On uniqueness of radial potentials for given Dirichlet spectra with distinct angular momenta

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Abstract

We consider an inverse spectral problem for radial Schrödinger operators with singular potentials. First, we show that the knowledge of the Dirichlet spectra for infinitely many angular momenta ℓ satisfying a Müntz-type condition uniquely determines the potential. Next, in a neighborhood of the zero potential, we prove local uniqueness from two Dirichlet spectra associated with distinct angular momenta in the cases $(\ell_1, \ell_2) = (0, 1)$, $(1, 2)$ and $(0, 3)$. Our approach relies on an explicit analysis of the associated singular differential equation, combined with the classical Kneser–Sommerfeld formula. These results sharpen a theorem of Carlson-Shubin (1994) and confirm, in the linearized setting and for these configurations, a conjecture originally formulated by Rundell and Sacks (2001).

1 Introduction

Inverse spectral problems for Schrödinger operators play a central role in quantum mechanics and mathematical physics, particularly in determining unknown potentials from spectral measurements. A classical example arises when the Schrödinger equation is considered in three spatial dimensions with a spherically symmetric potential in the unit ball of $\mathbb{R}^3$. In such cases, a standard separation of variables in spherical coordinates reduces the original partial differential equation to a family of one-dimensional singular Sturm-Liouville problems of the form

$$-\frac{d^2u}{dr^2} + \left(\frac{\ell(\ell+1)}{r^2} + q(r)\right) u = \lambda u\,, \qquad r \in (0,1)\,, \tag{1.1}$$

where $\ell \in \mathbb{N}$ is the angular momentum quantum number, and the radial potential q is assumed to be real-valued and square-integrable on $(0, 1)$. This equation is supplemented with a Dirichlet boundary condition at $r = 1$ and a regularity condition at the origin, typically of the form $u(r) = O(r)$ as $r \to 0$, which ensures square-integrability of the solution and reflects the physical finiteness of the wavefunction near the origin.

Such reduced radial problems are not only mathematically rich but also physically relevant. They appear in a variety of contexts including quantum scattering in bounded domains, the study of acoustic modes in stellar interiors, and in zonal decompositions of Laplace-type operators on spheres. See, for instance, [6, 27].

A foundational result in regular inverse spectral theory is the Borg–Levinson theorem, which asserts that a potential can be uniquely recovered from the knowledge of the Dirichlet spectrum together with the associated norming constants, (that is, the Neumann data of the associated eigenfunctions; see below for further details). In the setting of singular Sturm–Liouville equations such as (1.1), Carlson [6] extended this result to include centrifugal singularities, establishing uniqueness under suitable conditions on the spectrum and norming data.

However, these classical results rely crucially on the availability of norming constants. In contrast, the present work investigates a different question, physically more relevant: can one determine the potential q uniquely from the knowledge of the Dirichlet spectra corresponding to distinct values of ℓ, *without* any norming constants?

Before addressing this question, let us recall some spectral properties of the underlying operator. For each fixed angular momentum $\ell \in \mathbb{N}$ and potential $q \in L^2(0,1)$, the Dirichlet spectrum is given by the zeros of an entire function (the so-called regular solution; see Appendix A), and consists of a countable sequence of simple, real eigenvalues $\{\lambda_{\ell,n}(q)\}_{n\geq 1}$ tending to infinity. These properties are well established in the literature [5, 6] and form the basis for both direct and inverse spectral analysis in the radial setting.

This observation naturally raises the question of whether combining spectral data from distinct angular momenta can lead to uniqueness. Indeed, the spectrum corresponding to a single value of ℓ does not determine the potential uniquely: the associated isospectral set is locally infinite-dimensional in $L^2(0,1)$ [6, 11, 23, 28]. By contrast, Carlson and Shubin [5] showed that combining spectral data from two distinct angular momenta $\ell_1 \neq \ell_2$ significantly reduces the ambiguity. In particular, they proved that the corresponding isospectral set has finite local dimension when $\ell_2 - \ell_1$ is odd.

Rundell and Sacks [27] went further and conjectured that the potential is uniquely determined by the Dirichlet spectra corresponding to any two distinct values of ℓ, regardless of the parity of their difference. This reflects the fact that the angular dependence introduced via the centrifugal term $\ell(\ell+1)/r^2$ encodes genuinely distinct spectral information for each angular momentum channel, which can be leveraged to resolve the inverse problem even in the absence of norming constants.

Before turning to the more delicate case of two angular momenta, let us first consider the situation where Dirichlet spectra are known for infinitely many values of ℓ. Our first theorem is a global uniqueness result and concerns the case where one knows the Dirichlet spectra for infinitely many angular momenta ℓ satisfying a Müntz-type condition. In fact, this result is a direct consequence of Theorem 2.1 established in [24] in the framework of scattering theory; see also ([7], Theorem 1.4) for local uniqueness results and [9] for an extension to magnetic fields. The proof of Theorem 1.1 is given in Appendix A , where we establish a correspondence between Dirichlet spectral data and scattering phases.

Theorem 1.1 (Global uniqueness)**.** *Let $(\ell_k)_{k\geq 1}$ be a strictly increasing sequence of positive integers such that*

$$\sum_{k=1}^{\infty} \frac{1}{\ell_k} = +\infty\,.$$

Then the potential $q \in L^2(0,1)$ is uniquely determined by the Dirichlet spectra $\{\lambda_{\ell_k,n}(q)\}_{k,n\geq 1}$.

Having established this global uniqueness result in the case of infinitely many spectra, we now turn to the more delicate situation involving only finitely many angular momenta. Because of its singular structure and the nonlinear dependence of the spectrum on the potential, the full inverse problem poses significant analytical challenges. A natural first step is to consider its linearization around a reference potential, typically $q \equiv 0$. Let $\psi_{\ell,n}(x)$ denote the n-th normalized eigenfunction associated with angular momentum ℓ and potential q. Then the Fréchet derivative of the corresponding eigenvalue with respect to q is given by

$$D_q \lambda_{\ell,n}(q)\,\zeta = \int_0^1 \zeta(x)\,\psi_{\ell,n}(x)^2\,dx\,, \tag{1.2}$$

where ζ denotes a perturbation of the potential, (see [27]). The norming constants are defined by $\gamma_n := (\psi'_{\ell,n}(1))^2$. In the linearized setting, the requirement that two potentials q and $\check{q}$ yield the same eigenvalues for two angular momenta $\ell_1 \neq \ell_2$ implies that $\zeta = \check{q} - q$ must be orthogonal to the family of squared eigenfunctions corresponding to both ℓ_1 and ℓ_2, evaluated at $q = 0$.

