An inverse scattering problem for short-range systems in a time-periodic electric field.

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Abstract

We consider the time-dependent Hamiltonian $H(t) = \frac{1}{2}p^2 - E(t) \cdot x + V(t,x)$ on $L^2(I\!R^n)$, where the external electric field $E(t)$ and the short-range electric potential $V(t,x)$ are time-periodic with the same period. It is well-known that the short-range notion depends on the mean value E_0 of the external electric field. When $E_0 = 0$, we show that the high energy limit of the scattering operators determines uniquely $V(t,x)$. When $E_0 \neq 0$, the same result holds in dimension $n \geq 3$ for generic short-range potentials. In dimension $n = 2$, one has to assume a stronger decay on the electric potential.

1 Introduction.

In this note, we study an inverse scattering problem for a two-body short-range system in the presence of an external time-periodic electric field $E(t)$ and a time-periodic short-range potential $V(t,x)$ (with the same period T). For the sake of simplicity, we assume that the period $T = 1$.

The corresponding Hamiltonian is given on $L^2(I\!R^n)$ by :

$$H(t) = \frac{1}{2}p^2 - E(t) \cdot x + V(t,x), \tag{1.1}$$

where $p = -i\partial_x$. When $E(t) = 0$, the Hamiltonian $H(t)$ describes the dynamics of the hydrogen atom placed in a linearly polarized monochromatic electric field, or a light particle in the restricted three-body problem in which two other heavy particles are set on prescribed periodic orbits. When $E(t) = \cos(2\pi t)\ E$ with $E \in I\!R^n$, the Hamiltonian describes the well-known AC-Stark effect in the E-direction [7].

In this paper, we assume that the external electric field $E(t)$ satisfies :

$$(A_1) \qquad t \to E(t) \in L^1_{loc}(I\!R; I\!R^n) \quad , \quad E(t+1) = E(t)\ a.e\ .$$

Moreover, we assume that the potential $V \in C^\infty(I\!R \times I\!R^n)$, is time-periodic with period 1, and satisfies the following estimations :

$$(A_2) \qquad \forall \alpha \in I\!N^n,\ \forall k \in I\!N,\ |\ \partial_t^k \partial_x^\alpha V(t,x)\ | \leq\ C_{k,\alpha}\ < x >^{-\delta-|\alpha|},\ \text{with } \delta > 0,$$

where $< x >= (1+x^2)^{\frac{1}{2}}$. Actually, we can accommodate more singular potentials (see [10], [11], [12] for example) and we need (A_2) for only $k,\ \alpha$ with finite order . It is well-known that under assumptions $(A_1) - (A_2)$, $H(t)$ is essentially self-adjoint on $\mathcal{S}(I\!R^n)$ the Schwartz space, [16]. We denote $H(t)$ the self-adjoint realization with domain $D(H(t))$.

Now, let us recall some well-known results in scattering theory for time-periodic electric fields. We denote $H_0(t)$ the free Hamiltonian :

$$(1.2) \qquad H_0(t) = \frac{1}{2}p^2 - E(t)\cdot x\ ,$$

and let $U_0(t,s)$, (resp. $U(t,s)$) be the unitary propagator associated with $H_0(t)$, (resp. $H(t)$) (see section 2 for details).

For short-range potentials, the wave operators are defined for $s \in I\!R$ and $\Phi \in L^2(I\!R^n)$ by :

$$(1.3) \qquad W^\pm(s)\ \Phi = \lim_{t\to\pm\infty}\ U(s,t)\ U_0(t,s)\ \Phi.$$

We emphasize that the short-range condition depends on the value of the mean of the external electric field :

$$(1.4) \qquad E_0\ =\ \int_0^1\ E(t)\ dt\ .$$

• **The case $E_0 = 0$.**

By virtue of the Avron-Herbst formula (see section 2), this case falls under the category of two-body systems with time-periodic potentials and this case was studied by Kitada and Yajima ([10], [11]), Yokoyama [22].

We recall that for a unitary or self-adjoint operator U, $\mathcal{H}_c(U)$, $\mathcal{H}_{ac}(U)$, $\mathcal{H}_{sc}(U)$ and $\mathcal{H}_p(U)$ are, respectively, continuous, absolutely continuous, singular continuous and point spectral subspace of U.

