A LOCAL–TO–GLOBAL PROPAGATION PRINCIPLE FOR DIRICHLET–TO–NEUMANN MAPS

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Abstract. We establish four local-to-global propagation results for Dirichlet–to–Neumann maps. Our first two results are proved in the general setting of smooth compact Riemannian manifolds with boundary. The first shows that if two smooth Riemannian metrics coincide in a collar neighborhood of a connected boundary component Γ, then equality of the corresponding local Dirichlet–to–Neumann maps on a nonempty open subset of Γ propagates to equality of the associated global Dirichlet–to–Neumann maps on all of Γ. The proof combines unique continuation and self-adjointness arguments. The second replaces the geometric collar assumption by an exponential spectral assumption on the difference of the corresponding global Dirichlet–to–Neumann maps. The proof relies on the spectral unique continuation theory of Jerison–Lebeau, through the formulation of Le Rousseau–Lebeau. Our third and fourth results establish local-to-global propagation principles under Ingham-type quasi–analytic spectral assumptions. Assuming that the boundary manifold is respectively a compact Riemannian symmetric space or a compact quasi–analytic Riemannian manifold, they rely on the propagation theorems of Ganguly–Thangavelu and of Bhowmik–Pradhan. As an application, we consider a class of conformally warped product metrics. In this setting, the local Borg–Marchenko theorem and Weyl–Titchmarsh theory relate the required Ingham-type spectral decay to a suitable quasi–analytic boundary closeness of the conformal factors, yielding new local-to-global uniqueness results for Dirichlet–to–Neumann maps.

Contents

1. Introduction

Since Calderón's pioneering work [3], inverse boundary value problems have generated an extensive literature. One of the central questions is whether the coefficients of an elliptic equation, or the underlying geometry of a manifold, can be recovered from boundary measurements encoded by the Dirichlet–to–Neumann (DN) map.

The case of partial boundary measurements has attracted particular attention during the last two decades. For the conductivity and Schrödinger equations in Euclidean domains, important uniqueness results with partial data were obtained by Bukhgeim–Uhlmann [2], Kenig–Sjöstrand–Uhlmann [16], Isakov [11], Imanuvilov–Uhlmann–Yamamoto [9], and very recently Uhlmann–Wang [29]. Similar questions have also been investigated on Riemannian manifolds, notably by Kenig–Salo [14]. We also refer to the surveys [15, 28] for a broader overview of the subject.

The philosophy of the partial data Calderón problem is that measurements performed on a suitable open subset of the boundary should determine global information inside the manifold. In the present paper, we investigate a different but related question. Instead of asking whether local measurements determine the metric, we ask whether equality of local Dirichlet–to–Neumann maps forces equality of the corresponding global Dirichlet–to–Neumann maps, thereby reducing the inverse problem to a setting where stronger uniqueness results are available.

To investigate this issue, we first introduce the global Dirichlet–to–Neumann map associated with a smooth metric g. Let M be a smooth compact orientable connected manifold with smooth boundary. Let u be the solution of

$$-\Delta_g u = 0 \quad \text{on } M\,, \tag{1.1}$$

with boundary condition

$$u|_{\partial M} = \psi\,, \qquad \psi \in C^\infty(\partial M)\,.$$

The corresponding global Dirichlet–to–Neumann map is defined by

$$\Lambda_g \psi = \partial_\nu u\big|_{\partial M}\,, \tag{1.2}$$

where ν denotes the outward unit normal vector field along ∂M.

Given a nonempty open subset $\mathcal{O} \subset \partial M$, we define the local Dirichlet–to–Neumann map as follows. For $\psi \in C_c^\infty(\mathcal{O})$, let u be the solution of (1.1) with boundary condition

$$u = \psi \quad \text{on } \mathcal{O}\,, \qquad u = 0 \quad \text{on } \partial M \setminus \mathcal{O}\,.$$

The local Dirichlet–to–Neumann map is then given by

$$\Lambda_{g,\mathcal{O}}\, \psi = \partial_\nu u\big|_{\mathcal{O}}\,. \tag{1.3}$$

We first state a geometric local-to-global result in which the boundary may have several connected components. The point is that the propagation takes place along one fixed connected boundary component: equality of the local Dirichlet–to–Neumann maps on an arbitrary nonempty open subset of this component extends to the whole component, provided the two metrics agree in a collar neighborhood of it.

Theorem 1.1 (Geometric local-to-global propagation). *Let g and $\widetilde{g}$ be smooth Riemannian metrics on M. Assume that $g = \widetilde{g}$ in a collar neighborhood of the connected boundary component Γ. If the corresponding local Dirichlet–to–Neumann maps coincide on a nonempty open subset $\mathcal{O} \subset \Gamma$, then $\Lambda_{g,\Gamma} = \Lambda_{\widetilde{g},\Gamma}$ on the whole boundary component Γ.*

The proof combines boundary unique continuation with a self-adjointness argument. While the result may be well known to specialists, we have not been able to locate an explicit reference. We include it here because it isolates a simple local-to-global propagation mechanism based on self-adjointness, which will reappear in a different context in the proof of our main result.

As we shall see in the final part of the introduction (see (1.12)), in the warped product setting, the local Borg–Marchenko theorem shows that coincidence of the metrics in a collar neighborhood of the boundary is equivalent to a fixed exponential decay of the spectral coefficients of the difference of the corresponding Dirichlet–to–Neumann maps. This naturally leads to the following abstract propagation result, in which the geometric collar assumption of Theorem 1.1 is replaced by an exponential spectral assumption.

Theorem 1.2 (Exponential spectral propagation)**.** *Let M be a smooth compact orientable connected manifold with smooth connected boundary $K = \partial M$, and let $\mathcal{O} \subset K$ be a nonempty open subset. Let g and $\widetilde{g}$ be two smooth Riemannian metrics on M inducing the same Riemannian metric g_K on K. Let*

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

be the distinct eigenvalues of the Laplace–Beltrami operator Δ_{g_K} on K, and let

$$P_k : L^2(K) \longrightarrow E_{\lambda_k}$$

be the associated spectral projections. Assume that $\Lambda_{g,\mathcal{O}} = \Lambda_{\widetilde{g},\mathcal{O}}$ and that there exist constants $C > 0$ and $\varepsilon > 0$ such that

$$\|P_k(\Lambda_g - \Lambda_{\widetilde{g}})\|_{\mathcal{L}(L^2(K))} \leq Ce^{-\varepsilon\sqrt{\lambda_k}}\,, \qquad k \geq 0\,.$$

Then $\Lambda_g = \Lambda_{\widetilde{g}}$.

Remark 1.3. Set $A = \Lambda_g - \Lambda_{\widetilde{g}}$. Since the two metrics induce the same boundary metric on K, the two Dirichlet–to–Neumann maps have the same principal symbol. Hence $A \in \Psi^0(K)$, that is, A is a pseudodifferential operator of order 0. In particular, A extends to a bounded operator on $L^2(K)$. The theorem extends without change to manifolds with nonconnected boundary by considering each connected boundary component separately.

When the boundary K is a compact real-analytic Riemannian manifold, a theorem of Seeley [27] characterizes real analyticity by the exponential decay of the eigenfunction coefficients. First observe that our assumption on the spectral projections immediately implies that, with $f = (\Lambda_g - \Lambda_{\tilde{g}})\psi$, we have

$$\|P_k f\|_{L^2(K)} \leq Ce^{-\varepsilon\sqrt{\lambda_k}}\,, \qquad k \geq 0\,. \tag{1.4}$$

In particular, if (ϕ_ℓ) is an orthonormal basis of eigenvectors of the Laplace Beltrami operator $-\Delta_{g_K}$, we get $P_k\phi_\ell = \phi_\ell$ when λ_k is the eigenvalue associated with ϕ_ℓ ($k = k(\ell)$) and we obtain

$$\big|\langle f, \phi_\ell\rangle\big| \leq Ce^{-\varepsilon\sqrt{\lambda_{k(\ell)}}}\,, \tag{1.5}$$

which corresponds to Seeley's criterion on the individual eigenfunction coefficients.

Conversely, Seeley's hypothesis implies the exponential decay of the spectral projections up to the factor $\sqrt{d_k}$, where $d_k = \dim E_{\lambda_k} = O(\lambda_k^{(n-1)/2})$ by Weyl's law, with optimal remainder [7]. Hence this polynomial factor can be absorbed into the exponential decay by replacing ϵ with any constant $\epsilon' < \epsilon$.

Thus, in the real-analytic setting, Theorem 1.2 can be proved using the spectral characterization of analyticity together with the self-adjointness of $\Lambda_g - \Lambda_{\widetilde{g}}$, without appealing to the Jerison–Lebeau propagation theorem.

