An inverse scattering problem for the Schrödinger equation in a semiclassical process.

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Abstract

We study a direct and an inverse scattering problem for a pair of Hamiltonians $(H(h), H_0(h))$ on $L^2(I\!\!R^n)$, where $H_0(h) = -h^2\Delta$ and $H(h) = H_0(h) + V$, V is a short-range potential and h is the semiclassical parameter. First, we show that if two potentials are equal in the classical allowed region for a fixed non-trapping energy, the associated scattering matrices coincide up to $O(h^\infty)$ in $\mathcal{B}(L^2(S^{n-1}))$. Then, for potentials with a regular behaviour at infinity, we study the inverse scattering problem. We show that, in dimension $n \geq 3$, the knowledge of the scattering operators $S(h)$, $h \in]0,1]$, up to $O(h^\infty)$ in $\mathcal{B}(L^2(I\!\!R^n))$, and which are localized near a fixed energy $\lambda > 0$, determine the potential V at infinity.

1 Introduction.

In this paper, we study a direct and an inverse scattering problem for the pair of Hamiltonians $(H(h), H_0(h))$ on $L^2(I\!\!R^n)$, $n \geq 2$, where the free operator is $H_0(h) = -h^2\Delta,\ h \in]0,1]$ is the semiclassical parameter, and the perturbed Hamiltonian is given by :

$$H(h) = H_0(h) + V. \tag{1.1}$$

We assume that V is a real-valued smooth short-range potential satisfying :

$$(H_1) \qquad \forall \alpha \in I\!\!N^n \ , \ \exists\, C_\alpha > 0 \ , \ | \partial_x^\alpha V(x) | \leq C_\alpha < x >^{-\rho-|\alpha|} \ , \ \rho > 1.$$

Under the hypothesis (H_1), it is well-known that the wave operators :

$$W^{\pm}(h) \ = \ s - \lim_{t\to\pm\infty} \ e^{itH(h)}e^{-itH_0(h)}, \tag{1.2}$$

exist and are complete, i.e $Ran\ W^{\pm}(h) = \mathcal{H}_{ac}(H) =$ subspace of absolute continuity of $H(h)$.

Let $S(h)$ be the scattering operator defined by :

$$S(h) = W^{+*}(h)\ W^{-}(h). \tag{1.3}$$

Since $S(h)$ commutes with $H_0(h)$, we can define the scattering matrices $S(\lambda,h)$, $\lambda>0$, as unitary operators acting on $L^2(S^{n-1})$, where S^{n-1} is the unit sphere in $I\!R^n$. We denote by $\Phi_0(x,\lambda,\omega,h),\ (\lambda,\omega)\in]0,+\infty[\times S^{n-1}$, the generalized eigenfunction of $H_0(h)$:

$$\Phi_0(x,\lambda,\omega,h) = e^{\frac{i}{h}\sqrt{\lambda}x\cdot\omega}. \tag{1.4}$$

The unitary mapping $\mathcal{F}_0(h)$ defined by :

$$(\mathcal{F}_0(h)f)(\lambda,\omega) = (2\pi h)^{-\frac{n}{2}}\ \lambda^{\frac{n-2}{4}} \int_{I\!R^n} \overline{\Phi_0}(x,\lambda,\omega,h)\ f(x)\ dx\ , \tag{1.5}$$

gives the spectral representation for $H_0(h)$, i.e $H_0(h)$ is transformed into the multiplication by λ in the space $L^2\ (I\!R^+\ ;\ L^2(S^{n-1}))$.