In the unperturbed case $q \equiv 0$, the eigenfunctions of (1.1) can be expressed explicitly in terms of Bessel functions of the first kind. Specifically, for each $\ell \in \mathbb{N}$, the eigenfunctions take the form (up to normalization)

$$\psi_{\ell,n}(r) := \sqrt{\frac{\pi r}{2}}\, J_\nu(j_{\nu,n} r)\,, \qquad \nu = \ell + \tfrac{1}{2}\,, \quad n \geq 1\,,$$

where J_ν denotes the Bessel function of order ν, and $j_{\nu,n}$ is its n-th positive zero. These functions satisfy Dirichlet boundary conditions at $r = 1$, and regularity at the origin is ensured by the behavior $J_\nu(r) \sim r^\nu$ as $r \to 0$.

This leads to the problem of determining whether the span of the functions

$$\big\{\psi^2_{\ell,n}(x) \,:\, n \geq 1,\ \ell \in \{\ell_1, \ell_2\}\big\}$$

is dense in $L^2(0,1)$. Establishing this completeness property in the linearized setting is a first step in order to prove the local uniqueness of the potential near $q = 0$.

Even in the more favorable case where the angular momentum ℓ takes infinitely many values, it is not known whether the vector space spanned by the functions $\psi^2_{\ell,n}(x)$ is dense in $L^2(0,1)$, despite the fact that the problem is expected to be highly overdetermined. Although global uniqueness can now be derived by other methods, see Appendix A, the following result remains of independent interest.

Theorem 1.2 (Completeness). *Let $(\ell_k)_{k\geq 1}$ be a strictly increasing sequence of positive integers such that*

$$\sum_{k=1}^{\infty} \frac{1}{\ell_k} = +\infty\,.$$

Then the family

$$\big\{\psi^2_{\ell_k,n}(x) : k \geq 1,\ n \geq 1\big\}$$

is complete in $L^2(0,1)$.

The proof of this theorem is given in Section 3.2.

Independently of the completeness result, the asymptotic behavior of the eigenvalues imposes further constraints on admissible perturbations. Specifically, as shown in [28, Proposition 3.1] (see also [27, Eq. (2.10)]), for each fixed angular momentum ℓ, the following expansion holds uniformly on bounded subsets of $L^2(0,1)$:

$$\lambda_{\ell,n}(q) = \big(n + \tfrac{\ell}{2}\big)^2 \pi^2 + \int_0^1 q(x)\,dx - \ell(\ell+1) + \tilde{\lambda}_{\ell,n}(q)\,, \tag{1.3}$$

with $(\tilde{\lambda}_{\ell,n}(q) \in \ell^2_{\mathbb{R}}(\mathbb{N})$. Thus, if two potentials q and $\check{q}$ produce identical eigenvalues for large n, their difference $\zeta = \check{q} - q$ must satisfy

$$\int_0^1 \zeta(x)\,dx = 0\,.$$

This condition reflects the fact that the mean of the potential appears as the subleading term in the asymptotic expansion and must therefore be preserved under any isospectral deformation.

We conclude this section with the main result of the paper. It shows that, in three specific cases, the Dirichlet spectra corresponding to two distinct angular momenta uniquely determine the potential in a neighborhood of the zero potential in the $L^2(0,1)$ topology.

Theorem 1.3 (Local uniqueness near the zero potential). *For $(\ell_1, \ell_2) \in \{(0,1),(1,2),(0,3)\}$, the knowledge of the Dirichlet spectra corresponding to angular momenta ℓ_1 and ℓ_2 uniquely determines the potential $q \in L^2(0,1)$ in a L^2-neighborhood of the zero potential.*

Remark 1.4. *The case $(\ell_1, \ell_2) = (0,2)$ is also investigated in this paper. We show that the differential of the spectral map (see Section 2 for its definition) is injective at $q = 0$. We conjecture that this differential is an isomorphism. If this is the case, Theorem 1.3 would also hold for $(\ell_1, \ell_2) = (0,2)$ by the local inversion theorem. We leave this question as an open problem.*

Our approach is based on a detailed analysis of the spectral problem associated with (1.1) for varying values of ℓ, in combination with the classical Kneser–Sommerfeld expansion. This representation plays a central role in establishing connections between the spectral data corresponding to different angular momenta, and in proving completeness results that support uniqueness in the linearized regime. We believe that this uniqueness result holds for any pair of distinct angular momenta (ℓ_1, ℓ_2).

2 Strategy of the Proof

2.1 The spectral map

We define a spectral map involving the renormalized eigenvalues $\widetilde{\lambda}_{\ell,n}(q)$ defined by the asymptotic expansion (1.3) and associated with two distinct angular momenta. We then prove that this spectral map is an immersion at the zero potential and consequently locally injective at this point.

Let $\ell_1 \neq \ell_2$ be two fixed non-negative integers. Consider the spectral map

$$\mathcal{S}_{\ell_1,\ell_2} \,:\, L^2(0,1) \longrightarrow \mathbb{R} \times \ell^2_{\mathbb{R}}(\mathbb{N}) \times \ell^2_{\mathbb{R}}(\mathbb{N}),$$

defined by

$$\mathcal{S}_{\ell_1,\ell_2}(q) = \left(\int_0^1 q(x)\, dx \,,\ \big(\widetilde{\lambda}_{\ell_1,n}(q)\big)_{n\geq 1} ,\ \big(\widetilde{\lambda}_{\ell_2,n}(q)\big)_{n\geq 1} \right). \tag{2.1}$$

The following proposition, proved in [28, Theorems 2.3 and 4.1] and based on (1.2), states that the spectral map is real-analytic and provides an explicit expression for its Fréchet differential at the zero potential.

Before stating the proposition, we introduce the following notation.

Notation 2.1. *The spherical Bessel function j_ℓ is defined by*

$$j_\ell(z) = \sqrt{\frac{\pi z}{2}}\, J_\nu(z), \qquad \nu = \ell + \tfrac{1}{2},$$

where J_ν denotes the Bessel function of the first kind. For $n \in \mathbb{N}$, we set

$$g_{\ell,n}(x) = \frac{j_\ell(j_{\nu,n} x)}{\|\, j_\ell(j_{\nu,n}\, \cdot)\, \|_{L^2(0,1)}}.$$

Proposition 2.1. *Let $\ell_1 \neq \ell_2$ be two fixed non-negative integers. The spectral map*

$$\mathcal{S}_{\ell_1,\ell_2} : L^2(0,1) \to \mathbb{R} \times \ell^2_{\mathbb{R}}(\mathbb{N}) \times \ell^2_{\mathbb{R}}(\mathbb{N})$$

is real-analytic. Moreover, for every $\zeta \in L^2(0,1)$, its Fréchet differential at the zero potential is given by

$$d_0\mathcal{S}_{\ell_1,\ell_2}(\zeta) = \Big(\langle \zeta, 1\rangle\,, \quad \big(\langle \zeta, g^2_{\ell_1,n} - 1\rangle\big)_{n\geq 1}, \quad \big(\langle \zeta, g^2_{\ell_2,n} - 1\rangle\big)_{n\geq 1}\Big)\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(0,1)$.