We emphasize that the proof of Theorem 1.2 does not require any analyticity assumption on the boundary manifold K. The quasi-analytic propagation results established below are obtained under additional assumptions on K, namely that it is either a compact Riemannian symmetric space or a compact quasi-analytic Riemannian manifold. Their strength is that they replace the exponential spectral decay in Theorem 1.2 by the much weaker Ingham-type quasi-analytic decay [10]. It would be interesting to determine whether these additional assumptions on K are genuinely necessary, or whether the same quasi-analytic propagation principle remains valid for arbitrary smooth compact Riemannian manifolds, as is the case for the exponential propagation theorem of Jerison–Lebeau.

We now assume that the boundary manifold K is a compact (necessarily orientable) connected Riemannian symmetric space. Typical examples include flat tori, spheres, odd-dimensional real projective spaces, complex projective spaces, compact Lie groups, and finite products of such spaces. In this setting, Ganguly and Thangavelu [5] established an Ingham-type quasi-analytic propagation theorem, building on the classical work of Ingham [10]; see also Koosis [17] for an elegant exposition of Ingham's theory. This leads to the following local-to-global propagation result.

Theorem 1.4 (Ingham-type propagation on symmetric spaces)**.** *Let M be a smooth compact orientable connected manifold with smooth connected boundary $K = \partial M$, and let $\mathcal{O} \subset K$ be a nonempty open subset. Let g and $\widetilde{g}$ be two smooth Riemannian metrics on M inducing the same Riemannian metric g_K on K. Assume that (K, g_K) is a compact connected Riemannian symmetric space. Let*

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

be the distinct eigenvalues of the Laplace–Beltrami operator Δ_{g_K} on K , and let

$$P_k : L^2(K) \longrightarrow E_{\lambda_k}$$

be the associated spectral projections. Assume that $\Lambda_{g,\mathcal{O}} = \Lambda_{\widetilde{g},\mathcal{O}}$, and that there exist a constant $C > 0$ and a positive decreasing function

$$\theta(t) \longrightarrow 0 \quad \text{as } t \to +\infty\,, \qquad \int_T^{+\infty} \frac{\theta(t)}{t}\, dt = +\infty\,,$$

and

$$\|P_k(\Lambda_g - \Lambda_{\widetilde{g}})\|_{\mathcal{L}(L^2(K))} \leq C \exp\Big(-\sqrt{\lambda_k}\, \theta(\sqrt{\lambda_k})\Big)\,, \qquad k \geq 0\,. \tag{1.6}$$

Then $\Lambda_g = \Lambda_{\widetilde{g}}$.

Remark 1.5. The conclusion of Theorem 1.4 remains valid if the global spectral estimate (1.6) is replaced by the following local pointwise assumption: for every $\psi \in C_c^\infty(\mathcal{O})$ and every $\omega \in \mathcal{O}$, there exists a constant $C_{\omega,\psi} > 0$ such that

$$|P_k(\Lambda_g - \Lambda_{\widetilde{g}})\psi(\omega)| \leq C_{\omega,\psi}\, e^{-\sqrt{\lambda_k}\theta(\sqrt{\lambda_k})}\,, \qquad k \geq 0\,. \tag{1.7}$$

Indeed, the proof is unchanged: one simply uses the pointwise version, Theorem 1.4 of Ganguly and Thangavelu [5], instead of the L^2-version, Theorem 1.3. It would be interesting to investigate whether such pointwise estimates can be derived directly from local geometric assumptions. More precisely, if the two metrics coincide in a neighborhood of $\mathcal{O}$, then the operator $A = \Lambda_g - \Lambda_{\widetilde{g}}$ should possess strong microlocal regularity properties over $\mathcal{O}$. In the C^∞ category, this would suggest rapid decay of the corresponding spectral components. It would be interesting to investigate whether additional real-analytic regularity assumptions could lead to exponential pointwise spectral decay estimates, thereby providing a genuinely local route to the pointwise hypothesis above.

We next consider the case where the boundary manifold K is a compact quasi-analytic Riemannian manifold. We briefly recall the necessary background on quasi-analytic manifolds and Denjoy–Carleman classes in the proof of the following theorem. Our result relies on the recent propagation theorem of Bhowmik and Pradhan [1]. Following their work, we restrict ourselves to the family of Ingham weights

$$\theta_\delta(t) = \frac{1}{(\log t)^\delta}\,, \qquad 0 < \delta \leq 1\,. \tag{1.8}$$

Theorem 1.6 (Ingham-type propagation on quasi-analytic manifolds)**.** *Let M be a smooth compact orientable connected manifold with smooth connected boundary $K = \partial M$, and let $\mathcal{O} \subset K$ be a nonempty open subset. Let g and $\widetilde{g}$ be two smooth Riemannian metrics on M inducing the same Riemannian metric g_K on K. Assume in addition that (K, g_K) is a compact connected quasi-analytic Riemannian manifold. Let*

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

be the distinct eigenvalues of the Laplace–Beltrami operator Δ_{g_K} on K, and let

$$P_k : L^2(K) \longrightarrow E_{\lambda_k}$$

be the associated spectral projections. Assume that $\Lambda_{g,\mathcal{O}} = \Lambda_{\widetilde{g},\mathcal{O}}$, and that there exist constants $C > 0$ and $0 < \delta \leq 1$ such that

$$\|P_k(\Lambda_g - \Lambda_{\widetilde{g}})\|_{\mathcal{L}(L^2(K))} \leq C \exp\Big(-\sqrt{\lambda_k}\,\theta_\delta(\sqrt{\lambda_k})\Big)\,, \qquad k \geq 0\,.$$

Then $\Lambda_g = \Lambda_{\widetilde{g}}$.

We now turn to an application of the preceding propagation principles to a class of conformally warped product metrics. Our starting point is the geometric assumption in Theorem 1.1. Although natural from the viewpoint of boundary unique continuation, one may wonder whether the requirement that the two metrics coincide in a collar neighborhood of the boundary is really necessary, or whether it can be replaced by a weaker boundary closeness condition. In the warped product setting, the answer is provided by Weyl–Titchmarsh theory. Using the diagonalization of the Dirichlet–to–Neumann map together with the Weyl–Titchmarsh representation established in [4], we derive quantitative estimates for the spectral coefficients of the difference of two Dirichlet–to–Neumann maps from suitable boundary closeness conditions on the conformal factors. Combined with the abstract propagation principles established in the first part of the paper, these estimates yield new local-to-global uniqueness results in the warped product setting.

More precisely, we consider non-compact manifolds of the form

$$M = (0,1] \times K\,, \tag{1.9}$$

where $r \in (0,1]$ is a radial variable and K is either a compact orientable connected Riemannian symmetric space or a compact orientable connected quasi-analytic Riemannian manifold. The boundary of M is naturally identified with the boundary manifold

$$\partial M = \{1\} \times K \simeq K\,.$$

We endow M with the warped product metric

$$g = c(r)^4 \big(dr^2 + r^2 g_K\big)\,, \tag{1.10}$$

A natural question then is whether the manifold structure of M and the Riemannian metric g can be extended across $r = 0$ to provide a smooth compact Riemannian manifold $(\check{M}, \check{g})$ with connected boundary $\partial\check{M} = \{1\} \times K$ (which we identify with K). As we now briefly recall, the answer to this question is generally no, so that we need to allow for a geometric setting that is possibly singular at $r = 0$. First, at the level of the manifold structure, the extension of the differentiable structure of the non-compact manifold $M = (0,1] \times K$ across $r = 0$ as a compact orientable smooth manifold $\check{M}$ with connected boundary $\partial\check{M} = K$, in other words the "fillability" of K as the boundary of a compact orientable smooth manifold, puts significant topological conditions on K, such as the vanishing of all the Stiefel-Whitney numbers of its tangent bundle (these conditions are in fact necessary and sufficient if $\dim K \not\equiv 4 \mod 1,2,3$; the case $\dim K \equiv 4$ requires in addition the vanishing of the Pontrjagin numbers of the tangent bundle, see [21]). These imply for example that the Euler characteristic $\chi(K)$ should be even. In particular, among compact Riemannian symmetric spaces, they rule out examples such as the complex projective plane $\mathbb{P}_2(\mathbb{C})$, since $\chi(\mathbb{P}_2(\mathbb{C})) = 3$. This is why the range of r in our model is restricted to the semi-open interval $(0,1]$. Second, assuming that the topological conditions guaranteeing the "fillability" of K as the boundary of a compact orientable smooth manifold are satisfied, the additional requirement that the metric (1.10) extend smoothly across the point corresponding to $r = 0$ imposes rather restrictive conditions. For example, when $K = S^{d-1}$, smoothness of the metric at the origin requires that the transverse metric g_K be the standard round metric on the sphere and that all odd-order derivatives of the warping factor vanish at $r = 0$, namely

$$c^{(2k+1)}(0) = 0, \qquad k \geq 0,$$

see [22, Section 4.3.4]. In general, neither of these assumptions will be imposed here. Consequently, the metric (1.10) should be regarded as a possibly singular warped product metric near $r = 0$, so that our model generally gives rise to a metric cone with a singular vertex at $r = 0$, except in very special cases. This explains why the geometric applications below are formulated in this singular setting, rather than for a smooth compact completion.