Then, the scattering matrices are defined by :

$$\mathcal{F}_0(h)S(h)f(\lambda) = S(\lambda,h)\ (\mathcal{F}_0(h)f)(\lambda), \tag{1.6}$$

and we have the Kuroda's representation formula :

$$S(\lambda,h) = 1 - 2i\pi F_0(\lambda,h)VF_0(\lambda,h)^* + 2i\pi F_0(\lambda,h)VR(\lambda+i0,h)VF_0(\lambda,h)^*\ , \tag{1.7}$$

where for $f\in L^2_s(I\!R^n),\ s>\frac{1}{2}$, $F_0(\lambda,h)f(\omega) = (\mathcal{F}_0(h)f)(\lambda,\omega)$ and $R(\lambda+i0,h)$ is given by the well-known principle of limiting absorption. The kernel of $S(\lambda,h)-1$ is called the scattering amplitude.

We recall the non-trapping condition. Let $(z(t,x,\xi),\ \zeta(t,x,\xi))$ be the solution to the Hamilton equations :

$$\dot{z}(t,x,\xi) = 2\zeta(t,x,\xi)\ \ ,\ \ \dot{\zeta}(t,x,\xi) = -\nabla V(z(t,x,\xi)), \tag{1.8}$$

with the initial data $z(0,x,\xi)=x,\ \zeta(0,x,\xi)=\xi$.

We say that the energy λ is non-trapping, if for any $R\gg 1$ large enough, there exists $T=T(R)$ such that $\mid z(t,x,\xi)\mid > R$ for $\mid t\mid > T$, when $\mid x\mid < R$ and $\lambda = \xi^2+V(x)$.

When λ is a non-trapping energy, and if V satisfies (H_1) with $\rho > 0$, we have the following estimate when $h \to 0$, (see [13], [15]),

$$\forall s > \frac{1}{2} \quad , \quad \| < x >^{-s} \ R(\lambda + i0, h) \ < x >^{-s} \| = O(\frac{1}{h}). \tag{1.9}$$

The goal of this paper is to give a partial answer to the following conjecture. We denote by $\mathcal{B}(\mathcal{H})$ the space of bounded operators acting on a Hilbert space $\mathcal{H}$, and by $f_+(x) = max\ (f(x), 0)$.

Conjecture.

Let V_1 and V_2 be potentials satisfying (H_1) and let $S_j(\lambda, h)$, $j = 1, 2$, be the corresponding scattering matrices at a fixed non-trapping energy $\lambda > 0$.
Then the two following assertions are equivalent :

$$(i) \qquad (\lambda - V_1(x))_+ = (\lambda - V_2(x))_+.$$

$$(ii) \qquad S_1(\lambda, h) = S_2(\lambda, h) + O(h^\infty) \ \text{ in } \ \mathcal{B}(L^2(S^{n-1})) \quad , \quad h \to 0.$$

According to the author, the inverse scattering problem is not known in this semiclassical setting. Let us remark that it seems to be difficult to prove our conjecture for a fixed energy without assuming the non-trapping condition. The difficulty comes from the estimate of the resolvent $R(\lambda + i0, h)$ due to resonances converging exponentially to the real axis.

In this paper, we show in section 2 that $(i) \Longrightarrow (ii)$ and in section 3, we give a partial answer to the implication $(ii) \Longrightarrow (i)$ for a class of potentials which are regular at infinity.

2 The direct scattering.

In this section, we give a quite elementary proof of $(i) \Longrightarrow (ii)$. This proof is based on a suitable representation formula for the scattering matrices.

Proposition 1
Let V_1 and V_2 be potentials satisfying (H_1) and let $S_j(\lambda, h)$, $j = 1, 2$, be the corresponding scattering matrices at a fixed non-trapping energy $\lambda > 0$. Assume that $(\lambda - V_1(x))_+ = (\lambda - V_2(x))_+$.
Then, when $h \to 0$,

$$S_1(\lambda, h) = S_2(\lambda, h) + O(h^\infty) \text{ in } \mathcal{B}(L^2(S^{n-1})).$$

Proof :

• **Step 1 : a representation formula for the scattering matrices.**

Consider $\chi \in C_0^\infty(I\!R^n)$ such that $\chi(x) = 1$ for $\mid x \mid \leq R$, $R \gg 1$ and let $\zeta \in C_0^\infty([0,1[)$ such that $\zeta = 1$ in a small neighborhood of 0.

