Hölder Stability in the Inverse Steklov Problem for Radial Schrödinger operators and Quantified Resonances

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Abstract

In this paper, we obtain Hölder stability estimates for the inverse Steklov problem for Schrödinger operators corresponding to a special class of L^2 radial potentials on the unit ball. These results provide an improvement on earlier logarithmic stability estimates obtained in [8] in the case of the the Schrödinger operators related to deformations of the closed unit ball. The main tools involve a formula relating the difference of the Steklov spectra of the Schrödinger operators associated to the original and perturbed potential to the Laplace transform of the difference of the corresponding amplitude functions introduced in [17] and a key moment stability estimate due to Still [18]. It is noteworthy that with respect to the original Schrödinger operator, the type of perturbation being considered for the amplitude function amounts to the introduction of a finite number of negative eigenvalues and of a countable set of negative resonances which are quantified explicitly in terms of the eigenvalues of the Laplace-Beltrami operator on the boundary sphere.

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Contents

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1 Introduction

In a recent paper, [8], we have obtained a set of logarithmic stability estimates in the inverse Steklov problem for the Laplace-Beltrami operator on a class of warped product Riemannian manifolds defined on a d-dimensional closed ball. These manifolds can be thought of as deformations of the closed Euclidean d-ball in which the deformation is parametrized by the choice of radial warping function. The deformations considered in [8] include both the regular and singular cases of warped product metrics [15].

The approach taken in [8] was based on expressing the warped product metric in a coordinate system in which the metric takes the form of a conformal rescaling of the flat Euclidean metric. This enabled us by using the transformation law of the Laplace-Beltrami operator under conformal changes of the metric to reformulate the inverse Steklov problem for the Laplace-Beltrami operator on the original deformed closed ball as the inverse Steklov problem for a Schrödinger operator on the Euclidean ball, with a potential determined by the warping function of the original deformed ball and its derivatives. The logarithmic stability estimates that we obtained were thus of the nature of the estimates obtained by Alessandrini [1] and Novikov [14].

Our goal in the present paper is to improve the logarithmic stability results of [8] in a significant way by obtaining instead a set of *Hölder* stability estimates. In the approach we will take in the current paper, instead of starting from a deformed closed d-ball, we will reverse the initial step taken in [8] and start with a Schrödinger operator on the Euclidean d-ball, endowed with a radial potential. One reason for doing so is that the set of warped product metrics for which Hölder stability estimates can be obtained for the inverse Steklov problem will be significantly more restricted than the ones for which logarithmic stability estimates were obtained in [8]. Thus by considering Schrödinger operators, we will be broadening the range of inverse Steklov problems to which our Hölder stability results will apply. We emphasize nevertheless that the class of potentials for which we are able to establish Hölder stability for the inverse Steklov problem is still rather special and that any extension of our results to more general potentials may require stronger techniques than the ones we are using the present paper. We now proceed to describe the main results of our paper.

More precisely, on the d-dimensional closed Euclidean ball

$$M = (0,1] \times S^{d-1}\,, \tag{1.1}$$

where $d \geq 3$, we consider the Dirichlet problem for the Schrödinger operator with a potential q, given by

$$\begin{cases} -\triangle u + q\,u = 0, & \text{on } M\,, \\ u = \psi \in H^{\frac{1}{2}}(\partial M), & \text{on } \partial M\,. \end{cases} \tag{1.2}$$

When $q \in L^\infty(M)$ and $\lambda = 0$ is not a Dirichlet eigenvalue of the above Schrödinger operator, the Dirichlet problem (1.2) has a unique solution $u \in H^1(M)$. The Dirichlet-to-Neumann (DN) map Λ_q is then (formally) defined as an operator from $H^{1/2}(\partial M)$ to $H^{-1/2}(\partial M)$ by

$$\Lambda_q \psi = (\partial_\nu u)_{|\partial M}\,, \tag{1.3}$$

where u is the unique solution of (1.2) and $(\partial_\nu u)_{|\partial M}$ is the normal derivative of u with respect to the outer unit normal vector ν on ∂M.

The DN map thus defined is a self-adjoint operator on $L^2(\partial M, dS_g)$ where dS_g denotes the metric induced by the Euclidean metric on the boundary sphere $\partial M = S^{d-1}$. Its spectrum (the so-called *Steklov spectrum*) is discrete and accumulates at infinity. We shall thus denote the Steklov eigenvalues (counted with multiplicity)

$$0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_k \to \infty\,. \tag{1.4}$$

The Steklov spectrum will be the central object of study in this paper.

For the remainder of this paper, we shall assume that the potential is radial and we write $q = q(r)$, where r denotes the Euclidean distance to the origin. It will be convenient to replace the radial coordinate $r \in (0,1]$ by a new radial coordinate $x \in (0,\infty)$ defined by $x = -\log r$, in which case the the boundary of M now corresponds to $x = 0$. The Euclidean metric then takes the form

$$g = f(x)^4 (dx^2 + d\Omega^2)\,,$$

where $f(x) = \exp(-x/2)$ and $d\Omega^2$ denotes the round metric on the unit sphere S^{d-1}. The Dirichlet problem (1.2) gets transformed into

$$\begin{cases} [-\partial_x^2 - \triangle_S + Q(x)]v = -\frac{(d-2)^2}{4} v, & \text{on } M\,, \\ v = f^{d-2}\psi, & \text{on } \partial M\,, \end{cases} \tag{1.5}$$

where $\triangle_S$ denotes the Laplacian on the boundary sphere S^{d-1} and $Q(x) := e^{-2x} q(e^{-x})$.

Thanks to the spherical symmetry of the potential Q, we can use separation of variables and reduce (1.5) to an infinite sequence of radial ordinary differential equations. We thus let $\{Y_k,\, k \geq 0\}$ denote an orthonormal Hilbert basis of $L^2(S^{d-1})$ consisting of eigenfunctions of $\triangle_S$,

$$-\triangle_S Y_k = \alpha_k Y_k\,, \quad \alpha_k = k(k+d-2)\,.$$

Writing

$$v = \sum_{k \geq 0} v_k(x)\ Y_k\,, \tag{1.6}$$

we obtain an infinite sequence of ordinary differential equations on $(0, \infty)$ given by

$$-v_k'' + Qv_k = -(\alpha_k + \frac{(d-2)^2}{4})v_k = -\kappa_k^2 v_k\,, \tag{1.7}$$

where

$$\kappa_k := k + \frac{d-2}{2}\,, \quad k \geq 0\,.$$

As in [8], we introduce the Weyl-Titchmarsh (WT) function $M(z)$ associated to the Sturm-Liouville operator L on $(0, +\infty)$ given by

$$L = -\frac{d^2}{dx^2} + Q\,. \tag{1.8}$$

This function will play a central role in our subsequent analysis of the stability problem of the Steklov spectrum for our Schrödinger operator. We assume in this paper that

$$Q \in L^2(0, \infty)\,. \tag{1.9}$$

Under this assumption, it is well-known that L is of limit point-type at infinity, which means that for all $z \in \mathbb{C} \setminus [-\beta, \infty)$ with $\beta >> 1$, there exists, up to a zon-zero multiplicative constant, a unique solution $u(x, z)$ of

$$-u'' + Qu = zu \quad z \in \mathbb{C}\,, \tag{1.10}$$

which is L^2 at ∞. The Weyl-Titchmarsh function $M(z)$ is then defined by

$$M(z) := \frac{u'(0, z)}{u(0, z)} \text{ for all } z \in \mathbb{C} \setminus [-\beta, \infty)\,. \tag{1.11}$$

Of course, the square-integrability hypothesis (1.9) we made on the potential Q does not necessarily imply that the initial potential $q \in L^\infty(M)$. Thus, the previous definition we gave for the DN map is not directly applicable in this L^2 setting. We overcome this difficulty by exploiting the separation of variables and follow the procedure used in Section 2 of [8] to define the DN map; namely we expand the boundary data ψ in the Hilbert basis $\{Y_k\}_{k \geq 0}$ of $L^2(S^{d-1})$ as

$$\psi = \sum_{k \geq 0} \psi_k Y_k\,,$$

and define the DN map Λ_q as a sum of operators Λ_q^k by

$$\Lambda_q \psi = \sum_{k=0}^{\infty} (\Lambda_q^k \psi_k) Y_k\,. \tag{1.12}$$

The "diagonalized" DN operators Λ_q^k are then computed using the separation of variables to give

$$\Lambda_q^k \psi_k = -\frac{(d-2)}{2} v_k(0) - v_k'(0)\,, \tag{1.13}$$

where

$$-v_k'' + Qv_k = -\kappa_k^2 v_k \quad v_k(0) = \psi_k, \;\; v_k \in L^2 \text{ at } +\infty\,.$$