We have the following result ([10], [11], [21]) :

Theorem 1
Assume that hypotheses (A_1), (A_2) *are satisfied with* $\delta > 1$ *and with* $E_0 = 0$.
Then : (i) the wave operators $W^{\pm}(s)$ *exist for all* $s \in I\!R$.

(ii) $W^{\pm}(s+1) = W^{\pm}(s)$ *and* $U(s+1,s)\ W^{\pm}(s) = W^{\pm}(s)\ U_0(s+1,s)$.

(iii) Ran $(W^{\pm}(s)) = \mathcal{H}_{ac}(U(s+1,s))$ *and* $\mathcal{H}_{sc}(U(s+1,s)) = \emptyset$.

(iv) the purely point spectrum $\sigma_p(U(s+1,s))$ *is discrete outside* $\{1\}$.

Comments.

1 - The unitary operators $U(s+1,s)$ are called the Floquet operators and they are mutually equivalent. The Floquet operators play a central role in the analysis of time periodic systems. The eigenvalues of these operators are called Floquet multipliers. In [5], Galtbayar, Jensen and Yajima improve assertion (iv) : for $n = 3$ and $\delta > 2$, $\mathcal{H}_p(U(s+1,s))$ is finite dimensional.

2 - For general $\delta > 0$, $W^{\pm}(s)$ do not exist and we have to define other wave operators. In ([10], [11]), Kitada and Yajima have constructed modified wave operators $W^{\pm}_{HJ}$ by solving an Hamilton-Jacobi equation.

• **The case** $E_0 \neq 0$**.**

This case was studied by Moller [12] : using the Avron-Herbst formula, it suffices to examine Hamiltonians with a constant external electric field, (Stark Hamiltonians) : the spectral and the scattering theory for Stark Hamiltonians are well established [2]. In particular, a Stark Hamiltonian with a potential V satisfying (A_2) has no eigenvalues [2]. The following theorem, due to Moller [12], is a time-periodic version of these results.

Theorem 2
Assume that hypotheses (A_1), (A_2) *are satisfied with* $\delta > \frac{1}{2}$ *and with* $E_0 \neq 0$.
Then : (i) the Floquet operators $U(s+1,s)$ *have purely absolutely continuous spectrum.*

(ii) the wave operators $W^{\pm}(s)$ *exist for all* $s \in I\!R$ *and are unitary.*

(iii) $W^{\pm}(s+1) = W^{\pm}(s)$ *and* $U(s+1,s)\ W^{\pm}(s) = W^{\pm}(s)\ U_0(s+1,s)$.

The inverse scattering problem.

For $s \in I\!R$, we define the scattering operators $S(s) = W^{+*}(s)\ W^{-}(s)$. It is clear that the scattering operators $S(s)$ are periodic with period 1.

The inverse scattering problem consists to reconstruct the perturbation $V(s,x)$ from the scattering operators $S(s)$, $s \in [0,1]$.

In this paper, we prove the following result :

Theorem 3
Assume that $E(t)$ satisfies (A_1) and let $V_j,\ j = 1,2$ be potentials satisfying (A_2). We assume that $\delta > 1$ (if $E_0 = 0$), $\delta > \frac{1}{2}$ (if $E_0 \neq 0$ and $n \geq 3$), $\delta > \frac{3}{4}$ (if $E_0 \neq 0$ and $n = 2$). Let $S_j(s)$ be the corresponding scattering operators.
Then :

$$\forall s \in [0,1],\ S_1(s) = S_2(s) \implies V_1 = V_2 \ .$$

We prove Theorem 3 by studying the high energy limit of $[S(s), p]$, (Enss-Weder's approach [4]). We need $n \geq 3$ in the case $E_0 \neq 0$ in order to use the inversion of the Radon transform [6] on the orthogonal hyperplane to E_0. See also [15] for a similar problem with a Stark Hamiltonian.

We can also remark that if we know the free propagator $U_0(t,s)$, $s,t \in I\!R$, then by virtue of the following relation :

$$S(t) = U_0(t,s)\ S(s)\ U_0(s,t)\ , \tag{1.5}$$

the potential $V(t,x)$ is uniquely reconstructed from the scattering operator $S(s)$ at only one initial time.