For $j = 1,2$ and $\epsilon > 0$ one has by the intertwining property :

$$W_j^\pm(h)\ \zeta(\frac{H_0(h)-\lambda}{h^\epsilon}) = \zeta(\frac{H_j(h)-\lambda}{h^\epsilon})\ W_j^\pm(h),$$

where $W_j^\pm(h)$ are the Moeller wave operators associated with the pair $(H_j(h), H_0(h))$, $H_j(h) = H_0(h) + V_j(x)$. Then,

$$W_j^\pm(h)\ \zeta(\frac{H_0(h)-\lambda}{h^\epsilon})\ =\ s - \lim_{t\to\pm\infty}\ e^{itH_j(h)}\ \zeta(\frac{H_j(h)-\lambda}{h^\epsilon})\ (1-\chi)\ e^{-itH_0(h)}\ , \tag{2.1}$$

We denote by $J_j(h) = \zeta(\frac{H_j(h)-\lambda}{h^\epsilon})\ (1-\chi)$ and $T_j(h) = H_j(h)J_j(h) - J_j(h)H_0(h)$.

We have the representation formula for the scattering matrices, (see [5], for example) :

$$\begin{aligned} S_j(\lambda,h) = 1 -\ & 2i\pi\ F_0(\lambda,h)\ J_j^*(h)\ T_j(h)\ F_0^*(\lambda,h) \\ & + 2i\pi\ F_0(\lambda,h)\ T_j^*(h)\ R_j(\lambda+i0,h)\ T_j(h)\ F_0^*(\lambda,h), \end{aligned} \tag{2.2}$$

where $R_j(\lambda+i0,h)$ is the resolvent associated with the Hamiltonian $H_j(h)$.

• **Step 2 :**

For $\epsilon < \frac{1}{2}$, $\zeta(\frac{H_j(h)-\lambda}{h^\epsilon})$ is an h-pseudodifferential operator and his h-symbol is given by the standard functional calculus. Since $V_1 = V_2$ outside a compact set, choosing $R \gg 1$, we obtain :

$$\forall L > 0\ ,\ \| < x >^L (J_1(h) - J_2(h)) < x >^L \| =\ O(h^\infty). \tag{2.3}$$

In the same way, since $T_j(h) = \zeta(\frac{H_j(h)-\lambda}{h^\epsilon})\,([\chi, H_0(h)] + V_j\ (1-\chi))$, we have :

$$\forall L > 0\ ,\ \| < x >^L (T_1(h) - T_2(h)) < x >^L \| =\ O(h^\infty). \tag{2.4}$$

• **Step 3 :**

For $s > \frac{1}{2}$, $\|\ F_0(\lambda,h) < x >^{-s}\| = O(h^{-\frac{1}{2}})$. Moreover, since λ is a non trapping energy, we have the following estimate (see [2]) : for $j,k = 1,2$,

$$\| < x >^{-s}\ \ T_j^*(h)\ R_k(\lambda \pm i0,h)\ \ < x >^{-s}\| =\ O(h^{-1}). \tag{2.5}$$

Thus, by (2.3) – (2.5), one has :

$$(2.6)\quad S_1(\lambda,h)-S_2(\lambda,h)=2i\pi\ F_0(\lambda,h)\ T_1^*(h)\ (R_1(\lambda+i0,h)-R_2(\lambda+i0,h))\ T_2(h)\ F_0^*(\lambda,h) + O(h^\infty).$$

Using the resolvent identity, one obtains :

$$(2.7)\quad S_1(\lambda,h)-S_2(\lambda,h)=2i\pi\ F_0(\lambda,h)\ T_1^*(h)\ R_1(\lambda+i0,h)\ (V_2-V_1)\ R_2(\lambda+i0,h)\ T_2(h)\ F_0^*(\lambda,h) + O(h^\infty).$$