As was proved in [8], the operators Λ_q^k, $k \geq 0$ can be further simplified to take the form of multiplication operators by making use of the Weyl-Titchmarsh function M evaluated at the points $-\kappa_k^2$:

$$\Lambda_q^k \psi_k = \left(-\frac{(d-2)}{2} - M(-\kappa_k^2)\right)\psi_k\,, \tag{1.14}$$

and the Steklov spectrum $\{\sigma_k,\, k \geq 0\}$ is given in terms of the Weyl-Titchmarsh function M as

$$\sigma_k = -\frac{(d-2)}{4} - M(-\kappa_k^2)\,. \tag{1.15}$$

There is an important representation formula first obtained in [17] for the Weyl-Titchmarsh function in terms of the Laplace transform of a unique *amplitude function* A, under the hypothesis that $Q \in L^1(0,\infty)$:

$$M(-\kappa^2) = -\kappa - \int_0^\infty A(\alpha)e^{-2\kappa\alpha}d\alpha\,, \quad \forall \kappa > \frac{1}{2}||Q||_1\,. \tag{1.16}$$

We shall use a slightly refined version of this formula which applies in our L^2-setting and which will serve as the starting point of our formulation of the stability problem for the Steklov spectrum.

Let us now explain what we mean precisely by *local stability estimates*. The starting point is a *fixed potential* $Q \in L^2(0,\infty)$ which we perturb through the addition of a certain exponential series to its corresponding amplitude A. The set of admissible exponential series parametrizing the perturbations lies in an infinite-dimensional space. Using powerful results of Killip -Simon [13] we then show that $\tilde{Q} \in L^2(0,\infty)$.

Given now such a pair of potentials $Q, \tilde{Q}$, we assume that the difference between their corresponding Steklov spectra is uniformly bounded in absolute value by a small error $\epsilon > 0$,

$$|\sigma_k - \tilde{\sigma}_k| < \epsilon\,, \quad \forall k \geq 0\,. \tag{1.17}$$

Our main goal is to estimate the difference $Q - \tilde{Q}$ of these potentials. Again, our result is local: for any fixed parameter $T > 0$, we get stability estimates in the space $L^2(0,T)$. Roughly speaking, this means that

$$||\tilde{Q} - Q||_{L^2(0,T)} \leq C_T\; g(\epsilon)\,, \tag{1.18}$$

where $g(\epsilon) \to 0$ when $\epsilon \to 0$, and C_T is a constant only depending on T.

Now, we can state our main result in this paper :

Theorem 1.1. *Let $Q \in L^2(0,\infty)$ be a square-integrable potential with amplitude function A, and let $\delta \geq 3-d$ be any fixed parameter. Set $\mu_k := \lambda_k + \delta$ where $\lambda_k = 2k + d - 3 + \delta$ and let $\{c_k,\, k \geq 0\}$ be a sequence of real numbers such that*

- *i) $c_k \leq 0$ for all $k \geq 0$.*
- *ii) The power series $\sum_{k\geq 0} c_k t^{\lambda_k}$ has a radius of convergence $R > 1$.*

Then the function $\tilde{A}$ defined by

$$\tilde{A}(\alpha) = A(\alpha) + \sum_{k\geq 0} c_k e^{-\mu_k \alpha}\,, \quad \alpha > 0\,, \tag{1.19}$$

is the amplitude function of a potential $\tilde{Q} \in L^2(0,\infty)$.

Moreover, under the hypothesis (1.17), for any fixed $T > 0$, there exist a positive constant C_T and a Hölder exponent $\theta \in (0, \frac{1}{2})$ such that

$$||\tilde{Q} - Q||_{L^2(0,T)} \leq C_T\ \epsilon^{\theta}\,, \tag{1.20}$$

where

$$\theta = \frac{1}{2}\min(1, \frac{\log R}{\log(\frac{9M_0}{2})}), \quad M_0 = \max\{2, 4(d-3+\delta)+1\}\,. \tag{1.21}$$

Note that when the radius of convergence $R \geq \frac{9M_0}{2}$, we can take as Hölder exponent $\theta = \frac{1}{2}$. Note also that when the initial potential is the trivial potentiial $Q = 0$, the perturbed potentials $\tilde{Q}$ can be seen as a generalization of the so-called Bargmann potentials (see section 5 for details).

Finally, it is noteworthy that with respect to the original Schrödinger operator, the type of perturbation being considered for the amplitude function A amounts to the introduction of a finite number of negative eigenvalues $-\frac{\mu_k^2}{4}$ for $k = 1, ..., N$, (corresponding to the case where μ_k is negative), and of a countable set of real resonances $-\frac{|\mu_k|}{2}$ which are equally spaced on the negative real axis (for k greater than some k_0). These resonances are quantified explicitly in terms of the parameter δ and the eigenvalues of the Laplace Beltrami operator Δ_S on the boundary sphere.

2 Notation and set-up of the model

On the d-dimensional closed Euclidean ball

$$M = (0,1] \times S^{d-1}\,, \tag{2.22}$$

where $d \geq 3$, we consider the Dirichlet problem for the Schrödinger operator with potential $q \in L^\infty(M)$, given by

$$\left\{ \begin{array}{ll} -\triangle u + q\,u = 0, & \text{on } M\,, \\ u = \psi \in H^{\frac{1}{2}}(\partial M), & \text{on } \partial M\,. \end{array} \right. \tag{2.23}$$

Although we shall shortly make several assumptions about the potential q (including the requirement that it be radial), we begin by recalling a few general facts concerning the Dirichlet problem (2.23) for a general potential $q \in L^\infty(M)$. These will lead us to the definition of the Dirichlet-to-Neumann (DN) map and the associated Steklov spectrum, which will be central objects of study in this paper.

We first recall (see for example Theorem 8.3 in [11]) that if $q \in L^\infty(M)$ and zero is not a Dirichlet eigenvalue of the above Schrödinger operator (which is the case for example if $q \geq 0$), then the Dirichlet problem (2.23) has a unique solution $u \in H^1(M)$. The Dirichlet-to-Neumann (DN) map Λ_q is then defined as an operator from $H^{1/2}(\partial M)$ to $H^{-1/2}(\partial M)$ by

$$\Lambda_q \psi = (\partial_\nu u)_{|\partial M}\,, \tag{2.24}$$

where u is the unique solution of (2.23) and $(\partial_\nu u)_{|\partial M}$ is the normal derivative of u with respect to the outer unit normal vector ν on ∂M. Here $(\partial_\nu u)_{|\partial M}$ is generally defined in the weak sense as an element of $H^{-1/2}(\partial M)$ by

$$\langle \Lambda_q \psi | \phi \rangle = \int_M \langle du, dv \rangle \, dVol \, ,$$

for any $\psi \in H^{1/2}(\partial M)$ and $\phi \in H^{1/2}(\partial M)$ such that u is the unique solution of (2.23) and v is any element of $H^1(M)$ such that $v_{|\partial M} = \phi$. It is easily checked that if ψ is sufficiently smooth, we have

$$\Lambda_q \psi = g(\nu, \nabla u)_{|\partial M} = du(\nu)_{|\partial M} = \nu(u)_{|\partial M} \, ,$$

so that the expression in local coordinates for the normal derivative is then given by

$$\partial_\nu u = \nu^i \partial_i u \, . \tag{2.25}$$

It is well known that the DN map is a pseudo-differential operator of order 1 which is self-adjoint on $L^2(\partial M, dS_g)$ where dS_g denotes the metric induced by the Euclidean metric on the boundary $\partial M = S^{d-1}$. Therefore, the DN map Λ_q has a real and discrete spectrum accumulating at infinity, known as the Steklov spectrum. We shall denote the Steklov eigenvalues (counted with multiplicity) by

$$0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_k \to \infty \, . \tag{2.26}$$

We refer the reader to [12] and references therein for an excellent survey of the known results on the Steklov spectrum.