In [21], Yajima proves uniqueness for the case of time-periodic potential with the condition $\delta > \frac{n}{2} + 1$ and with $E(t) = 0$ by studying the scattering matrices in a high energy regime.

In [20], for a time-periodic potential that decays exponentially at infinity, Weder proves uniqueness at a fixed quasi-energy.

Note also that inverse scattering for long-range time-dependent potentials without external electric fields was studied by Weder [18] with the Enss-Weder time-dependent method, and by Ito for time-dependent electromagnetic potentials for Dirac equations [8].

2 Proof of Theorem 3.

2.1 The Avron-Herbst formula.

First, let us recall some basic definitions for time-dependent Hamiltonians. Let $\{H(t)\}_{t \in I\!R}$ be a family of selfadjoint operators on $L^2(I\!R^n)$ such that $\mathcal{S}(I\!R^n) \subset D(H(t))$ for all $t \in I\!R$.

Definition.
We call *propagator* a family of unitary operators on $L^2(I\!R^n)$, $U(t,s),\ t,s \in I\!R$ such that :

1 - $U(t,s)$ is a strongly continuous fonction of $(t,s) \in I\!R^2$.
2 - $U(t,s)\ U(s,r) = U(t,r)$ for all $t,s,r \in I\!R$.
3 - $U(t,s)\,(\mathcal{S}(I\!R^n)) \subset \mathcal{S}(I\!R^n)$ for all $t,s \in I\!R$.
4 - If $\Phi \in \mathcal{S}(I\!R^n)$, $U(t,s)\Phi$ is continuously differentiable in t and s and satisfies :

$$i\ \frac{\partial}{\partial t} U(t,s)\ \Phi = H(t)\ U(t,s)\ \Phi\ ,\quad i\ \frac{\partial}{\partial s} U(t,s)\ \Phi = -U(t,s)\ H(s)\ \Phi\ .$$

To prove the existence and the uniqueness of the propagator for our Hamiltonians $H(t)$, we use a generalization of the Avron-Herbst formula close to the one given in [3].
In [12], the author gives, for $E_0 \neq 0$, a different formula which has the advantage to be time-periodic. To study our inverse scattering problem, we use here a different one, which is defined for all E_0. We emphasize that with our choice, $c(t)$ (see below for the definition of $c(t)$) is also periodic with period 1; in particular $c(t) = O(1)$.

The basic idea is to generalize the well-known Avron-Herbst formula for a Stark Hamiltonian with a constant electric field E_0, [2]; if we consider the Hamiltonian B_0 on $L^2(I\!R^n)$,

$$B_0 = \frac{1}{2}p^2 - E_0 \cdot x \ , \tag{2.1}$$

we have the following formula :

$$e^{-itB_0} = e^{-i\frac{E_0^2}{6}t^3} \ e^{itE_0\cdot x} \ e^{-i\frac{t^2}{2}E_0\cdot p} \ e^{-it\frac{p^2}{2}} \ . \tag{2.2}$$

In the next definition, we give a similar formula for time-dependent electric fields.

Definition.
We consider the family of unitary operators $T(t)$, for $t \in I\!R$:

$$T(t) = \ e^{-ia(t)} \ e^{-ib(t)\cdot x} \ e^{-ic(t)\cdot p} \ ,$$

where :

$$b(t) = \ - \int_0^t \ (E(s) - E_0) \ ds - \int_0^1 \int_0^t \ (E(s) - E_0) \ ds \ dt \ . \tag{2.3}$$

$$c(t) = \ - \int_0^t \ b(s) \ ds \ . \tag{2.4}$$

$$a(t) = \ \int_0^t \ \left(\frac{1}{2} \ b^2(s) - E_0 \cdot c(s)\right) \ ds \ . \tag{2.5}$$

Lemma 4
The family $\{H_0(t)\}_{t\in I\!R}$ has an unique propagator $U_0(t,s)$ defined by :

$$U_0(t,s) = \ T(t) \ e^{-i(t-s)B_0} \ T^*(s) \ . \tag{2.6}$$

Proof.
We can always assume $s = \ 0$ and we make the following ansatz :

$$U_0(t,0) = \ e^{-ia(t)} \ e^{-ib(t)\cdot x} \ e^{-ic(t)\cdot p} \ e^{-itB_0} \ . \tag{2.7}$$