Then,

$$(2.8)\quad S_1(\lambda,h)-S_2(\lambda,h)=2i\pi\ F_0(\lambda,h)\ T_1^*(h)\ R_1(\lambda+i0)\ (V_2-V_1)\ \zeta(\frac{H_2(h)-\lambda}{h^\epsilon})\ R_2(\lambda+i0)\ \ ([\chi,H_0(h)]+V_2\ (1-\chi))\ \ F_0^*(\lambda,h)+O(h^\infty).$$

• **Step 4 :**

By the above considerations, it suffices to show that :

$$(2.9)\quad ||\ \ (V_2-V_1)\ \zeta(\frac{H_2(h)-\lambda}{h^\epsilon})<x>^s||=O(h^\infty).$$

We recall that $(\lambda-V_1(x))_+ = (\lambda-V_2(x))_+$, so V_1-V_2 has a compact support, and for $j=1,2$, we can denote :

$$(2.10)\quad \Sigma_\lambda=\{\ z=(x,\xi)\in I\!\!R^{2n}\ \ :\ \xi^2+V_j(x)=\lambda\}.$$

Since λ is a non-trapping energy, Σ_λ is a smooth $(2n-1)$-submanifold, so we can work with local coordinates. Let (χ_j) be a C^∞ partition of unity in $T^*I\!\!R^n$ such that $\forall z=(z_1,...,z_{2n})\in$ *Supp* $\chi_j\cap\Sigma_\lambda$, $z_{2n}=g_j(z_1,...,z_{2n-1})$ (for example) where g_j is a smooth function defined on a neighborhood of *Supp* χ_j.
For $z=(z_1,...,z_{2n})\in$ *Supp* χ_j, we denote by $z_\perp=z_{2n}-g_j(z_1,...,z_{2n-1})$.
Since *Supp* (V_1-V_2) is a compact set, it suffices to show that for a finite number of j,

$$(2.11)\quad ||\ \ (V_2-V_1)\ Op_h(\chi_j)\ \zeta(\frac{H_2(h)-\lambda}{h^\epsilon})<x>^s||=O(h^\infty).$$

By the hypothesis,

$$(2.12)\quad |\ (V_1-V_2)\ \chi_j(z)\ |=\ O\,(z_\perp^\infty)\quad,\quad z_\perp\to 0.$$

On the other hand, the left symbol of the pseudifferential operator $\zeta(\frac{H_2(h)-\lambda}{h^\epsilon})$ is supported in $A_h:=\{\ (x,\xi)\ :|\ \xi^2+V_2(x)-\lambda\ |\le h^\epsilon\ \}$.
It is easy to see that if $z=(x,\xi)\in A_h\cap$ *Supp* χ_j, then $|\ z_\perp\ |=O(h^\epsilon)$. So, (2.11) follows from (2.12) and the standard h-pseudodifferential calculus. □

3 The inverse scattering.

3.1 Introduction.

In this section, we give a partial answer to $(ii) \Longrightarrow (i)$ for a class of potentials which are regular at infinity. But first, let us recall some inverse results in the semiclassical setting.

In [12], for potentials satisfying (H_1) with $\rho > n$, Novikov shows, without the non-trapping condition and using the $\overline{\partial}$-method, that if $S_1(\lambda, h) = S_2(\lambda, h)$, $\forall h \in]0,1[$, then $V_1 = V_2$. It is clear that this approach is not adapted to study our conjecture in the semiclassical setting since we saw in section 2 that we can not obtain any information on the potential outside the classical allowed region.