As of now and for the remainder of this paper, we shall assume that the potential is radial and write $q = q(r)$, where r denotes the Euclidean distance to the origin. It will be convenient for our subsequent analysis to replace the radial coordinate $r \in (0, 1]$ by a new radial coordinate $x \in (0, \infty)$ defined by $x = -\log r$, in which case the the boundary of M now corresponds to $x = 0$. The Euclidean metric

$$g = dr^2 + r^2 d\Omega^2 \, ,$$

where $d\Omega^2$ denotes the round metric on the unit sphere S^{d-1}, then takes the form

$$g = f(x)^4 (dx^2 + d\Omega^2) \, ,$$

where $f(x) = \exp(-x/2)$, and the Dirichlet problem (2.23) gets transformed into

$$\begin{cases} [-\partial_x^2 - \triangle_S + Q(x)] v = -\frac{(d-2)^2}{4} v, & \text{on } M \, , \\ v = f^{d-2} \psi, & \text{on } \partial M \, , \end{cases} \tag{2.27}$$

where $\triangle_S$ denotes the Laplacian on the boundary sphere S^{d-1} and $Q(x) := e^{-2x} q(e^{-x})$.

2.1 Separation of variables

Our first step in the analysis of (2.27) is to exploit the spherical symmetry of the potential Q in order to separate variables and reduce (2.27) to an infinite sequence of radial ordinary differential equations. We thus let $\{Y_k, \, k \geq 0\}$ denote an orthonormal Hilbert basis of $L^2(S^{d-1})$ consisting of eigenfunctions of $\triangle_S$,

$$-\triangle_S Y_k = \alpha_k Y_k \, , \quad \alpha_k = k(k + d - 2) \, ,$$

where the eigenvalue α_k is of multiplicity $\binom{k+d-1}{d-1} - \binom{k+d-3}{d-1}$, and let

$$\kappa_k := k + \frac{d-2}{2}, \quad k \geq 0.$$

We separate variables by letting

$$v = \sum_{k\geq 0} v_k Y_k, \tag{2.28}$$

where $v_k = v_k(x)$ depends only on the radial variable $x \in (0, \infty)$, which gives rise upon substitution of (2.28) into (2.27) to the infinite sequence of ordinary differential equations on $(0, \infty)$ given by

$$-v_k'' + Qv_k = -(\alpha_k + \frac{(d-2)^2}{4})v_k = -\kappa_k^2 v_k. \tag{2.29}$$

For our subsequent analysis of the stability problem, it will be very useful to complexify the spectral parameter κ_k and consider instead the ordinary differential equation

$$-v'' + Qv = zv \quad z \in \mathbb{C}, \tag{2.30}$$

where z is now to be thought of as a complex spectral parameter.

2.2 The Weyl-Titchmarsh function

We now introduce the Weyl-Titchmarsh function $M(z)$ associated to the Sturm-Liouville operator

$$L = -\frac{d^2}{dx^2} + Q, \tag{2.31}$$

defined by the left-hand side of (2.30). This function will play a central role in our subsequent analysis of the stability problem of the Steklov spectrum for our Schrödinger operator.

We first recall that in order for the Weyl-Titchmarsh function to be well-defined, we need to assume that L is of limit point-type at infinity, meaning that for all $z \in \mathbb{C} \setminus [-\beta, \infty)$ with $\beta >> 1$, there exists, up to a non-zero multiplicative constant, a unique solution $u(x, z)$ of (2.30) which is L^2 at ∞. The Weyl-Titchmarsh function $M(z)$ is then defined by

$$M(z) := \frac{u'(0, z)}{u(0, z)} \text{ for all } z \in \mathbb{C} \setminus [-\beta, \infty). \tag{2.32}$$

It is easy to show that this property will be guaranteed if

$$Q \in L^2(0, \infty), \tag{2.33}$$

a property which we will require Q to satisfy from now onwards. Indeed, from (1.5) in [17], we know that L will be of limit point-type at infinity if

$$\beta_2 := \sup_{y>0} \int_y^{y+1} \max\{Q(x), 0\} dx < \infty. \tag{2.34}$$

But by the Cauchy-Schwarz inequality, we have

$$\sup_{y>0} \int_y^{y+1} \max\{Q(x), 0\} dx \leq \sup_{y>0} \int_y^{y+1} |Q(x)| dx \leq ||Q||_2, \tag{2.35}$$

which shows that (2.33) ensures indeed the property that L is of limit point-type at infinity.

2.3 The DN map and the Weyl-Titchmarsh function

We begin by remarking that the square-integrability hypothesis (2.33) made on the potential Q appearing in the radial Sturm-Liouville operator L does not necessarily imply that the potential q of the initial Schrödinger operator considered in the set-up of Section 2 will satisfy the condition $q \in L^\infty(M)$, which we assumed in order to define the DN map Λ_q. In other words, the definition we gave for the DN map may not be applicable in the L^2 setting defined by the condition (2.33).

We may nevertheless circumvent this difficulty by exploiting the separation of variables worked out in Section 2.1 and following the procedure used in Section 2 of [8] to define the DN map. Indeed, by expanding the boundary data ψ in the Hilbert basis $\{Y_k\}_{k\geq 0}$ of $L^2(S)$ as

$$\psi = \sum_{k\geq 0} \psi_k Y_k \,,$$

the DN map Λ_q is represented as a sum of operators Λ_q^k by

$$\Lambda_q \psi = \sum_{k=0}^{\infty} (\Lambda_q^k \psi_k) Y_k \,, \tag{2.36}$$

where the "diagonalized" DN operators Λ_q^k are computed using the separation of variables to give (see (2.16) in [8])

$$\Lambda_q^k \psi_k = -\frac{(d-2)}{2} v_k(0) - v_k'(0) \,, \tag{2.37}$$

where

$$-v_k'' + Q v_k = -(\mu_k + \frac{(d-2)^2}{4}) v_k = -\kappa_k^2 v_k \quad v_k(0) = \psi_k, \;\; v_k \in L^2 \text{ at } +\infty \,.$$

As shown in the calculation leading from (2.14) to (2.15) in [8], the form given in (2.37) for the operators Λ_q^k, $k \geq 0$ can be further simplified to obtain an expression of these operators as multiplication operators:

$$\Lambda_q^k \psi_k = \left(-\frac{(d-2)}{2} - M(-\kappa_k^2) \right) \psi_k \,. \tag{2.38}$$

Finally, we recall from Lemma 2.3 in [8] the explicit formula giving the Steklov spectrum $\{\sigma_k, \, k \geq 0\}$ in terms of the Weyl-Titchmarsh function M,

$$\sigma_k = -\frac{(d-2)}{4} - M(-\kappa_k^2) \,. \tag{2.39}$$

The latter formula will serve as the starting point of our formulation of the stability problem for the Steklov spectrum.

2.4 The amplitude A of a radial potential and the Weyl-Titchmarsh function

As stated in the Introduction, central to our analysis of the stability problem lies a remarkable representation formula first obtained in [17] (under the hypothesis that $Q \in L^1(0,\infty)$) for the Weyl-Titchmarsh function of the Sturm-Liouville operator L in terms of a the Laplace transform of an *amplitude function* A. We shall be using a slightly refined version of this formula which applies to the class of L^2 potentials considered in our paper.