Since on the Schwartz space, $U_0(t,0)$ must satisfy :

$$i \frac{\partial}{\partial t} U_0(t,0) = H_0(t)\ U_0(t,0)\ , \tag{2.8}$$

the functions $a(t),\ b(t), c(t)$ solve :

$$\dot{b}(t) = -E(t) + E_0,\ \dot{c}(t) = -b(t),\ \dot{a}(t) = \frac{1}{2}\ b^2(t) - E_0 \cdot c(t). \tag{2.9}$$

We refer to [3] for details and [12] for a different formula. □

In the same way, in order to define the propagator corresponding to the family $\{H(t)\}$, we consider a Stark Hamiltonian with a time-periodic potential $V_1(t,x)$, (we recall that $c(t)$ a is C^1-periodic function) :

$$B(t) = B_0 + V_1(t,x) \ \text{ where } \ V_1(t,x) = e^{ic(t)\cdot p}\ V(t,x)\ e^{-ic(t)\cdot p} = V(t, x+c(t)). \tag{2.10}$$

Then, $B(t)$ has an unique propagator $R(t,s)$, (see [16] for the case $E_0 = 0$ and [12] for the case $E_0 \neq 0$). It is easy to see that the propagator $U(t,s)$ for the family $\{H(t)\}$ is defined by :

$$U(t,s) = T(t)\ R(t,s)\ T^*(s). \tag{2.11}$$

Comments.

Since the Hamiltonians $H_0(t)$ and $H(t)$ are time-periodic with period 1, one has for all $t, s \in I\!R$:

$$U_0(t+1, s+1) = U_0(t,s) \quad , \quad U(t+1,s+1) = U(t,s)\ . \tag{2.12}$$

Thus, the wave operators satisfy $W^{\pm}(s+1) = W^{\pm}(s)$.

2.2 The high energy limit of the scattering operators.

In this section, we study the high energy limit of the scattering operators by using the well-known Enss-Weder's time-dependent method [4]. This method can be used to study Hamiltonians with electric and magnetic potentials on $L^2(I\!R^n)$ [1], the Dirac equation [9], the N-body case [4], the Stark effect ([15], [17]), the Aharonov-Bohm effect [18].

In [13], [14] a stationary approach, based on the same ideas, is proposed to solve scattering inverse problems for Schrödinger operators with magnetic fields or with the Aharonov-Bohm effect.

Before giving the main result of this section, we need some notation.

- Φ, Ψ are the Fourier transforms of functions in $C_0^\infty(I\!R^n)$.
- $\omega \in S^{n-1} \cap \Pi_{E_0}$ is fixed, where Π_{E_0} is the orthogonal hyperplane to E_0.
- $\Phi_{\lambda,\omega} = e^{i\sqrt{\lambda}x.\omega}\Phi, \ \Psi_{\lambda,\omega} = e^{i\sqrt{\lambda}x.\omega}\Psi$.

We have the following high energy asymptotics where $< \ , \ >$ is the usual scalar product in $L^2(I\!R^n)$:

Proposition 5
Under the assumptions of Theorem 3, we have for all $s \in [0,1]$,

$$< [S(s),p] \ \Phi_{\lambda,\omega} \ , \ \Psi_{\lambda,\omega} > = \ \lambda^{-\frac{1}{2}} \ < \left(\int_{-\infty}^{+\infty} \ \partial_x V(s, x+t\omega) \ dt \right) \ \Phi \ , \ \Psi > + o \ (\lambda^{-\frac{1}{2}}) \ .$$

Comments.
Actually, for the case $n = 2, \ E_0 \neq 0$ and $\delta > \frac{3}{4}$, Proposition 5 is also valid for $\omega \in S^{n-1}$ satisfying $| \ \omega \cdot E_0 \ | < | \ E_0 \ |$, (see ([18], [15]).
Then, Theorem 3 follows from Proposition 5 and the inversion of Radon transform ([6] and [15], Section 2.3).

Proof of Proposition 5.

For example, let us prove Proposition 5 for the case $E_0 \neq 0$ and $n \geq 3$, the other cases are similar. More precisely, see [18] for the case $E_0 = 0$, and for the case $n = 2, \ E_0 \neq 0$, see ([17], Theorem 2.4) and ([15], Theorem 4).

Step 1.

