A SHARP REGULARITY THRESHOLD FOR UNIQUENESS IN RIEMANNIAN CALDERÓN-TYPE PROBLEMS

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Abstract. We prove a sharp regularity threshold for uniqueness in two anisotropic Calderón-type inverse problems in dimension $n \geq 3$. The main setting is the Riemannian Schrödinger problem with fixed scalar potential: for a prescribed nonconstant analytic function V, we study whether the Dirichlet-to-Neumann map of $-\Delta_g + V$ on a domain $\Omega \subset \mathbb{R}^n$ determines the unknown metric g. The natural gauge is the group of boundary-fixing diffeomorphisms preserving V. We show that, while analytic metrics are uniquely determined modulo this gauge by a minor adaptation of the Lassas–Uhlmann reconstruction theorem, uniqueness fails densely in every non-analytic Gevrey class G^σ, $\sigma > 1$. In fact, our counterexamples are not isometric in the sense that they are not connected by the pushforward of any diffeomorphism of $\overline{\Omega}$. We also prove the analogous sharp threshold for the anisotropic Calderón problem at fixed nonzero frequency, thereby upgrading the previously known finite-regularity counterexamples to Gevrey and C^∞ regularity. The two constructions use different scalar mechanisms: for fixed potentials, the nonconstant potential itself provides a local coordinate, while at nonzero frequency one uses a compactly supported prescribed-Jacobian lemma in Gevrey spaces. Thus analyticity is the exact threshold for uniqueness in both problems.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded connected C^∞ domain. This paper proves sharp regularity thresholds for uniqueness in two anisotropic Calderón-type inverse boundary value problems with zeroth-order terms. The first is a Riemannian Schrödinger problem with a fixed scalar potential and unknown metric. The second is the anisotropic Calderón problem at a fixed nonzero frequency, viewed in conductivity variables. In both settings, analytic coefficients are uniquely determined modulo the natural gauge, while nonuniqueness is dense in every non-analytic Gevrey class G^σ, $\sigma > 1$, and hence in C^∞.

We start with the fixed-potential problem. Let g be a Riemannian metric on $\overline{\Omega}$, and let V be a fixed real-valued scalar potential. We consider the boundary value problem

$$(-\Delta_g + V)u = 0 \quad \text{in } \Omega\,, \qquad u|_{\partial\Omega} = f\,. \tag{1.1}$$

If 0 is not in the Dirichlet spectrum $\sigma_{\mathrm{D}}(-\Delta_g + V)$, then for every $f \in H^{1/2}(\partial\Omega)$ there is a unique weak solution. The associated Dirichlet-to-Neumann map is defined by the bilinear form

$$H^{\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \ni (f,h) \mapsto \langle \Lambda_{g,V} f, h\rangle = \int_\Omega \langle \nabla u, \nabla v\rangle_g \, dV_g + \int_\Omega V\, u\, v\, dV_g\,, \tag{1.2}$$

where u satisfies (1.1) and $v \in H^1(\Omega)$ has trace h. For smooth coefficients and smooth boundary data this agrees with the classical normal derivative $\partial_{\nu_g} u|_{\partial\Omega}$.

The natural gauge in this problem is not the full anisotropic Calderón gauge. If $\Psi : \overline{\Omega} \to \overline{\Omega}$ is a diffeomorphism equal to the identity on $\partial\Omega$, then

$$\Lambda_{\Psi_* g,\, V\circ\Psi^{-1}} = \Lambda_{g,V}.$$

Thus, when the potential V is fixed *a priori*, the boundary data are invariant only under boundary-fixing diffeomorphisms satisfying

$$V \circ \Psi = V\,.$$

If $V \equiv 0$, this is the full boundary-fixing diffeomorphism gauge of the classical anisotropic Calderón problem. If V is nonconstant, the gauge is typically much smaller. One should note, however, that the counterexamples to uniqueness constructed below are in fact non-isometric in a stronger sense: the two metrics are not connected by the pushforward of any diffeomorphism of $\overline{\Omega}$, whether or not it fixes the boundary or preserves V.

To our knowledge, this fixed-potential inverse problem has not previously been isolated as a separate global Calderón-type problem. The metric is unknown, while the scalar potential is prescribed. This places the problem between the classical anisotropic Calderón problem, where the geometry is unknown and there is no zeroth-order term, and the inverse Schrödinger problem on a fixed geometry, where the metric is known and the potential is unknown. Existing anisotropic Schrödinger results usually concern recovery of the potential on a fixed or geometrically structured background, for instance in conformally transversally anisotropic geometries [16, 17, 20]. Here the potential is fixed and the geometry varies.

We use the standard PDE convention for Gevrey spaces. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\sigma \geq 1$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we write

$$|\alpha| = \alpha_1 + \cdots + \alpha_n\,, \qquad \alpha! = \alpha_1! \cdots \alpha_n!\,,$$

and

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}\,, \qquad \partial_i = \frac{\partial}{\partial x_i}\,.$$

The Gevrey class $G^\sigma(\Omega)$ consists of all functions $f \in C^\infty(\Omega)$ such that, for every compact set $K \subset \Omega$, there exist constants $C > 0$ and $R > 0$ such that

$$\sup_{x \in K} |\partial^\alpha f(x)| \leq C R^{|\alpha|} (\alpha!)^\sigma\,,$$

for every multi-index $\alpha \in \mathbb{N}^n$.

Thus $G^1(\Omega) = C^\omega(\Omega)$ is the analytic class. If Ω is a bounded C^∞ domain, we define $G^\sigma(\overline{\Omega})$ as the space of restrictions to $\overline{\Omega}$ of G^σ functions defined in a neighborhood of $\overline{\Omega}$. Accordingly, a metric $g \in G^\sigma(\overline{\Omega})$ means that the coefficients of g in the ambient Euclidean coordinates belong to $G^\sigma(\overline{\Omega})$.

For $\sigma > 1$, the class is nonquasianalytic; in particular, compactly supported G^σ functions exist. This localization property is the essential reason that the constructions below work for every $\sigma > 1$, but not in the analytic class.

The Gevrey norms used throughout the paper are introduced in Section 3.1; they induce the topology on the Gevrey spaces considered below. In the C^∞ setting, we use the usual Fréchet topology. With these conventions in mind, our first main theorem is the following.

Theorem 1.1 (Nonuniqueness for fixed nonconstant potentials)**.** *Let $n \geq 3$, and let $\Omega \subset \mathbb{R}^n$ be a bounded connected smooth domain. Let*

$$(g, V) \in C^\omega(\overline{\Omega}) \qquad (\textit{resp. } (g, V) \in C^\infty(\overline{\Omega})),$$

where g is a Riemannian metric on $\overline{\Omega}$ and V is a nonconstant scalar potential. Assume that

$$0 \notin \sigma_{\mathrm{D}}(-\Delta_g + V).$$

Then, for every $\sigma > 1$, every G^σ-neighborhood of g (resp. every C^∞-neighborhood of g) contains infinitely many pairs of metrics

$$g_1, g_2 \in G^\sigma(\overline{\Omega}) \qquad (\textit{resp. } g_1, g_2 \in C^\infty(\overline{\Omega}))$$

such that

$$0 \notin \sigma_{\mathrm{D}}(-\Delta_{g_j} + V), \qquad j = 1, 2,$$

$$\Lambda_{g_1, V} = \Lambda_{g_2, V},$$

but g_1 and g_2 are not isometric in the strong sense: there is no diffeomorphism $\Psi : \overline{\Omega} \to \overline{\Omega}$ such that $g_2 = \Psi_* g_1$.

Remark 1.2. When (g,V) is analytic, the notion of G^σ neighborhood can be made more precise by stating that for any $\sigma>1$, there exists $\tau>0$ with $|g|_{\sigma,\tau,\overline{\Omega}}<\infty$, (see Subsection 3.1 for the definition of this seminorm) and there exist two sequences of uniformly elliptic metrics

$$(g_1^k)_{k\geq 1}, \qquad (g_2^k)_{k\geq 1}\subset G^\sigma(\overline{\Omega}),$$

such that, for $j=1,2$,

$$|g_j^k-g|_{\sigma,\tau,\overline{\Omega}}\longrightarrow 0 \qquad \text{as } k\to\infty,$$

and, for every $k\geq 1$,

$$0\notin\sigma_{\mathrm{D}}(-\Delta_{g_j^k}+V), \qquad j=1,2,$$

$$\Lambda_{g_1^k,V}=\Lambda_{g_2^k,V},$$

while the metrics g_1^k and g_2^k are not isometric.

Remark 1.3. The restriction $n\geq 3$ is natural, since the conformal change

$$g_c=c^{4/(n-2)}g$$

appearing in Section 2.3 becomes singular in dimension $n=2$. Moreover, the proof of the previous theorem still works if the pair (g,V) belongs only to a Gevrey class $G^\sigma(\overline{\Omega})$ with $\sigma>1$. Indeed, the argument only relies on the stability of Gevrey classes under composition together with an inverse mapping theorem in Gevrey classes; see, for instance, [32, Remark 1.4.7] and [21]. In particular, any sufficiently small nonconstant perturbation of the potential (both in size and support) still destroys uniqueness in the Calderón problem.

By contrast, the analytic endpoint is rigid up to the natural gauge, as shown by a minor adaptation of a result of Lassas–Uhlmann [23] that we present in Appendix A.

Theorem 1.4 (Analytic uniqueness for fixed potentials)**.** *Assume that Ω has real-analytic boundary. Consider a scalar potential $V\in C^\omega(\overline{\Omega})$, and let $g_1,g_2\in C^\omega(\overline{\Omega})$ be uniformly elliptic real-analytic metrics. Assume that $0\notin\sigma_{\mathrm{D}}(-\Delta_{g_j}+V)$ for $j=1,2$. If $\Lambda_{g_1,V}=\Lambda_{g_2,V}$, then there exists a real-analytic diffeomorphism $\Psi:\overline{\Omega}\to\overline{\Omega}$ with $\Psi|_{\partial\Omega}=\mathrm{Id}$ such that $g_2=\Psi_*g_1$ and $V\circ\Psi=V$.*

Let us briefly explain the mechanism behind Theorem 1.1. Since V is analytic and nonconstant, there is an interior box $Q\Subset\Omega$ where $dV\neq 0$. In this box, V can be used as one coordinate. We choose a non-negative compactly supported Gevrey function $u\in G_c^\sigma(Q)$ and set $c_\varepsilon=1+\varepsilon u$ with $\varepsilon>0$ small enough. The conformal perturbation is $g_{2,\varepsilon}=c_\varepsilon^{4/(n-2)}g$. If $z=c_\varepsilon v$, the equation $(-\Delta_{g_{2,\varepsilon}}+V)v=0$ is transformed into

$$\left(-\Delta_g+Vc_\varepsilon^{4/(n-2)}+\frac{\Delta_g c_\varepsilon}{c_\varepsilon}\right)z=0.$$

The conformal perturbation alone does not preserve the effective potential. Indeed, unless $c_\varepsilon\equiv 1$, the transformed potential

$$Vc_\varepsilon^{4/(n-2)}+\frac{\Delta_g c_\varepsilon}{c_\varepsilon}$$

cannot coincide with V, (see Lemma 2.1). Thus the compensating diffeomorphism Ψ_ϵ should satisfy

$$V\circ\Psi_\varepsilon=Vc_\varepsilon^{4/(n-2)}+\frac{\Delta_g c_\varepsilon}{c_\varepsilon}.$$

The right-hand side is equal to V outside Q and is close to V for ϵ small. Since V is a coordinate in Q, this equation is solved explicitly by changing only the V-coordinate. In particular, no prescribed-Jacobian theorem is needed in the fixed-potential construction. The non-isometry is detected by the total volume: if $g_{1,\varepsilon}=(\Psi_\varepsilon)_*g$, then $\mathrm{Vol}_{g_{1,\varepsilon}}(\Omega)=\mathrm{Vol}_g(\Omega)$, while $dV_{g_{2,\varepsilon}}=c_\varepsilon^{2n/(n-2)}\,dV_g$, and $u\geq 0$ is chosen so that the total volume changes.

We now turn to the fixed nonzero-frequency anisotropic Calderón problem on a bounded smooth domain $\Omega \subset \mathbb{R}^n$. This is the direct continuation of the problem studied in [12]. For a uniformly elliptic symmetric matrix-valued conductivity γ, write

$$\mathcal{L}_\gamma u = -\operatorname{div}(\gamma\nabla u) = -\nabla\cdot(\gamma\nabla u). \tag{1.3}$$

Given $\lambda \in \mathbb{R}\setminus\sigma_{\mathrm{D}}(\mathcal{L}_\gamma)$, we consider

$$\mathcal{L}_\gamma u = \lambda u \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = f.$$

As in the Schrödinger case (see (1.2)) the Dirichlet-to-Neumann map is defined weakly by

$$\langle \Lambda_{\gamma,\lambda} f, h\rangle = \int_\Omega \gamma\nabla u\cdot\nabla v\,dx - \lambda\int_\Omega u\,v\,dx\,,$$

where $v \in H^1(\Omega)$ has trace h. For smooth coefficients and smooth boundary data this agrees with the classical conormal derivative $\gamma\nabla u\cdot\nu|_{\partial\Omega}$.