We now state the linearized uniqueness result in terms of the spectral map and its differential.

Theorem 2.2 (Injectivity of the differential of the spectral map)**.** *Let $\ell_1 \neq \ell_2$ be two distinct non-negative integers, and consider the spectral map*

$$\mathcal{S}_{\ell_1,\ell_2} : L^2(0,1) \to \mathbb{R} \times \ell^2_{\mathbb{R}}(\mathbb{N}) \times \ell^2_{\mathbb{R}}(\mathbb{N})\,.$$

Then the Fréchet differential at the zero potential $q = 0$

$$d_0\mathcal{S}_{\ell_1,\ell_2} \;:\; L^2(0,1) \to \mathbb{R} \times \ell^2_{\mathbb{R}}(\mathbb{N}) \times \ell^2_{\mathbb{R}}(\mathbb{N})\,,$$

is injective, provided that $(\ell_1, \ell_2) \in \{(0,1)\,,\,(0,2)\,,\,(1,2)\,,\,(0,3)\}$.

The proof of Theorem 2.2 is given in Sections 7-9.

As a byproduct of our analysis, we obtain the following theorem in the case $\ell_1 = 0$ or 1 and arbitrary ℓ_2.

Theorem 2.3 (Finite-dimensional kernel)**.** *Let $\ell_1 = 0$ or 1, and $\ell_2 \geq 1$. Consider*

$$\mathcal{S}_{\ell_1,\ell_2} : L^2(0,1) \longrightarrow \mathbb{R} \times \ell^2_{\mathbb{R}}(\mathbb{N}) \times \ell^2_{\mathbb{R}}(\mathbb{N})\,.$$

Then the Fréchet differential of $\mathcal{S}_{\ell_1,\ell_2}$ at the zero potential $q = 0$ has a finite-dimensional kernel.

Sketch of proof. Our approach shows that any element of this kernel satisfies a homogeneous linear ODE on $(0,1)$, whose order and coefficients depend only on ℓ_2. The cases $(\ell_1,\ell_2) = (0,1)\,,\,(0,2)\,,\,(1,2)$ and $(0,3)$ are treated in detail in this paper, and the extension to arbitrary ℓ_2 follows without additional difficulty. Consequently, the kernel is spanned by finitely many fundamental solutions and is therefore finite-dimensional. □

Remark 2.4. *This result extends Theorem 1 of Shubin Christ [29] in the case $(\ell_1,\ell_2) = (0,1)$, as well as the result of Carlson–Shubin [5, Theorem 1.1] in the case where $\ell_1 - \ell_2$ is odd, at the level of the Fréchet differential at the zero potential $q = 0$. In these works, the authors proved that the Fréchet differential $d_q\mathcal{S}_{\ell_1,\ell_2}$ has a finite-dimensional kernel for arbitrary potentials q. We conjecture that the same conclusion holds for all pairs of distinct nonnegative integers $\ell_1 \neq \ell_2$.*

We conclude this section with the following result, whose proof is given in Appendix B.

Theorem 2.5 (Closed range)**.** *Let $(\ell_1,\ell_2) \in \{(0,1)\,,\,(1,2)\,,\,(0,3)\}$. Then the differential of the spectral map at the origin, $d_0\mathcal{S}_{\ell_1,\ell_2}$, has closed range. More precisely, in the case $(\ell_1,\ell_2) = (0,1)$, it is surjective, and its range coincides with the entire space*

$$\mathbb{R} \times \ell^2_{\mathbb{R}}(\mathbb{N}) \times \ell^2_{\mathbb{R}}(\mathbb{N})\,.$$

Consequently, in the case $(0,1)$, the standard local inverse function theorem in Banach spaces, together with Theorem 2.2 and Theorem 2.5, implies Theorem 1.3. More precisely, in this case the spectral map provides a local real-analytic coordinate system.

The cases $(1,2)$ and $(0,3)$ are slightly different. For this reason, we recall below the following local injectivity result, which is a direct consequence of the mean value theorem and the open mapping theorem (see, for instance, [1], Theorem 2.5.10).

Proposition 2.6 (Local injectivity)**.** *Let X and Y be Banach spaces, and let*

$$\mathcal{S} : U \subset X \longrightarrow Y$$

be a $\mathcal{C}^1$ map defined on an open neighborhood U of a point $x_0 \in X$. Assume that the Fréchet differential $d_{x_0}\mathcal{S} : X \to Y$ is injective and has closed range. Then there exists a neighborhood $V \subset U$ of x_0 such that $\mathcal{S}$ is injective on V.

Consequently, Theorem 1.3 follows from the previous proposition together with Theorem 2.2 and Theorem 2.5.

3 The Kneser–Sommerfeld formula

The so-called Kneser–Sommerfeld expansion is a classical identity involving Bessel functions and their zeros. It appears in Watson's treatise [34] (Section 15.42), where it is stated as a Fourier–Bessel series identity. However, as first noted by Buchholz in 1947 [3], and more recently clarified by Martin [19], the formula given in Watson is incorrect: it is missing an integral term on the right-hand side.

This correction was independently rediscovered in 1981 by Kobayashi [13], who analyzed the same identity and showed that Watson's version could not be valid in general. However, a proper historical credit for the correction belongs to Buchholz, whose 1947 paper already pointed out the failure and provided a corrected structure.

In his detailed analysis, Martin [19] confirms that a different version - found in earlier works by Kneser [14] for $\nu = 0$, Sommerfeld [30] for integer values of ν, and Carslaw [4] - is in fact the correct identity. Surprisingly, Watson does not include this valid form, despite its mathematical consistency and symmetry.

The corrected version of the Kneser–Sommerfeld expansion reads:

$$\sum_{n=1}^{\infty} \frac{J_\nu(xj_{\nu,n})J_\nu(Xj_{\nu,n})}{(z^2 - j_{\nu,n}^2)\,[J_\nu'(j_{\nu,n})]^2} = \frac{\pi}{4J_\nu(z)} J_\nu(xz)\,[J_\nu(z)Y_\nu(Xz) - Y_\nu(z)J_\nu(Xz)]\ , \tag{3.1}$$

valid for $0 \le x \le X \le 1$, $\nu \in \mathbb{R}$, $z \neq j_{\nu,n}$, where J_ν and Y_ν are the Bessel functions of the first and second kinds, respectively, and $j_{\nu,n}$ denotes the n-th positive zero of J_ν.