In order to state this formula, we first recall from Theorem 2.1 in [17] that if $Q \in L^1(0,\infty)$, then the Weyl-Titchmarsh function M may be expressed in the form of the Laplace transform of an amplitude function A by

$$M(-\kappa^2) = -\kappa - \int_0^\infty A(\alpha) e^{-2\kappa\alpha} d\alpha\,, \quad \forall \kappa > \frac{1}{2}||Q||_1\,. \tag{2.40}$$

It was proved in [9] (in the remark following (1.17)) that the above equality also holds for all $\kappa \in \mathbb{C}$ such that $\mathrm{Re}\,\kappa > \frac{1}{2}||Q||_1$. In [3] (see Section 5, Algorithm 1, point 2), it is proved that if $\beta_2 < \infty$, where β_2 is defined in (2.34), then the integral in (2.40) is absolutely convergent for $\mathrm{Re}\,\kappa > 2\max\{\sqrt{2\beta_2}, e\beta_2\}$. But we saw in (2.35) that for square-integrable potentials Q, one has the estimate $\beta_2 \le ||Q||_2 < \infty$. It therefore follows from (2.40) that the Weyl-Titchmarsh function M admits the representation

$$M(-\kappa^2) = -\kappa - \int_0^\infty A(\alpha) e^{-2\kappa\alpha} d\alpha\,, \quad \forall\,\mathrm{Re}\,\kappa > 2\max\{\sqrt{2\beta_2}, e\beta_2\}\,. \tag{2.41}$$

3 The problem of stability

3.1 Statement of the problem and strategy

The stability problem may be stated as follows in general terms: Given a pair of potentials $Q, \tilde{Q}$ such that the difference between their corresponding Steklov spectra is uniformly bounded in absolute value by a small error $\epsilon > 0$,

$$|\sigma_k - \tilde{\sigma}_k| < \epsilon\,, \quad \forall k \ge 0\,, \tag{3.1}$$

what can we say about the difference $Q - \tilde{Q}$ of these potentials? As a first step, we can use the expression (2.39) of the Steklov spectrum in terms of the Weyl-Titchmarsh function M and Simon's representation formula (2.41) for M in terms of the Laplace transform of the amplitude function A to reformulate the condition (3.1) in terms of A. We have

$$|\sigma_k - \tilde{\sigma}_k| = |M(-\kappa_k^2) - \tilde{M}(-\kappa_k^2)| = |\int_0^\infty (A(\alpha) - \tilde{A}(\alpha)) e^{-2\kappa_k\alpha} d\alpha|\,,$$

and making the change of variables $\alpha = -\log t$, our hypothesis (3.1) on the difference of the Steklov spectra takes the form

$$|\int_0^1 t^{-\delta}\big(A(-\log t) - \tilde{A}(-\log t)\big)\, t^{2k+d-3+\delta} dt| < \epsilon\,, \tag{3.2}$$

where δ will be a fixed real parameter that will be properly chosen later. We can see that (3.2) is effectively a Hausdorff moment problem, and thus one should not expect better stability results than the logarithmic stability estimates of the type obtained in [8] for the Steklov spectra of deformed balls, or in [1] and [14] for the Steklov spectra of certain Schrödinger operators. Nevertheless, as we shall explain in Section 3.2 below, one can approach the stability problem from a different starting point by working directly with perturbations of Simon's amplitude function A by a certain families of exponential series obtained from power series of Müntz type. We shall see that his leads in turn to Hölder type stability results which are significantly stronger than the logarithmic stability results mentioned earlier, albeit at the cost of restricting the class of potentials to a somewhat small subset of the set of square integrable potentials. More precisely, given the amplitude function A associated to a square-integrable potential function Q by (2.41), our strategy will consist in defining a perturbed amplitude function $\tilde{A}$ as in (3.5) and then use an important stability estimate due to Still (Theorem 2 in [18]) to obtain a Hölder estimate on $\tilde{A} - A$. This will be the substance of Section 3.2. The next step, worked out in Section 4, will consist in using the powerful results of [13] to construct an L^2 potential $\tilde{Q}$ associated to the perturbed amplitude $\tilde{A}$. Finally, we shall use the methods of boundary control theory of [3] to estimate the difference $\tilde{Q} - Q$.

It is noteworthy that with respect to the original Schrödinger operator, the type of perturbation being considered for the amplitude function A amounts to the introduction of a finite number of negative eigenvalues (corresponding to the choice of a negative real parameter δ) and of a countable set of resonances on the negative real axis, which admit a precise quantitative expression through to the eigenvalues μ_k of the Laplace Beltrami operator Δ_S on the boundary sphere.

3.2 Improved stability by Still's method - first main result

Let $Q \in L^2(0, \infty)$ be a given fixed potential and let A be the corresponding amplitude function, given by the representation formula (2.40).

Having in mind the inequality (3.2), we set for $k \geq 0$,

$$\lambda_k := 2k + d - 3 + \delta\,, \tag{3.3}$$

where $\delta \geq 3 - d$ is an arbitrary fixed real parameter (so that $\lambda_k \geq 0$) and

$$h(t) = t^{-\delta}\left(\tilde{A}(-\log t) - A(-\log t)\right). \tag{3.4}$$

We define formally a new amplitude $\tilde{A}$ by adding to A a power series

$$\tilde{A}(\alpha) = A(\alpha) + \sum_{k\geq 0} c_k e^{-(\lambda_k+\delta)\alpha}\,, \tag{3.5}$$

or equivalently

$$h(t) = \sum_{k\geq 0} c_k t^{\lambda_k}\,. \tag{3.6}$$

We assume that the series defining $h(t)$ has a radius of convergence $R > 1$, so that $h \in C^0([0,1])$. Furthermore we assume that h is such that the estimate (3.2) holds, that is

$$\left|\int_0^1 h(t)\, t^{\lambda_k} dt\right| \leq \epsilon\,, \quad \forall k \geq 0\,. \tag{3.7}$$

Our goal for this section is to obtain a good approximation of Hölder or Lipschitz type for $||h||_2^2$, under the above assumptions. We shall do so by using Theorem 2 in the paper [18] by Still and the procedure used in Section 4.3 of [8]. In order to do so we first recall some of the notation used in [8].

Theorem 3.1. *Given $\epsilon > 0$ and $R > 1$ as above and letting $M_0 = \max\{2, 4(d-3+\delta)+1\}$, we have, for some universal constant $B > 0$, the estimate*

$$||h||_2^2 \leq B^2\epsilon + R^{1-d}\epsilon^{\frac{\log R}{\log(\frac{9M_0}{2})}}\,. \tag{3.8}$$

We note that the estimate (3.8) is generally a Hölder type estimate for $||h||_2^2$, but that if $R > \frac{9M_0}{2}$, this estimate is Lipschitz.

Proof. Given a sequence $\Lambda_\infty := (\lambda_n)_{n\geq 0}$ of integers such that $0 \leq \lambda_0 < \lambda_1 < \cdots$ and $\lambda_k \to \infty$ as $k \to \infty$, we define for fixed $n \geq 1$ the finite sequence

$$\Lambda_n := 0 \leq \lambda_0 < \lambda_1 < ... < \lambda_n \,, \tag{3.9}$$

giving rise to the vector space $\mathcal{M}(\Lambda)$ of "Müntz polynomials of degree λ_n":

$$\mathcal{M}(\Lambda_n) = \{P : \; P(t) = \sum_{k=0}^{n} a_k \, t^{\lambda_k}\} \,. \tag{3.10}$$

Recall that according to the Müntz-Szász's Theorem, if Λ_∞ is a sequence of positive real numbers as above, then span $\{t^{\lambda_0}, t^{\lambda_1}, ...\}$ is dense in $L^2([0,1])$ if and only if

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty. \tag{3.11}$$

We remark that if $\lambda_0 = 0$, the denseness of the Müntz polynomials in $C^0([0,1])$ in the sup norm is also characterized by (3.11).

Given now a function f in $C^0([0,1])$ or in $L^2([0,1])$, the error of approximation of f with respect to $\mathcal{M}(\Lambda_n)$ is defined by

$$E_p(f, \Lambda_n) := \inf_{P \in \mathcal{M}(\Lambda_n)} \; ||f - P||_p \,, \tag{3.12}$$

where $p = 2$ or $p = \infty$ depending on whether $f \in C^0([0,1])$ or $f \in L^2([0,1])$. For our application, we have $\lambda_k := 2k + d - 3 + \delta$, giving $\lambda_{k+1} - \lambda_k = 2 > 0$, so by Theorem 2 of [18], we know that

$$E_\infty(h, \Lambda_n) \leq C R^{-\lambda_{n+1}} \,, \tag{3.13}$$

for some positive constant C.