This inverse problem is a variant of Calderón's inverse boundary value problem [9], originally motivated by electrical impedance tomography (EIT); see also the surveys [34, 38]. Although the physical EIT problem corresponds to $\lambda = 0$, nonzero frequencies arise naturally in inverse medium, inverse scattering, and viscoelasticity models; see for instance [4, 8, 16, 28]. Related frequency-dependent or partial-data questions appear in [7].

At zero frequency, the anisotropic Calderón problem has the full boundary-fixing diffeomorphism gauge. If Ψ is a diffeomorphism of $\overline{\Omega}$ equal to the identity on $\partial\Omega$, then $\Lambda_{\Psi_*\gamma,0} = \Lambda_{\gamma,0}$, where

$$(\Psi_*\gamma)(\Psi(x)) = \frac{D\Psi(x)\,\gamma(x)\,D\Psi(x)^{\mathsf{T}}}{\det D\Psi(x)}.$$

At nonzero frequency this gauge is smaller. If u solves $-\nabla\cdot((\Psi_*\gamma)\nabla u) = \lambda u$, then $u\circ\Psi$ solves $-\nabla\cdot(\gamma\nabla(u\circ\Psi)) = \lambda\det D\Psi\,(u\circ\Psi)$. Thus, for $\lambda\neq 0$, the equation is preserved at the same frequency only when $\det D\Psi = 1$. We write

$$\mathrm{SDiff}(\overline{\Omega}) = \Big\{\Psi\in\mathrm{Diff}^\infty(\overline{\Omega}) :\ \Psi|_{\partial\Omega} = \mathrm{Id},\ \det D\Psi = 1\Big\}.$$

The natural gauge for nonzero frequency problem for the Calderón problem is thus: if $\Psi \in \mathrm{SDiff}(\overline{\Omega})$, one has

$$\Lambda_{\Psi_*\gamma,\lambda} = \Lambda_{\gamma,\lambda}\,.$$

The classical zero-frequency anisotropic Calderón problem is known to have positive uniqueness results in several important settings. In dimension two, global uniqueness for isotropic conductivities was proved by Nachman [27] and extended to lower regularity in [6]; the anisotropic two-dimensional problem can be reduced to the isotropic one through isothermal coordinates [3, 36]. In dimension $n \geq 3$, the isotropic problem was solved in the smooth case by Sylvester and Uhlmann [35], with later extensions to rougher conductivities including [5, 10, 22]. For anisotropic conductivities, or equivalently Riemannian metrics, the full smooth global problem remains open in dimension $n \geq 3$. Positive results are known in the real-analytic category [23, 24] and in several conformally transversally anisotropic geometries [16, 17, 20]. Counterexamples and obstructions are known in singular settings and for certain partial-data configurations; see for instance [13–15, 18, 19] and the references therein.

The previous paper [12] showed that the modified nonzero-frequency uniqueness statement modulo $\mathrm{SDiff}(\Omega)$ fails in every finite C^k class. More precisely, for each finite k, one can construct non-isometric C^k conductivities, close to a prescribed smooth background, with identical DN maps at a fixed nonzero frequency. The construction uses a conformal rescaling of the conductivity, a prescribed-Jacobian diffeomorphism, and a determinant invariant to rule out equivalence.

Our second main theorem upgrades this result to a sharp infinite-regularity statement. Here, Sym_n^+ denotes the cone of positive definite symmetric $n\times n$ matrices.

Theorem 1.5 (Nonuniqueness at fixed nonzero frequency)**.** *Let* $n \geq 3$ *, and let* $\Omega \subset \mathbb{R}^n$ *be a bounded connected smooth domain. Let*

$$\gamma \in C^\omega(\overline{\Omega}; \mathrm{Sym}_n^+) \qquad \textit{resp. } \gamma \in C^\infty(\overline{\Omega}; \mathrm{Sym}_n^+)$$

be a uniformly elliptic conductivity. Let $\lambda_0 \in \mathbb{R} \setminus \{0\}$ *satisfy*

$$\lambda_0 \notin \sigma_{\mathrm{D}}(\mathcal{L}_\gamma).$$

Then, for every $\sigma > 1$*, every* G^σ*-neighborhood of* γ *(resp. every* C^∞*-neighborhood of* γ*) contains infinitely many pairs of uniformly elliptic conductivities*

$$\gamma_1, \gamma_2 \in G^\sigma(\overline{\Omega}; \mathrm{Sym}_n^+) \qquad (\textit{resp. } \gamma_1, \gamma_2 \in C^\infty(\overline{\Omega}; \mathrm{Sym}_n^+))$$

such that

$$\lambda_0 \notin \sigma_{\mathrm{D}}(\mathcal{L}_{\gamma_j}), \qquad j = 1, 2,$$

$$\Lambda_{\gamma_1,\lambda_0} = \Lambda_{\gamma_2,\lambda_0},$$

but γ_1 *and* γ_2 *are not isometric in the strong sense: there is no diffeomorphism* $\Psi : \overline{\Omega} \to \overline{\Omega}$ *such that* $\gamma_2 = \Psi_* \gamma_1$*.*

Again the analytic endpoint is rigid.

Theorem 1.6 (Analytic uniqueness at fixed nonzero frequency)**.** *Assume that* Ω *has real-analytic boundary. Consider conductivities* $\gamma_1, \gamma_2 \in C^\omega(\overline{\Omega}; \mathrm{Sym}_n^+)$ *and let* $\lambda_0 \in \mathbb{R} \setminus \{0\}$ *satisfy* $\lambda_0 \notin \sigma_{\mathrm{D}}(\mathcal{L}_{\gamma_j})$ *for* $j = 1, 2$*. If* $\Lambda_{\gamma_1,\lambda_0} = \Lambda_{\gamma_2,\lambda_0}$*, then there exists a real-analytic diffeomorphism* $\Psi \in \mathrm{SDiff}(\overline{\Omega})$ *such that* $\gamma_2 = \Psi_* \gamma_1$*.*

Let us emphasize the relation between the two settings. The fixed-potential theorem is not a reformulation of the nonzero-frequency theorem. If one rewrites the fixed-frequency conductivity equation in terms of the associated metric g_γ (see Subsection 2.1), the potential becomes $V_\gamma = -\lambda_0 |g_\gamma|^{-1/2}$, which depends on the unknown metric. In the fixed-potential theorem, by contrast, V is prescribed independently of g. Thus, although the two results concern different inverse problems, they reveal the same sharp regularity threshold: in the analytic class $G^1 = C^\omega$, uniqueness holds modulo the natural gauge invariance, whereas nonuniqueness occurs in every Gevrey class G^σ with $\sigma > 1$, as well as in the smooth category C^∞.

The fixed-frequency construction also uses a compactly supported prescribed-Jacobian lemma. The prescribed-Jacobian problem goes back to Dacorogna–Moser [11], while Rivière–Ye [29] obtained sharp Hölder estimates for the deviation of the associated diffeomorphism from the identity.

We do not use a global Gevrey version of that theory. Since our density is supported in an interior box, the Jacobian step reduces to a compactly supported same-class Gevrey statement with radius loss: solve $\nabla \cdot X = f$ explicitly in the box and run a localized Moser flow.

Thus the two results illustrate the same phenomenon through different scalar mechanisms. In the fixed-potential problem, the scalar correction is the prescription of $V \circ \Psi$, solved using V as a local coordinate. In the fixed-frequency problem, the scalar correction is the prescription of $\det D\Psi$, solved by a localized Jacobian construction. In both cases a zeroth-order term breaks the full anisotropic gauge. Once that gauge is broken, analyticity remains rigid, while every non-analytic Gevrey class admits dense nonuniqueness.

The paper is organized as follows. Section 2 recalls the metric–conductivity dictionary, the fixed-potential gauge, and the conformal identities for both problems. Section 3 develops the Gevrey tools, including the compactly supported prescribed-Jacobian lemma. Section 4 proves Theorem 1.1. Section 5 constructs the fixed-frequency counterexamples. Section 6 proves non-isometry via the determinant invariant. Appendix A recalls analytic uniqueness for Theorems 1.4 and 1.6.

2. Transformation laws and conformal identities

This section recalls the elementary geometric identities used in the two constructions. We first discuss the fixed-potential problem, where the unknown is the metric g. We then recall the corresponding formulas for conductivities at fixed nonzero frequency.

Throughout, $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a bounded smooth domain. All diffeomorphisms are understood to be smooth diffeomorphisms of the compact manifold with boundary $\overline{\Omega}$, equal to the identity on $\partial\Omega$ when this is stated. The identities below are written for smooth objects; the corresponding weak formulations follow by the same change-of-variables arguments.

2.1. **The metric–conductivity dictionary.** Since throughout this paper we work on an open subset $\Omega \subset \mathbb{R}^n$ equipped with the ambient Euclidean coordinates, all tensorial expressions below are written in these fixed coordinates. If g is a Riemannian metric on Ω, we associate to it the conductivity

$$\gamma_g^{ij} = |g|^{1/2} g^{ij}.$$

Equivalently, in intrinsic terms, the corresponding conductivity operator coincides with the Laplace–Beltrami operator:

$$\mathcal{L}_{\gamma_g} = |g|^{1/2}\, \Delta_g,$$

where $\mathcal{L}_\gamma$ was introduced in (1.3).

$$-\Delta_g u = -|g|^{-1/2} \partial_i (\gamma_g^{ij} \partial_j u)\,.$$

Conversely, if γ is a positive-definite symmetric conductivity, then the corresponding metric g_γ is determined by

$$\gamma^{ij} = |g_\gamma|^{1/2} g_\gamma^{ij}\,.$$

Equivalently,

$$|g_\gamma|^{1/2} = (\det \gamma)^{1/(n-2)}\,,$$

and

$$g_\gamma^{ij} = (\det \gamma)^{-1/(n-2)} \gamma^{ij}\,.$$

Thus the Riemannian volume density of g_γ is

$$dV_{g_\gamma} = (\det \gamma)^{\frac{1}{n-2}}\, dx, \qquad dx = dx_1 \wedge \cdots \wedge dx_n.$$

This explains why the determinant functional

$$\gamma \mapsto \mathcal{I}(\gamma) = \int_\Omega (\det \gamma)^{1/(n-2)}\, dx \tag{2.1}$$

is simply the Riemannian volume of (Ω, g_γ) .

2.2. **The fixed-potential gauge.** Let g be a Riemannian metric and V a scalar potential. We write

$$P_{g,V} := -\Delta_g + V\,.$$

The Dirichlet-to-Neumann weak form was defined in (1.2).

Let $\Psi : \overline{\Omega} \to \overline{\Omega}$ be a diffeomorphism. The pushforward metric $\Psi_* g$ is defined by

$$(\Psi_* g)_{\Psi(x)}(\eta, \zeta) = g_x(D\Psi^{-1}\eta, D\Psi^{-1}\zeta).$$

Equivalently, if u and φ belong to $C_c^\infty(\Omega)$, and $w = u \circ \Psi^{-1}$, then

$$\int_\Omega \langle \nabla w, \nabla \varphi \rangle_{\Psi_* g}\, dV_{\Psi_* g} = \int_\Omega \langle \nabla u, \nabla (\varphi \circ \Psi) \rangle_g\, dV_g.$$

The potential transforms as a scalar:

$$(\Psi_* V)(\Psi(x)) = V(x) \text{ or } \Psi_* V = V \circ \Psi^{-1}\,.$$

Therefore

$$\Lambda_{\Psi_* g,\, \Psi_* V} = \Lambda_{g,V}$$

whenever $\Psi|_{\partial\Omega} = \mathrm{Id}$.

Consequently, if the potential V is fixed *a priori*, the natural gauge group is

$$\mathrm{Diff}_0(\overline{\Omega}; V) := \left\{ \Psi \in \mathrm{Diff}^\infty(\overline{\Omega}) : \ \Psi|_{\partial\Omega} = \mathrm{Id},\ V \circ \Psi = V \right\}.$$

This is the full boundary-fixing diffeomorphism group when V is constant, and is typically much smaller when V is nonconstant.