The identity (3.1) plays a fundamental role in the analysis of our inverse spectral problem.

3.1 An integral Green identity

In the next subsection, we will present a first completeness theorem for the inverse problem under consideration. The argument relies solely on the moment formula (3.4) together with the Müntz–Szász theorem, and yields uniqueness under a rather restrictive vanishing condition on the moments.

So, we begin by considering a function $\zeta \in L^2(0,1)$ that is orthogonal to the family of squared eigenfunctions associated with a fixed angular momentum $\ell \in \mathbb{N}$ and vanishing potential. That is,

$$\int_0^1 x\,\zeta(x)\,[J_\nu(j_{\nu,n}x)]^2\,dx = 0\,, \qquad \text{for all } n \geq 1\,, \tag{3.2}$$

where $\nu = \ell + \frac{1}{2}$ and $j_{\nu,n}$ denotes the n-th positive zero of the Bessel function J_ν.

Our goal is to exploit the structure of the family $\left\{[J_\nu(j_{\nu,n}x)]^2\right\}_{n\geq 1}$ using the Kneser–Sommerfeld expansion (3.1), and to derive an additional integral identity satisfied by ζ.

To that end, we consider a weighted sum of the orthogonality relations (3.2) over all $n \geq 1$. By exchanging the order of summation and integration (justified, for instance, by Fubini's theorem), we obtain for all z not equal to any zero $j_{\nu,n}$:

$$\int_0^1 x\,\zeta(x)\,J_\nu(xz)\,[J_\nu(z)\,Y_\nu(xz) - Y_\nu(z)\,J_\nu(xz)]\,dx = 0\,. \tag{3.3}$$

The expression inside the integral corresponds, up to a multiplicative constant, to the kernel appearing in the corrected Kneser–Sommerfeld expansion on the diagonal $x = X$ multiplied by $J_\nu(z)$. By a standard continuity argument, this identity extends to all complex values of z.

Remark 3.1. *The integral kernel appearing in* (3.3) *coincides with the gradient with respect to q of the so-called regular solution evaluated at $q = 0$ and $x = 1$ (see [28], Proposition 2.2). This observation is consistent with Lemma A.3 of the present paper.*

3.2 A first completeness result

In this subsection we prove Theorem 1.2, the completeness result stated in the introduction. Our strategy is to establish first an integral identity, which will then allow us to deduce a moment condition. Combining this with the classical Müntz–Szász theorem ultimately leads to the desired completeness result, which appears as a direct consequence of the final theorem in this section.

Recall that, as $x \to 0$, the Bessel functions admit the following asymptotic behavior (see [17]):

$$J_\nu(x) \sim \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu\,, \quad Y_\nu(x) \sim -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu \quad \text{for } \nu > 0\,.$$

Thus, taking the limit $z \to 0$ in (3.3), we obtain the following closed-form expression:

$$\int_0^1 x\;\zeta(x)\;\left(1 - x^{2\nu}\right)dx = 0\,. \tag{3.4}$$

We can now state a first completeness result based on the classical Müntz–Szász theorem ([20, 21, 31]). It is worth emphasizing that, in the following theorem, no zero-mean condition is required for the function ζ.

Theorem 3.2 (A completeness result)**.** *Let $\zeta \in L^2(0,1)$ be a real-valued function satisfying (3.2) for an infinite increasing sequence $\nu_k = \{\ell_k + \frac{1}{2}\}$ of positive half-integers such that*

$$\sum_{k=1}^{\infty} \frac{1}{\ell_k} = +\infty\,.$$

Then $\zeta = 0$ almost everywhere in $(0,1)$.

Proof. Let $\{\ell_k\} \subset \mathbb{N}^*$ be the sequence of positive integers appearing above, with $\sum_{k=1}^{\infty} \frac{1}{\ell_k} = +\infty$, and apply identity (3.4) with $\nu_k = \ell_k + \frac{1}{2}$. This yields, as $k \to +\infty$,

$$\int_0^1 x\,\zeta(x)\,dx = 0\,.$$

Plugging this back into (3.4), we obtain the moment identities

$$\int_0^1 \zeta(x)\,x^{2\ell_k+2}\,dx = 0 \quad \text{for all } k \in \mathbb{N}\,.$$

By the classical Müntz–Szász theorem, this implies that $\zeta = 0$ almost everywhere in $(0,1)$. □

As explained at the beginning of this section, Theorem 3.2 immediately yields Theorem 1.2, thereby completing the proof of the first completeness result.

4 Transformation Operators

Transformation operators play a central role in our inverse spectral analysis. For the case $\ell = 1$, such operators were first introduced by Guillot and Ralston [11] in their study of radial Schrödinger operators. These operators were also used by C. Shubin Christ in [29]. Carlson and Shubin [5] extended this construction to more general settings, and Rundell and Sacks [27] developed a systematic framework valid for all integer orders $\ell \geq 1$. Their method rewrites inner products involving Bessel kernels in terms of trigonometric ones by means of suitable *index-reduction* operators.

We recall below two key lemmas due to Rundell and Sacks [27] (also used by Serier [28]). These lemmas provide the basic tools for reducing the Bessel case to the trigonometric one and will be crucial for the fine analysis of isospectral sets developed later.

Notation 4.1. *Let $\nu = \ell + \frac{1}{2}$ for $\ell \geq 0$ and $x \in [0,1]$. Define*

$$\Phi_\ell(x) \;=\; \frac{\pi x}{2}\,J_\nu(x)^2\,, \qquad \Psi_\ell(x) \;=\; -\,\frac{\pi x}{2}\,J_\nu(x)\,Y_\nu(x)\,,$$

where J_ν, Y_ν are the ordinary Bessel functions.

Lemma 4.1. *[Rundell–Sacks [27]] For each integer $\ell \geq 1$ and $x \in (0,1)$, define the* index-reduction operator

$$S_\ell[f](x) \;=\; f(x) - 4\ell\,x^{2\ell-1}\int_x^1 t^{-2\ell}\,f(t)\,dt\,. \tag{4.1}$$

Then:

(i) S_ℓ is bounded on $L^2(0,1)$.