We have, denoting by π_n the orthogonal projection onto the subspace $\mathcal{M}(\Lambda_n)$,

$$||h||_2^2 = ||\pi_n(h)||_2^2 + ||h - \pi_n(h)||_2^2 \,. \tag{3.14}$$

Our next step is to combine the estimate (4.57) from [8] Section 4.3 of and the estimate (3.13) to obtain an estimate for the norm of $\pi_n(h)$. In order to do so, we use the Gram-Schmidt process to obtain polynomials $(L_m(t))$ with $L_0(t) = 1$, and for $m \geq 1$,

$$L_m(t) = \sum_{j=0}^{m} C_{mj} t^{\lambda_j} \,, \tag{3.15}$$

where we have set

$$C_{mj} = \sqrt{2\lambda_m + 1} \; \frac{\prod_{r=0}^{m-1} (\lambda_j + \lambda_r + 1)}{\prod_{r=0, r\neq j}^{m} (\lambda_j - \lambda_r)} \,. \tag{3.16}$$

The family $(L_m(t))$ defines an orthonormal Hilbert basis of $L^2([0,1])$. We may now recall the estimate (4.57) from [8],

$$||\pi_n(h)||_2^2 \leq \epsilon^2 \; \sum_{k=0}^{n} \left(\sum_{p=0}^{k} |C_{kp}| \right)^2 \,, \tag{3.17}$$

which gives immediately

$$||h||_2^2 \leq \epsilon^2 \sum_{k=0}^{n} \left(\sum_{p=0}^{k} |C_{kp}| \right)^2 + CR^{-\lambda_{n+1}} , \tag{3.18}$$

using (3.13) and the inequality

$$||h - \pi_n(h)||_2^2 = E_2(h, \Lambda_n) \leq E_\infty(h, \Lambda_n) .$$

Now, according to the estimate (4.72) of [8], we have

$$||\pi_n h||_2^2 \leq B^2 \epsilon^2 \ g(n)^2 , \tag{3.19}$$

where B is a positive constant and $g : [0, +\infty[$ is a monotone increasing function defined for $t \in [0, +\infty)$ by

$$g(t) = \frac{3}{2} \frac{1}{\sqrt{\left(\frac{9M_0}{2}\right)^2 - 1}} \sqrt{2t+1} \left(\frac{9M_0}{2} \right)^{t+1} \quad , \quad M_0 = \max\{2, 4(d-3+\delta)+1\} . \tag{3.20}$$

Now, repeating the steps that lead from the inequalities (4.72) to (4.73) in [8], we choose n as a function of ϵ so as to control the norm of the projection $||\pi_n h||_2^2$ of h and thus set $n(\epsilon) := [\ (g^{-1}(\frac{1}{\sqrt{\epsilon}}))]$ where square brackets denote the integral part function. Since g is a monotone increasing function, we have

$$g(n(\epsilon)) \leq \frac{1}{\sqrt{\epsilon}} , \tag{3.21}$$

so using (3.19) we obtain immediately:

$$||\pi_{n(\epsilon)} h||_2^2 \leq B^2 \epsilon. \tag{3.22}$$

Our next task is now to estimate the size of $n(\epsilon)$ relative to ϵ so as to obtain the Hölder or Lipschitz estimate we seek for $||h||_2^2$. From (3.20), we obtain that

$$g(t) \sim (t+1) \log(\frac{9M_0}{2}) ,$$

as $t \to \infty$, which combined with (3.21) leads to

$$n(\epsilon) = \frac{\log(\frac{1}{\sqrt{\epsilon}})}{\log(\frac{9M_0}{2})} . \tag{3.23}$$

Plugging this into (3.18) gives

$$||h||_2^2 \leq B^2 \epsilon + CR^{-\lambda_{n(\epsilon)+1}} . \tag{3.24}$$

Now, using the expression $\lambda_k = 2k + d - 3 + \delta$ and (3.23), we have

$$R^{-\lambda_{n(\epsilon)+1}} = R^{1-d-\delta} R^{-2n(\epsilon)} \sim R^{1-d-\delta} R^{-\frac{2 \log \frac{1}{\sqrt{\epsilon}}}{\log(\frac{9M_0}{2})}} \sim R^{1-d-\delta} e^{\frac{\log \epsilon}{\log(\frac{9M_0}{2})} \log R} \sim R^{1-d-\delta} \epsilon^{\frac{\log R}{\log(\frac{9M_0}{2})}} . \tag{3.25}$$

Substituting (3.25) into (3.24), we obtain

$$||h||_2^2 \leq B^2 \epsilon + R^{1-d-\delta} \epsilon^{\frac{\log R}{\log(\frac{9M_0}{2})}} . \tag{3.26}$$

In terms of the amplitude function A in the variable $\alpha \in (0,\infty)$, using the relation

$$||h||_2^2 = \int_0^1 t^{-2\delta}(A(-\log t) - \tilde{A}(-\log t))^2 \, dt \,,$$

we obtain

$$\int_0^\infty e^{(2\delta-1)\alpha}(A(\alpha) - \tilde{A}(\alpha))^2 \; d\alpha \leq B^2\epsilon + R^{1-d-\delta}\epsilon^{\frac{\log R}{\log(\frac{9M_0}{2})}} \,. \tag{3.27}$$

□

4 From the perturbed amplitude $\tilde{A}$ to a potential $\tilde{Q} \in L^2(0,\infty)$

4.1 Statement of the main result

Our objective for Section 4 is to establish a result on the existence of square-integrable potentials $\tilde{Q}$ associated to perturbed amplitudes $\tilde{A}$ as defined in (3.5). As we shall see, this will require a few additional hypotheses on the perturbation of the amplitude function A given by (3.5) and thus on the perturbation of the starting potential Q.

We set for $k \geq 0$,

$$\mu_k := \lambda_k + \delta = 2k + d - 3 + 2\delta \tag{4.1}$$

so that

$$\tilde{A}(\alpha) = A(\alpha) + \sum_{k\geq 0} c_k e^{-\mu_k \alpha} \,, \quad \alpha > 0 \,. \tag{4.2}$$

Thus, in dimension d greater than 3, μ_k may be a negative real; so we split the series in (4.2) as

$$\tilde{A}(\alpha) = A(\alpha) + \sum_{k=0}^{N-1} c_k e^{-\mu_k \alpha} + \sum_{k\geq N} c_k e^{-\mu_k \alpha} \,, \quad \alpha > 0 \,, \tag{4.3}$$

such that for $k = 0, ..., N-1,\ \mu_k < 0$ and for $k \geq N+1,\ \mu_k \geq 0$, with the convention that the first sum in (4.3) does not appear if all the μ_k's are positive (i.e if $N = 0$).

Now, it is convenient to rewrite (4.3) as:

$$\tilde{A}(\alpha) - A(\alpha) = \sum_{k=0}^{N-1} 2c_k \; \sinh(|\mu_k|\alpha) + \sum_{k\geq 0} c_k e^{-|\mu_k| \; \alpha}. \tag{4.4}$$

In this form, we note that the perturbation $\tilde{A}(\alpha) - A(\alpha)$ has exactly the same expression as the amplitude given in [9], Eq. (11.9), modulo the fact we consider a convergent series instead a finite sum, and that the coefficients $|\mu_k|$ for $k = 0, ..., N-1$ appear simultaneously in the first finite sum and also in the convergent series.

As we shall see in the next section, the first finite sum in the (RHS) of (4.4) corresponds to the introduction of negative eigenvalues $-\frac{\mu_k^2}{4}$ for $k = 0, ..., N-1$, whereas the second sum corresponds to the introduction of real resonances $-\frac{|\mu_k|}{2}$ for $k \geq 0$.

We state this result in the form of a theorem, that will be proved in Section 4.3 below by applying Theorem 1.2 from the paper [13] by Killip and Simon:

Theorem 4.1. *Let $Q \in L^2(0,\infty)$ be a square-integrable potential with amplitude function A, let $\{c_k,\, k \geq 0\}$ be a sequence of real numbers such that*

- *i) For all $k \geq 0$, $c_k \leq 0$.*
- *ii) the power series $\sum_{k\geq 0} c_k t^{\lambda_k}$ has a radius of convergence $R > 1$.*

Then the function $\tilde{A}$ defined by

$$\tilde{A}(\alpha) = A(\alpha) + \sum_{k\geq 0} c_k e^{-\mu_k \alpha}, \quad \alpha > 0, \tag{4.5}$$

is the amplitude function of a potential $\tilde{Q} \in L^2(0,\infty)$.