Since $c(t)$ is periodic, $c(t) = O(1)$. Then, $V_1(t,x)$ is a short-range perturbation of B_0, and we can define the usual wave operators for the pair of Hamiltonians $(B(t), B_0)$:

(2.13) $$\Omega^{\pm}(s) = \ \mathrm{s} - \lim_{t\to\pm\infty} \ R(s,t) \ e^{-i(t-s)B_0} \ .$$

Consider also the scattering operators $S_1(s) = \ \Omega^{+*}(s) \ \Omega^-(s)$. By virtue of (2.6) and (2.11), it is clear that :

(2.14) $$S(s) = T(s) \ S_1(s) \ T^*(s) \ .$$

Using the fact that $e^{-ib(s)\cdot x} \ p \ e^{ib(s)\cdot x} = p + b(s)$, we have :

(2.15) $$[S(s),p] = [S(s), p + b(s)] = T(s) \ [S_1(s),p] \ T^*(s) \ .$$

Thus,

(2.16) $$< [S(s),p] \ \Phi_{\lambda,\omega} \ , \ \Psi_{\lambda,\omega} >= < [S_1(s),p] \ T^*(s) \ \Phi_{\lambda,\omega} \ , \ T^*(s) \ \Psi_{\lambda,\omega} > .$$

On the other hand,

(2.17) $$T^*(s)\ \Phi_{\lambda,\omega} = \ e^{i\sqrt{\lambda}x.\omega}\ e^{ic(s)\cdot(p+\sqrt{\lambda}\omega)}\ e^{ib(s)\cdot x}\ e^{ia(s)}\ \Phi.$$

So, we obtain :

(2.18) $$< [S(s),p]\ \Phi_{\lambda,\omega}\ ,\ \Psi_{\lambda,\omega} > = < [S_1(s),p]\ f_{\lambda,\omega},\ g_{\lambda,\omega}\ >,$$

where

(2.19) $$f = \ e^{ic(s)\cdot p}\ e^{ib(s)\cdot x}\ \Phi \text{ and } g = \ e^{ic(s)\cdot p}\ e^{ib(s)\cdot x}\ \Psi\ .$$

Clearly, $f,\ g$ are the Fourier transforms of functions in $C_0^\infty(I\!R^n)$.

- **Step 2 : Modified wave operators.**

Now, we follow a strategy close to [15] for time-dependent potentials. First, let us define a free-modified dynamic $U_D(t,s)$ by :

(2.20) $$U_D(t,s) = e^{-i(t-s)B_0}\ e^{-i\int_0^{t-s}\ V_1(u+s,up'+\frac{1}{2}u^2E_0)\ du}\ ,$$

where p' is the projection of p on the orthogonal hyperplane to E_0.

We define the modified wave operators :

(2.21) $$\Omega_D^\pm(s) = \mathrm{s}-\lim_{t\to\pm\infty}\ R(s,t)\ U_D(t,s)\ .$$

It is clear that :

(2.22) $$\Omega_D^\pm(s) = \Omega^\pm(s)\ e^{-ig^\pm(s,p')}\ ,$$

where

(2.23) $$g^\pm(s,p')\ =\ \int_0^{\pm\infty}\ V_1(u+s,up'+\frac{1}{2}u^2E_0)\ du\ .$$

Thus, if we set $S_D(s) = \Omega_D^{+*}(s)\Omega_D^-(s)$, one has :

(2.24) $$S_1(s) = e^{-ig^+(s,p')}\ S_D(s)\ e^{ig^-(s,p')}$$

- **Step 3 : High energy estimates.**

Denote $\rho = min\ (1,\delta)$. We have the following estimations, (the proof is exactly the same as in ([15], Lemma 3) for time-independent potentials).