(ii) Its Hilbert adjoint is

$$S_\ell^*[g](x) \;=\; g(x) - \frac{4\ell}{x^{2\ell}}\int_0^x t^{2\ell-1}\,g(t)\,dt\,. \tag{4.2}$$

(iii) It induces a Banach isomorphism

$$L^2(0,1) \;\xrightarrow{\sim}\; \{x \mapsto x^{2\ell}\}^{\perp},$$

with inverse

$$A_\ell[g](x) \;=\; g(x) - \frac{4\ell}{x^{2\ell+1}} \int_0^x t^{2\ell}\, g(t)\, dt\,. \tag{4.3}$$

(iv) If $g = S_\ell[f]$, then f and g satisfy

$$f^{(2\ell)}(x) + \frac{4\ell}{x}\, f^{(2\ell-1)}(x) \;=\; g^{(2\ell)}(x)\,. \tag{4.4}$$

(v) The operators commute pairwise:

$$S_\ell\, S_m \;=\; S_m\, S_\ell\,, \quad \forall\, \ell, m \in \mathbb{N}\,.$$

(vi) The functions Φ_ℓ and Ψ_ℓ satisfy

$$\Phi_\ell = -\, S_\ell^*\big[\Phi_{\ell-1}\big]\,, \qquad\qquad \Psi_\ell = -\, S_\ell^*\big[\Psi_{\ell-1}\big], \tag{4.5}$$

$$\Phi_\ell' = -\, A_\ell\big[\Phi_{\ell-1}'\big]\,, \qquad\qquad \Psi_\ell' = -\, A_\ell\big[\Psi_{\ell-1}'\big]\,. \tag{4.6}$$

We will also need the following complementary result (see Lemma 3.4 in [27]).

Lemma 4.2. *If for $\ell \geq 1$, f and $g \in L^2(0,1)$, it holds $g = S_\ell^*[f]$ then, in $\mathcal{D}'(0,1)$, we have*

$$g^{(2\ell+1)}(x) + \frac{4\ell}{x} g^{(2\ell)}(x) = f^{(2\ell+1)}(x)\,.$$

Proof. We follow the strategy of [27] for the proof of (4.4). We start from

$$g(x) = f(x) - 4\ell x^{-2\ell} \int_0^x s^{2\ell-1} f(s)\, ds\,, \tag{4.7}$$

and differentiate once to obtain

$$g'(x) = f'(x) + 8\ell^2 x^{-2\ell-1} \int_0^x s^{2\ell-1} f(s)\, ds - \frac{4\ell}{x} f(x)\,. \tag{4.8}$$

We then eliminate the integral term by considering $\frac{2\ell}{x} \times (4.7) + (4.8)$:

$$\frac{2\ell}{x} g(x) + g'(x) = \frac{2\ell}{x} f(x) + f'(x) - \frac{4\ell}{x} f(x)\,.$$

This leads to

$$2\ell g(x) + x g'(x) = -2\ell f(x) + x f'(x)\,.$$

Differentiating again, we get

$$(2\ell+1) g'(x) + x g''(x) = (-2\ell+1) f'(x) + x f''(x)\,,$$

and iterating k times we obtain

$$(2\ell+k) g^{(k)}(x) + x g^{(k+1)}(x) = (-2\ell+k) f^{(k)}(x) + x f^{(k+1)}(x)\,.$$

Taking $k = 2\ell$ and dividing by x achieves the proof of the lemma. □

We now consider the composite operator T_ℓ, obtained by composing the index-reduction operators $S_1, \ldots, S_\ell$, which carries Bessel kernels to trigonometric ones.

Lemma 4.3. *[Rundell–Sacks [27], Serier [28]] Let us define T_ℓ by $T_1 = S_1$ and for $\ell \geq 2$ by*

$$T_\ell \;=\; (-1)^{\ell+1}\, S_\ell\, S_{\ell-1} \cdots S_1\,.$$

Then:

(i) T_ℓ is a bounded and injective operator on $L^2(0,1)$. For any $\zeta \in L^2(0,1)$ and $z \in \mathbb{C}$,

$$\int_0^1 \big[2\,\Phi_\ell(zt) - 1\big]\,\zeta(t)\,dt = \int_0^1 \cos(2zt)\,T_\ell[\zeta](t)\,dt\,, \tag{4.9}$$

$$\int_0^1 \Psi_\ell(zt)\,\zeta(t)\,dt = -\tfrac{1}{2}\int_0^1 \sin(2zt)\,T_\ell[\zeta](t)\,dt\,. \tag{4.10}$$

(ii) Its adjoint satisfies

$$T_\ell^*[\cos(2zx)] \;=\; 2\,\Phi_\ell(zx) - 1\,, \qquad T_\ell^*\big[-\tfrac{1}{2}\sin(2zx)\big] \;=\; \Psi_\ell(zx)\,,$$

with $\ker(T_\ell^*) = \operatorname{span}\{x^2, x^4, \ldots, x^{2\ell}\}$.

(iii) T_ℓ provides a Banach isomorphism

$$L^2(0,1) \;\xrightarrow{\sim}\; \big(\ker T_\ell^*\big)^\perp\,,$$

whose inverse is

$$B_\ell[f] \;=\; (-1)^{\ell+1}\,A_\ell\,A_{\ell-1}\cdots A_1[f]\,.$$

Moreover,

$$\Phi_\ell'(zx) \;=\; B_\ell[-\sin(2zx)]\,, \qquad \Psi_\ell'(zx) \;=\; B_\ell[-\cos(2zx)]\,.$$

5 Green's identity via transformation operators

In this section, we shall use the transformation operators T_ℓ to reformulate relation (3.3) in a more convenient form. First, it is straightforward to verify that,

$$J_\nu(xz)^2 = \frac{2}{\pi\,z\,x}\,\Phi_\ell(zx)\,, \qquad J_\nu(xz)\,Y_\nu(xz) = -\frac{2}{\pi\,z\,x}\,\Psi_\ell(zx)\,.$$

Substituting these into (3.3) gives

$$J_\nu(z)\int_0^1 \Psi_\ell(zx)\,\zeta(x)\,dx \;+\; Y_\nu(z)\int_0^1 \Phi_\ell(zx)\,\zeta(x)\,dx \;=\; 0\,. \tag{5.1}$$

Moreover, for functions ζ orthogonal to the constants on $[0,1]$, i.e. satisfying $\int_0^1 \zeta(x)\,dx = 0$, it follows from (5.1) that

$$2\,J_\nu(z)\int_0^1 \Psi_\ell(zx)\,\zeta(x)\,dx \;+\; Y_\nu(z)\int_0^1 \big(2\,\Phi_\ell(zx) - 1\big)\,\zeta(x)\,dx \;=\; 0\,. \tag{5.2}$$