2.3. **The conformal identity for the fixed-potential problem.** Let

$$g_c = c^{4/(n-2)} g \text{ with } c > 0\,.$$

Then

$$dV_{g_c} = c^{2n/(n-2)}\, dV_g, \qquad \langle \nabla v, \nabla\varphi\rangle_{g_c}\, dV_{g_c} = c^2 \langle \nabla v, \nabla \varphi\rangle_g\, dV_g\,.$$

Equivalently, in divergence form, for $v \in C_c^\infty$, $-\Delta_{g_c} v + V v = 0$ is the same as

$$-\operatorname{div}_g(c^2 \nabla v) + V c^{2n/(n-2)} v = 0\,.$$

Set $z := c\, v$. Using

$$\operatorname{div}_g(c^2 \nabla v) = c\, \Delta_g z - z\, \Delta_g c\,,$$

we obtain that

$$(-\Delta_{g_c} + V)v = 0 \text{ iff } \left(-\Delta_g + V c^{4/(n-2)} + \frac{\Delta_g c}{c}\right) z = 0\,. \tag{2.2}$$

Thus a conformal perturbation of the metric changes the potential from V to $V c^{4/(n-2)} + \Delta_g c / c$, which we call an effective potential.

Lemma 2.1 (Localized conformal factors change the effective potential)**.** *Let $n \geq 3$. Let $\Omega \subset \mathbb{R}^n$ be connected, let g be a smooth Riemannian metric on $\overline{\Omega}$, and let $V \in C^\infty(\overline{\Omega})$. Let $c \in C^\infty(\overline{\Omega})$ be positive and assume that $c = 1$ and $\partial_{\nu_g} c = 0$ on a nonempty open subset of $\partial\Omega$. Set $g_c = c^{4/(n-2)} g$. If, after the conjugation $z = c\, v$, the Schrödinger operator on (Ω, g_c) has the same effective potential as the operator on (Ω, g) in the sense that*

$$V c^{4/(n-2)} + \frac{\Delta_g c}{c} = V \quad \text{in } \Omega\,, \tag{2.3}$$

then $c \equiv 1$ on Ω. In particular, any nontrivial conformal factor c which equals 1 near $\partial\Omega$ necessarily changes the effective potential away from V.

Proof. The condition (2.3) is equivalent to

$$\Delta_g c + V \left(c^{(n+2)/(n-2)} - c \right) = 0\,. \tag{2.4}$$

Set $d = c - 1$. Then d vanishes on a nonempty open subset of $\partial\Omega$, and since $\Delta_g c = \Delta_g d$, equation (2.4) gives

$$\Delta_g d + V \left(c^{(n+2)/(n-2)} - c \right) = 0 \qquad \text{in } \Omega.$$

Consider the scalar function

$$\theta(a) := \frac{a^{(n+2)/(n-2)} - a}{a - 1}, \qquad a > 0,\ a \neq 1.$$

The apparent singularity at $a = 1$ is removable, so that $\theta \in C^\infty((0, +\infty))$. Since $c > 0$ and $c \in C^\infty(\overline{\Omega})$, it follows that

$$q(x) := V(x)\, \theta(c(x))$$

belongs to $C^\infty(\overline{\Omega})$. Therefore,

$$\Delta_g d + q(x) d = 0 \qquad \text{in } \Omega.$$

Since q is bounded, boundary unique continuation for linear second-order elliptic equations implies that a solution whose Cauchy data d and $\partial_{\nu_g} d$ are zero on a nonempty open subset of $\partial\Omega$ must vanish identically on the connected domain Ω, (see, for instance, [37] or Subsection 5.4 of [26]). Hence $d \equiv 0$ and therefore $c \equiv 1$. ☐

Proposition 2.2 (Fixed-potential conformal-diffeomorphism identity)**.** *Let g be a smooth metric, let V be a smooth potential, let $c > 0$ be smooth with $c = 1$ near $\partial\Omega$, and let $\Psi \in \mathrm{Diff}^\infty(\overline{\Omega})$ satisfy $\Psi = \mathrm{Id}$ near $\partial\Omega$. Assume that*

$$V \circ \Psi = V c^{4/(n-2)} + \frac{\Delta_g c}{c} \quad \text{in } \Omega\,. \tag{2.5}$$

Then

$$\Lambda_{c^{4/(n-2)} g, V} = \Lambda_{\Psi_* g, V}\,. \tag{2.6}$$

Moreover, 0 is a Dirichlet eigenvalue of $-\Delta_{c^{4/(n-2)} g} + V$ if and only if 0 is a Dirichlet eigenvalue of $-\Delta_{\Psi_* g} + V$.

Proof. Let $v \in C^\infty(\overline{\Omega})$ solve $(-\Delta_{c^{4/(n-2)} g} + V)v = 0$ and $f := v|_{\partial\Omega}$. Set $z = c\,v$. Since $c = 1$ near $\partial\Omega$, z has the same boundary trace f. By (2.2)–(2.5),

$$(-\Delta_g + V \circ \Psi) z = 0\,.$$

Therefore $w = z \circ \Psi^{-1}$ solves $(-\Delta_{\Psi_* g} + V)w = 0$. Since $\Psi = \mathrm{Id}$ near $\partial\Omega$, $w|_{\partial\Omega} = f$. This gives a bijection of solution spaces, and hence the spectral assertion.

The equality of DN maps follows from the same calculation in the weak form. If $r \in H^1(\Omega)$ has boundary trace h, test the $(\Psi_* g, V)$-equation with $(c\,r) \circ \Psi^{-1}$. Changing variables gives

$$\int_\Omega \langle \nabla w, \nabla((c\,r) \circ \Psi^{-1})\rangle_{\Psi_* g}\, dV_{\Psi_* g} = \int_\Omega \langle \nabla z, \nabla(c\,r)\rangle_g\, dV_g\,,$$

and

$$\int_\Omega V w ((c\,r) \circ \Psi^{-1})\, dV_{\Psi_* g} = \int_\Omega (V \circ \Psi) c^2 v r\, dV_g\,.$$

Comparing this with the DN form for $c^{4/(n-2)}\, g$, the difference is

$$\int_\Omega \langle \nabla c, \nabla(c\,v\,r)\rangle_g\, dV_g + \int_\Omega \Big[(V \circ \Psi)c^2 - V c^{2n/(n-2)}\Big]\, v r\, dV_g\,.$$

Since $c = 1$ near the boundary, integration by parts gives

$$\int_\Omega \langle \nabla c, \nabla(c\,v\,r)\rangle_g\, dV_g = -\int_\Omega (\Delta_g c)\, c\,v\,r\, dV_g\,.$$

Multiplying the compatibility condition (2.5) by c^2 yields

$$(V \circ \Psi)c^2 - V c^{2n/(n-2)} = c\,\Delta_g c\,,$$

since $4/(n-2) + 2 = 2n/(n-2)$. Therefore the sum of the gradient term and the potential-difference term vanishes identically, and (2.6) follows. ☐

Remark 2.3. Lemma 2.1 shows that the diffeomorphism in Proposition 2.2 plays a genuine compensating role and is not merely the standard boundary-fixing diffeomorphism invariance of the problem. Indeed, writing

$$V_c := V c^{4/(n-2)} + \frac{\Delta_g c}{c},$$

one cannot have $V_c = V$ for a nontrivial localized conformal factor c satisfying $c = 1$ on an open subset of Ω (for instance, on a boundary collar). Thus a localized conformal deformation necessarily changes the effective potential. The diffeomorphism Ψ is introduced precisely to compensate for this change through the compatibility condition

$$V \circ \Psi = V_c,$$

which is the fixed-potential analogue of the Jacobian compensation mechanism appearing in the fixed-frequency conductivity construction.

2.4. **Pushforward of a conductivity.** Let $\Psi : \overline{\Omega} \to \overline{\Omega}$ be a diffeomorphism. The pushforward conductivity is

$$(\Psi_*\gamma)(\Psi(x)) = \frac{D\Psi(x)\,\gamma(x)\,D\Psi(x)^{\mathsf{T}}}{|\det D\Psi(x)|}\,. \tag{2.7}$$

Equivalently,

$$\Psi_*\gamma = \left(\frac{D\Psi\,\gamma\,D\Psi^{\mathsf{T}}}{|\det D\Psi|}\right) \circ \Psi^{-1}\,.$$

The following elementary transformation law is the basis of the compensating mechanism used in the construction.

Lemma 2.4 (Fixed-frequency transformation law)**.** *Let u solve*

$$-\nabla \cdot ((\Psi_*\gamma)\nabla u) = \lambda u \quad \textit{in } \Omega\,. \tag{2.8}$$

Set $\widetilde{u} = u \circ \Psi$. Then

$$-\nabla \cdot (\gamma \nabla \widetilde{u}) = \lambda |\det D\Psi|\,\widetilde{u} \quad \textit{in } \Omega\,. \tag{2.9}$$

Conversely, if $\widetilde{u}$ solves (2.9), then $u = \widetilde{u} \circ \Psi^{-1}$ solves (2.8).

Proof. Let $v \in C_0^\infty(\Omega)$ and set $\widetilde{v} = v \circ \Psi$. Changing variables $y = \Psi(x)$, one obtains

$$\int_\Omega (\Psi_*\gamma)(y)\nabla_y u(y) \cdot \nabla_y v(y)\,dy = \int_\Omega \gamma(x)\nabla_x \widetilde{u}(x) \cdot \nabla_x \widetilde{v}(x)\,dx.$$

On the other hand,

$$\int_\Omega \lambda u(y)v(y)\,dy = \int_\Omega \lambda |\det D\Psi(x)|\,\widetilde{u}(x)\widetilde{v}(x)\,dx\,.$$

The weak formulation gives the result. ☐

As an immediate consequence, if $|\det D\Psi| \equiv 1$, then Ψ preserves the equation at the same frequency. Thus the natural gauge group at $\lambda \neq 0$ can then be written as

$$\mathrm{SDiff}(\overline{\Omega}) = \left\{\Psi : \overline{\Omega} \to \overline{\Omega} :\ \Psi|_{\partial\Omega} = \mathrm{Id}\,,\ \det D\Psi = 1\right\}.$$

Proposition 2.5 (Invariance under $\mathrm{SDiff}(\bar{\Omega})$)**.** *If $\Psi \in \mathrm{SDiff}(\bar{\Omega})$ and $\lambda \notin \sigma_{\mathrm{D}}(\mathcal{L}_\gamma)$, then*

$$\Lambda_{\Psi_*\gamma,\lambda} = \Lambda_{\gamma,\lambda}\,.$$

Proof. Since $\det D\Psi = 1$, Lemma 2.4 gives a one-to-one correspondence between solutions of $\mathcal{L}_{\Psi_*\gamma} u = \lambda u$ and solutions of $\mathcal{L}_\gamma(u \circ \Psi) = \lambda(u \circ \Psi)$. Because $\Psi = \mathrm{Id}$ on $\partial\Omega$, the boundary traces agree. The weak DN forms are also equal by the same change of variables. ☐

For a general boundary-fixing diffeomorphism, $|\det D\Psi|$ appears in the equation. The conformal construction below is designed to produce precisely this factor.

2.5. **The conformal identity.** Let $c > 0$ be smooth, and let v be a function. Set $z = c\,v$. Then

$$\nabla \cdot (c^2\gamma\nabla v) = c\,\nabla \cdot (\gamma\nabla z) - z\,\nabla \cdot (\gamma\nabla c)\,.$$

Equivalently, if

$$-\nabla \cdot (c^2\gamma\nabla v) = \lambda v\,,$$

then $z = c\,v$ satisfies

$$-\nabla \cdot (\gamma\nabla z) = \left(\frac{\lambda}{c^2} - \frac{1}{c}\nabla \cdot (\gamma\nabla c)\right) z\,. \tag{2.10}$$

Now suppose that c, f, and λ satisfy the compatibility equation

$$\nabla \cdot (\gamma\nabla c) + \lambda\left(c - \frac{1}{c} + cf\right) = 0\,. \tag{2.11}$$

Dividing by c, this gives

$$\frac{1}{c}\nabla\cdot(\gamma\nabla c) = -\lambda\left(1-\frac{1}{c^2}+f\right).$$

Substituting into (2.10) yields

$$-\nabla\cdot(\gamma\nabla z) = \lambda(1+f)z\,. \tag{2.12}$$

Thus the conformal factor converts the equation for $c^2\gamma$ at frequency λ into an equation for γ with density $(1+f)$ on the right-hand side. Equations of this type were for example studied by Alessandrini in connection with Courant's nodal domain theorem; see [1].

2.6. **The conformal-diffeomorphism identity.** The next proposition is the algebraic core of the construction.