4.2 The spectral measure

In order to proceed with the proof of Theorem 4.1, we need to first compute the difference of the spectral measures of Q and $\tilde{Q}$ in terms of the data contained in the perturbed amplitude $\tilde{A}$. In the first instance we will work heuristically so as to set the stage for the mathematical objects at play.

Let us first recall from [10] that the Weyl-Titchmarsh function M is a function of Herglotz type, meaning that for all $z \in \mathbb{C}$ such that $\mathrm{Im} z > 0$ we have $\mathrm{Im}\, M(z) > 0$. This implies that we have the representation formula

$$M(z) = c + \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{1}{1+\lambda^2}\right) d\rho(\lambda),$$

where $d\rho(\lambda)$ is the (positive) spectral measure associated to (2.30). It can be constructed by taking the following weak limit (in the distributional sense)

$$d\rho(E) = \operatorname*{w-lim}_{\epsilon\downarrow 0} \frac{1}{\pi} \mathrm{Im}\,(M(E+i\epsilon))\, dE. \tag{4.6}$$

We denote by $\tilde{M}(-\kappa^2)$ the putative Weyl-Titchmarsh function associated with the amplitude $\tilde{A}(\alpha)$, so thanks to (2.41), we have (formally and for suitable κ),

$$\tilde{M}(-\kappa^2) - M(-\kappa^2) = -\int_0^\infty \left(\tilde{A}(\alpha) - A(\alpha)\right) e^{-2\kappa\alpha}\, d\alpha. \tag{4.7}$$

Now, using (4.4), we easily write the difference $\tilde{M}(-\kappa^2) - M(-\kappa^2)$ as

$$\tilde{M}(-\kappa^2) - M(-\kappa^2) = -2\sum_{k=0}^{N-1} c_k \, \frac{|\mu_k|}{4\kappa^2 - \mu_k^2} \; - \sum_{k\geq 0} \frac{c_k}{2\kappa + |\mu_k|}. \tag{4.8}$$

We note that the above series is indeed convergent since by hypothesis the power series $\sum_{k\geq 0} c_k t^{\lambda_k}$ has radius of convergence $R > 1$ so that in particular $\sum_{k\geq 0} |c_k| < \infty$.

Now, using (4.6), we are able to define the difference of the spectral measures $d\tilde{\rho}(E) - d\rho(E)$ (for more details, we refer the reader to ([9], Eqs. (11.7)-(11.9), p.637).

For $E \geq 0$,

$$d\tilde{\rho}(E) = d\rho(E) - \frac{2}{\pi}\sum_{k\geq 0} c_k \, \frac{\sqrt{E}}{4E + \mu_k^2}\, dE, \tag{4.9}$$

and for $E < 0$,

$$d\tilde{\rho}(E) = d\rho(E) - \frac{1}{2}\sum_{k=0}^{N-1} c_k \ |\mu_k| \ \delta(\cdot + \frac{\mu_k^2}{4}) \ dE, \tag{4.10}$$

where $\delta(\cdot - a)$ stands for the usual delta distribution, centered at the point a.

As was explained in ([9], Section 11), this corresponds to the introduction of a finite number of negative eigenvalues $-\frac{\mu_k^2}{4}$ for $k = 0, ..., N-1$, and of real resonances $-\frac{|\mu_k|}{2}$ for $k \geq 0$.

Now, we can explain precisely our strategy : in the next section we show, using the Killip-Simon conditions, that under the hypotheses of Theorem 4.1, there exists a potential $\tilde{Q} \in L^2(0,\infty)$ associated to the above spectral measure $d\tilde{\rho}(E)$ allowing us to define rigourously the associated Weyl-Titchmarsh function $\tilde{M}(z)$ for $z \in \mathbb{C}\backslash[-\tilde{\beta}, +\infty[$ for $\tilde{\beta} >> 1$. The amplitude function associated to $\tilde{M}(z)$ is automatically given by $\tilde{A}(\alpha)$ thanks to the uniqueness of the inverse Laplace transform and analytic continuation.

4.3 The Killip-Simon conditions and the proof of Theorem 4.1

As earlier, we now apply Theorem 1.2 from the paper [13] by Killip and Simon in order to prove Theorem 4.1 establishing existence of a potential $\tilde{Q} \in L^2(0,\infty)$ associated to the perturbed amplitude $\tilde{A}$. Besides the positivity of the perturbed measure $d\tilde{\rho}$, there are four conditions stated in the theorem of Killip and Simon that we need to verify on $d\tilde{\rho}$ in order for their theorem to apply. In what follows, we state these conditions and show they are satisfied under the hypotheses of Theorem 4.1.

- Positivity of the measure $d\tilde{\rho}(E)$:

  This follows immediately on account of (4.9), (4.10), and the hypothesis $c_k \leq 0\,, \forall\, k \geq 0$.

- Weyl condition: The Weyl condition on $d\tilde{\rho}$ states that the support of $d\tilde{\rho}$ should decompose as

  $$\mathrm{supp}\, d\tilde{\rho} = [0,\infty) \cup \{\tilde{E}_j\}_{j=1}^N\,, \quad \text{with } \tilde{E}_1 < \tilde{E}_2 < \cdots < 0\,, \quad \text{with } \tilde{E}_j \to 0 \text{ if } N = \infty\,.$$

  Again this is immediate from the fact that the measure $d\rho$ associated to Q satisfies the Weyl condition, from the identities (4.9), (4.10) and from the fact that our perturbation is only adding real resonances and a finite number of negative eigenvalues eigenvalues $-\frac{{\mu_k}^2}{4}$ for $k = 0, ..., N-1$.

- Normalization: We need to verify that $d\tilde{\rho}$ satisfies a certain estimate whose formulation requires the introduction of the Hardy-Littlewood maximal function of a measure. The argument is more elaborate and we present it therefore is greater detail. Following [13], we introduce a measure $d\nu$ on $(1,\infty)$ parametrized by k, with $E = k^2$,

  $$\frac{d\nu}{dk} = \mathrm{Im}\big(M(k^2 + i0)\big) - k\,,$$

  which gives on account of (4.7) with $\kappa = -ik$,

  $$\frac{d\tilde{\nu}}{dk} = \frac{d\nu}{dk} - 2\sum_{n\geq 0} c_n \frac{k}{4k^2 + \mu_n^2}\,.$$

  We then define the Hardy-Littlewood maximal function M_s of the measure ν by

  $$(M_s\nu)(x) = \sup_{0<L\leq 1} \frac{1}{2L}|\nu|([x-L, x+L])\,.$$

and compute $|\tilde{\nu}|([x-L,x+L])$:

$$|\tilde{\nu}|([x-L,x+L]) = |\nu|([x-L.x+L]) - 2\sum_{n\geq 0} c_n \int_{x-L}^{x+L} \frac{k}{4k^2+\mu_n^2}\,dk\,.$$

Since

$$\int_{x-L}^{x+L} \frac{k}{4k^2+\mu_n^2}\,dk = \frac{1}{8}\log\frac{(x+L)^2+\mu_n^2}{(x-L)^2+\mu_n^2}\,,$$

we obtain

$$|\tilde{\nu}|([x-L,x+L]) = |\nu|([x-L.x+L]) - 2\sum_{n\geq 0} \frac{c_n}{8}\log\frac{(x+L)^2+\mu_n^2}{(x-L)^2+\mu_n^2}\,.$$

Taking $x = k >> 1$, the normalization condition we need to verify is

$$\int_1^{+\infty} \log\left[1+\left(\frac{(M_s\tilde{\nu})(k)}{k}\right)^2\right]k^2\,dk < \infty\,. \tag{4.11}$$

We have

$$\log\frac{(k+L)^2+\mu_n^2}{(k-L)^2+\mu_n^2} = \log\big(1+\frac{4kL}{(k-L)^2+\mu_n^2}\big) = \log(1+\mathcal{O}(\frac{L}{k}))\,,$$

uniformly in n and $L\in(0,1]$. But by condition ii) in Theorem 4.1, we know that the series $\sum_{n\geq 0}|c_n|$ is convergent, so it follows that

$$|\tilde{\nu}|([x-L,x+L]) = |\nu|([x-L.x+L]) + \mathcal{O}(\frac{L}{k})\,, \quad k\to\infty\,,$$

uniformly in $L\in(0,1]$. Therefore we have

$$(M_s\tilde{\nu})(k) = (M_s\nu)(k) + \mathcal{O}(\frac{1}{k})\,. \tag{4.12}$$

Now, using the inequality

$$\log\big(1+(x+y)^2\big) \leq C\big(x^2+\log(1+y^2)\big)\,,$$

for some constant $C>0$, we obtain using (4.12) the estimate

$$\begin{aligned}\log\left[1+\left(\frac{(M_s\tilde{\nu})(k)}{k}\right)^2\right] &= \log\left[1+\left(\frac{(M_s\nu)(k)+\mathcal{O}(\frac{1}{k})}{k}\right)^2\right]\\ &\leq C\left[\mathcal{O}(\frac{1}{k^4}) + \log\left[1+\left(\frac{(M_s\nu)(k)}{k}\right)^2\right]\right],\end{aligned}$$

which implies the normalization condition (4.11) after integration over $\mathbb{R}$, using the fact that ν is associated to a potential $Q\in L^2(0,\infty)$.