We now invoke Lemma 4.3(i) together with the identity

$$Y_\nu(x) = (-1)^{\,\ell+1} J_{-\nu}(x)$$

to deduce that, for such ζ, (5.2) can be equivalently written as

$$J_\nu(z) \int_0^1 T_\ell[\zeta](x)\, \sin(2zx)\, dx \;+\; (-1)^{\,\ell}\, J_{-\nu}(z) \int_0^1 T_\ell[\zeta](x)\, \cos(2zx)\, dx \;=\; 0\,. \tag{5.3}$$

We will now employ the half-integer Bessel functions and their associated polynomials P_ℓ and Q_ℓ (see [2, 10.1.19–20]). For $\ell = 0, 1, 2, \ldots$ and $z \in \mathbb{C}$,

$$J_{\ell+\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\, \Big[P_\ell\big(\tfrac{1}{z}\big)\, \sin z \;-\; Q_{\ell-1}\big(\tfrac{1}{z}\big)\, \cos z \Big]\,, \tag{5.4}$$

$$J_{-\,\ell-\frac{1}{2}}(z) = (-1)^{\,\ell} \sqrt{\frac{2}{\pi z}}\, \Big[P_\ell\big(\tfrac{1}{z}\big)\, \cos z \;+\; Q_{\ell-1}\big(\tfrac{1}{z}\big)\, \sin z \Big]\,. \tag{5.5}$$

The polynomials $P_\ell(t)$ and $Q_\ell(t)$, each of degree ℓ in t, satisfy the three-term recursion

$$P_{\ell+1}(t) = (2\ell+1)\, t\, P_\ell(t) \;-\; P_{\ell-1}(t)\,, \qquad \ell \geq 1, \tag{5.6}$$

$$Q_{\ell+1}(t) = (2\ell+3)\, t\, Q_\ell(t) \;-\; Q_{\ell-1}(t)\,, \qquad \ell \geq 0, \tag{5.7}$$

with initial conditions

$$P_0(t) = 1\,, \quad P_1(t) = t\,, \qquad Q_{-1}(t) = 0\,, \quad Q_0(t) = 1\,.$$

Note that $P_\ell(t)$ and $Q_\ell(t)$ are even polynomials if ℓ is even, and odd polynomials if ℓ is odd.

As a concrete example, the simplest half-integer Bessel functions are

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\, \sin z\,, \qquad J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\, \cos z\,. \tag{5.8}$$

The next case corresponds to order $\pm\frac{3}{2}$:

$$J_{\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left(\frac{\sin z}{z} - \cos z \right)\,, \qquad J_{-\frac{3}{2}}(z) = \sqrt{\frac{2}{\pi z}} \left(-\frac{\cos z}{z} - \sin z \right)\,. \tag{5.9}$$

The first few polynomials $P_\ell(t)$ and $Q_\ell(t)$ generated by the recursion relations are:

$$\begin{aligned} P_0(t) &= 1\,, & Q_{-1}(t) &= 0\,, \\ P_1(t) &= t\,, & Q_0(t) &= 1\,, \\ P_2(t) &= 3t^2 - 1\,, & Q_1(t) &= 3t\,, \\ P_3(t) &= 15t^3 - 6t\,, & Q_2(t) &= 15t^2 - 1\,. \end{aligned} \tag{5.10}$$

From the foregoing we obtain the following:

Proposition 5.1. *Let $\zeta \in L^2(0,1)$ be orthogonal to the constants and suppose further that for $\nu = \ell + \frac{1}{2}$,*

$$\int_0^1 x\,\zeta(x)\,J_\nu\big(j_{\nu,n}\,x\big)^2\,dx \;=\; 0\,, \qquad \text{for all } n \geq 1\,.$$

Then for every $z \in \mathbb{C}$ and every integer $\ell \geq 1$,

$$\int_0^1 T_\ell[\zeta](x)\,\Big[P_\ell\big(\tfrac{1}{z}\big)\,\cos(z((2x-1)) \;-\; Q_{\ell-1}\big(\tfrac{1}{z}\big)\,\sin(z(2x-1))\Big]\,dx \;=\; 0\,. \tag{5.11}$$

Proof. The result follows immediately from (5.3), (5.4) and (5.5), by a straightforward application of elementary trigonometric identities. □

Remark 5.2. *In the foregoing we have relied on the transformation operators S_ℓ and T_ℓ defined for $\ell \geq 1$. For the case $\ell = 0$, (i.e $\nu = \frac{1}{2}$), one obtains an analogous reduction by using the explicit formulas (5.8). It follows that*

$$\Phi_0(x) = \frac{1-\cos(2x)}{2}\,, \qquad \Psi_0(x) = \tfrac{1}{2}\sin(2x)\,.$$

Hence, under the hypotheses of Proposition 5.1, it follows from (5.2) *that*

$$\int_0^1 \zeta(x)\,\cos\big(z(2x-1)\big)\,dx = 0 \quad \text{for all } z \in \mathbb{C}\,. \tag{5.12}$$

In other words, if one conventionally sets $T_0 = \mathrm{Id}$, then relation (5.11) *also extends to the case $\ell = 0$.*

Remark 5.3. *Note that if one multiplies* (5.11) *by z^ℓ and then sets $z = 0$, it follows that for $\ell \geq 1$,*

$$\int_0^1 T_\ell[\zeta](x)\,dx = 0\,.$$

6 From Green's integral identities to differential relations

Let $\zeta \in L^2(0,1)$ be orthogonal to the constants and satisfy the hypotheses of Proposition 5.1. Then, for each integer $\ell \geq 0$, we shall show that the integral identity (5.11) gives rise to a differential relation satisfied by the transformed function $T_\ell[\zeta]$.

We introduce the reflection symmetry about the midpoint $x = \frac{1}{2}$, defined by the involutive map

$$\sigma(x) := 1 - x\,.$$

This transformation naturally arises in several parts of the analysis, particularly when exploiting parity properties. Throughout the paper, we refer to this symmetry simply as σ.

6.1 The case $\ell = 0$

By standard Fourier series theory, the relation (5.12) implies that ζ is an odd function with respect to the midpoint $x_0 = \frac{1}{2}$, i.e.

$$\zeta(x) = -\,\zeta\big(\sigma(x)\big) \quad \text{for almost every } x \in [0,1]\,.$$

6.2 The case $\ell = 1$

According to (5.10), and setting $g = T_1(\zeta) = S_1(\zeta)$, equation (5.11) for $\ell = 1$ can be written as

$$\int_0^1 g(x) \left[\tfrac{1}{z} \cos\big(z(2x-1)\big) - \sin\big(z(2x-1)\big) \right] dx = 0 \,. \tag{6.1}$$

Now, we define

$$G(x) = \int_0^x g(t)\, \mathrm{d}t \,.$$

By Remark 5.3, $G(1) = 0$, so using integration by parts, we deduce

$$\int_0^1 \big[\, 2G(x) - g(x) \big] \sin\big(z(2x-1)\big)\, \mathrm{d}x = 0. \tag{6.2}$$

In other words, $2G(x) - g(x)$ is an even function with respect to $x = \frac{1}{2}$.