For potentials with compact support, an inverse scattering problem close to our conjecture is studied in [1]. Assuming that $\lambda > \sup_{x \in \mathbb{R}^n} V_+(x)$ and that the metric $g(x) = (\lambda - V(x))\, dx^2$ is conformal to the Euclidian, Alexandrova shows in [1] that the scattering amplitude in the semiclassical regime, (actually, the scattering relations), determines V uniquely. This result is obtained by comparing the inverse scattering problem for the Schrödinger equation with the problem of the wave equation with variable speed for which the result is well-known. The scattering relations for the metric $g(x)$ determine the boundary distance function uniquely, and then we determine g and therefore $V(x)$. This method can not be used for general potentials satisfying (H_1).

Now, let us explain more precisely the setting of this section. We consider the class of potentials V which are asymptotic sums of homogenous terms at infinity, i.e. we assume that $V \in C^\infty(\mathbb{R}^n)$ and satisfies the regular behaviour at infinity :

$$(H_2) \qquad V(x) \simeq \sum_{j=1}^{\infty} V_j(x)$$

where the $V_j(x)$ are homogeneous functions of order $-\rho_j$ with $1 < \rho_1 < \rho_2 < \cdots$.
Hypothesis (H_2) means that for any $N \geq 2$ and $\alpha \in \mathbb{N}^n$, there exists $C_{\alpha,N}$ such that :

$$(3.1) \qquad |\, \partial_x^\alpha \, (V(x) - \sum_{j=1}^{N-1} V_j(x)) \,| \leq C_{\alpha,N} \, < x >^{-\rho_N - |\alpha|} .$$

Inverse scattering at a fixed energy with regular potentials at infinity and when the Planck's constant $h = 1$ was first studied in [6] with $\rho_j = j+1$. In [16], Weder and Yafaev study a similar problem for general degrees of homogeneity ρ_j. They show that the complete asymptotic expansion of the potential is determined by the singularities in the forward direction of the scattering amplitude. In particular, they obtain the following result :

Let V^1, V^2 be potentials satisfy the assumption (H_2) and let $S_j(\lambda)$, $j = 1,2$, be the corresponding scattering matrices. Assume that the kernel of $S_1(\lambda) - S_2(\lambda)$ belongs to $C^\infty(S^{n-1} \times S^{n-1})$, then $V^1 - V^2$ belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$.

If we want to study the inverse scattering problem in the semiclassical setting with a fixed non-trapping energy and for potentials having a regular behaviour at infinity, it seems to the author that the technics developed in [16] are not well adapted with a semiclassical parameter. Indeed, as it was pointed in ([13], [14]), for potentials satisfying (H_1) with $\rho > n$, the forward amplitude $f(\lambda, \omega, \omega, h)$ is of order $O(h^{-\mu})$ with $\mu = \dfrac{(n-1)(n+1)}{2(\rho-1)}$ and $f(\lambda, \theta, \omega, h)$ with $\theta \neq \omega$ is of order $O\ (1)$. Thus, the scattering amplitude has a strong peak in a neighborhood of the forward direction. We also mention the works ([7], [8], [17]) where the semiclassical asymptotics of the scattering amplitude are obtained.

Moreover, let us emphasize that the knowledge of $S(\lambda, h)$ in the norm operator sense does not give enough precise information on his kernel. These are the reasons why we prefer to work with an arbitrary small interval of positives energies and we use the approach given in ([9], [10], [11]) which is a time-independent version of [3].

So, our goal in this section is to recover all functions V_j from the knowledge of the scattering operator $S(h)$ localized near a fixed energy λ, up to $O(h^\infty)$ in the sense of the operator norm in $L^2(I\!R^n)$. Since we do not work with a fixed energy, but in an averaged sense, we emphasize that the non-trapping condition is not necessary.

In order to localize the scattering operator near a fixed energy $\lambda > 0$, we introduce a cut-off function $\chi \in C_0^\infty(]0, +\infty[)$, $\chi = 1$ in a neighborhood of $\lambda > 0$.