We thus consider metrics of the form

$$g = c(r)^4 (dr^2 + r^2 g_K)\,, \qquad \widetilde{g} = \widetilde{c}(r)^4 (dr^2 + r^2 g_K)\,,$$

where g_K is a Riemannian metric on a closed orientable manifold K. The warped product structure allows one to separate variables and to diagonalize the Dirichlet–to–Neumann maps in the eigenbasis of the Laplace–Beltrami operator on K. In particular, if $c(1) = \widetilde{c}(1)$, then, as explained above, the difference $A = \Lambda_g - \Lambda_{\widetilde{g}}$ is a bounded operator on $L^2(K)$ and is diagonal with respect to the spectral decomposition of the Laplace–Beltrami operator on K, namely

$$A = \sum_{k \geq 0} a_k P_k\,. \tag{1.11}$$

A careful reading of the proof of the local Borg–Marchenko theorem ([4, Theorem 1.4]) shows that the exponential decay

$$|a_k| \leq C e^{-\epsilon \sqrt{\lambda_k}}\,, \tag{1.12}$$

is equivalent to the coincidence of the conformal factors in a collar neighborhood of the boundary. Consequently, in this particular setting, the exponential spectral assumption of the previous theorem is equivalent to the collar assumption of Theorem 1.1.

Our main objective is to identify geometric boundary closeness conditions that give rise to the quasi-analytic spectral estimates appearing in Theorems 1.4 and 1.6. More precisely, we seek conditions on the conformal factors that imply the decay estimate

$$|a_k| \leq C e^{-\sqrt{\lambda_k}\theta(\sqrt{\lambda_k})}\,,$$

where θ is an admissible Ingham weight in the symmetric-space setting, and

$$\theta(t) = \theta_\delta(t) = \frac{1}{(\log t)^\delta}\,, \qquad 0 < \delta \leq 1\,,$$

in the quasi-analytic setting. The local-to-global propagation results then follow immediately from Theorems 1.4 and 1.6. To formulate the corresponding geometric application, it is convenient to introduce the logarithmic variable

$$x = -\log r, \qquad r = e^{-x}\,,$$

which identifies M with the infinite cylinder $[0, \infty) \times K$. In these coordinates, the metrics take the form

$$g = f(x)^4 (dx^2 + g_K)\,, \qquad \widetilde{g} = \widetilde{f}(x)^4 (dx^2 + g_K)\,, \tag{1.13}$$

where

$$f(x) = e^{-x/2} c(e^{-x})\,, \qquad \widetilde{f}(x) = e^{-x/2} \widetilde{c}(e^{-x})\,. \tag{1.14}$$

We first consider the symmetric-space setting, so that Theorem 1.4 applies. The key point is to relate the geometric boundary closeness of the conformal factors to the spectral assumption in Theorem 1.4. This is achieved by combining the local Borg–Marchenko theorem with quantitative Weyl–Titchmarsh estimates. The abstract propagation result of Theorem 1.4 then immediately yields the following local-to-global uniqueness theorem.

Theorem 1.7 (Symmetric-space case). *Let (K, g_K) be a compact orientable connected Riemannian symmetric space, and let*

$$g = c(r)^4(dr^2 + r^2 g_K)\,, \qquad \widetilde{g} = \widetilde{c}(r)^4(dr^2 + r^2 g_K)$$

be two warped product metrics on $M = (0,1] \times K$, where $c, \widetilde{c} \in C^m([0,1])$, $m \geq 2$, are positive functions. Define

$$f(x) = e^{-x/2}c(e^{-x})\,, \qquad \widetilde{f}(x) = e^{-x/2}\widetilde{c}(e^{-x})\,.$$

Assume that there exist $C > 0$ and $\varepsilon > 0$ such that

$$|f^{(j)}(x) - \widetilde{f}^{(j)}(x)| \leq Ce^{\Phi(x)}\,, \qquad j = 0,1,2, \quad x \in (0,\varepsilon],$$

where

$$\Phi(x) = \inf_{t \geq T}\big(2xt - t\theta(t)\big)\,,$$

and $\theta : [T,\infty) \to (0,\infty)$ is a positive decreasing function satisfying

$$\theta(t) \longrightarrow 0\,, \qquad \int_T^\infty \frac{\theta(t)}{t}\,dt = +\infty\,.$$

Let $\mathcal{O} \subset K$ be a nonempty open subset. If $\Lambda_{g,\mathcal{O}} = \Lambda_{\widetilde{g},\mathcal{O}}$, then $\Lambda_g = \Lambda_{\widetilde{g}}$.

The assumptions of Theorem 1.7 deserve some further comments. The function $\Phi(x)$ plays a central role in the proof. It is introduced so that the weighted Laplace transform

$$I(\rho) = \int_0^\varepsilon e^{-2\rho x}e^{\Phi(x)}\,dx \tag{1.15}$$

satisfies the estimate

$$I(\rho) \leq e^{-\rho\theta(\rho)}\,, \qquad \rho \geq T\,, \tag{1.16}$$

(see Lemma 5.1). Furthermore, the assumptions on θ imply that

$$\Phi(x) \longrightarrow -\infty\,, \qquad x \to 0^+, \tag{1.17}$$

(see Lemma 5.2). Thus the condition

$$|f^{(j)}(x) - \widetilde{f}^{(j)}(x)| \leq Ce^{\Phi(x)}\,, \qquad j = 0,1,2, \tag{1.18}$$

imposes a quantitative boundary closeness condition on f and $\widetilde{f}$, whose strength depends on the choice of θ. A typical example is obtained by choosing

$$\theta(t) = \frac{1}{\log t}\,. \tag{1.19}$$

In this case, a simple minimization yields

$$\Phi(x) \sim -\frac{4}{e}\,x^2 e^{\frac{1}{2x}}\,, \qquad x \to 0^+\,. \tag{1.20}$$

Since $x = -\log r \sim 1 - r$ as $r \to 1^-$, this corresponds to a boundary closeness condition of the form

$$|c(r) - \widetilde{c}(r)| \leq C\exp\left(-\frac{4}{e}(1-r)^2 e^{\frac{1}{2(1-r)}}\right), \qquad r \to 1^-. \tag{1.21}$$

If c and $\widetilde{c}$ are C^∞ up to the boundary, such a condition implies that their difference is flat at $r = 1$. In particular, if the conformal factors belong to a quasi-analytic class, for instance if they are real-analytic on $[0,1]$, then this already implies $c = \widetilde{c}$ on $[0,1]$, and no further argument is required. However, the boundary estimate (1.21) should not be confused with a quasi-analytic regularity assumption. Indeed, if $\chi \in C_c^\infty([0,1])$ satisfies $\chi \equiv 1$ in a neighborhood of $r = 1$, then replacing $c - \widetilde{c}$ by $\chi(c - \widetilde{c})$ leaves the estimate (1.21) unchanged. The resulting functions vanish identically on a nonempty interior region and therefore cannot, in general, belong to a quasi–analytic class unless they vanish identically.

A noteworthy feature of the present theorem is therefore that no quasi-analytic regularity is assumed on the conformal factors. The assumptions are compatible with conformal factors of finite regularity, for instance C^m, $m \geq 2$, where neither flatness propagation nor quasi-analytic continuation can be invoked.

Finally, the example (1.19) may be viewed as a borderline admissible choice since

$$\int_T^\infty \frac{dt}{t\log t} = +\infty\,, \qquad \int_T^\infty \frac{dt}{t(\log t)^{1+\varepsilon}} < \infty\,, \quad \varepsilon > 0\,. \tag{1.22}$$

Hence $\theta(t) = 1/\log t$ is, up to slowly varying factors, the fastest decay compatible with the quasi–analyticity condition.