6.3 The case $\ell \geq 2$

Before stating the result, we briefly recall the notion of symmetry for distributions.

Definition 6.1. *Let $\sigma(x) := 1 - x$ denote the reflection with respect to the midpoint of the interval $[0, 1]$. This map induces a natural action on distributions $T \in \mathcal{D}'(0, 1)$ via pullback, defined by duality:*

$$\langle \sigma_* T, \varphi \rangle := \langle T, \varphi \circ \sigma \rangle, \qquad \textit{for all } \varphi \in \mathcal{D}(0, 1).$$

We say that a distribution $T \in \mathcal{D}'(0, 1)$ is even *if $\sigma_* T = T$, and* odd *if $\sigma_* T = -T$.*

Remark 6.2. *There is a classical characterization of even and odd distributions on the interval $(0, 1)$ using Fourier series adapted to the symmetry with respect to the midpoint $x = \frac{1}{2}$.*
Consider the orthonormal basis of $L^2(0, 1)$ formed by the functions

$$\phi_n(x) := \sqrt{2} \cos\big(\pi n(2x-1)\big), \quad n \geq 0 \;\; \textit{and} \;\; \psi_n(x) := \sqrt{2} \sin\big(\pi n(2x-1)\big), \quad n \geq 1 \,.$$

Each ϕ_n is even with respect to $x = \frac{1}{2}$, while each ψ_n is odd.
Then, a distribution $T \in \mathcal{D}'(0, 1)$ is odd with respect to $x = \frac{1}{2}$ if and only if

$$\langle T, \phi_n \rangle = 0 \qquad \textit{for all } n \geq 1 \,.$$

Similarly, T is even if and only if $\langle T, \psi_n \rangle = 0$ for all $n \geq 1$.
Equivalently, a distribution $T \in \mathcal{D}'(0, 1)$ is odd with respect to the midpoint $x = \frac{1}{2}$ if

$$\langle T, \cos\big(z(2x-1)\big) \rangle = 0 \qquad \textit{for all } z \in \mathbb{C} \,,$$

and it is even if

$$\langle T, \sin\big(z(2x-1)\big) \rangle = 0 \qquad \textit{for all } z \in \mathbb{C} \,.$$

We now introduce the following transformed polynomials:

$$\widetilde{P}_\ell(z) := z^\ell P_\ell\left(\frac{1}{z}\right), \qquad \widetilde{Q}_{\ell-1}(z) := z^\ell Q_{\ell-1}\left(\frac{1}{z}\right),$$

where $P_\ell(t)$ and $Q_{\ell-1}(t)$ are as previously defined.

By construction, $\widetilde{P}_\ell(z)$ is an even polynomial for all ℓ, while $\widetilde{Q}_{\ell-1}(z)$ is always odd, regardless of the parity of ℓ. We list below the first few examples:

$$\widetilde{P}_1(z) = 1, \qquad \widetilde{Q}_0(z) = z,$$
$$\widetilde{P}_2(z) = 3 - z^2, \qquad \widetilde{Q}_1(z) = 3z,$$
$$\widetilde{P}_3(z) = 15 - 6z^2, \qquad \widetilde{Q}_2(z) = 15z - z^3.$$

We also introduce the differential operator

$$\widetilde{D} := \frac{1}{2i}\frac{d}{dx},$$

so that for any polynomial $R(z) = \sum_k a_k z^k$, the corresponding operator $R(\widetilde{D})$ acts on a function f via

$$R(\widetilde{D})f(x) = \sum_k a_k (\widetilde{D}^k f)(x).$$

We now proceed by multiplying the identity (5.11) by z^ℓ. Under the assumptions of Lemma 5.1, we obtain the following identity.

Lemma 6.3. *Let $\zeta \in L^2(0,1)$ satisfy the hypotheses of Lemma 5.1, and let $g := T_\ell[\zeta]$. Then, one has for all $z \in \mathbb{C}$,*

$$\int_0^1 g(x) \cdot \left[\widetilde{P}_\ell(\widetilde{D}) + i\,\widetilde{Q}_{\ell-1}(\widetilde{D})\right] \cos(z(2x-1))\,dx = 0.$$

Remark 6.4. *As a consistency check, let us consider the case $\ell = 1$. In this case, one has*

$$\widetilde{P}_1(\widetilde{D}) + i\,\widetilde{Q}_0(\widetilde{D}) = 1 + \frac{1}{2}\frac{d}{dx}.$$

Thus, we obtain:

$$\int_0^1 g(x)\left(1 + \frac{1}{2}\frac{d}{dx}\right)\cos\big(z(2x-1)\big)\,dx = \int_0^1 g(x)\,\left[\cos\big(z(2x-1)\big) - z\sin\big(z(2x-1)\big)\right]\,dx = 0,$$

as already observed in (6.1).

In particular, Lemma 6.3 implies that the distribution

$$\left(\widetilde{P}_\ell(\widetilde{D}) - i\,\widetilde{Q}_{\ell-1}(\widetilde{D})\right) g(x)$$

is odd with respect to the midpoint $x = \frac{1}{2}$. This follows from the fact that $\widetilde{P}_\ell$ is an even polynomial, while $\widetilde{Q}_{\ell-1}$ is odd.

We are now left with understanding more precisely the differential operator

$$\left(\widetilde{P}_\ell(\widetilde{D}) - i\,\widetilde{Q}_{\ell-1}(\widetilde{D})\right).$$

Using the recursion relations satisfied by the polynomials P_ℓ and Q_ℓ, and introducing

$$\widetilde{A}_\ell(z) := \widetilde{P}_\ell(z) - i\,\widetilde{Q}_{\ell-1}(z)\,,$$

we obtain the following lemma.