We show that, for $n \geq 3$, we obtain the complete asymptotic expansion of the potential from the knowledge of $S(h)\chi(H_0(h))$ up to $O(h^\infty)$ in $\mathcal{B}(L^2(I\!R^n))$, when $h \to 0$. More precisely, we determine the asymptotics expansion of the potential at infinity, by studying the asymptotics of :

$$F(h) = < S(h)\chi(H_0(h))\Phi_{h,\omega}, \Psi_{h,\omega} > , \tag{3.2}$$

where $<,>$ is the usual scalar product in $L^2(I\!R^n)$, and $\Phi_{h,\omega}$, $\Psi_{h,\omega}$ are suitable test functions. We need $n \geq 3$ in order to use the uniqueness for the Radon transform in hyperplanes.

3.2 Semiclassical asyptotics.

3.2.1 Definition of the test functions.

First, let us define the unitary dilation operator $U(h^\delta)$, $\delta > 0$, on $L^2(I\!R^n)$ by :

$$U(h^\delta)\ \Phi(x) = h^{\frac{n\delta}{2}}\ \Phi(h^\delta x). \tag{3.3}$$

We also need an energy cut-off $\chi_0 \in C_0^\infty(I\!R^n)$ such that $\chi_0(\xi) = 1$ if $|\ \xi\ | \leq 1$, $\chi_0(\xi) = 0$ if $|\ \xi\ | \geq 2$.

For $\omega \in S^{n-1}$, we write $x \in I\!R^n$ as $x = y + t\omega$, $y \in \Pi_\omega =$ orthogonal hyperplane to ω and we consider :

$$X_\omega = \{x = y + t\omega \in I\!R^n\ :\ |\ y\ | \geq 1\ \}. \tag{3.4}$$

For $\delta > \dfrac{1}{\rho_1 - 1}$ and $\epsilon < 1 + \delta$, we define :

$$\Phi_{h,\omega} = e^{\frac{i}{h}\sqrt{\lambda} x.\omega} \ U(h^{\delta}) \ \chi_0(h^{\epsilon} D) \ \Phi \ , \tag{3.5}$$

where $\Phi \in C_0^{\infty}(X_\omega)$ and $D = -i\nabla$, ($\Psi_{h,\omega}$ is defined in the same way with $\Psi \in C_0^{\infty}(X_\omega)$).

3.2.2 Semiclassical asymptotics for the scattering operator.

In this section, we prove the following theorem :

Theorem 2
Let V be a potential satisfying (H_2). Then, there exists an increasing positive sequence $(\nu_k)_{k\geq 1}$, (depending only on δ), with $\nu_1 = \delta(\rho_1 - 1) - 1$ and $\lim_{k\to +\infty} \nu_k = +\infty$ such that :

$$< (S(h) - 1) \ \chi(H_0(h)) \ \Phi_{h,\omega} \ , \ \Psi_{h,\omega} > \ \simeq \ \sum_{k=1}^{\infty} h^{\nu_k} \ < \Phi \ , \ A_k(x,\omega,D) \ \Psi > \ , \tag{3.6}$$

when $h \to 0$, where $A_j(x,\omega,D)$ are differential operators defined in a recursive way.
Moreover, $\forall j \geq 1, \exists \ k_j \geq 1$, (with $k_1 = 1$), such that :

$$A_{k_j}(x,\omega,D) = \frac{i}{2\sqrt{\lambda}} \int_{-\infty}^{+\infty} V_j(x + t\omega) \ dt \ + \ B_j(x,\omega,D), \tag{3.7}$$

with $B_1 = 0$ and for $j \geq 2$, $B_j(x,\omega,D)$ is a differential operator only depending on the functions V_k, $1 \leq k \leq j-1$.

3.2.3 Proof of Theorem 2.

Step 1 :

Let us begin by an elementary lemma.

