alpha}$ in order to show that $\bar{G} = \bar{\tilde{G}}$. For this, we use

Lemma 3.4. *The conformal factor α satisfies the elliptic PDE*

$$-\Delta_{g_0}\alpha - Q_{g_0,\bar{\phi}_i}\alpha = 0, \tag{3.92}$$

where

$$Q_{g_0,\bar{\phi}_i} = \sum_{i=1}^{3} g_0^{ii}\left[\frac{\partial_{ii}^2 \log\det g_0}{4} + \frac{\partial_i \log\det g_0}{8} + \frac{(\partial_i \log\det g_0)^2}{16} + \bar{\phi}_i\right].$$

Proof. We start from the PDE (3.89) satisfied by the conformal factor $\bar{c}$ and we recall that this PDE comes from the generalized Robertson-Condition (2.2), *i.e.*

$$\sum_{i=1}^{3} \bar{H}_i^{-2}\left(\frac{\partial_i\bar{\Gamma}_i}{2} - \frac{(\bar{\Gamma}_i)^2}{4} - \bar{\phi}_i\right) = 0.$$

Recalling that $\bar{H}_i^2 = \frac{\alpha}{s^{i1}}$, a direct calculation shows that α satisfies

$$\sum_{i=1}^{3} s^{i1}\left(-\partial_{ii}^2\alpha - \frac{1}{2}\partial_i\alpha\right) + Q\alpha = 0, \tag{3.93}$$

where

$$Q = \sum_{i=1}^{3} s^{i1}\left(\frac{\partial_{ii}^2 \log s^{11}s^{21}s^{31}}{4} + \frac{\partial_i \log s^{11}s^{21}s^{31}}{8} - \frac{(\partial_i \log s^{11}s^{21}s^{31})^2}{16} - \bar{\phi}_i\right). \tag{3.94}$$

We observe then that

$$s^{11}s^{21}s^{31} = \frac{1}{\det g_0}, \quad \sum_{i=1}^{3} s^{i1}\left(-\partial_{ii}^2\alpha - \frac{1}{2}\partial_i\alpha\right) = -\Delta_{g_0}.$$

Hence we obtain easily from (3.93) and (3.94) the elliptic PDE (3.92) satisfied by α. □

Thanks to (3.44), (3.84), (3.90) and (3.91), we see that the conformal factors α and $\tilde{\alpha}$ satisfy the *same* second order elliptic PDE (3.92). Recalling that the last two columns of the Stäckel matrices (3.87) coincide, it is easy to see from the definitions

$$\alpha = \bar{c}^4 \det \bar{S}, \quad \tilde{\alpha} = \bar{\tilde{c}}^4 \det \bar{\tilde{S}},$$

and from the boundary determination results from Section 3.2 (see in particular (3.14) and (3.24)), that α and $\tilde{\alpha}$ have the same Cauchy data at $u^1 = 0$, *i.e.*

$$\alpha(0) = \tilde{\alpha}(0), \quad \dot{\alpha}(0) = \dot{\tilde{\alpha}}(0).$$

Hence, classical unique continuation (see for instance [42]) gives

$$\alpha = \tilde{\alpha}, \quad \text{on } \Omega.$$

Consequently, we have shown

$$\bar{G} = \bar{\tilde{G}},$$

or equivalently,

$$G = \tilde{G}, \quad \text{up to a change of variables of the form (1.18).}$$

This finishes the proof of our uniqueness result in the anisotropic Calderón problem on three-dimensional conformally Stäckel cylinders.

4 Some perspectives

The results in this paper could be extended in several directions.

1. There exists a theory of *non-orthogonal* Stäckel manifolds (in the sense that the metrics are non-diagonal) for which the HJ and Helmholtz equations admit a complete set of classical separated solutions at all energies [2, 3, 52, 54]. In particular, these non-orthogonal Stäckel manifolds contain (and generalize enormously) the well-known family of Kerr black holes in General Relativity and their Riemannian counterparts. It would be interesting

a) to extend this theory to the case of HJ and Helmholtz equation at fixed energy using the notion of R-separability following the lines of [11],

b) to address the question of uniqueness for the anisotropic Calderón problem in this non-orthogonal setting.

2. The methods employed in this paper should work in more general situations in which the Laplace equation could be separated with respect to one variable only (and not all the variables as in the present case). Such models have been studied recently by us in [18] and named conformally Painlevé manifolds. This class of manifolds contains Riemannian manifolds of dimension n for which the geodesic flow is not completely integrable, but rather possesses $1 \leq r < n-1$ *hidden symmetries*, that is conformal Killing tensors of rank two satisfying certain additional assumptions. In such manifolds, the HJ and Laplace equations can be separated in groups of variables, leading to r coupled PDEs.

We intend in the near future to study the anisotropic Calderón problem on conformally Painlevé manifolds.

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