Proposition 2.6 (Conformal-diffeomorphism mechanism)**.** *Let* $c>0$*,* f*, and* Ψ *be smooth and suppose*

$$\Psi|_{\partial\Omega} = \mathrm{Id}, \qquad \det D\Psi = 1+f>0.$$

Assume also that

$$c=1, \qquad \gamma\nabla c\cdot\nu = 0 \quad \text{on } \partial\Omega\,, \tag{2.13}$$

and that (2.11) *holds in* Ω*. If* $\lambda\notin\sigma_{\mathrm{D}}(\mathcal{L}_{c^2\gamma})$*, then* $\lambda\notin\sigma_{\mathrm{D}}(\mathcal{L}_{\Psi_*\gamma})$ *and*

$$\Lambda_{c^2\gamma,\lambda} = \Lambda_{\Psi_*\gamma,\lambda}\,. \tag{2.14}$$

Proof. Let v solve $-\nabla\cdot(c^2\gamma\nabla v)=\lambda v$ with $v|_{\partial\Omega}=h$. Set $z=c\,v$. Since $c=1$ on $\partial\Omega$, we have $z|_{\partial\Omega}=h$. By (2.10)–(2.12),

$$-\nabla\cdot(\gamma\nabla z) = \lambda(1+f)z.$$

Since $\det D\Psi = 1+f$, Lemma 2.4 implies that $w=z\circ\Psi^{-1}$ solves $-\nabla\cdot((\Psi_*\gamma)\nabla w)=\lambda w$. Moreover, $\Psi=\mathrm{Id}$ on $\partial\Omega$, so $w|_{\partial\Omega}=h$. This gives a bijection between solutions for $\mathcal{L}_{c^2\gamma}$ and $\mathcal{L}_{\Psi_*\gamma}$, hence the spectral assertion.

It remains to compare the DN forms. Let $r\in H^1(\Omega)$ have trace k. For the $\Psi_*\gamma$-problem, use the test function $\omega=(c\,r)\circ\Psi^{-1}$. Then $\omega|_{\partial\Omega}=k$. Changing variables gives

$$\int_\Omega(\Psi_*\gamma)\nabla w\cdot\nabla\omega\,dy = \int_\Omega \gamma\nabla z\cdot\nabla(c\,r)\,dx\,,$$

and

$$\int_\Omega w\,\omega\,dy = \int_\Omega z\,c\,r\,(1+f)\,dx\,.$$

Therefore

$$\langle\Lambda_{\Psi_*\gamma,\lambda}h,k\rangle = \int_\Omega\gamma\nabla(c\,v)\cdot\nabla(c\,r)\,dx - \lambda\int_\Omega(1+f)c^2vr\,dx\,.$$

We compare this with

$$\langle\Lambda_{c^2\gamma,\lambda}h,k\rangle = \int_\Omega c^2\gamma\nabla v\cdot\nabla r\,dx - \lambda\int_\Omega vr\,dx\,.$$

The difference of the two expressions is

$$\int_\Omega\gamma\nabla c\cdot\nabla(c\,v\,r)\,dx - \lambda\int_\Omega((1+f)c^2-1)v\,r\,dx\,. \tag{2.15}$$

Using $\gamma\nabla c\cdot\nu=0$ on $\partial\Omega$, integration by parts gives

$$\int_\Omega\gamma\nabla c\cdot\nabla(c\,v\,r)\,dx = -\int_\Omega\nabla\cdot(\gamma\nabla c)\,c\,v\,r\,dx\,.$$

The compatibility equation (2.11) gives

$$-\nabla\cdot(\gamma\nabla c)\,c = \lambda((1+f)c^2-1)\,,$$

hence the difference (2.15) is zero, and therefore (2.14) holds. ☐

In our application, $c = 1$ in a full collar of $\partial\Omega$, so both boundary conditions in (2.13) are automatic.

2.7. **Scaling the frequency.** We shall also use the following elementary observation. For any constant $s > 0$,

$$\mathcal{L}_{s\gamma} = s\mathcal{L}_\gamma ,$$

which implies

$$\mathcal{L}_{s\gamma} u = \lambda_0 u \quad \iff \quad \mathcal{L}_\gamma u = \frac{\lambda_0}{s} u .$$

At the level of DN maps,

$$\Lambda_{s\gamma,\lambda_0} = s\,\Lambda_{\gamma,\lambda_0/s} . \tag{2.16}$$

Thus, if two conductivities a_1, a_2 satisfy $\Lambda_{a_1,\lambda_\varepsilon} = \Lambda_{a_2,\lambda_\varepsilon}$ and if $s_\varepsilon = \lambda_0/\lambda_\varepsilon > 0$, then

$$\Lambda_{s_\varepsilon a_1,\lambda_0} = \Lambda_{s_\varepsilon a_2,\lambda_0} .$$

This allows us to first construct equality at a nearby frequency λ_ε, and then rescale both conductivities to recover the prescribed frequency λ_0.

3. Gevrey preliminaries and a local prescribed-Jacobian lemma

This section contains the only Gevrey-specific part of the proof. For details on Gevrey spaces, see e.g. [32]. Our main result is a compactly supported prescribed-Jacobian lemma: if a small density perturbation h is supported in an interior box $Q \Subset \Omega$, has zero integral, and is Gevrey of order $\sigma > 1$, then one can find a Gevrey diffeomorphism Ψ, equal to the identity near $\partial\Omega$, such that $\det D\Psi = 1 + h$.

The point is that this is a local statement: we do not solve a global divergence problem on Ω, and no Gevrey regularity of the boundary is used.

3.1. **Gevrey norms.** Let $\Omega \subset \mathbb{R}^n$ be an open set, let $\sigma \geq 1$, and let $\tau > 0$. For every compact set $K \Subset \Omega$ and every $u \in C^\infty(\Omega)$, we define the Gevrey seminorm

$$|u|_{\sigma,\tau,K} = \sum_{\alpha\in\mathbb{N}^n} \frac{\tau^{|\alpha|}}{(\alpha!)^\sigma} |\partial^\alpha u|_{L^\infty(K)}. \tag{3.1}$$

For vector- and matrix-valued functions, the seminorm is defined by summing over the components.

We say that $u \in G^\sigma(\Omega)$ if, for every compact set $K \Subset \Omega$, there exists $\tau > 0$ such that $|u|_{\sigma,\tau,K} < \infty$. Equivalently, $u \in G^\sigma(\Omega)$ if for every compact set $K \Subset \Omega$, there exist constants $C > 0$ and $R > 0$ such that

$$|\partial^\alpha u(x)| \leq CR^{|\alpha|}(\alpha!)^\sigma ,$$

for every multi-index $\alpha \in \mathbb{N}^n$ and every $x \in K$. In particular, $G^1(\Omega) = C^\omega(\Omega)$ is the analytic class.

If $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, the space $G^\sigma(\overline{\Omega})$ denotes the space of restrictions to $\overline{\Omega}$ of G^σ functions defined in some neighborhood of $\overline{\Omega}$.

Fix an open neighborhood U of $\overline{\Omega}$ and let $\tau > 0$. For a function $u \in C^\infty(U)$, we consider the Gevrey seminorm (3.1) on $\overline{\Omega}$,

$$|u|_{\sigma,\tau,\overline{\Omega}} < \infty .$$

This seminorm induces a natural metric topology on the corresponding Gevrey class. In the statements below, a "G^σ-neighborhood" always refers to a neighborhood with respect to one of these seminorm topologies.

If $Q \Subset \Omega$, then $G^\sigma_c(Q)$ denotes the space of compactly supported G^σ functions in Q.

The parameter τ is called a Gevrey radius. In the estimates below, differentiation, composition, and flow maps may require decreasing τ.

3.2. **Basic Gevrey calculus.** We record the elementary estimates used later, (see [32]).

First, for $\sigma \geq 1$, the Gevrey class G^σ is an algebra. More precisely, for every pair (K, τ), one has

$$|u\, v|_{\sigma,\tau,K} \leq |u|_{\sigma,\tau,K} |v|_{\sigma,\tau,K} \,. \tag{3.2}$$

Second, if $|h|_{\sigma,\tau,K} < 1$, then

$$\left| \frac{1}{1+h} \right|_{\sigma,\tau,K} \leq \frac{1}{1 - |h|_{\sigma,\tau,K}} \,,$$

by the Neumann series and the algebra estimate.

More generally, if $F(z) = \sum_{m \geq 0} a_m z^m$ is analytic for $|z| < R$, and if $|h|_{\sigma,\tau,K} < R$, then $F(h) \in G^\sigma_\tau(K)$ and

$$|F(h)|_{\sigma,\tau,K} \leq \sum_{m \geq 0} |a_m| \, |h|^m_{\sigma,\tau,K} \,.$$

This will be used for powers such as $(1 + \varepsilon u)^\alpha$.

Third, derivatives cost radius, namely if $0 < \tau' < \tau$, then

$$|\partial_j u|_{\sigma,\tau',K} \leq C_{\sigma,\tau,\tau'} |u|_{\sigma,\tau,K} \,,$$

where one may take

$$C_{\sigma,\tau,\tau'} = \frac{1}{\tau} \sup_{m \geq 1} m^\sigma \left(\frac{\tau'}{\tau} \right)^{m-1} < \infty \,.$$

This radius loss is important later when estimating terms such as $\nabla \cdot (\gamma \nabla c_\varepsilon)$.

Finally, for $\sigma > 1$, compactly supported G^σ cutoffs exist. This follows from the Denjoy–Carleman non-quasianalyticity condition for the weight $M_k = (k!)^\sigma$, (see [33] for instance):

$$\sum_{k \geq 1} \frac{M_{k-1}}{M_k} = \sum_{k \geq 1} \frac{1}{k^\sigma} < \infty \,.$$

This is the only essential use of the strict inequality $\sigma > 1$. At $\sigma = 1$, the class is analytic, and compactly supported analytic functions vanish identically.

3.3. **A compactly supported right inverse for divergence on a box.** A standard approach to the divergence equation consists in solving the Neumann problem

$$\Delta u = f, \qquad \partial_\nu u = 0,$$

and then setting $X = \nabla u$. This yields Gevrey regularity under appropriate assumptions on the boundary regularity, but does not provide any control on the support of X, even when f is compactly supported. To retain precise support properties, we instead use an explicit localized construction on a box. Let $Q = I_1 \times \cdots \times I_n \Subset \Omega$ with $I_j = (a_j, b_j)$. Choose one-dimensional cutoffs $\theta_j \in G^\sigma_c(I_j)$ with $\int_{I_j} \theta_j(s)\, ds = 1$. After decreasing the Gevrey radius if necessary, assume $\Theta_j := |\theta_j|_{\sigma,\tau,I_j} < \infty$.

We first recall the elementary antiderivative estimate. If

$$(\mathcal{I}_j g)(x) = \int_{a_j}^{x_j} g(x_1, \ldots, s, \ldots, x_n)\, ds \,,$$

then

$$|\mathcal{I}_j g|_{\sigma,\tau,\bar{Q}} \leq (|I_j| + \tau) |g|_{\sigma,\tau,\bar{Q}} \,. \tag{3.3}$$

Lemma 3.1 (Divergence primitive on a box)**.** *Let $h \in G^\sigma_c(Q)$ with $\int_Q h\, dx = 0$. If $|h|_{\sigma,\tau,\bar{Q}} < \infty$, then there exists $X = \mathcal{B}_Q h \in G^\sigma_c(Q; \mathbb{R}^n)$ such that $\nabla \cdot X = h$. Moreover,*

$$|X|_{\sigma,\tau,\bar{Q}} \leq C_{\mathrm{div}}(Q, \tau) |h|_{\sigma,\tau,\bar{Q}} \,.$$

In dimension one, $C_{\mathrm{div}}(I_1, \tau) = |I_1| + \tau$. If $Q = I_1 \times Q'$, then

$$C_{\mathrm{div}}(Q, \tau) = (|I_1| + \tau)(1 + |I_1| \Theta_1) + |I_1| \Theta_1 C_{\mathrm{div}}(Q', \tau) \,.$$

Proof. We argue by induction on n. Write $x = (x_1, x')$, with $Q' = I_2 \times \cdots \times I_n$, and define the fiber average $H(x') = \int_{I_1} h(s, x')\, ds$. Then $\int_{Q'} H(x')\, dx' = 0$ and $|H|_{\sigma,\tau,\bar{Q}'} \le |I_1||h|_{\sigma,\tau,\bar{Q}}$. Set $\widetilde{h}(x_1, x') = h(x_1, x') - \theta_1(x_1)H(x')$. For each fixed x', $\int_{I_1} \widetilde{h}(s, x')\, ds = 0$. Define

$$X_1(x_1, x') = \int_{a_1}^{x_1} \widetilde{h}(s, x')\, ds\,.$$

Because the fiber integral of $\widetilde{h}$ vanishes, X_1 vanishes near both endpoints of I_1. Since h, H, and θ_1 are compactly supported in their respective variables, $X_1 \in G^\sigma_c(Q)$. Also $\partial_{x_1} X_1 = \widetilde{h}$. By (3.3) and (3.2),

$$|X_1|_{\sigma,\tau,\bar{Q}} \le (|I_1| + \tau)(1 + |I_1|\Theta_1)|h|_{\sigma,\tau,\bar{Q}}\,.$$

By induction, there exists $W \in G^\sigma_c(Q'; \mathbb{R}^{n-1})$ such that

$$\nabla_{x'} \cdot W = H\,, \qquad |W|_{\sigma,\tau,\bar{Q}'} \le C_{\mathrm{div}}(Q', \tau)|H|_{\sigma,\tau,\bar{Q}'}\,.$$

For $j = 2, \ldots, n$, define $X_j(x_1, x') = \theta_1(x_1)W_j(x')$. Then $\sum_{j=2}^n \partial_{x_j} X_j = \theta_1 H$, and therefore $\nabla \cdot X = \widetilde{h} + \theta_1 H = h$. The norm estimate follows from the estimates above. ☐

We shall also use the following support consequence of the construction. If $Q_0 \Subset Q$ is fixed and h is supported in Q_0, then, after the one-dimensional cutoffs θ_j have been fixed, the vector field $\mathcal{B}_Q h$ is supported in a compact set $K_0 \Subset Q$ depending only on Q_0, Q, and on the chosen cutoffs, but not on h.