Remark 1.8. While the conclusion of the theorem is an equality of global Dirichlet–to–Neumann maps, the global uniqueness result of [4] then yields $g = \widetilde{g}$. However, the novelty of the theorem is not this uniqueness statement itself, but rather the propagation mechanism showing that equality of the local Dirichlet–to–Neumann maps on an arbitrary nonempty open subset $\Gamma \subset K$ forces equality of the corresponding global Dirichlet–to–Neumann maps.

A distinctive feature of Theorem 1.7 is that the local-to-global propagation does not stem from the usual unique continuation machinery for partial differential equations. Instead, it is achieved through a purely spectral argument combining Weyl–Titchmarsh asymptotics with a quasi-analytic propagation theorem of Ingham's type. To the best of our knowledge, this mechanism is new in the context of Calderón-type inverse problems.

The previous theorem relies on the fact that the transversal manifold K is a compact orientable Riemannian symmetric space, and hence possesses an intrinsic real-analytic structure. Using a recent result of Bhowmik and Pradhan [1], we show that the same conclusion remains valid under the much weaker assumption that K is merely a compact orientable quasi–analytic Riemannian manifold.

Our next theorem provides the corresponding quasi–analytic analogue of Theorem 1.7. For technical reasons, and in order to keep the presentation as simple as possible, we restrict ourselves to the particular class of weights

$$\theta(t) = \frac{1}{(\log t)^\delta}\,, \qquad 0 < \delta \leq 1\,. \tag{1.23}$$

Theorem 1.9 (Quasi-analytic case)**.** *Let (K, g_K) be a compact connected quasi–analytic Riemannian manifold, and let $c, \widetilde{c} \in C^m([0,1])$, $m \geq 2$, be positive functions. Consider the metrics*

$$g = c(r)^4(dr^2 + r^2 g_K)\,, \qquad \widetilde{g} = \widetilde{c}(r)^4(dr^2 + r^2 g_K)$$

on $M = (0,1] \times K$. Define

$$f(x) = e^{-x/2}c(e^{-x}), \qquad \widetilde{f}(x) = e^{-x/2}\widetilde{c}(e^{-x})\,.$$

For $0 < \delta \leq 1$, we introduce

$$\Phi_\delta(x) = \inf_{t\geq T}\left(2xt - \frac{t}{(\log t)^\delta}\right)\,.$$

Assume that there exist $C > 0$ and $\varepsilon > 0$ such that

$$|f^{(j)}(x) - \widetilde{f}^{(j)}(x)| \leq Ce^{\Phi_\delta(x)}\,, \qquad j = 0,1,2, \quad x \in (0,\varepsilon]\,.$$

Let $\mathcal{O} \subset K$ be a nonempty open subset. If $\Lambda_{g,\mathcal{O}} = \Lambda_{\widetilde{g},\mathcal{O}}$, then $\Lambda_g = \Lambda_{\widetilde{g}}$.

Remark 1.10. A straightforward minimization shows that

$$\Phi_\delta(x) \sim -\delta(2x)^{1+1/\delta}\exp\Big((2x)^{-1/\delta}\Big)\,, \qquad x \to 0^+\,.$$

Consequently, the assumptions of Theorem 1.9 require the conformal factors to be extremely close near the boundary $r = 1$, although this closeness is substantially weaker than coincidence in a collar neighborhood.

Beyond the warped product setting, the present work suggests that Weyl–Titchmarsh theory may serve as a bridge between geometric boundary closeness conditions and abstract quasi-analytic propagation principles. It would be interesting to investigate whether similar mechanisms exist in more general geometric settings.

The paper is organized as follows. Section 2 is devoted to the proof of the geometric local-to-global propagation theorem. In Section 3, we prove the exponential spectral propagation theorem using the spectral unique continuation theorem of Jerison–Lebeau in the formulation of Le Rousseau–Lebeau. Section 4 establishes the quasi–analytic propagation principle underlying the proofs of Theorems 1.4 and 1.6. The remaining sections introduce the warped product framework together with the associated Weyl–Titchmarsh theory, establish the required quantitative spectral decay estimates, and combine them with the abstract propagation results to obtain the corresponding local-to-global uniqueness theorems.

2. A GEOMETRIC PROPAGATION ARGUMENT

In this section, we prove Theorem 1.1. We retain the notation and assumptions introduced there. The geometric hypothesis that the two metrics coincide in a collar neighborhood of the boundary component Γ is illustrated in Figure 1.

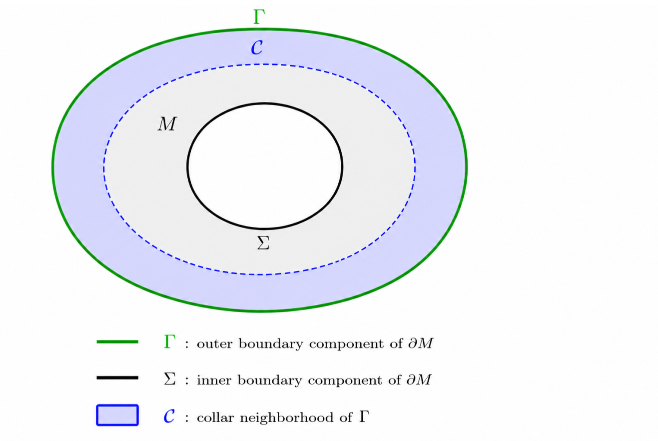

FIGURE 1. A collar neighborhood $\mathcal{C}$ of the boundary component Γ.

Proof. Since $g = \widetilde{g}$ in the collar neighborhood $\mathcal{C}$, we denote their common value by

$$g_0 := g = \widetilde{g} \text{ in } \mathcal{C}\,.$$

Consequently, g and $\widetilde{g}$ induce the same boundary metric, boundary measure, and outward unit normal vector field on Γ. We denote by $\langle \cdot, \cdot \rangle$ the associated $L^2(\Gamma)$ inner product. Define

$$A := \Lambda_{g,\Gamma} - \Lambda_{\widetilde{g},\Gamma}\,. \tag{2.1}$$

Since both Dirichlet–to–Neumann maps are self-adjoint with respect to $\langle \cdot, \cdot \rangle$, the operator A is self-adjoint on $L^2(\Gamma)$. Let $f \in C_c^\infty(\mathcal{O})$. Let u and $\widetilde{u}$ solve

$$\begin{cases} -\Delta_g u = 0 & \text{in } M\,, \\ u = f & \text{on } \Gamma\,, \\ u = 0 & \text{on } \partial M \setminus \Gamma\,, \end{cases} \qquad \begin{cases} -\Delta_{\widetilde{g}} \widetilde{u} = 0 & \text{in } M\,, \\ \widetilde{u} = f & \text{on } \Gamma\,, \\ \widetilde{u} = 0 & \text{on } \partial M \setminus \Gamma\,. \end{cases} \tag{2.2}$$

Define $w := u - \widetilde{u}$. Since u and $\widetilde{u}$ are harmonic with respect to the same metric g_0 in the collar neighborhood $\mathcal{C}$, it follows that

$$-\Delta_{g_0} w = 0 \text{ in } \mathcal{C}\,. \tag{2.3}$$

Moreover, $w|_\Gamma = 0$. The local DN equality gives

$$\partial_\nu w = 0 \text{ on } \mathcal{O}\,. \tag{2.4}$$

By boundary unique continuation for elliptic equations (see [8], Section 28 or [26]), it follows that $w \equiv 0$ in $\mathcal{C}$. In particular,

$$Af = \partial_\nu w|_\Gamma = 0 \text{ on } \Gamma\,. \tag{2.5}$$

We have therefore shown that

$$Af = 0 \text{ on } \Gamma\,, \qquad f \in C_c^\infty(\mathcal{O})\,. \tag{2.6}$$

We now use self-adjointness. Let $F \in C^\infty(\Gamma)$. For every $h \in C_c^\infty(\mathcal{O})$, we have

$$\langle AF, h\rangle = \langle F, Ah\rangle = 0\,. \tag{2.7}$$

Hence $AF = 0$ on $\mathcal{O}$. Repeating the preceding boundary unique-continuation argument with boundary datum F, we obtain $AF = 0$ on Γ. Since $F \in C^\infty(\Gamma)$ was arbitrary, this proves $A = 0$, that is, $\Lambda_{g,\Gamma} = \Lambda_{\widetilde{g},\Gamma}$.