Lemma 3
$\forall h \ll 1$ small enough, we have :

$$\chi(H_0(h))\Phi_{h,\omega} \ = \ \Phi_{h,\omega}. \tag{3.8}$$

Proof
We easily obtain :

$$\mathcal{F}\ [\chi(H_0(h))\Phi_{h,\omega}](\xi) \ = \ h^{-\frac{n\delta}{2}} \chi((h\xi)^2)\chi_0(h^{\epsilon-\delta-1}(h\xi - \sqrt{\lambda}\omega))\mathcal{F}\Phi(h^{-\delta-1}(h\xi - \sqrt{\lambda}\omega)), \tag{3.9}$$

where $\mathcal{F}$ is the usual Fourier transform. Then, on $Supp\ \chi_0$, we have $\mid h\xi - \sqrt{\lambda}\omega \mid \leq 2h^{1+\delta-\epsilon}$. Since $\epsilon < 1+\delta$, we have for h small enough, $\chi((h\xi)^2) = 1$. □

Then, by Lemma 3, we obtain,

$$F(h) =< W^-(h)\Phi_{h,\omega}\ ,\ W^+(h)\Psi_{h,\omega} >, \tag{3.10}$$

and an easy calculation gives :

$$F(h) =< \Omega^-(h,\omega)\chi_0(h^\epsilon D)\Phi\ ,\ \Omega^+(h,\omega)\chi_0(h^\epsilon D)\Psi >, \tag{3.11}$$

where

$$\Omega^{\pm}(h,\omega) = s - \lim_{t\to\pm\infty}\ e^{itH(h,\omega)}e^{-itH_0(h,\omega)}, \tag{3.12}$$

with

$$H_0(h,\omega) = (D + \sqrt{\lambda}h^{-(1+\delta)}\omega)^2, \tag{3.13}$$

and

$$H(h,\omega) = H_0(h,\omega) + h^{-2(1+\delta)}V(h^{-\delta}x). \tag{3.14}$$

So, by (3.11), we have to find the asymptotics of $\Omega^{\pm}(h,\omega)\chi_0(h^\epsilon D)\Phi$. For the sake of simplicity, we only treate the case (+).

Step 2 :

Now, we follow the same strategy as in [11]. In order to be self-contained, let us briefly recall the basic ideas.

We construct, for a suitable sequence (ν_k) defined below, a modifier $J^+(h,\omega)$ as a pseudodifferential operator close to Isozaki-Kitada's construction [5] , (actually, $J^+(h,\omega)$ will be a differential operator), and having the asymptotic expansion :

$$J^+(h,\omega) = \ op\left(1 + \sum_{k\geq 1}\ h^{\nu_k}\ d_k^+(x,\xi,\omega)\right). \tag{3.15}$$

We denote :

$$T^+(h,\omega) = H(h,\omega)J^+(h,\omega) - J^+(h,\omega)H_0(h,\omega). \tag{3.16}$$

Assume that the modifier $J^+(h,\omega)$ satisfies :

$$\Omega^+(h,\omega)\ =\ s - \lim_{t\to+\infty}\ e^{itH(h,\omega)}\ J^+(h,\omega)\ e^{-itH_0(h,\omega)}. \tag{3.17}$$

If (3.17) is satisfied, differentiating and integrating, we have :

$$\Omega^+(h,\omega) - J^+(h,\omega)\ =\ i\ \int_0^{+\infty}\ e^{itH(h,\omega)}\ T^+(h,\omega)\ e^{-itH_0(h,\omega)}\ dt. \tag{3.18}$$

So, the idea is to construct the modifier $J^+(h,\omega)$, (i.e on has to choose ν_k and d_k^+) in such a way (3.17) is satisfied and with $T^+(h,\omega) = O(h^\infty)$. Then, roughly speaking, by (3.18), we will be able to show that :

$$\Omega^+(h,\omega) - J^+(h,\omega) \ = O(h^\infty). \tag{3.19}$$

Thus, using (3.19) the asymptotic expansion of $\Omega^+(h,\omega)\chi_0(h^\epsilon D)\Phi$ will come from the asymptotics of $J^+(h,\omega)\chi_0(h^\epsilon D)\Phi$ which is explicit.