3.4. **Gevrey flows with radius loss.** We use one classical fact from Gevrey calculus: Gevrey vector fields generate Gevrey flows, with a loss of radius in the norms. We discuss this fact here. In particular, stability under composition, inversion, and ODE flows follows from [31]; the continuity of the composition operator is used in the proof below.

Lemma 3.2 (Gevrey flow theorem with radius loss)**.** *Let $\sigma > 1$. Let*

$$K_0 \Subset K_1 \Subset \mathbb{R}^n$$

be compact sets, in the sense that K_0 is contained in the interior of K_1, and let $0 < r < \tau$. There exist constants

$$\delta_{\mathrm{fl}} > 0, \qquad C_{\mathrm{fl}} > 0,$$

depending only on $n, \sigma, K_0, K_1, \tau, r$, with the following property.

Let $A = A(t, x)$ be a G^σ vector field on a neighborhood of $[0, 1] \times \mathbb{R}^n$, and assume that

$$\mathrm{supp}_x\, A(t, \cdot) \subset K_0$$

for every $t \in [0, 1]$. Write $A_t(x) = A(t, x)$, and suppose

$$M := \int_0^1 |A_t|_{\sigma,\tau,K_1}\, dt \le \delta_{\mathrm{fl}}.$$

Then the non-autonomous flow

$$\dot{\Phi}_t(x) = A_t(\Phi_t(x)), \qquad \Phi_0(x) = x,$$

exists globally on $[0, 1]$, equals the identity outside K_0, and satisfies

$$\sup_{0 \le t \le 1} |\Phi_t - \mathrm{Id}\,|_{\sigma,r,K_0} \le C_{\mathrm{fl}} M.$$

Since $\Phi_t - \mathrm{Id}$ is supported in K_0, the same estimate holds on any fixed compact set containing K_0, with the same right-hand side. The inverse maps Φ_t^{-1} are also G^σ, possibly after decreasing the radius once more.

Proof. The Gevrey regularity of the flow is standard from the Denjoy–Carleman ODE theorem. Since $A(t, x)$ is Gevrey jointly in (t, x), the non-autonomous equation can be written as an autonomous equation in the variables (t, x),

$$\frac{d}{ds}(t(s), x(s)) = (1, A(t(s), x(s))).$$

The vector field $(1, A)$ is Gevrey, hence its flow is Gevrey wherever it exists.

It remains to prove the quantitative radius-loss estimate. We use the following local composition estimate. Since $K_0 \Subset K_1$ and $0 < r < \tau$, there exist constants $\rho > 0$ and $C_{\mathrm{comp}} > 0$, depending only on $n, \sigma, K_0, K_1, \tau, r$, such that, whenever

$$|\xi|_{\sigma,r,K_0} \leq \rho$$

and $(\mathrm{Id} + \xi)(K_0) \subset K_1$, one has

$$|F \circ (\mathrm{Id} + \xi)|_{\sigma,r,K_0} \leq C_{\mathrm{comp}} |F|_{\sigma,\tau,K_1}$$

for all $F \in G^\sigma_\tau(K_1; \mathbb{R}^n)$. By decreasing ρ, the condition $(\mathrm{Id} + \xi)(K_0) \subset K_1$ follows from the C^0-part of the bound $|\xi|_{\sigma,r,K_0} \leq \rho$. This is the usual continuity of the composition operator with loss of radius, applied on nested compact sets.

Write

$$\Phi_t = \mathrm{Id} + \xi_t.$$

For $x \in K_0$,

$$\xi_t(x) = \int_0^t A_s \circ (\mathrm{Id} + \xi_s)(x)\, ds.$$

As long as

$$\sup_{0 \leq s \leq t} |\xi_s|_{\sigma,r,K_0} \leq \rho,$$

the composition estimate gives

$$|\xi_t|_{\sigma,r,K_0} \leq C_{\mathrm{comp}} \int_0^t |A_s|_{\sigma,\tau,K_1}\, ds \leq C_{\mathrm{comp}} M.$$

Choose

$$\delta_{\mathrm{fl}} \leq \frac{\rho}{2C_{\mathrm{comp}}}.$$

Then the bootstrap closes and

$$\sup_{0 \leq t \leq 1} |\Phi_t - \mathrm{Id}\,|_{\sigma,r,K_0} \leq C_{\mathrm{comp}} M.$$

Renaming C_{comp} as C_{fl} gives the desired estimate.

Since A_t is supported in K_0, the flow is the identity outside K_0. The inverse flow is obtained by solving the backward non-autonomous equation, which has the same Gevrey regularity and the same type of estimate. ☐

3.5. **Local prescribed Jacobian in Gevrey class.**

Lemma 3.3 (Compactly supported Gevrey Jacobian lemma)**.** *Let $Q_0 \Subset Q \Subset \Omega$ be open boxes, let $\sigma > 1$, and let*

$$h \in G^\sigma_c(Q_0), \qquad \int_Q h\, dx = 0.$$

Fix $\tau > 0$ such that $|h|_{\sigma,\tau,\bar{Q}} < \infty$, and let C_{div} be the constant from Lemma 3.1, applied on the box Q.

After fixing the one-dimensional cutoffs in Lemma 3.1, there exist compact sets

$$K_0 \Subset K_1 \Subset Q$$

depending only on Q_0, Q, and on the chosen cutoffs, such that

$$\operatorname{supp} h \cup \operatorname{supp} \mathcal{B}_Q h \subset K_0$$

for every $h \in G^\sigma_c(Q_0)$. Let $0 < r < \tau$, and let δ_{fl}, C_{fl} be the constants from Lemma 3.2 for these compact sets K_0, K_1.

There exists

$$\eta = \eta(Q_0, Q, \sigma, \tau, r) > 0$$

such that, if

$$|h|_{\sigma,\tau,\bar{Q}} \leq \eta,$$

then there is a diffeomorphism

$$\Psi \in \mathrm{Diff}^{G^\sigma}(\overline{\Omega}, \overline{\Omega})$$

satisfying

$$\Psi = \mathrm{Id} \quad \textit{near } \partial\Omega, \qquad \det D\Psi = 1 + h \quad \textit{in } \Omega.$$

One may take

$$\eta = \min\left\{\frac{1}{2}, \frac{\delta_{\mathrm{fl}}}{2C_{\mathrm{div}}}\right\}.$$

Moreover,

$$|\Psi - \mathrm{Id}\,|_{\sigma,r,K_0} \le 2C_{\mathrm{fl}} C_{\mathrm{div}} |h|_{\sigma,\tau,\bar{Q}}.$$

Equivalently, since $\Psi - \mathrm{Id}$ is supported in K_0, the same estimate holds on any fixed compact set containing K_0.

Proof. By Lemma 3.1, choose

$$X = \mathcal{B}_Q h \in G^\sigma_c(Q; \mathbb{R}^n)$$

such that

$$\nabla \cdot X = h$$

and

$$|X|_{\sigma,\tau,\bar{Q}} \le C_{\mathrm{div}} |h|_{\sigma,\tau,\bar{Q}}.$$

By the support property stated in the lemma,

$$\mathrm{supp}\, h \cup \mathrm{supp}\, X \subset K_0.$$

Extend h and X by zero outside Q. Define

$$\rho_t = 1 + (1-t)h, \qquad A_t = \frac{X}{\rho_t}.$$

Since

$$|h|_{\sigma,\tau,\bar{Q}} \le \frac{1}{2},$$

the reciprocal estimate gives

$$|\rho_t^{-1}|_{\sigma,\tau,\bar{Q}} \le 2.$$

Hence

$$|A_t|_{\sigma,\tau,K_1} \le |A_t|_{\sigma,\tau,\bar{Q}} \le 2C_{\mathrm{div}} |h|_{\sigma,\tau,\bar{Q}}.$$

Moreover $A(t,x)$ is real-analytic in t and Gevrey in x, hence Gevrey jointly in (t,x), and

$$\mathrm{supp}_x\, A(t,\cdot) \subset K_0.$$

Therefore

$$\int_0^1 |A_t|_{\sigma,\tau,K_1}\, dt \le 2C_{\mathrm{div}} |h|_{\sigma,\tau,\bar{Q}} \le \delta_{\mathrm{fl}}.$$

Lemma 3.2 gives a Gevrey flow φ_t, equal to the identity outside K_0, satisfying

$$\dot{\varphi}_t = A_t(\varphi_t), \qquad \varphi_0 = \mathrm{Id},$$

and

$$|\varphi_1 - \mathrm{Id}\,|_{\sigma,r,K_0} \le 2C_{\mathrm{fl}} C_{\mathrm{div}} |h|_{\sigma,\tau,\bar{Q}}.$$

Since $K_0 \Subset Q \Subset \Omega$, the flow is equal to the identity near $\partial\Omega$.

It remains to compute the Jacobian. We have

$$\partial_t \rho_t = -h, \qquad \nabla \cdot (\rho_t A_t) = \nabla \cdot X = h.$$

Thus

$$\partial_t \rho_t + \nabla \cdot (\rho_t A_t) = 0.$$

Let

$$J_t(x) = \det D\varphi_t(x).$$

The standard Moser identity gives

$$\frac{d}{dt}\left[\rho_t(\varphi_t(x)) J_t(x)\right] = 0.$$

Hence

$$\rho_t(\varphi_t(x)) \det D\varphi_t(x) = \rho_0(x) = 1 + h(x).$$

At $t = 1$, $\rho_1 \equiv 1$, and therefore

$$\det D\varphi_1 = 1 + h.$$

Set $\Psi = \varphi_1$. ☐

Remark 3.4. The strict inequality $\sigma > 1$ enters only through localization. We need nonzero compactly supported Gevrey functions in two places: to choose $u \in G^\sigma_c(Q_0)$ so that the conformal factor $c_\varepsilon = (1+\varepsilon u)^\alpha$ equals 1 near $\partial\Omega$, and to choose the one-dimensional cutoffs $\theta_j \in G^\sigma_c(I_j)$ used in the box right inverse for divergence. The remaining Gevrey operations also hold locally in the analytic class. What fails at $\sigma = 1$ is the existence of nontrivial compactly supported analytic functions.

4. The fixed-potential problem

In this section We prove Theorem 1.1. For simplicity, we only treat the analytic case; the C^∞ case is completely analogous up to minor modifications. The proof uses only the elementary Gevrey calculus of Section 3 and the fixed-potential conformal identity established in Proposition 2.2. Unlike the fixed-frequency construction, no prescribed-Jacobian theorem is needed.

Throughout this section, $g \in C^\omega(\overline{\Omega})$ is a uniformly elliptic metric, and $V \in C^\omega(\overline{\Omega})$ is a scalar potential. We assume that V is nonconstant and that

$$0 \notin \sigma_{\mathrm{D}}(-\Delta_g + V)\,. \tag{4.1}$$

Since g is analytic on $\overline{\Omega}$, it belongs to $G^\sigma(\overline{\Omega})$ for every $\sigma > 1$. Fix such a $\sigma > 1$ once and for all. The goal is to construct metrics $g_{1,\varepsilon}, g_{2,\varepsilon} \in G^\sigma(\overline{\Omega})$ arbitrarily close to g, with identical DN maps for the fixed potential V, but not related by any diffeomorphism.