□

3. An exponential spectral propagation argument

In this section, we prove Theorem 1.2. The geometric collar assumption is now replaced by an exponential decay condition on the spectral components of the difference of the Dirichlet–to–Neumann maps. We shall use the exponential spectral unique continuation theorem of Jerison–Lebeau [13]. We also refer to the interpolation inequality of Lebeau–Zuazua [20, (5.6)]; see also Lebeau–Robbiano [19] and Le Rousseau–Lebeau [18, Proposition 5.6]. We recall that

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \tag{3.1}$$

are the distinct eigenvalues of the Laplace–Beltrami operator on K , and

$$P_k : L^2(K) \longrightarrow E_{\lambda_k} \tag{3.2}$$

is the associated spectral projection. Assume that

$$g = \widetilde{g} \text{ on } K = \partial M\,,$$

and set

$$A = \Lambda_g - \Lambda_{\widetilde{g}}\,. \tag{3.3}$$

Since $\Lambda_{g,\mathcal{O}} = \Lambda_{\widetilde{g},\mathcal{O}}$, we have, for every $\psi \in C_c^\infty(\mathcal{O})$,

$$A\psi = 0 \text{ on } \mathcal{O}\,. \tag{3.4}$$

Assume moreover that there exist constants $C > 0$ and $\varepsilon > 0$ such that

$$\|P_k A\|_{\mathcal{L}(L^2(K))} \le Ce^{-\varepsilon\sqrt{\lambda_k}}, \qquad k \ge 0\,. \tag{3.5}$$

Then

$$\|P_k(A\psi)\|_{L^2(K)} \le Ce^{-\varepsilon\sqrt{\lambda_k}}\|\psi\|_{L^2(K)}\,, \tag{3.6}$$

so that the spectral components of $A\psi$ satisfy a fixed exponential decay. Since $A\psi = 0$ on $\mathcal{O}$, all the assumptions of Proposition 5.6 in Le Rousseau–Lebeau [18] are fulfilled. Therefore,

$$A\psi = 0 \text{ on } K\,. \tag{3.7}$$

Now, since A is self-adjoint, the same argument as in the proof of Theorem 1.1 yields

$$\langle Au, \psi\rangle = \langle u, A\psi\rangle = 0\,,$$

for every $u \in C^\infty(K)$ and every $\psi \in C_c^\infty(\mathcal{O})$. Hence

$$Au = 0 \text{ on } \mathcal{O}.$$

Moreover, by (3.5),

$$\|P_k(Au)\|_{L^2(K)} \le Ce^{-\varepsilon\sqrt{\lambda_k}}\|u\|_{L^2(K)}\,. \tag{3.8}$$

Proposition 5.6 of [18] applies once again and yields

$$Au = 0 \text{ on } K\,. \tag{3.9}$$

Since this holds for every $u \in C^\infty(K)$, we conclude that $\Lambda_g = \Lambda_{\widetilde{g}}$. This completes the proof.

4. The quasi–analytic propagation principle

In this section, we prove Theorems 1.4 and 1.6. The common idea is to replace the exponential spectral propagation used in Theorem 1.2 by an Ingham-type quasi–analytic propagation theorem. We first treat the case where the boundary K is a compact Riemannian symmetric space, relying on the theorem of Ganguly–Thangavelu. We then turn to compact quasi–analytic Riemannian manifolds, using the recent propagation result of Bhowmik–Pradhan.

4.1. **Propagation on symmetric spaces.**

In this subsection, we assume that K is a compact connected Riemannian symmetric space. Typical examples include

$$K = \mathbb{S}^d\,, \qquad K = \mathbb{T}^d\,, \qquad K = \mathbb{RP}^d\,, \tag{4.1}$$

more generally compact rank-one symmetric spaces and finite products of such spaces. We recall that

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

denote the distinct eigenvalues of the Laplace–Beltrami operator on K, and

$$P_k : L^2(K) \longrightarrow E_{\lambda_k}$$

the corresponding spectral projections.

The following propagation theorem is Theorem 1.3 of Ganguly and Thangavelu [5].

Proposition 4.1. *Let $\mathcal{O} \subset K$ be a nonempty open subset and let $F \in L^2(K)$. Assume that $F = 0$ on $\mathcal{O}$. Suppose that there exists a positive decreasing function*

$$\theta : [T, +\infty) \to (0, +\infty)\,,$$

such that

$$\theta(t) \to 0 \quad \text{as } t \to +\infty\,, \qquad \int_T^\infty \frac{\theta(t)}{t}\, dt = +\infty\,.$$

Assume moreover that

$$\|P_k F\|_{L^2(K)} \le C e^{-\sqrt{\lambda_k}\,\theta(\sqrt{\lambda_k})}\,, \qquad k \ge 0\,.$$

Then $F \equiv 0$ on K.

With Proposition 4.1 at hand, the proof of Theorem 1.4 is identical to that of Theorem 1.2: one simply replaces the exponential propagation result of Le Rousseau–Lebeau by Proposition 4.1. We omit the details.

4.2. **Propagation on quasi–analytic manifolds.**

We now turn to the more general setting where K is a compact quasi–analytic Riemannian manifold. In contrast with the symmetric space case, the proof relies on the recent work of Bhowmik and Pradhan [1], which extends the quasi–analytic propagation theorem of Ganguly and Thangavelu to arbitrary compact quasi–analytic manifolds.

We begin by recalling the classical notion of a Denjoy–Carleman class on a connected open set $\Omega \subset \mathbb{R}^n$. Let $M = (M_k)_{k\ge 0}$ be an increasing logarithmically convex sequence of positive real numbers satisfying $M_0 = 1$. The associated Denjoy–Carleman class $C_M(\Omega)$ consists of all functions $f \in C^\infty(\Omega)$ such that, for every compact subset $X \subset \Omega$, there exist positive constants C and h satisfying

$$\sup_{x\in X} |\partial^\alpha f(x)| \le C h^{|\alpha|} M_{|\alpha|}\,, \qquad \alpha \in \mathbb{N}_0^n\,. \tag{4.2}$$

The class $C_M(\Omega)$ is said to be *quasi–analytic* if every function $f \in C_M(\Omega)$ whose derivatives of all orders vanish at one point is necessarily identically zero in Ω. The celebrated Denjoy–Carleman theorem [24] asserts that $C_M(\Omega)$ is quasi–analytic if and only if

$$\sum_{k=1}^{\infty} \frac{M_{k-1}}{M_k} = +\infty\,. \tag{4.3}$$

An important example is the real–analytic class, which is obtained by choosing $M_k = k!$.

We assume from now on that $M = (M_k)_{k\geq 0}$ is a quasi-analytic weight sequence associated with a regular sequence $m_k = \frac{M_k}{k!}$ in the sense of Bhowmik and Pradhan [1]. A compact Riemannian manifold K is called *quasi–analytic* if it admits an atlas whose transition maps belong locally to the Denjoy–Carleman class C_M, and whose metric coefficients belong locally to the same class. Following [1], we introduce the global Denjoy–Carleman class associated with the Laplace–Beltrami operator. This definition is intrinsic and therefore does not depend on the choice of local coordinates:

$$C_M(K) = \left\{ f \in C^\infty(K) : \|(-\Delta_K)^k f\|_{L^2(K)} \leq M_{2k}\, \|f\|_{L^2(K)}\,, \quad k \geq 0 \right\}. \tag{4.4}$$

When K is a real–analytic manifold and $M_k = k!$, this definition coincides with the usual class of real–analytic functions (see [1, Remark 1.10]). The following proposition, which is Corollary 1.13 of Bhowmik and Pradhan [1], is the only property of the class $C_M(K)$ that we shall use.

Proposition 4.2. *Let K be a compact connected quasi–analytic Riemannian manifold. If $f \in C_M(K)$ vanishes on a measurable subset of K of positive measure, then $f \equiv 0$.*

Proof of Theorem 1.6. We adapt the proof of [1, Theorem 1.14] to our setting. Set $A = \Lambda_g - \Lambda_{\widetilde{g}}$. Let $\psi \in C_c^\infty(\mathcal{O})$ and put $F = A\psi$. Since the local Dirichlet–to–Neumann maps coincide, we have $F = 0$ on $\mathcal{O}$. By the spectral decay assumption of Theorem 1.6,

$$\sum_{k\geq 0} \|P_k F\|^2_{L^2(K)} \exp\Big(\sqrt{\lambda_k}\,\theta_\delta(\sqrt{\lambda_k})\Big) \leq C\|F\|^2_{L^2(K)}\,. \tag{4.5}$$

Define

$$W(t) = \exp\left(\frac{t}{2\log^\delta(e+t)}\right)\,, \qquad t \geq 0\,. \tag{4.6}$$

Then (4.5) becomes

$$\sum_{k\geq 0} \|P_k F\|^2_{L^2(K)} W(\sqrt{\lambda_k})^2 \leq C\|F\|^2_{L^2(K)}\,. \tag{4.7}$$