- Lieb-Thirring condition: The Lieb-Thirring condition $\sum_j |\tilde{E}_j|^{3/2}<\infty$ is trivially satisfied here as was the case for the Weyl condition since all we are doing is to add a finite number of negative eigenvalues.

- Quasi-Szegö condition: This condition states that if $d\rho_0$ is the free spectral measure, that is the spectral measure associated to the zero potential $Q \equiv 0$, then

$$\int_0^\infty \log\Big[\frac{1}{4}\frac{d\tilde\rho}{d\rho_0} + \frac{1}{2} + \frac{1}{4}\frac{d\rho_0}{d\tilde\rho}\Big]\sqrt{E}\,dE < \infty\,. \tag{4.13}$$

Again, the verification of this condition for the perturbed measure $d\tilde\rho$ is more elaborate and we therefore present it in greater detail. The spectral spectral measure $d\rho_0$ has the expression

$$d\rho_0(E) = \frac{1}{\pi}\,\chi_{(0,\infty)}(E)\sqrt{E}\,dE\,,$$

so that using (4.9), we have

$$d\tilde\rho(E) - d\rho_0(E) = d\rho(E) - d\rho_0(E) - \frac{2}{\pi}\sum_{n\geq 0} c_n \frac{\sqrt{E}}{4E+\mu_n^2}\,dE\,. \tag{4.14}$$

We now use (4.6) to express the spectral measure $d\rho$ in terms of the Jost function $\psi(x,\kappa)$ associated to the potential Q, where we let $E = -\kappa^2$. Recall that the Weyl-Titchmarsh function M is given by

$$M(-\kappa^2) = \frac{\psi'(0,\kappa)}{\psi(0,\kappa)}\,, \tag{4.15}$$

so that using (4.15), we obtain

$$\begin{aligned}\operatorname{Im} M(-\kappa^2) &= \frac{1}{2i}\left(M(-\kappa^2) - \overline{M(-\kappa^2)}\right) = \frac{1}{2i}\left(\frac{\psi'(0,\kappa)}{\psi(0,\kappa)} - \frac{\overline{\psi'(0,\kappa)}}{\overline{\psi(0,\kappa)}}\right)\\ &= \frac{1}{2i}\,\frac{W(\overline{\psi},\psi)}{|\psi|^2}(0,\kappa)\,,\end{aligned} \tag{4.16}$$

where W denotes the Wronskian. But ψ and $\overline{\psi}$ are solutions of the same linear second-order ODE since the potential Q and the spectral parameter $E = -\kappa^2$ are both real. It follows that the Wronskian $W(\overline{\psi},\psi)$ is independent of x. Now since ψ is the Jost function, we have, in terms of the parameter $k = i\kappa$ introduced in the normalization condition the asymptotics

$$\psi(x,\kappa) \simeq e^{-\kappa x} = e^{ikx}\,, \quad \text{for } x\to\infty\,,$$

so that

$$W(\overline{\psi},\psi) = 2ik\,.$$

Substituting the latter into (4.16), we obtain

$$\operatorname{Im} M(-\kappa^2) = \frac{i\kappa}{|\psi(0,\kappa)|^2}\,,$$

which plugged into in (4.6) gives for $E > 0$

$$d\rho(E) = \frac{1}{\pi}\,\frac{\sqrt{E}}{|\psi(0,\sqrt{E})|^2}\,dE\,. \tag{4.17}$$

But (4.14) gives

$$\frac{d\tilde{\rho}}{d\rho_0}(E) = \frac{d\rho}{d\rho_0}(E) - 2\sum_{n\geq 0} c_n \frac{1}{4E+\mu_n^2}\,,$$

which combined with (4.17) implies

$$\frac{d\tilde{\rho}}{d\rho_0}(E) = \frac{1}{|\psi(0,\sqrt{E})|^2} - 2\sum_{n\geq 0} \frac{c_n}{4E+\mu_n^2}\,. \tag{4.18}$$

We now analyze the asymptotics of $\frac{d\tilde{\rho}}{d\rho_0}(E)$ in the limit $E \to \infty$. On the one hand we have

$$\frac{1}{4E+\mu_n^2} = \frac{1}{4E}\big(1 + \mathcal{O}(\frac{{\mu_n}^2}{E})\big)\,,$$

and on the other hand we know that since the radius of convergence R of the series $\sum_{n\geq 0} c_n t^{\lambda_n}$ satisfies $R > 1$ and since $\lambda_n = \mathcal{O}(n)$, the series $\sum_{n\geq 0} |c_n|\mu_n^2$ is convergent. The identity (4.18) implies

$$\frac{d\tilde{\rho}}{d\rho_0}(E) = \frac{1}{|\psi(0,\sqrt{E})|^2} - \big(\frac{1}{2}\sum_{n\geq 0} c_n\big)\frac{1}{E} + \mathcal{O}(\frac{1}{E^2})\,.$$

Using the asymptotics on the modulus of the Jost function given by

$$|\psi(0,\sqrt{E})| = 1 + \frac{a}{E} + \mathcal{O}(\frac{1}{E^2})\,,$$

where a is a real constant, we obtain that

$$\frac{d\tilde{\rho}}{d\rho_0}(E) = 1 + \frac{b}{E} + \mathcal{O}(\frac{1}{E^2})\,, \tag{4.19}$$

for some real constant b, which implies in turn that

$$\frac{d\rho_0}{d\tilde{\rho}} = 1 - \frac{b}{E} + \mathcal{O}(\frac{1}{E^2})\,. \tag{4.20}$$

Using (4.19) and (4.20), we obtain

$$\frac{1}{4}\frac{d\tilde{\rho}}{d\rho_0} + \frac{1}{2} + \frac{1}{4}\frac{d\rho_0}{d\tilde{\rho}} = 1 + \mathcal{O}(\frac{1}{E^2})\,, \tag{4.21}$$

which implies that the Quasi-Szegö condition (4.13) is satisfied since (4.21) implies that

$$\log\big[\frac{1}{4}\frac{d\tilde{\rho}}{d\rho_0} + \frac{1}{2} + \frac{1}{4}\frac{d\rho_0}{d\tilde{\rho}}\big] = \mathcal{O}(\frac{1}{E^2})\,.$$

5 A few examples : Bargmann potentials

In this section, we consider perturbations of the potential $Q(x) = 0$, (with the associated amplitude function $A(\alpha) = 0$), and we give examples of amplitudes $\tilde{A}(\alpha)$ for which we can calculate explicitly the associated potentials. These examples are borrowed from ([9], section 11).

5.1 First example

We define for $\alpha \geq 0$,

$$\tilde{A}(\alpha) = 2(\gamma^2 - \beta^2)\ e^{-2\gamma\alpha}, \tag{5.1}$$

where $\beta > 0$ and $\gamma \in [0, \beta[$. Of course, it corresponds to a Müntz series of the type (4.2) (with a single term) with $c_0 = 2(\gamma^2 - \beta^2) < 0$ and $\mu_0 = 2\gamma \geq 0$.