4.1. **Choosing a submersion box.**

Since V is analytic and nonconstant, dV is not identically zero in Ω. Hence there is a point $x_0 \in \Omega$ such that $dV(x_0) \neq 0$. After relabeling the Euclidean coordinates, we may assume $\partial_{x_1} V(x_0) \neq 0$. Choose open sets $Q_0 \Subset Q \Subset U \Subset \Omega$ with Q, Q_0 boxes, such that $\partial_{x_1} V \neq 0$ on U. After shrinking U, the map

$$F : U \to F(U), \qquad F(x) = (V(x), x_2, \ldots, x_n),$$

is an analytic diffeomorphism onto its image.

Choose

$$u \in G^\sigma_c(Q_0)\,, \qquad u \geq 0\,, \qquad u \not\equiv 0\,.$$

Then

$$\int_\Omega u \, dV_g > 0.$$

For sufficiently small $\varepsilon > 0$, define

$$c_\varepsilon = 1 + \varepsilon u\,.$$

Then $c_\varepsilon > 0$, $c_\varepsilon = 1$ outside Q_0, and $c_\varepsilon - 1 \in G^\sigma_c(Q_0)$.

4.2. **The scalar correction.**

Define

$$T_\varepsilon = V c_\varepsilon^{\frac{4}{n-2}} + \frac{\Delta_g c_\varepsilon}{c_\varepsilon}\,.$$

Since $c_\varepsilon = 1$ outside Q_0, and all derivatives of c_ε vanish outside Q_0, we have

$$T_\varepsilon = V \quad \text{outside } Q_0\,. \tag{4.2}$$

Moreover, by the Gevrey calculus of Section 3, after possibly decreasing the Gevrey radius, $T_\varepsilon - V = O(\varepsilon)$ in G^σ, and in particular in $C^1(\overline{\Omega})$[1]. Here we use that $g \in G^\sigma$, that g is uniformly elliptic, and that differentiation costs only a loss of Gevrey radius.

We now define the compensating diffeomorphism. On U, set

$$\Psi_\varepsilon(x) = F^{-1}(T_\varepsilon(x), x_2, \dots, x_n)\,.$$

Outside U, set $\Psi_\varepsilon(x) = x\,.$ This is well-defined for all sufficiently small ε. Indeed, T_ε is C^1-close to V, and the perturbation is supported in $Q_0 \Subset U$, so the point $(T_\varepsilon(x), x_2, \dots, x_n)$ remains in $F(U)$. Also, by (4.2), the definition agrees with the identity near ∂U, so it glues smoothly to the identity outside U.

Since F^{-1} is analytic and $T_\varepsilon \in G^\sigma$, we have $\Psi_\varepsilon \in \mathrm{Diff}^{G^\sigma}(\overline{\Omega})$, after possibly shrinking ε. Moreover, $\Psi_\varepsilon = \mathrm{Id}$ near $\partial\Omega$, and $\Psi_\varepsilon \to \mathrm{Id}$ in every smaller Gevrey radius norm. The map is a diffeomorphism because it is C^1-close to the identity.

By construction,

$$V \circ \Psi_\varepsilon = T_\varepsilon\,.$$

Equivalently,

$$V \circ \Psi_\varepsilon = V c_\varepsilon^{4/(n-2)} + \frac{\Delta_g c_\varepsilon}{c_\varepsilon}\,. \tag{4.3}$$

This is the fixed-potential analogue of the scalar compatibility equation in the fixed-frequency construction.

4.3. **Equality of DN maps.**

Define

$$g_{2,\varepsilon} = c_\varepsilon^{4/(n-2)} g, \qquad g_{1,\varepsilon} = (\Psi_\varepsilon)_* g\,.$$

Since $c_\varepsilon = 1$ near $\partial\Omega$ and $\Psi_\varepsilon = \mathrm{Id}$ near $\partial\Omega$, Proposition 2.2 applies with $c = c_\varepsilon$ and $\Psi = \Psi_\varepsilon$. The compatibility condition (4.3) gives

$$\Lambda_{g_{2,\varepsilon},V} = \Lambda_{g_{1,\varepsilon},V}\,.$$

The spectral condition is also preserved. Since $g_{2,\varepsilon} \to g$ as $\varepsilon \to 0$ in the C^1-topology, and (4.1) holds, standard spectral stability gives $0 \notin \sigma_{\mathrm{D}}(-\Delta_{g_{2,\varepsilon}} + V)$ for all sufficiently small ε. The solution correspondence in Proposition 2.2 then gives $0 \notin \sigma_{\mathrm{D}}(-\Delta_{g_{1,\varepsilon}} + V)\,.$

Both $g_{1,\varepsilon}$ and $g_{2,\varepsilon}$ belong to $G^\sigma(\overline{\Omega})\,.$ For $g_{2,\varepsilon}$ this follows from the algebra and analytic functional calculus in G^σ. For $g_{1,\varepsilon}$, it follows from the Gevrey stability of composition, multiplication, and inverse maps. Moreover, $g_{j,\varepsilon} \to g$ for $j = 1, 2$ in every smaller Gevrey radius norm, and hence in every C^m norm.

4.4. **Non-isometry.**

We now show that $g_{1,\varepsilon}$ and $g_{2,\varepsilon}$ are not related by any diffeomorphism of $\overline{\Omega}$.

To this end we use the invariant is given by the total Riemannian volume $\mathrm{Vol}_g(\Omega) = \int_\Omega dV_g$. It is invariant under pushforward by any diffeomorphism. Since $g_{1,\varepsilon} = (\Psi_\varepsilon)_* g$, we have

$$\mathrm{Vol}_{g_{1,\varepsilon}}(\Omega) = \mathrm{Vol}_{(\Psi_\varepsilon)_* g}(\Omega) = \mathrm{Vol}_g(\Omega)\,.$$

Since $g_{2,\varepsilon} = c_\varepsilon^{4/(n-2)} g\,,$ $c_\varepsilon = 1 + \varepsilon u$, and $u \geq 0$ with $u \not\equiv 0$, we have

$$dV_{g_{2,\varepsilon}} = c_\varepsilon^{2n/(n-2)}\, dV_g\,,$$

with $c_\varepsilon^{2n/(n-2)} \geq 1$ on Ω and strict inequality on a set of positive measure. Therefore

$$\mathrm{Vol}_{g_{2,\varepsilon}}(\Omega) > \mathrm{Vol}_g(\Omega) = \mathrm{Vol}_{g_{1,\varepsilon}}(\Omega)\,.$$

Consequently, there is no diffeomorphism Φ of $\overline{\Omega}$ such that $g_{2,\varepsilon} = \Phi_* g_{1,\varepsilon}$ [2]

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[1] In the smooth case, the same argument shows that T_ε is merely a C^∞ function and that $T_\varepsilon - V = O(\varepsilon)$ in the C^∞ topology.

[2] Alternatively, one could appeal to [25, Proposition 3.3], under the additional assumption that Ψ restricts to the identity on $\partial\Omega\,.$

Taking a sequence $\varepsilon_j \downarrow 0$ gives infinitely many pairwise distinct pairs. This proves Theorem 1.1.

4.5. **Remarks on the assumptions.** The construction uses the hypothesis that V is nonconstant only to find one interior box $Q \Subset \Omega$ on which $dV \neq 0$. No global condition is imposed on V: it need not be Morse, it need not have regular level sets globally, and no sign condition is required.

If $V \equiv 0$, the gauge becomes the full boundary-fixing diffeomorphism group of the classical anisotropic Calderón problem. That problem is of a different nature, and the present construction does not apply. More generally, when V is constant, the scalar equation $V \circ \Psi = T_\varepsilon$ supplies no local coordinate, and the mechanism above is unavailable. In fact, if one attempts to solve $V \circ \Psi = T_\varepsilon$ with a constant potential under the boundary conditions $c = 1$ and $\partial_{\nu_g} c = 0$ on $\partial\Omega$, then c must be trivial. Indeed, for $V \equiv \lambda$, the condition that the effective potential remains equal to λ reads

$$\Delta_g c + \lambda \left(c^{\frac{n+2}{n-2}} - c \right) = 0\,.$$

Arguing exactly as in Lemma 2.1, unique continuation from the boundary implies that $c \equiv 1$. Therefore, the conformal degree of freedom cannot compensate for the lack of a local coordinate in the constant-potential case.

We conclude with a simple concrete example illustrating the previous construction.

Example 4.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $g \in G^\omega(\overline{\Omega})$ be a uniformly elliptic metric, and consider the linear potential

$$V(x) = a x_1\,, \qquad a \neq 0\,.$$

We recall the construction used in the proof. Choose

$$u \in G^\sigma_c(\Omega)\,, \qquad u \geq 0\,, \qquad u \not\equiv 0\,,$$

and define for sufficiently small $\epsilon > 0$

$$c_\varepsilon(x) = 1 + \varepsilon u(x)\,.$$

Then

$$g_{2,\varepsilon} = c_\varepsilon^{\frac{4}{n-2}} g\,.$$

Since

$$F(x) = (a x_1, x_2, \dots, x_n)\,,$$

the associated diffeomorphism is explicitly given by

$$\Psi_\varepsilon(x) = \left(x_1 c_\varepsilon(x)^{\frac{4}{n-2}} + \frac{\varepsilon}{a} \frac{\Delta_g u(x)}{1 + \varepsilon u(x)}\,, x_2, \dots, x_n \right)\,.$$

Hence

$$g_{1,\varepsilon} = (\Psi_\varepsilon)_* g\,.$$

5. THE FIXED NONZERO-FREQUENCY CONSTRUCTION

In this section, we only prove Theorem 1.5 in the analytic setting. Thus, throughout the section, the conductivity γ is assumed to be analytic on $\overline{\Omega}$. In particular,

$$\gamma \in G^\sigma(\overline{\Omega}; \mathrm{Sym}_n^+) \qquad \text{for every } \sigma > 1\,.$$

We fix such a $\sigma > 1$ once and for all. The proof in the C^∞ setting is identical up to minor modifications.

5.1. **A constrained Gevrey test function.** We now choose the compactly supported Gevrey perturbation which will enter the conformal factor. The construction requires two moment conditions. The first is used in the expansion of the normalized frequency. The second removes the first-order term in the determinant invariant used in Section 6 to prove non-isometry.

Fix open boxes

$$Q_0 \Subset Q \Subset \Omega.$$

The Gevrey test function will be supported in Q_0, while the Jacobian correction will be carried out in the larger box Q. Throughout this subsection,

$$\gamma \in G^\sigma(\overline{\Omega}; \mathrm{Sym}_n^+), \qquad \sigma > 1\,.$$

Set

$$w(x) := (\det \gamma(x))^{1/(n-2)}\,. \tag{5.1}$$

Since γ is uniformly positive definite and Gevrey, w is also Gevrey on $\overline{\Omega}$. We write the Dirichlet energy

$$\mathcal{E}_\gamma(v) = \int_\Omega \gamma \nabla v \cdot \nabla v \, dx\,.$$

Lemma 5.1 (Prescribing large energy with two moments)**.** *There exists $q_0 > 0$ such that for every $q > q_0$ there is a real-valued function $u \in G_c^\sigma(Q_0)$ satisfying*

$$\|u\|_{L^2(\Omega)} = 1\,, \tag{5.2}$$

$$\int_\Omega u\, dx = 0\,, \qquad \int_\Omega u\, w\, dx = 0\,, \tag{5.3}$$

$$\mathcal{E}_\gamma(u) = q\,. \tag{5.4}$$

Proof. For $v \in G_c^\sigma(Q_0)$, let $\mathcal{M}(v) = \left(\int_{Q_0} v\, dx, \int_{Q_0} w\, v\, dx\right)$ and $Y = \ker \mathcal{M} \cap G_c^\sigma(Q_0)$. Since $G_c^\sigma(Q_0)$ is infinite-dimensional and $\mathcal{M}$ has rank at most two, Y is infinite-dimensional.

Let $r = \operatorname{rank} \mathcal{M}$. Choose $p_1, \dots, p_r \in G_c^\sigma(Q_0)$ such that the corresponding $r \times r$ moment matrix is invertible. More precisely, if

$$L_1(v) = \int_{Q_0} v\, dx\,, \qquad L_2(v) = \int_{Q_0} w\, v\, dx\,,$$

then the moment matrix is

$$A = (a_{ij})_{1 \le i,j \le r}\,, \qquad a_{ij} = L_i(p_j)\,.$$

Then there is a finite-rank projection

$$P : G_c^\sigma(Q_0) \to Y$$

of the form

$$Pv = v - \sum_{j=1}^r c_j(v) p_j\,,$$

where the coefficients $c_j(v)$ depend linearly on $\mathcal{M}(v)$ and are uniquely determined by the condition $\mathcal{M}(Pv) = 0$. Then there is a finite-rank projection $P : G_c^\sigma(Q_0) \to Y$ of the form $Pv = v - \sum_{j=1}^r c_j(v) p_j$, where the coefficients $c_j(v)$ depend linearly on $\mathcal{M}(v)$.