As in [1, Theorem 1.14], one can show that

$$\sup_{t\geq 0} \frac{t^k}{W(t)} \leq C^k k!\,(\log(e+k))^{\delta k}\,,$$

where $C > 0$ is independent of k. Let

$$M_k = C^k k!\,(\log(e+k))^{\delta k}\,, \qquad k \geq 0\,.$$

Then, by Plancherel's identity and (4.7),

$$\begin{aligned}\|(-\Delta_K)^p F\|^2_{L^2(K)} &= \sum_{k\geq 0} \lambda_k^{2p} \|P_k F\|^2_{L^2(K)} \\ &= \sum_{k\geq 0} \frac{\lambda_k^{2p}}{W(\sqrt{\lambda_k})^2} W(\sqrt{\lambda_k})^2 \|P_k F\|^2_{L^2(K)} \\ &\leq \left(\sup_{t\geq 0} \frac{t^{4p}}{W(t)^2}\right) \sum_{k\geq 0} W(\sqrt{\lambda_k})^2 \|P_k F\|^2_{L^2(K)} \\ &\leq C\, M_{2p}^2 \,\|F\|^2_{L^2(K)}\,.\end{aligned} \tag{4.8}$$

Consequently,

$$\|(-\Delta_K)^p F\|_{L^2(K)} \leq C\, M_{2p}\, \|F\|_{L^2(K)}\,.$$

Hence

$$F \in C_{M'}(K)\,,$$

where $M' = (C\, M_{2p})_{p\geq 0}$.
By [1, Proof of Theorem 1.14], the sequence $M' = (CM_{2p})_{p\geq 0}$ is again a regular quasi-analytic weight sequence. Therefore, Corollary 1.13 of Bhowmik and Pradhan applies to the class $C_{M'}(K)$. Since $F \in C_{M'}(K)$ and F vanishes on the nonempty open set $\mathcal{O}$, and hence on a measurable subset of positive measure, we conclude that $F \equiv 0$ on K. The remainder of the proof is identical to that of Theorem 1.2. Using the self-adjointness of A, one first obtains $Au = 0$ on $\mathcal{O}$ for every $u \in C^\infty(K)$. Applying the preceding argument to $F = Au$ then yields $Au = 0$ on K . Hence $A = 0$, that is, $\Lambda_g = \Lambda_{\widetilde{g}}$. □

5. Geometric applications to warped product metrics

In this section, we investigate the warped product model introduced in the introduction. The purpose is to translate geometric boundary closeness conditions on the conformal factors into the spectral hypotheses appearing in the abstract propagation theorems of Section 4. This is achieved by combining the diagonalization of the Dirichlet–to–Neumann map with Weyl–Titchmarsh theory. Besides yielding the geometric applications announced in the introduction, this approach illustrates the bridge between geometry and quasi-analytic propagation developed in the present paper.

5.1. **Geometric framework.** Throughout this section, we consider a class of d-dimensional manifolds with boundary of the form

$$M = (0,1] \times K\,, \tag{5.1}$$

where $d \geq 3$ and where K is a compact, connected and orientable $(d-1)$-dimensional Riemannian manifold. The manifold M has a single boundary component,

$$\partial M = \{1\} \times K\,,$$

which will be identified with K .

We equip M with a warped product metric of the form

$$g = c(r)^4 \big(dr^2 + r^2 g_K\big)\,, \tag{5.2}$$

where g_K denotes the Riemannian metric on K , and where

$$c : [0,1] \longrightarrow (0,\infty)$$

is a positive function of class C^m, with $m \geq 2$.

As explained in the introduction, the geometric setting described above allows for both regular and singular Riemannian structures. For the analysis of the Dirichlet–to–Neumann map, it is convenient to introduce the logarithmic variable

$$x = -\log r, \qquad x \in [0, +\infty)\,.$$

In these coordinates, the manifold acquires the cylindrical representation

$$g = f(x)^4 \big(dx^2 + g_K\big)\,, \tag{5.3}$$

where

$$f(x) = c(e^{-x})\, e^{-x/2}\,.$$

Since c admits a Taylor expansion of order m at the origin, one obtains the asymptotic behavior

$$f(x) = e^{-x/2} + \sum_{k=1}^{m} c_k e^{-(k+\frac{1}{2})x} + o\Big(e^{-(m+\frac{1}{2})x}\Big)\,, \qquad x \to +\infty\,, \tag{5.4}$$

for suitable real coefficients c_k .

The simplest example in this class is obtained by taking $K = S^{d-1}$ equipped with its round metric and choosing $c(r) \equiv 1$. In that case, g coincides with the Euclidean metric on the unit ball of $\mathbb{R}^d$, while $f(x) = e^{-x/2}$. More generally, when $K = S^{d-1}$, the metrics (5.2) may be viewed as conformal radial deformations of the Euclidean ball, the deformation being entirely encoded by the conformal factor c.

5.2. **The Dirichlet–to–Neumann map and the Steklov spectrum.** We now introduce the Dirichlet–to–Neumann operator associated with the warped product manifolds introduced above.

Given $\psi \in H^{1/2}(K)$, we consider the boundary value problem

$$\begin{cases} -\Delta_g u = 0 & \text{on } M\,, \\ \quad\ u = \psi & \text{on } \partial M\,. \end{cases} \tag{5.5}$$

The Dirichlet–to–Neumann map is first defined in the weak sense by the Green identity

$$\langle \Lambda_g \psi, \varphi \rangle = \int_M \langle du, dv \rangle_g \, dV_g\,,$$

for every $\varphi \in H^{1/2}(K)$ and every function $v \in H^1(M)$ satisfying

$$v|_{\partial M} = \varphi\,,$$

where u is the unique solution of (5.5) with boundary value ψ. When the metric, the boundary data and the solution are sufficiently regular, this weak definition agrees with the classical expression

$$\Lambda_g \psi = \partial_\nu u\big|_{\partial M}\,, \tag{5.6}$$

where ν denotes the outward unit normal vector field along ∂M. Indeed, in the regular case, the endpoint $r = 0$ corresponds to a single point p in the smooth compact completion $\check{M}$, so that $M = \check{M} \setminus \{p\}$. Since $\dim \check{M} \geq 3$, the point $\{p\}$ has zero H^1-capacity. Hence

$$H^1(\check{M}) = H^1_0(\check{M} \setminus \{p\})\,, \tag{5.7}$$

where $H^1_0(\check{M} \setminus \{p\})$ denotes the closure of $C^\infty_c(\check{M} \setminus \{p\})$ in the H^1-norm. Therefore, the integration by parts formula on M follows by density from the same formula for compactly supported smooth functions on $\check{M} \setminus \{p\}$. In particular, the removed point p produces no additional boundary contribution.

The operator Λ_g is self-adjoint on $L^2(\partial M, dS_g)$ and has compact resolvent. Its spectrum, referred to as the *Steklov spectrum* of (M, g), therefore consists of a discrete sequence of real eigenvalues, counted with multiplicities,

$$0 = \sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \cdots\,, \qquad \sigma_j \to +\infty\,. \tag{5.8}$$

We denote by

$$0 = \tau_0 < \tau_1 < \tau_2 < \cdots \tag{5.9}$$

the distinct Steklov eigenvalues, that is, the spectrum of Λ_g without multiplicities. We refer to [6] for a general introduction to Steklov spectral theory.