Thus, we take $\delta = \gamma + \frac{3-d}{2} \geq 3 - d$ since $d \geq 3$ by hypothesis. It is known that

$$\tilde{Q}(x) = -8\beta^2 \left(\frac{\beta - \gamma}{\beta + \gamma}\right)\ \frac{e^{-2\beta x}}{(1 + \frac{\beta-\gamma}{\beta+\gamma}\ e^{-2\beta x})^2} \tag{5.2}$$

The associated Jost function is given in the variable $\kappa = -ik$ by

$$\psi(0, \kappa) = \frac{\kappa + \gamma}{\kappa + \beta}, \tag{5.3}$$

(see [9], case 2, p. 636) and is holomorphic in Re $\kappa > -\beta$. The unique root of the Jost function is given by $\kappa = -\gamma$ which is a real resonance.

5.2 Second example

For $\alpha > 0$, we define the amplitude

$$\tilde{A}(\alpha) = -\frac{2c_1}{\kappa_1}\ \sinh(2\kappa_1 \alpha), \tag{5.4}$$

where $c_1 > 0$ is a normalization constant and $\kappa_1 > 0$. It corresponds to a Müntz series with two terms and with two μ_k of different sign. The associated potential is given by

$$\tilde{Q}(x) = -2\frac{d^2}{dx^2}\ \log\left(1 + \frac{c_1}{\kappa_1^2}\ \int_0^x \sinh^2(\kappa_1 y)\ dy\right), \tag{5.5}$$

ahe Jost function has the form in the κ variable

$$\psi(0, \kappa) = \frac{\kappa - \kappa_1}{\kappa + \kappa_1}, \tag{5.6}$$

(see [9], case 1, p. 635). We note that the Jost function is vanishing at $\kappa = \kappa_1$ wich correspond to the single negative eigenvalue $-\kappa_1^2$.

6 Gel'fand-Levitan equations and local stability estimates

In this section, we deduce from the estimates for the difference of the amplitudes $A - \tilde{A}$ obtained in Section 4 a set of new Hölder *local* stability estimates for the difference of the associated potentials $Q - \tilde{Q}$. By local stability, we mean that we are able to control the norm $||Q - \tilde{Q}||_{L^2(0,T)}$ with respect to ϵ, if the Steklov spectra of the underlying Schrödinger operators are close up to ϵ as in (1.17), (T being any fixed positive parameter).

More precisely, we assume here that the potential $Q \in L^2(0,\infty)$, (and thus its associated amplitude A), is *fixed*, and that $\tilde{Q} \in L^2(0,\infty)$ belongs to the infinite dimensional class, denoted $\mathcal{C}_Q$, defined above, that is we assume that the associated amplitude to $\tilde{Q}$ has the form

$$\tilde{A}(\alpha) = A(\alpha) + \sum_{k\geq 0} c_k e^{-\mu_k \alpha}\,, \quad \alpha > 0\,, \tag{6.1}$$

where $\mu_k = 2k + d - 3 + 2\delta$ and $\delta \geq 3 - d$. Moreover, one assumes that $c_k \leq 0$ for all $k \geq 0$ and the power series $\sum_{k\geq 0} c_k t^{\lambda_k}$ has a radius of convergence $R > 1$.

To obtain these local stability estimates, we shall make intensive use the local version of the classical Gel'fand-Levitan equations, (see for instance ([2], Eq. (2.24)) which we recall here. For $0 \leq x \leq t \leq T$, we consider the integral equation

$$V(x,t) + \int_x^T K(t,s)V(x,s)\ ds = -K(x,t), \tag{6.2}$$

where the integral kernel $K(t,s)$ is given by

$$K(t,s) = p(2T - t - s) - p(|t-s|), \tag{6.3}$$

and

$$p(t) = -\frac{1}{2}\int_0^{\frac{t}{2}} A(\alpha)\ d\alpha. \tag{6.4}$$

These integral equations are uniquely solvable for all $x \in (0,T)$ and we can recover the underlying potential using the relation:

$$Q(T-x) = -2\frac{d}{dx}\left(V(x,x)\right). \tag{6.5}$$

An easy calculation shows that

$$\begin{aligned} \frac{d}{dx}\left(V(x,x)\right) &= p(2T-x)V(x,x) + 2p'(2T-2x) - \int_x^T \left(p(2T-x-s) - p(s-x)\right)\ \frac{\partial V}{\partial x}(x,s)\ ds \\ &\quad - \int_x^T \left(p'(2T-x-s) - p'(s-x)\right)\ V(x,s)\ ds. \end{aligned} \tag{6.6}$$

Let us begin with an elementary result:

Lemma 6.1. *Under the hypotheses of Theorem 1.1, there exists a constant C_T depending only on T such that*

$$||p - \tilde{p}||_{(C^0(0,2T),||\cdot||_\infty)} \leq C_T\ f(\epsilon), \tag{6.7}$$

where

$$f(\epsilon) = \left(B^2\epsilon + R^{1-d}\ \epsilon^{\frac{\log R}{\log(\frac{9M_0}{2})}}\right)^{\frac{1}{2}} \tag{6.8}$$

Proof. This is an immediate application of (3.27) and the Cauchy-Schwartz inequality. □

Now, let us introduce some notation to simplify the presentation below. In what follows, the parameters x and T are assumed to be fixed and t is a variable lying in the interval $[x, T]$. We denote by K the integral operator on $L^2(x, T)$ with kernel $K(t, s)$,

$$Kf(t) = \int_x^T K(t, s)\ f(s)\ ds, \tag{6.9}$$

and set

$$d(t) := p(t - x) - p(2T - x - t). \tag{6.10}$$

Thus, the solution $V(x, .)$ of the integral equation (6.2) can be written as

$$V := V(x, .) = (I + K)^{-1} d. \tag{6.11}$$

Using (6.11) and the usual resolvent identity, one obtains

$$\tilde{V} - V = (I + \tilde{K})^{-1} \left(\tilde{d} - d + (K - \tilde{K})(I + K)^{-1} d \right). \tag{6.12}$$

By Lemma 6.1, one has the uniform estimate for $t, s \in [0, T]$,

$$|\tilde{d}(t) - d(t)| \leq C_T\ f(\epsilon)\ \ ,\ \ |\tilde{K}(t, s) - K(t, s)| \leq C_T\ f(\epsilon), \tag{6.13}$$

thus using Schur's lemma, one gets

$$||\tilde{K} - K|| \leq C_T\ f(\epsilon), \tag{6.14}$$

in the sense of the operator norm on $L^2(x, T)$. As a consequence for $\epsilon > 0$ sufficiently small, the operator $I + (I + K)^{-1}(\tilde{K} - K)$ is invertible, and using again the resolvent identity, one obtains easily

$$(I + \tilde{K})^{-1} = \left(I + (I + K)^{-1}(\tilde{K} - K) \right)^{-1} (I + K)^{-1}. \tag{6.15}$$

It follows that, for $\epsilon << 1$, the operator norm of $(I + \tilde{K})^{-1}$ is uniformly bounded:

$$||(I + \tilde{K})^{-1}|| \leq 2\ ||(I + K)^{-1}||. \tag{6.16}$$

Thus, thanks to (6.12) , (6.14) and (6.16), one has:

$$||\tilde{V} - V||_{L^2(x,T)} \leq C_T\ f(\epsilon) \tag{6.17}$$

In the same way, differentiating the integral equation (6.2) with respect to x, one obtains:

$$\frac{\partial V}{\partial x}(x, t) + \int_x^T K(t, s) \frac{\partial V}{\partial x}(x, s) = -p'(t - x) + p'(2T - x - t) + K(t, x) V(x, x), \tag{6.18}$$

and by the same argument, we get immediately

$$||\frac{\partial \tilde{V}}{\partial x} - \frac{\partial V}{\partial x}||_{L^2(x,T)} \leq C_T\ f(\epsilon). \tag{6.19}$$

Finally, using (6.6), (6.17) and (6.19), mimicking the above arguments, one has for all $0 \leq x \leq T$,

$$||\frac{d}{dx} \left(\tilde{V}(x, x) \right) - \frac{d}{dx} \left(V(x, x) \right) ||_{L^2(x,T)} \leq C_T\ f(\epsilon). \tag{6.20}$$

Then taking $x = 0$ and using (6.5), we see that

$$||\tilde{Q} - Q||_{L^2(x,T)} \leq C_T\ f(\epsilon), \tag{6.21}$$

and the proof of Theorem 1.1 is complete.

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