Now, let us determine the modifier $J^+(h,\omega)$. A direct calculation shows that the symbol of $T^+(h,\omega)$ is given by :

$$T^+(x,\xi,h,\omega) = h^{-1-\delta} \left(- \sum_{k\geq 1} 2i\sqrt{\lambda}\omega\cdot\nabla d_k \ h^{\nu_k} + (2i\xi\cdot\nabla d_k + \Delta d_k) \ h^{\nu_k+1+\delta} + \sum_{k\geq 1} V_k \ h^{-(1+\delta)+\delta\rho_k} + \sum_{j,k\geq 1} V_j \ d_k \ h^{-(1+\delta)+\delta\rho_j+\nu_k} \right). \tag{3.20}$$

In order to obtain $T^+(h,\omega) = O(h^\infty)$, we choose $\nu_1 = \delta(\rho_1 - 1) - 1$ and we solve the transport equation :

$$\omega.\nabla d_1^+(x,\omega) = \frac{1}{2i\sqrt{\lambda}} V_1(x). \tag{3.21}$$

The solution of (3.21) is given by :

$$d_1^+(x,\omega) = \frac{i}{2\sqrt{\lambda}} \int_0^{+\infty} \ V_1(x+t\omega) \ dt. \tag{3.22}$$

Then, we choose $\nu_2 = min\ (\nu_1 + 1 + \delta,\ -(1+\delta) + \delta\rho_1,\ 2\nu_1)$ and we solve the corresponding transport equation in order to construct d_2. The functions d_k and the coefficients ν_k for $k \geq 3$ are determined in this recursive way.

We have the following proposition, (the proof is the same as in [11]) :

Proposition 4

$$\Omega^+(h,\omega)\chi_0(h^\epsilon D) \ \Phi = \lim_{t\to+\infty} e^{itH(h,\omega)} \ J^+(h,\omega) \ e^{-itH_0(h,\omega)}\chi_0(h^\epsilon D) \ \Phi. \tag{3.23}$$

$$|| \ (\Omega^+(h,\omega) - J^+(h,\omega)) \ \chi_0(h^\epsilon D) \ \Phi \ ||= \ O \ (h^\infty). \tag{3.24}$$

Then, Theorem 3 follows from Proposition 3. We refer the reader to [11] for a complete exposition.

3.2.4 Applications.

Let V^j, $j = 1, 2$, be two potentials satisfying (H_2) and let $S_j(h)$, $j = 1, 2$ be the scattering operator associated with the pair $(H_0(h) + V^j, H_0(h))$.

We have the following result :

Corollary 5
For $n \geq 3$, assume that in $\mathcal{B}(L^2(I\!R^n))$,

(3.25) $$S_1(h)\chi(H_0(h)) = S_2(h)\chi(H_0(h)) + O(h^\infty) \ , \ h \to 0.$$

Then : $V^1 - V^2 \in \mathcal{S}(I\!R^n)$.

Proof :
Corollary 5 follows from Theorem 2, the uniqueness for the Radon transform in hyperplanes [4], and the fact that $|| \ \Phi_{h,\omega} \ ||$, (resp. $|| \ \Psi_{h,\omega} \ ||$) are uniformly bounded with respect to h. We refer to [11] for the details. □

Comments.
It is not difficult to generalize the previous results to the case of long-range potentials, i.e for potentials satisfying (H_2) with $\rho_1 > 0$, using modified wave operators, close to Isozaki-Kitada's ones [4], (see [11], Theorem 8, for details).

Acknowledgments.
The author is grateful to Xue Ping Wang who suggests him to study the inverse scattering problem in the semiclassical setting. The author wants also to thank warmly Jean-François Bony for his valuable help to prove Theorem 1 in Section 2.

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