Lemma 6
For $\lambda >> 1$, we have :

(i) $$||\ \Big(V_1(t,x) - V_1(t,(t-s)p'+\frac{1}{2}(t-s)^2E_0)\Big)\ U_D(t,s)\ e^{ig^\pm(s,p')}f_{\lambda,\omega}\ ||$$
$$\leq\ C\ (1+|\ (t-s)\sqrt{\lambda}\ |)^{-\frac{1}{2}-\rho}\ .$$

(ii) $$||\ \Big(R(t,s)\Omega_D^\pm(s) - U_D(t,s)\Big)\,e^{ig^\pm(s,p')}f_{\lambda,\omega}\ ||=\ O\ (\lambda^{-\frac{1}{2}})\ ,\ \text{uniformly for } t,\ s\in I\!R\ .$$

- **Step 4.**

We denote $F(s,\lambda,\omega) =< [S_1(s),p]\ f_{\lambda,\omega}\ ,\ g_{\lambda,\omega} >$. Using (2.24), we have :

$$\begin{aligned} F(s,\lambda,\omega) &= < [e^{-ig^+(s,p')}\ S_D(s)\ e^{ig^-(s,p')},p]\ f_{\lambda,\omega}\ ,\ g_{\lambda,\omega} > \\ &= < [S_D(s),p]\ e^{ig^-(s,p')} f_{\lambda,\omega}\ ,\ e^{ig^+(s,p')} g_{\lambda,\omega} > \\ &= < [S_D(s)-1, p-\sqrt{\lambda}\omega]\ e^{ig^-(s,p')} f_{\lambda,\omega}\ ,\ e^{ig^+(s,p')} g_{\lambda,\omega} > \\ &= < (S_D(s)-1)\ e^{ig^-(s,p')} (pf)_{\lambda,\omega}\ ,\ e^{ig^+(s,p')} g_{\lambda,\omega} > \\ &- \ < (S_D(s)-1)\ e^{ig^-(s,p')} f_{\lambda,\omega}\ ,\ e^{ig^+(s,p')} (pg)_{\lambda,\omega} > \\ &:= F_1(s,\lambda,\omega) - F_2(s,\lambda,\omega). \end{aligned}$$

First, let us study $F_1(s,\lambda,\omega)$. Writing $S_D(s)-1 = (\Omega_D^+(s)-\Omega_D^-(s))^*\ \Omega_D^-(s)$ and using

$$(2.25)\quad \Omega_D^+(s) - \Omega_D^-(s) \ = \ i \int_{-\infty}^{+\infty} R(s,t)\ \Big(V_1(t,x) - V_1(t,(t-s)p' + \frac{1}{2}(t-s)^2 E_0) \Big)\ U_D(t,s)\ dt\ ,$$

we obtain :

$$(2.26)\qquad S_D(s)-1 = -i \int_{-\infty}^{+\infty} U_D(t,s)^*\ \Big(V_1(t,x) - V_1(t,(t-s)p' + \frac{1}{2}(t-s)^2 E_0) \Big)\ R(t,s)\ \Omega_D^-(s)\ dt\ .$$

Thus,

$$\begin{aligned} F_1(s,\lambda,\omega) &= -i \int_{-\infty}^{+\infty} < R(t,s)\ \Omega_D^-(s)\ e^{ig^-(s,p')} (pf)_{\lambda,\omega}\ , \\ &\quad \Big(V_1(t,x) - V_1(t,(t-s)p' + \frac{1}{2}(t-s)^2 E_0) \Big)\ U_D(t,s)\ e^{ig^+(s,p')} g_{\lambda,\omega} >\ dt \\ &= -i \int_{-\infty}^{+\infty} < U_D(t,s)\ e^{ig^-(s,p')} (pf)_{\lambda,\omega}\ , \\ &\quad \Big(V_1(t,x) - V_1(t,(t-s)p' + \frac{1}{2}(t-s)^2 E_0) \Big)\ U_D(t,s)\ e^{ig^+(s,p')} g_{\lambda,\omega} >\ dt \\ &\quad +\ R_1(s,\lambda,\omega)\ , \end{aligned}$$

where :

$$\begin{aligned} (2.27)\qquad R_1(s,\lambda,\omega) \ &= \ -i \int_{-\infty}^{+\infty} < \Big(R(t,s)\ \Omega_D^-(s) - U_D(t,s) \Big)\ e^{ig^-(s,p')} (pf)_{\lambda,\omega}\ , \\ &\quad \Big(V_1(t,x) - V_1(t,(t-s)p' + \frac{1}{2}(t-s)^2 E_0) \Big)\ U_D(t,s)\ e^{ig^+(s,p')} g_{\lambda,\omega} >\ dt\ . \end{aligned}$$

By Lemma 6, it is clear that $R_1(s,\lambda,\omega) = O\ (\lambda^{-1})$. Thus, writing $t = \frac{\tau}{\sqrt{\lambda}} + s$, we obtain :