5.3. **Separation of variables.** We next rewrite the harmonic equation (5.5) in the cylindrical variable

$$x = -\log r \in [0, +\infty)\,.$$

Introducing

$$v = f^{d-2} u\,,$$

a direct computation shows that the equation $-\Delta_g u = 0$ becomes

$$\Big(-\partial_x^2 - \Delta_{g_K} + q_f(x)\Big) v = -\frac{(d-2)^2}{4}\, v\,, \tag{5.10}$$

where the effective potential q_f is defined by

$$q_f(x) = \frac{(f^{d-2})''(x)}{f^{d-2}(x)} - \frac{(d-2)^2}{4}\,. \tag{5.11}$$

In terms of the original radial variable $r = e^{-x}$, the potential can be written as

$$q_f(x) = (d-2)\left((d-3) r^2 \left(\frac{c'(r)}{c(r)}\right)^2 + r^2 \frac{c''(r)}{c(r)} + (d-1) r \frac{c'(r)}{c(r)}\right)\,. \tag{5.12}$$

Moreover, the asymptotic expansion (5.4) implies

$$q_f(x) = O(e^{-x}), \qquad x \to +\infty\,. \tag{5.13}$$

The product structure of (M, g) allows us to separate variables. Let

$$-\Delta_{g_K} Y_j = \mu_j Y_j\,, \qquad j \geq 0, \tag{5.14}$$

where $\{Y_j\}_{j\geq 0}$ is an orthonormal basis of eigenfunctions of $L^2(K, dV_{g_K})$. Recall that the distinct eigenvalues of $-\Delta_{g_K}$ are denoted by

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\,, \tag{5.15}$$

while

$$0 = \mu_0 \leq \mu_1 \leq \mu_2 \leq \cdots \tag{5.16}$$

denotes the spectrum counted with multiplicities. Expanding

$$v(x, \omega) = \sum_{j \geq 0} v_j(x) Y_j(\omega), \tag{5.17}$$

equation (5.10) reduces to the family of one-dimensional equations

$$-v_j'' + q_f(x) v_j = -\kappa_j^2 v_j\,, \qquad x \in (0, +\infty)\,, \tag{5.18}$$

where

$$\kappa_j = \sqrt{\mu_j + \frac{(d-2)^2}{4}}\,. \tag{5.19}$$

This naturally leads to the Schrödinger operator

$$H = -\frac{d^2}{dx^2} + q_f(x) \tag{5.20}$$

acting on the half-line, and to the associated spectral equation

$$-v'' + q_f(x) v = z v\,, \qquad z \in \mathbb{C}\,. \tag{5.21}$$

Let $\{C_0(\cdot, z), S_0(\cdot, z)\}$ be the fundamental system of solutions of (5.21) normalized by

$$C_0(0, z) = 1\,, \qquad C_0'(0, z) = 0\,, \qquad S_0(0, z) = 0\,, \qquad S_0'(0, z) = 1\,. \tag{5.22}$$

Their Wronskian is therefore given by

$$W(C_0, S_0) = 1\,. \tag{5.23}$$

Since $q_f \in L^1(0,+\infty)$ by (5.13), the operator H is in the limit-point case at infinity. Consequently, for every $z \in \mathbb{C}$, there exists, up to a multiplicative constant, a unique solution $S_\infty(\cdot, z)$ of (5.21) belonging to L^2 in a neighborhood of $+\infty$, (see [23], Theorem XI.57 where our spectral parameter $z = k^2$). Writing

$$S_\infty(x,z) = A(z)\big(C_0(x,z) + M(z)S_0(x,z)\big)\,, \tag{5.24}$$

defines the Weyl–Titchmarsh function $M(z)$. Using (5.23), we obtain

$$M(z) = \frac{S'_\infty(0,z)}{S_\infty(0,z)}\,. \tag{5.25}$$

Finally, the Dirichlet–to–Neumann map is diagonal with respect to the Hilbert basis of harmonics $\{Y_j\}_{j\geq 0}$. Thus, if $\psi = \sum_{j\geq 0} \psi_j Y_j$, then

$$\Lambda_g \psi = \sum_{j\geq 0} (\Lambda_g^{(j)} \psi_j)\, Y_j\,, \tag{5.26}$$

where $\Lambda_g^{(j)}$ is the scalar by which Λ_g acts on $\mathrm{span}\{Y_j\}$. The computation is identical to the one carried out in [4, Section 2]. If $\mu_j = \lambda_k$, with μ_j counted with multiplicities and λ_k denoting a distinct eigenvalue, then

$$\Lambda_g^{(j)} = \frac{(d-2)f'(0)}{f^3(0)} - \frac{M(-\rho_k^2)}{f^2(0)}\,, \tag{5.27}$$

where

$$\rho_k = \sqrt{\lambda_k + \frac{(d-2)^2}{4}}, \qquad k \geq 0\,. \tag{5.28}$$

Since the coefficient in (5.27) depends only on the corresponding distinct eigenvalue λ_k, the operator Λ_g acts by scalar multiplication on each eigenspace

$$E_{\lambda_k} = \ker(-\Delta_{g_K} - \lambda_k)\,. \tag{5.29}$$

Hence

$$\Lambda_g = \sum_{k\geq 0} \left(\frac{(d-2)f'(0)}{f^3(0)} - \frac{M(-\rho_k^2)}{f^2(0)} \right) P_k\,. \tag{5.30}$$

Since the Weyl–Titchmarsh function M is strictly increasing, (see Lemma 2.3 in [4]), the distinct Steklov eigenvalues are given by

$$\tau_k = \frac{(d-2)f'(0)}{f^3(0)} - \frac{M(-\rho_k^2)}{f^2(0)}\,, \qquad k \geq 0\,. \tag{5.31}$$

5.4. **A weighted Laplace estimate.** The following elementary estimate is the key analytic ingredient in the proof of the Weyl–Titchmarsh estimate established in the next subsection. It explains the particular choice of the weight Φ appearing in Theorem 1.7.

Let $\theta : [T, \infty) \longrightarrow (0, \infty)$ be a decreasing function such that

$$\lim_{t\to\infty} \theta(t) = 0\,.$$

Define

$$\Phi(x) = \inf_{t\geq T} \big(2xt - t\theta(t)\big)\,, \qquad x > 0\,, \tag{5.32}$$

and

$$I(\rho) = \int_0^{\varepsilon} e^{-2\rho x} e^{\Phi(x)}\, dx\,, \tag{5.33}$$

where, without loss of generality, we assume that $0 < \varepsilon \le 1$. We have the following result:

Lemma 5.1. *For every $x > 0$, the quantity $\Phi(x)$ is finite. Moreover,*

$$I(\rho) \le e^{-\rho\,\theta(\rho)}\,, \qquad \rho \ge T\,.$$

Proof. Fix $x > 0$. Since $\theta(t) \to 0$ as $t \to +\infty$, there exists $T_x \ge T$ such that

$$\theta(t) \le x\,, \qquad t \ge T_x\,.$$

Hence

$$2xt - t\theta(t) = t(2x - \theta(t)) \ge xt, \qquad t \ge T_x\,.$$

It follows that the infimum defining $\Phi(x)$ is finite. Next, by definition of Φ,

$$\Phi(x) \le 2\rho x - \rho\theta(\rho), \qquad \rho \ge T\,.$$

Therefore

$$e^{-2\rho x} e^{\Phi(x)} \le e^{-\rho\,\theta(\rho)}\,.$$

Integrating over $(0, \varepsilon)$ yields the desired estimate. □

5.5. **A Weyl–Titchmarsh estimate.** We now recall several estimates proved in [4] that will be used in the proof of our quasi-analytic propagation result. The key point is that the difference of the Weyl–Titchmarsh functions can be represented as a Laplace transform involving the difference of the effective potentials $q_{\tilde f}$ and q_f.

According to Lemma 4.4 of [4] (with $p = 1$), there exists a kernel

$$K_1 : \{(x, t)\,;\, 0 \le t \le x\} \longrightarrow \mathbb{R}$$

such that, for every $0 < \alpha < 1$, there exists a constant $C_\alpha > 0$ satisfying

$$\left|\partial_x^k \partial_t^\ell K_1(x, t)\right| \le C_\alpha\, e^{-\alpha x}\,, \qquad 0 \le t \le x\,, \qquad k, \ell \le m - 1\,. \tag{5.34}$$

We then define the Volterra operator B by

$$h \mapsto Bh(x) = h(x) + \int_0^x K_1(x, t) h(t)\, dt\,. \tag{5.35}$$

Lemma 4.5 of [4] asserts that, for all sufficiently large ρ,

$$S_\infty(0, -\rho^2)\widetilde{S}_\infty(0, -\rho^2)\big(M(-\rho^2) - \widetilde{M}(-\rho^2)\big) = \int_0^\infty e^{-2\rho x} B[q_{\tilde f} - q_f](x)\, dx\,. \tag{5.36}$$

Moreover, Corollary 4.3 of [4] yields

$$S_\infty(0, -\rho^2) = 1 + O(\rho^{-1})\,, \qquad \widetilde{S}_\infty(0, -\rho^2) = 1 + O(\rho^{-1})\,, \qquad \rho \to +\infty\,. \tag{5.37}$$

Combining (5.36) and (5.37), we obtain the existence of positive constants C and ρ_0 such that

$$\left|M(-\rho^2) - \widetilde{M}(-\rho^2)\right| \le C \left| \int_0^\infty e^{-2\rho x} B[q_{\tilde f} - q_f](x)\, dx \right|\,, \text{ for } \rho \ge \rho_0\,. \tag{5.38}$$