Choose $\eta \in G_c^\sigma(Q_0)$, $\eta \not\equiv 0$, and define $v_N(x) = \eta(x)\cos(N x_1)$ for $N \to +\infty$. By the Riemann–Lebesgue lemma, $\mathcal{M}(v_N) \to 0$, hence $c_j(v_N) \to 0$ and $Pv_N = v_N + o_{L^2}(1)$. In particular,

$$\|Pv_N\|^2_{L^2(\Omega)} = \|v_N\|^2_{L^2(\Omega)} + o(1) = \frac{1}{2}\int_{Q_0} \eta^2\, dx + o(1)\,,$$

so $\|Pv_N\|_{L^2}$ stays bounded away from zero.

Since $\partial_1 v_N = (\partial_1 \eta)\cos(N x_1) - N\eta \sin(N x_1)$ and the remaining derivatives are $O(1)$,

$$\mathcal{E}_\gamma(v_N) = N^2 \int_{Q_0} \gamma^{11}(x) \eta(x)^2 \sin^2(N x_1)\, dx + O(N)\,.$$

By Riemann–Lebesgue,

$$\int_{Q_0} \gamma^{11}\eta^2 \sin^2(Nx_1)\,dx = \frac{1}{2}\int_{Q_0} \gamma^{11}\eta^2\,dx + o(1)\,,$$

and since γ is uniformly positive definite and $\eta \not\equiv 0$, $\int_{Q_0} \gamma^{11}\eta^2\,dx > 0$. Hence $\mathcal{E}_\gamma(v_N) = \frac{N^2}{2}\int_{Q_0} \gamma^{11}\eta^2\,dx + o(N^2)$. The finite-rank correction does not change the leading term: $Pv_N - v_N = -\sum_{j=1}^r c_j(v_N)p_j$ with $c_j(v_N) \to 0$, so $\|\nabla(Pv_N - v_N)\|_{L^2} = o(1)$ and $\mathcal{E}_\gamma(Pv_N) = \mathcal{E}_\gamma(v_N) + o(N^2)$.

With $z_N = Pv_N/\|Pv_N\|_{L^2}$, we have $z_N \in Y$, $\|z_N\|_{L^2} = 1$, and $\mathcal{E}_\gamma(z_N) \to +\infty$. Choose $u_0 \in Y$ with $\|u_0\|_{L^2} = 1$ and set $q_0 > \mathcal{E}_\gamma(u_0)$. For $q > q_0$, choose N so large that $\mathcal{E}_\gamma(z_N) > q$. Since both u_0 and z_N lie in Y, the normalized path

$$\theta \mapsto \frac{(1-\theta)u_0 + \theta z_N}{\|(1-\theta)u_0 + \theta z_N\|_{L^2}}\,, \qquad 0 \le \theta \le 1\,,$$

is well defined for N so large that $z_N \neq -u_0$, and remains in Y. Along this path the energy is continuous. It starts below q and ends above q, hence by the intermediate value theorem there exists $u \in Y$ with $\|u\|_{L^2} = 1$ and $\mathcal{E}_\gamma(u) = q$. □

Let $\lambda_0 \in \mathbb{R} \setminus \{0\}$. Choose $q > q_0$ as in Lemma 5.1. If $\lambda_0 > 0$, impose additionally $q > 2\lambda_0$. Avoid also the at most one value of q for which α below equals $\frac{1}{2} - \frac{1}{n}$. Define

$$\alpha = \frac{\lambda_0}{q - 2\lambda_0}\,. \tag{5.5}$$

Then $2\alpha + 1 = q/(q - 2\lambda_0)$ and hence

$$\frac{\alpha}{2\alpha+1}\,q = \lambda_0\,.$$

With the above choice of q, we have $2\alpha + 1 > 0$, $\alpha \neq 0$, and $\alpha \neq \frac{1}{2} - \frac{1}{n}$. If $\lambda_0 > 0$, then $\alpha > 0$; if $\lambda_0 < 0$, then $-\frac{1}{2} < \alpha < 0$.

By Lemma 5.1, choose $u \in G^\sigma_c(Q_0)$ satisfying (5.2)–(5.4). These are the only properties of u used in the construction.

5.2. **Conformal factor, frequency normalization, and Jacobian correction.**

We now construct the two conductivities with identical DN maps. Throughout, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain; $\gamma \in G^\sigma(\overline{\Omega}; \mathrm{Sym}^+_n)$ with $\sigma > 1$; and $\lambda_0 \notin \sigma_{\mathrm{D}}(\mathcal{L}_\gamma) \cup \{0\}$.

Let $Q_0 \Subset Q \Subset \Omega$ be the boxes from Subsection 5.1. We use the function $u \in G^\sigma_c(Q_0)$ chosen there, satisfying (5.2)–(5.4) with w as in (5.1) and α as in (5.5).

For $\varepsilon > 0$ sufficiently small, define the conformal factor

$$c_\varepsilon = (1 + \varepsilon u)^\alpha\,. \tag{5.6}$$

Since u is compactly supported in Q_0, we have $c_\varepsilon = 1$ outside Q_0. In particular, $c_\varepsilon = 1$ near $\partial\Omega$, and $c_\varepsilon - 1 \in G^\sigma_c(Q_0)$. Thus

$$\gamma\nabla c_\varepsilon \cdot \nu = 0 \quad \text{on } \partial\Omega\,.$$

After fixing any sufficiently small Gevrey radius, $c_\varepsilon \to 1$ in that norm as $\varepsilon \to 0$; in particular there exist $\tau > 0$ and C with $|c_\varepsilon - 1|_{\sigma,\tau,\bar{Q}} \le C\varepsilon$ for all small ε, by the analytic functional calculus in Gevrey spaces applied to $z \mapsto (1+z)^\alpha$.

Next, we define the associated frequency by

$$\lambda_\varepsilon = \frac{\int_\Omega \gamma\nabla c_\varepsilon \cdot \nabla c_\varepsilon / c_\varepsilon^2\,dx}{\int_\Omega (c_\varepsilon^{-2} - 1)\,dx}\,. \tag{5.7}$$

On the one hand, $\nabla c_\varepsilon / c_\varepsilon = \alpha\varepsilon\nabla u/(1 + \varepsilon u)$, so

$$\int_\Omega \gamma\nabla c_\varepsilon \cdot \nabla c_\varepsilon / c_\varepsilon^2\,dx = \alpha^2\varepsilon^2 \int_{Q_0} \gamma\nabla u \cdot \nabla u\,dx + O(\varepsilon^3) = \alpha^2 q\,\varepsilon^2 + O(\varepsilon^3)\,,$$

where $q = \mathcal{E}_\gamma(u)$ as in (5.4). On the other hand, $c_\varepsilon^{-2} = (1+\varepsilon u)^{-2\alpha}$ expands as

$$c_\varepsilon^{-2} - 1 = -2\alpha\varepsilon u + \alpha(2\alpha+1)\varepsilon^2 u^2 + O(\varepsilon^3)\,.$$

Using (5.6), (5.3) and (5.2), the denominator is therefore nonzero for all sufficiently small ε

$$\int_\Omega (c_\varepsilon^{-2} - 1)\,dx = \alpha(2\alpha+1)\varepsilon^2 + O(\varepsilon^3)\,.$$

It follows that

$$\lambda_\varepsilon = \frac{\alpha^2 q}{\alpha(2\alpha+1)} + O(\varepsilon) = \frac{\alpha}{2\alpha+1} q + O(\varepsilon) = \lambda_0 + O(\varepsilon)\,.$$

In particular, for ε small, $\lambda_\varepsilon \neq 0$ and λ_ε has the same sign as λ_0. Since $c_\varepsilon^2\gamma \to \gamma$ in the C^1-topology and $\lambda_\varepsilon \to \lambda_0$ with λ_0 outside the Dirichlet spectrum of $\mathcal{L}_\gamma$, standard spectral stability gives $\lambda_\varepsilon \notin \sigma_{\mathrm{D}}(\mathcal{L}_{c_\varepsilon^2\gamma})$ for all sufficiently small ε.

We now introduce the adapted density

$$f_\varepsilon = -\frac{1}{\lambda_\varepsilon c_\varepsilon}\nabla\cdot(\gamma\nabla c_\varepsilon) + c_\varepsilon^{-2} - 1\,.$$

Then $f_\varepsilon \in G_c^\sigma(Q_0)$. Since $c_\varepsilon = 1$ outside Q_0, the support of f_ε is contained in Q_0, and $|f_\varepsilon|_{\sigma,\tau,\bar{Q}} \le C\varepsilon$ for small ε, by the algebra property and derivative estimates with radius loss. By construction,

$$\nabla\cdot(\gamma\nabla c_\varepsilon) + \lambda_\varepsilon(c_\varepsilon - c_\varepsilon^{-1} + c_\varepsilon f_\varepsilon) = 0\,.$$

Also $\int_\Omega f_\varepsilon\,dx = 0$: since $c_\varepsilon = 1$ near $\partial\Omega$,

$$\int_\Omega c_\varepsilon^{-1}\nabla\cdot(\gamma\nabla c_\varepsilon)\,dx = \int_\Omega \gamma\nabla c_\varepsilon\cdot\nabla c_\varepsilon/c_\varepsilon^2\,dx\,,$$

hence $\int_\Omega f_\varepsilon\,dx = 0$ by (5.7). Since $f_\varepsilon = O(\varepsilon)$ in G^σ, we have $1 + f_\varepsilon > 0$ for ε small.

Since $f_\varepsilon \in G_c^\sigma(Q_0)$, $\int_Q f_\varepsilon\,dx = 0$, and $|f_\varepsilon|_{\sigma,\tau,\bar{Q}}$ is small, Lemma 3.3, applied to the pair $Q_0 \Subset Q$, gives a diffeomorphism

$$\Psi_\varepsilon \in \mathrm{Diff}^{G^\sigma}(\overline{\Omega},\overline{\Omega})$$

such that

$$\Psi_\varepsilon = \mathrm{Id} \quad \text{near } \partial\Omega, \qquad \det D\Psi_\varepsilon = 1 + f_\varepsilon.$$

Moreover $\Psi_\varepsilon \to \mathrm{Id}$ in every smaller Gevrey radius norm.

The hypotheses of Proposition 2.6 hold with $c = c_\varepsilon$, $f = f_\varepsilon$, $\lambda = \lambda_\varepsilon$, $\Psi = \Psi_\varepsilon$. Hence

$$\Lambda_{c_\varepsilon^2\gamma,\lambda_\varepsilon} = \Lambda_{(\Psi_\varepsilon)_*\gamma,\lambda_\varepsilon}\,. \tag{5.8}$$

To pass from the nearby frequency λ_ε to the prescribed frequency λ_0, set $s_\varepsilon = \lambda_0/\lambda_\varepsilon$. For ε small, $s_\varepsilon > 0$ and $s_\varepsilon \to 1$. Define $\beta_\varepsilon = s_\varepsilon\gamma$,

$$\gamma_{2,\varepsilon} = c_\varepsilon^2\beta_\varepsilon\,, \qquad \gamma_{1,\varepsilon} = (\Psi_\varepsilon)_*\beta_\varepsilon\,. \tag{5.9}$$

Since $\beta_\varepsilon = s_\varepsilon\gamma$ and pushforward commutes with multiplication by the scalar s_ε, $\gamma_{1,\varepsilon} = s_\varepsilon(\Psi_\varepsilon)_*\gamma$. By (2.16),

$$\Lambda_{s_\varepsilon a,\lambda_0} = s_\varepsilon\,\Lambda_{a,\lambda_\varepsilon} \quad \text{whenever } s_\varepsilon = \lambda_0/\lambda_\varepsilon\,.$$

Applying this to both sides of (5.8) yields $\Lambda_{\gamma_{2,\varepsilon},\lambda_0} = \Lambda_{\gamma_{1,\varepsilon},\lambda_0}$. Moreover $\lambda_0 \notin \sigma_{\mathrm{D}}(\mathcal{L}_{\gamma_{2,\varepsilon}})$ because $\mathcal{L}_{\gamma_{2,\varepsilon}}u = \lambda_0 u$ is equivalent to $\mathcal{L}_{c_\varepsilon^2\gamma}u = \lambda_\varepsilon u$. The same holds for $\gamma_{1,\varepsilon}$ by the diffeomorphism equivalence.