(2.28) $$F_1(s,\lambda,\omega) \ = -\frac{i}{\sqrt{\lambda}} \int_{-\infty}^{+\infty} \ < U_D(\frac{\tau}{\sqrt{\lambda}}+s,s)\ e^{ig^-(s,p')}\ (pf)_{\lambda,\omega}\ ,$$
$$\left(V_1(\frac{\tau}{\sqrt{\lambda}}+s,x) - V_1(\frac{\tau}{\sqrt{\lambda}}+s,\frac{\tau}{\sqrt{\lambda}}p'+\frac{\tau^2}{2\lambda}E_0)\right)$$
$$U_D(\frac{\tau}{\sqrt{\lambda}}+s,s)\ e^{ig^+(s,p')} g_{\lambda,\omega} > \ d\tau + \ O\ (\lambda^{-1})\ .$$

Denote by $f_1(\tau,s,\lambda,\omega)$ the integrand of the (R.H.S) of (2.28). By Lemma 6 (i),

(2.29) $$|\ f_1(\tau,s,\lambda,\omega)\ | \leq C\ (1+ \mid \tau \mid)^{-\frac{1}{2}-\rho}\ .$$

So, by Lebesgue's theorem, to obtain the asymptotics of $F_1(s,\lambda,\omega)$, it suffices to determine $\lim\limits_{\lambda\to+\infty}\ f_1(\tau,s,\lambda,\omega)$.

Let us denote :

(2.30) $$U^{\pm}(t,s,p') = e^{i\ \int_t^{\pm\infty}\ V_1(u+s,up'+\frac{1}{2}u^2E_0)\ du}.$$

We have :

(2.31) $$f_1(\tau,s,\lambda,\omega) \ = < e^{-i\frac{\tau}{\sqrt{\lambda}}B_0}\ U^-(\frac{\tau}{\sqrt{\lambda}},s,p')\ (pf)_{\lambda,\omega}\ ,$$
$$\left(V_1(\frac{\tau}{\sqrt{\lambda}}+s,x) - V_1(\frac{\tau}{\sqrt{\lambda}}+s,\frac{\tau}{\sqrt{\lambda}}p'+\frac{\tau^2}{2\lambda}\ E_0)\right)\ e^{-i\frac{\tau}{\sqrt{\lambda}}B_0}\ U^+(\frac{\tau}{\sqrt{\lambda}},s,p')\ g_{\lambda,\omega} > \ .$$

Using the Avron-Herbst formula (2.2), we deduce that :

(2.32) $$f_1(\tau,s,\lambda,\omega) \ = < e^{-i\frac{\tau}{2\sqrt{\lambda}}p^2}\ U^-(\frac{\tau}{\sqrt{\lambda}},s,p')\ (pf)_{\lambda,\omega}\ ,$$
$$\left(V_1(\frac{\tau}{\sqrt{\lambda}}+s,x+\frac{\tau^2}{2\lambda}\ E_0) - V_1(\frac{\tau}{\sqrt{\lambda}}+s,\frac{\tau}{\sqrt{\lambda}}p'+\frac{\tau^2}{2\lambda}\ E_0)\right)\ e^{-i\frac{\tau}{2\sqrt{\lambda}}p^2}\ U^+(\frac{\tau}{\sqrt{\lambda}},s,p')\ g_{\lambda,\omega} > \ .$$

Then, we obtain :

(2.33) $$f_1(\tau,s,\lambda,\omega) \ = < e^{-i\frac{\tau}{2\sqrt{\lambda}}(p+\sqrt{\lambda}\omega)^2}\ U^-(\frac{\tau}{\sqrt{\lambda}},s,p'+\sqrt{\lambda}\omega)\ pf\ ,$$
$$\left(V_1(\frac{\tau}{\sqrt{\lambda}}+s,x+\frac{\tau^2}{2\lambda}\ E_0) - V_1(\frac{\tau}{\sqrt{\lambda}}+s,\frac{\tau}{\sqrt{\lambda}}(p'+\sqrt{\lambda}\omega)+\frac{\tau^2}{2\lambda}\ E_0)\right)$$
$$e^{-i\frac{\tau}{2\sqrt{\lambda}}(p+\sqrt{\lambda}\omega)^2}\ U^+(\frac{\tau}{\sqrt{\lambda}},s,p'+\sqrt{\lambda}\omega)\ g > \ .$$