We now estimate the right-hand side of (5.38). We first deal with the contribution of the interval (ϵ, ∞). Recall that

$$q_f(x) = O(e^{-x})\,, \qquad x \to +\infty. \tag{5.39}$$

Since q_f and $q_{\tilde f}$ belong to $C^{m-2}([0, \infty))$ and satisfy (5.39), we have

$$q_{\tilde f} - q_f \in H_\delta\,, \qquad 0 < \delta < 2\,, \qquad H_\delta = \left\{ q\,;\, \int_0^\infty |q(x)|^2 e^{\delta x}\, dx < \infty \right\}. \tag{5.40}$$

Moreover, by Proposition 4.6 of [4], for every $0 < \delta < 1$, the Volterra operator B is an isomorphism $B : H_\delta \longrightarrow H_\delta$. Consequently, for $0 < \delta < 1$, $B[q_{\widetilde{f}} - q_f] \in H_\delta$. Therefore, by the Cauchy–Schwarz inequality,

$$\left| \int_\varepsilon^\infty e^{-2\rho x} B[q_{\widetilde{f}} - q_f](x)\, dx \right| \leq \left(\int_\varepsilon^\infty e^{-(4\rho+\delta)x}\, dx \right)^{1/2} \left\| B[q_{\widetilde{f}} - q_f] \right\|_{H_\delta} . \tag{5.41}$$

Thus,

$$\left| \int_\varepsilon^\infty e^{-2\rho x} B[q_{\widetilde{f}} - q_f](x)\, dx \right| \leq C e^{-2\rho\varepsilon}, \qquad \rho > 0 . \tag{5.42}$$

We now estimate the contribution of the interval $(0, \varepsilon)$ in the right-hand side of (5.38). To this end, we assume that

$$\left| \partial_x^j (f - \widetilde{f})(x) \right| \leq C e^{\Phi(x)} , \qquad 0 < x < \varepsilon , \qquad j = 0, 1, 2. \tag{5.43}$$

Consequently, since q_f depends smoothly on f, f', and f'', and since f and $\widetilde{f}$ are bounded away from zero, there exists $C > 0$ such that

$$|q_{\widetilde{f}}(x) - q_f(x)| \leq C e^{\Phi(x)} , \qquad 0 < x < \epsilon . \tag{5.44}$$

Moreover, by (5.34), the kernel K_1 is bounded on $\{0 \leq t \leq x \leq \varepsilon\}$. Hence, using (5.35),

$$|B[q_{\widetilde{f}} - q_f](x)| \leq |q_{\widetilde{f}}(x) - q_f(x)| + C \int_0^x |q_{\widetilde{f}}(t) - q_f(t)|\, dt .$$

Since Φ is increasing, (5.44) yields

$$|q_{\widetilde{f}}(t) - q_f(t)| \leq C e^{\Phi(x)} , \qquad 0 < t < x ,$$

and therefore

$$|B[q_{\widetilde{f}} - q_f](x)| \leq C e^{\Phi(x)} , \qquad 0 < x < \varepsilon . \tag{5.45}$$

It follows that

$$\left| \int_0^\varepsilon e^{-2\rho x} B[q_{\widetilde{f}} - q_f](x)\, dx \right| \leq C \int_0^\varepsilon e^{-2\rho x} e^{\Phi(x)}\, dx .$$

Using Lemma 5.1, we obtain

$$\left| \int_0^\varepsilon e^{-2\rho x} B[q_{\widetilde{f}} - q_f](x)\, dx \right| \leq C e^{-\rho\,\theta(\rho)} , \qquad \rho \geq T . \tag{5.46}$$

Combining (5.38), (5.42) and (5.46), we obtain

$$\left| M(-\rho^2) - \widetilde{M}(-\rho^2) \right| \leq C \big(e^{-\rho\,\theta(\rho)} + e^{-2\rho\varepsilon} \big) .$$

Since $\theta(\rho) \to 0$ as $\rho \to +\infty$, it follows that $e^{-2\rho\varepsilon} = O(e^{-\rho\,\theta(\rho)})$. Hence there exist positive constants C and ρ_0 such that

$$\left| M(-\rho^2) - \widetilde{M}(-\rho^2) \right| \leq C e^{-\rho\,\theta(\rho)} , \text{ for } \rho \geq \rho_0 . \tag{5.47}$$

5.6. **Proof of Theorem 1.7.** We now prove the geometric application announced in Theorem 1.7. By Theorem 1.4, it is enough to establish the required Ingham-type spectral decay estimate for the difference of the Dirichlet–to–Neumann maps. We first prove an elementary property of the weight function Φ.

Lemma 5.2. *Let*

$$\Phi(x) = \inf_{t \geq T} (2xt - t\theta(t)) ,$$

where $\theta : [T, +\infty) \to (0, +\infty)$ is a positive decreasing function satisfying

$$\int_T^\infty \frac{\theta(t)}{t}\, dt = +\infty .$$

Then

$$\Phi(x) \to -\infty \ \text{as } x \to 0^+ .$$

Proof. Let $A > 0$. We first observe that

$$\sup_{t\geq T} t\theta(t) = +\infty .$$

Indeed, otherwise there would exist $C > 0$ such that

$$t\theta(t) \leq C, \qquad t \geq T,$$

and therefore

$$\int_T^\infty \frac{\theta(t)}{t}\, dt \leq C \int_T^\infty \frac{dt}{t^2} < +\infty ,$$

which contradicts the assumption. Hence we may choose $t_A \geq T$ such that

$$t_A\theta(t_A) \geq 2A.$$

For $x > 0$ small enough so that

$$2x \leq \frac{\theta(t_A)}{2} ,$$

we get

$$\Phi(x) \leq 2xt_A - t_A\theta(t_A) \leq -\frac{1}{2}t_A\theta(t_A) \leq -A .$$

Since $A > 0$ is arbitrary, this proves that $\Phi(x) \to -\infty \qquad \text{as } x \to 0^+ .$ □

Now, assume that

(5.48) $$\left|\partial_x^j(f - \widetilde{f})(x)\right| \leq Ce^{\Phi(x)} , \qquad 0 < x < \varepsilon, \qquad j = 0, 1, 2.$$

By the previous lemma,

$$e^{\Phi(x)} \longrightarrow 0 \qquad \text{as } x \to 0^+ .$$

Hence (5.48) implies

(5.49) $$f^{(j)}(0) = \widetilde{f}^{(j)}(0) , \qquad j = 0, 1, 2.$$

Let

(5.50) $$F = (\Lambda_g - \Lambda_{\widetilde{g}})\psi , \qquad \psi \in C_c^\infty(\mathcal{O}) .$$

If the local Dirichlet–to–Neumann maps coincide on $\mathcal{O}$, we have $F = 0$ on $\mathcal{O}$. Combining the diagonalization formula for the Dirichlet–to–Neumann map with the boundary identities (5.49), we obtain

(5.51) $$P_kF = -\frac{1}{f^2(0)}\Big(M(-\rho_k^2) - \widetilde{M}(-\rho_k^2)\Big)P_k\psi .$$

where

$$\rho_k = \sqrt{\lambda_k + \frac{(d-2)^2}{4}} .$$

Hence,

(5.52) $$\|P_kF\|_{L^2(K)} = \frac{1}{f^2(0)}\left|M(-\rho_k^2) - \widetilde{M}(-\rho_k^2)\right| \|P_k\psi\|_{L^2(K)} .$$

Using the Weyl–Titchmarsh estimate (5.47), we deduce

(5.53) $$\|P_kF\|_{L^2(K)} \leq Ce^{-\rho_k\theta(\rho_k)}\|P_k\psi\|_{L^2(K)} \leq Ce^{-\rho_k\theta(\rho_k)}\|\psi\|_{L^2(K)} .$$

As $\rho_k \sim \sqrt{\lambda_k}$, possibly replacing θ by a smaller function with the same properties, we obtain

(5.54) $$\|P_kF\|_{L^2(K)} \leq Ce^{-\sqrt{\lambda_k}\theta(\sqrt{\lambda_k})} .$$

Therefore F satisfies the assumptions of Proposition 4.1. Since $F = 0$ on $\mathcal{O}$, we conclude that $F \equiv 0$ on K .

The remainder of the argument is identical to the final self-adjointness argument in the proof of Theorem 1.2. Since $f(0) = \widetilde{f}(0)$, the operator $A = \Lambda_g - \Lambda_{\widetilde{g}}$ is self-adjoint on $L^2(K, dS_K)$. Arguing exactly as in the proof of Theorem 1.2, we conclude that

$$Au = 0\,, \qquad u \in C^\infty(K)\,.$$

Therefore, $\Lambda_g = \Lambda_{\widetilde{g}}$, which completes the proof.

5.7. **Proof of Theorem 1.9.** The proof is identical to that of Theorem 1.7, replacing Theorem 1.4 by Theorem 1.6.

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