It remains to verify the Gevrey regularity and the convergence $\gamma_{j,\varepsilon} \to \gamma$. Both $\gamma_{1,\varepsilon}$ and $\gamma_{2,\varepsilon}$ belong to $G^\sigma(\overline{\Omega};\mathrm{Sym}_n^+)$. For $\gamma_{2,\varepsilon}$ this follows from $\gamma_{2,\varepsilon} = s_\varepsilon c_\varepsilon^2\gamma$. For $\gamma_{1,\varepsilon}$, use the pushforward formula (2.7). The Gevrey class is stable under multiplication, determinant, reciprocal, composition, and inverse maps, with loss of radius. Since $c_\varepsilon \to 1$, $s_\varepsilon \to 1$, and $\Psi_\varepsilon \to \mathrm{Id}$ in smaller Gevrey norms, $\gamma_{j,\varepsilon} \to \gamma$ for $j = 1, 2$ in every smaller Gevrey radius, hence in every C^m norm.

6. Non-isometry

We prove that the conductivities constructed in Section 5 are not connected by the pushforward of a diffeomorphism. Recall (5.9), $\beta_\varepsilon = s_\varepsilon \gamma$, $c_\varepsilon = (1+\varepsilon u)^\alpha$ from (5.6), and w from (5.1). The moment and slope constraints from Lemma 5.1 and (5.5) are in force.

Consider the determinant invariant $\mathcal{I}$ from (2.1). If $\Psi : \overline{\Omega} \to \overline{\Omega}$ is a diffeomorphism, then

$$\det((\Psi_*\kappa)(\Psi(x))) = |\det D\Psi(x)|^{2-n} \det \kappa(x)\,.$$

Therefore $(\det((\Psi_*\kappa)(\Psi(x))))^{1/(n-2)} = |\det D\Psi(x)|^{-1} (\det \kappa(x))^{1/(n-2)}$, and changing variables $y = \Psi(x)$ gives

$$\mathcal{I}(\Psi_*\kappa) = \mathcal{I}(\kappa)\,. \tag{6.1}$$

Suppose, for contradiction, that $\gamma_{1,\varepsilon}$ and $\gamma_{2,\varepsilon}$ are isometric, so there exists $\Phi \in \mathrm{Diff}(\overline{\Omega})$ with $\gamma_{2,\varepsilon} = \Phi_*\gamma_{1,\varepsilon}$. Hence $\mathcal{I}(\gamma_{2,\varepsilon}) = \mathcal{I}(\gamma_{1,\varepsilon})$. Since $\gamma_{1,\varepsilon} = (\Psi_\varepsilon)_*\beta_\varepsilon$, (6.1) gives $\mathcal{I}(\gamma_{1,\varepsilon}) = \mathcal{I}(\beta_\varepsilon)$. On the other hand, $\gamma_{2,\varepsilon} = c_\varepsilon^2 \beta_\varepsilon$, so $\det \gamma_{2,\varepsilon} = c_\varepsilon^{2n} \det \beta_\varepsilon$ and

$$\mathcal{I}(\gamma_{2,\varepsilon}) = \int_\Omega c_\varepsilon^{2n/(n-2)} (\det \beta_\varepsilon)^{1/(n-2)}\, dx\,.$$

Since $\beta_\varepsilon = s_\varepsilon \gamma$, $(\det \beta_\varepsilon)^{1/(n-2)} = s_\varepsilon^{n/(n-2)} w$. Thus $\mathcal{I}(\gamma_{2,\varepsilon}) = \mathcal{I}(\gamma_{1,\varepsilon})$ implies

$$\int_\Omega (c_\varepsilon^{2n/(n-2)} - 1) w\, dx = 0\,. \tag{6.2}$$

Since $c_\varepsilon = (1+\varepsilon u)^\alpha$, expand uniformly on Ω:

$$c_\varepsilon^{2n/(n-2)} = (1+\varepsilon u)^{2\alpha n/(n-2)} = 1 + \frac{2\alpha n}{n-2}\varepsilon u + \frac{\alpha n}{n-2}\left(\frac{2\alpha n}{n-2} - 1\right)\varepsilon^2 u^2 + O(\varepsilon^3)\,.$$

Substituting into (6.2) and using $\int_\Omega u\, w\, dx = 0$ yields

$$\frac{\alpha n}{n-2}\left(\frac{2\alpha n}{n-2} - 1\right)\varepsilon^2 \int_\Omega u^2 w\, dx + O(\varepsilon^3) = 0\,.$$

But $w > 0$ and $u \not\equiv 0$, hence $\int_\Omega u^2 w\, dx > 0$. Moreover $\alpha \neq 0$ and $\frac{2\alpha n}{n-2} - 1 \neq 0$ because $\alpha \neq \frac{1}{2} - \frac{1}{n}$. Thus the coefficient of ε^2 is nonzero, a contradiction for all sufficiently small $\varepsilon > 0$. Hence $\gamma_{1,\varepsilon}$ and $\gamma_{2,\varepsilon}$ are not isometric.

Finally, the pairs may be chosen infinitely many and distinct: the quantity $\mathcal{I}(\gamma_{2,\varepsilon}) - \mathcal{I}(\gamma_{1,\varepsilon})$ has expansion

$$s_\varepsilon^{n/(n-2)} \frac{\alpha n}{n-2}\left(\frac{2\alpha n}{n-2} - 1\right)\varepsilon^2 \int_\Omega u^2 w\, dx + O(\varepsilon^3)\,,$$

with nonzero leading coefficient, hence along a sequence $\varepsilon_j \downarrow 0$ for which this scalar is strictly monotone the corresponding pairs are distinct. This completes the proof of Theorem 1.5.

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Appendix A. Analytic reconstruction for Schrödinger pairs

The uniqueness statements in the introduction are direct consequences of the analytic reconstruction method of Lassas–Uhlmann [23], with the standard lower-order modification needed for Schrödinger operators. The purpose of this appendix is only to spell out this reduction and to identify the corresponding gauges in the two problems considered in the paper. We do not repeat the full statements of Theorems 1.4 and 1.6.

Throughout the appendix, $\Omega \subset \mathbb{R}^n$, $n \geq 3$, has real-analytic boundary, and all metrics and potentials are assumed real analytic in a neighborhood of $\overline{\Omega}$. Diffeomorphisms are real-analytic diffeomorphisms of $\overline{\Omega}$ fixing $\partial\Omega$ pointwise.

A.1. The analytic reconstruction input.

We use the following analytic reconstruction principle. Let g_1, g_2 be real-analytic Riemannian metrics and let V_1, V_2 be real-analytic scalar potentials. Assume that zero is not a Dirichlet eigenvalue of either operator

$$-\Delta_{g_j} + V_j\,, \qquad j = 1, 2\,.$$

If

$$\Lambda_{g_1,V_1} = \Lambda_{g_2,V_2}\,,$$

then there exists a boundary-fixing real-analytic diffeomorphism Ψ such that

$$g_2 = \Psi_* g_1\,, \qquad V_2 = V_1 \circ \Psi^{-1}\,.$$

Equivalently, $V_2(\Psi(x)) = V_1(x)\,.$

This is the analytic uniqueness theorem of Lassas–Uhlmann [23] with a zeroth-order term included. The lower-order term does not change the analytic continuation mechanism. We recall the relevant points.

First, the full symbol of the Dirichlet-to-Neumann operator determines the boundary jets of the analytic metric in boundary normal coordinates. If the potential is unknown, its boundary jets are also determined by the lower-order terms in the same symbol expansion. This is the usual analytic boundary determination step, in the spirit of Lee–Uhlmann [24].

Second, after the boundary jets are identified, the coefficients can be analytically extended to a collar outside the boundary. Since zero is not a Dirichlet eigenvalue, the Dirichlet Green kernel $G(x, y)$ of $-\Delta_g + V$ exists. The DN map determines $G(x, y)$ for (x, y) in the exterior collar by solving the corresponding boundary value problem, exactly as in the Laplace–Beltrami case [23].

Third, for fixed y, the function $G(\cdot, y)$ is real analytic away from y, since it solves an elliptic equation with analytic coefficients. The Green functions are then analytically continued through the manifold by the Lassas–Uhlmann sheaf construction [23]. Unique continuation and the singularity of the Green kernel show that these Green functions separate points, thereby allowing one to reconstruct the analytic structure of the manifold.

Finally, the leading singularity of the Green kernel determines the metric. Once g is known, the potential is recovered from the equation

$$-\Delta_{g,x} G(x, y) + V(x) G(x, y) = 0\,, \qquad x \neq y\,.$$

For each fixed x, choose $y \neq x$ sufficiently close to x so that $G(x, y) \neq 0$, which is possible because the Green kernel has a nonzero leading singularity near the pole. Then

$$V(x) = \frac{\Delta_{g,x} G(x, y)}{G(x, y)}.$$

Thus the analytic Schrödinger pair (g, V) is determined modulo the natural boundary-fixing analytic diffeomorphism gauge.

A.2. **Consequence for fixed potentials.** We now apply the preceding reconstruction principle to the fixed-potential problem. Here the two Schrödinger pairs are (g_1, V) and (g_2, V). If $\Lambda_{g_1,V} = \Lambda_{g_2,V}$, then the analytic reconstruction principle gives a boundary-fixing analytic diffeomorphism Ψ such that

$$g_2 = \Psi_* g_1, \qquad V = V \circ \Psi^{-1}.$$

Equivalently, $V \circ \Psi = V$. This is exactly the natural gauge of the fixed-potential problem. Hence analytic metrics are uniquely determined by the fixed-potential DN map modulo boundary-fixing analytic diffeomorphisms preserving V. This proves the analytic endpoint asserted in Theorem 1.4.

A.3. **Consequence for fixed nonzero frequency.** We next explain how the same analytic reconstruction principle yields the fixed nonzero-frequency uniqueness theorem in conductivity variables. We briefly recall the metric–conductivity correspondence introduced in Subsection 2.1. Given a uniformly elliptic conductivity γ, the associated metric g_γ is defined by

$$\gamma^{ij} = |g_\gamma|^{1/2} g_\gamma^{ij}\,.$$

Equivalently,

$$|g_\gamma|^{1/2} = (\det \gamma)^{1/(n-2)}\,.$$

With this notation, the fixed-frequency conductivity equation

$$-\nabla \cdot (\gamma \nabla u) = \lambda_0 u$$

is equivalent to

$$-\Delta_{g_\gamma} u = \lambda_0 |g_\gamma|^{-1/2} u\,.$$

Thus it can be written as the zero-energy Schrödinger equation $(-\Delta_{g_\gamma} + V_\gamma)u = 0$ with

$$V_\gamma = -\lambda_0\, |g_\gamma|^{-1/2} = -\lambda_0 (\det \gamma)^{-1/(n-2)}\,.$$

The weak DN forms coincide in the sense that

$$\int_\Omega \gamma \nabla u \cdot \nabla v \, dx - \lambda_0 \int_\Omega u\, v\, dx = \int_\Omega \langle \nabla u, \nabla v \rangle_{g_\gamma} dV_{g_\gamma} + \int_\Omega V_\gamma\, u\, v\, dV_{g_\gamma}\,.$$

Now assume that γ_1, γ_2 are real-analytic conductivities and $\Lambda_{\gamma_1,\lambda_0} = \Lambda_{\gamma_2,\lambda_0}$ with $\lambda_0 \neq 0$. Applying the analytic reconstruction principle to the Schrödinger pairs $(g_{\gamma_1}, V_{\gamma_1})$ and $(g_{\gamma_2}, V_{\gamma_2})$, we obtain a boundary-fixing analytic diffeomorphism Ψ such that $g_{\gamma_2} = \Psi_* g_{\gamma_1}$ and $V_{\gamma_2} = V_{\gamma_1} \circ \Psi^{-1}$. The metric identity translates back into $\gamma_2 = \Psi_* \gamma_1$.

It remains to identify the determinant of Ψ. Since $g_{\gamma_2} = \Psi_* g_{\gamma_1}$, the Riemannian volume densities satisfy

$$|g_{\gamma_2}|^{1/2}(\Psi(x)) \det D\Psi(x)| = |g_{\gamma_1}^{1/2}(x)\,.$$

On the other hand, equality of the reconstructed potentials gives

$$-\lambda_0 |g_{\gamma_2}|^{-1/2}(\Psi(x)) = -\lambda_0 |g_{\gamma_1}|^{-1/2}(x)\,.$$

Because $\lambda_0 \neq 0$, this implies $|g_{\gamma_2}|^{1/2}(\Psi(x)) = |g_{\gamma_1}|^{1/2}(x)$. Substituting this into the volume-density transformation law yields $\det D\Psi(x) = 1$. Therefore $\Psi \in \mathrm{SDiff}(\Omega)$ and $\gamma_2 = \Psi_* \gamma_1$. This proves the analytic endpoint asserted in Theorem 1.6.

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