Since

(2.34) $$e^{-i\frac{\tau}{2\sqrt{\lambda}}(p+\sqrt{\lambda}\omega)^2} \ = \ e^{-i\frac{\tau\sqrt{\lambda}}{2}}\ e^{-i\tau\omega.p}\ e^{-i\frac{\tau}{2\sqrt{\lambda}}p^2}\ ,$$

we have

$$(2.35)\quad f_1(\tau,s,\lambda,\omega) \ = < e^{-i\frac{\tau}{2\sqrt{\lambda}}p^2}\ U^-(\frac{\tau}{\sqrt{\lambda}},s,p'+\sqrt{\lambda}\omega)\ \ pf\ ,$$

$$\left(V_1(\frac{\tau}{\sqrt{\lambda}}+s,x+\tau\omega+\frac{\tau^2}{2\lambda}\ E_0)-V_1(\frac{\tau}{\sqrt{\lambda}}+s,\frac{\tau}{\sqrt{\lambda}}(p'+\sqrt{\lambda}\omega)+\frac{\tau^2}{2\lambda}\ E_0)\right)$$

$$e^{-i\frac{\tau}{2\sqrt{\lambda}}p^2}\ U^+(\frac{\tau}{\sqrt{\lambda}},s,p'+\sqrt{\lambda}\omega)\ g > \ .$$

Since $\mid V_1(u+s,u(p'+\sqrt{\lambda}\omega)+\frac{1}{2}u^2E_0))\mid \leq\ C\ (u^2+1)^{-\delta}\in L^1(I\!R^+,\ du)$, it is easy to show (using Lebesgue's theorem again) that :

$$(2.38)\qquad s-\lim_{\lambda\to+\infty}\ U^{\pm}(\frac{\tau}{\sqrt{\lambda}},s,p'+\sqrt{\lambda}\omega)\ =1\ .$$

Then,

$$(2.39)\qquad \lim_{\lambda\to+\infty}\ f_1(\tau,s,\lambda,\omega)\ \ = < pf\ ,\ (V_1(s,x+\tau\omega)-V_1(s,\tau\omega))\ \ g > \ .$$

So, we have obtained :

$$(2.40)\qquad F_1(s,\lambda,\omega)\ \ =\ \ -\frac{i}{\sqrt{\lambda}}\ \ < pf,\ \left(\int_{-\infty}^{+\infty}\ (V_1(s,x+\tau\omega)-V_1(s,\tau\omega))\ d\tau\right)\ g > \ +o(\frac{1}{\sqrt{\lambda}}).$$

In the same way, we obtain

$$(2.41)\qquad F_2(s,\lambda,\omega)\ \ =\ \ -\frac{i}{\sqrt{\lambda}}\ \ < f,\ \left(\int_{-\infty}^{+\infty}\ (V_1(s,x+\tau\omega)-V_1(s,\tau\omega))\ d\tau\right)\ pg > \ +o(\frac{1}{\sqrt{\lambda}})\ ,$$

so

$$(2.42)\qquad F(s,\lambda,\omega)\ \ =\ \ F_1(s,\lambda,\omega)-F_2(s,\lambda,\omega)$$

$$(2.43)\qquad =\ \ \frac{1}{\sqrt{\lambda}}\ \ < f,\ \left(\int_{-\infty}^{+\infty}\ \partial_x V_1(s,x+\tau\omega)\ d\tau\right)\ g > \ +o(\frac{1}{\sqrt{\lambda}})\ .$$

Using (2.19) and $\partial_x V(s,x+\tau\omega)=e^{-ic(s)\cdot p}\ \partial_x V_1(s,x+\tau\omega)\ e^{ic(s)\cdot p}$, we obtain :

$$(2.44)\qquad F(s,\lambda,\omega)\ \ =\ \ \frac{1}{\sqrt{\lambda}}\ \ < \Phi,\ \left(\int_{-\infty}^{+\infty}\ \partial_x V(s,x+\tau\omega)\ d\tau\right)\ \Psi > \ +o(\frac{1}{\sqrt{\lambda}})\ .\ \Box$$

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