Lemma 6.5. *For every integer $\ell \geq 1$, the polynomials $\widetilde{A}_\ell(z)$ satisfy the recursion relation*

$$\widetilde{A}_{\ell+1}(z) - (2\ell+1)\widetilde{A}_\ell(z) + z^2\widetilde{A}_{\ell-1}(z) = 0\,,$$

with initial values

$$\widetilde{A}_0(z) = 1\,, \qquad \widetilde{A}_1(z) = 1 - iz\,.$$

At this stage, the polynomials $\widetilde{A}_\ell(z)$ have complex coefficients. To simplify the situation, we set $A_\ell(t) = \tilde{A}_\ell(\frac{t}{2i})$. We obtain the following result, which also holds for $\ell = 0$ and $\ell = 1$:

Theorem 6.6. *Let $\{A_\ell(t)\}_{\ell\in\mathbb{N}}$ be the sequence of polynomials defined recursively by*

$$A_0(t) = 1\,, \qquad A_1(t) = 1 - \frac{t}{2}\,,$$

and for all $\ell \geq 1$,

$$A_{\ell+1}(t) = (2\ell+1)\,A_\ell(t) + \frac{t^2}{4}\,A_{\ell-1}(t)\,.$$

Suppose that ζ satisfies the hypotheses of Lemma 5.1. Then, in the sense of distributions, the function

$$A_\ell\big(D\big)\big[T_\ell[\zeta]\big]\,,$$

where $D = \frac{d}{dx}$, is odd with respect to the midpoint $x = \frac{1}{2}$.

Remark 6.7. *The next polynomials in the sequence are given by*

$$A_2(t) \;=\; \tfrac{1}{4}t^2 - \tfrac{3}{2}t + 3\,, \qquad A_3(t) \;=\; -\tfrac{1}{8}t^3 + \tfrac{3}{2}t^2 - \tfrac{15}{2}t + 15\,.$$

Remark 6.8. *The polynomials A_ℓ are well known. More precisely, if θ_ℓ denotes the* reverse Bessel polynomial *(also called the* reverse Bessel polynomial of degree ℓ*), then*

$$A_\ell(t) = \theta_\ell\left(-\frac{t}{2}\right), \qquad \ell \in \mathbb{N},$$

see, e.g., [22, 18.34]. In particular, A_ℓ admits the explicit expansion

$$A_\ell(t) = \sum_{k=0}^{\ell} \frac{(\ell+k)!}{(\ell-k)!\,k!}\,\frac{(-1)^{\ell-k}}{2^\ell}\,t^{\ell-k}\,. \tag{6.3}$$

Moreover, A_ℓ satisfies the following second order differential equation

$$t\,A_\ell''(t) + \big(t - 2\ell\big)A_\ell'(t) - \ell\,A_\ell(t) = 0\,. \tag{6.4}$$

7 Uniqueness result in the case $(\ell_1, \ell_2) = (0, 1)$

In this section, we assume that ζ satisfies the hypotheses of Proposition 5.1 for $\ell = 0$ and $\ell = 1$ and we aim to show that ζ vanishes almost everywhere on $(0,1)$. For $\ell = 0$, this assumption simply implies that ζ is odd with respect to the midpoint $x = \frac{1}{2}$. For $\ell = 1$, it means by Theorem 6.6 that the function $A_1(D)\big[T_1[\zeta]\big]$ is odd in the sense of distributions.

The aim is to prove that $y = \zeta'$ is a solution of a linear second-order differential equation on $(0,1)$ and satisfies the conditions

$$y\big(\tfrac{1}{2}\big) = y'\big(\tfrac{1}{2}\big) = 0 .$$

It then follows that y vanishes identically on $(0,1)$, and consequently ζ vanishes almost everywhere on $(0,1)$, since it is an odd function.

A straightforward computation yields

$$2A_1(D)\big[T_1[\zeta]\big](x) = 2A_1(D)\big[S_1[\zeta]\big](x) = -\zeta'(x) + \left(2 - \frac{4}{x}\right)\zeta(x) - 4(2x-1)\int_x^1 \frac{\zeta(t)}{t^2}\,dt . \tag{7.1}$$

We further compute, for all $x \in (0,1)$,

$$G(x) := D^2 A_1(D)\big[T_1[\zeta]\big](x) = A_1(D)\big[D^2 S_1[\zeta]\big](x) = A_1(D)\left[\zeta'' + \frac{4}{x}\zeta'\right](x)$$

where we used Lemma 4.1 (iv) with $l = 1$. Setting $y := \zeta'$, we obtain for all $x \in (0,1)$,

$$G(x) = -\frac{1}{2}y''(x) + \left(1 - \frac{2}{x}\right)y'(x) + \left(\frac{4}{x} + \frac{2}{x^2}\right)y(x) . \tag{7.2}$$

Recalling that G is odd on $(0,1)$, it satisfies

$$G(x) + G(1-x) = 0 \tag{7.3}$$

in the sense of distributions. Since y is even, the identities (7.2) and (7.3) imply that y satisfies a linear second-order differential equation on $(0,1)$. Moreover, since ζ is odd, $y' = \zeta''$ is also odd and $y'\left(\frac{1}{2}\right) = 0$. Finally, evaluating (7.1) at $x = \frac{1}{2}$, we obtain

$$y\big(\tfrac{1}{2}\big) = \zeta'\big(\tfrac{1}{2}\big) = -2A_1(D)\big[T_1[\zeta]\big]\big(\tfrac{1}{2}\big) - 6\zeta\big(\tfrac{1}{2}\big) = 0$$

since both ζ and $A_1(D)\big[T_1[\zeta]\big]$ are odd functions on $(0,1)$.

The function y thus satisfies a linear second-order differential equation on $(0,1)$, together with the conditions $y\big(\frac{1}{2}\big) = 0$ and $y'\big(\frac{1}{2}\big) = 0$. By the Cauchy–Lipschitz theorem, we conclude that $y \equiv 0$. Therefore $\zeta' = y = 0$, so ζ is constant. Since ζ is odd with respect to $x = \frac{1}{2}$, this constant must vanish, and $\zeta \equiv 0$. This completes the proof of the following completeness result.

Theorem 7.1. *Assume that $\zeta \in L^2(0,1)$ satisfies the hypotheses of Proposition 5.1 for $\ell = 0$ and $\ell = 1$. Then $\zeta = 0$ almost everywhere on $(0,1)$.*

In particular, this implies that the differential spectral map associated with the pair $(\ell_1, \ell_2) = (0,1)$ is injective.

8 Uniqueness result in the case $(\ell_1, \ell_2) = (0, 2)$

Throughout this subsection, we assume that ζ satisfies the hypotheses of Proposition 5.1 for both $\ell = 0$ and $\ell = 2$ and we aim to show that ζ vanishes almost everywhere on $(0,1)$. As before, the condition for $\ell = 0$ implies that ζ is odd with respect to the midpoint $x = \frac{1}{2}$ and for $\ell = 2$, it means by Theorem 6.6 that the function $A_2(D)\big[T_2[\zeta]\big]$ is odd in the sense of distributions.

In order to prove that ζ vanishes almost everywhere on $(0,1)$, we follow the same strategy as in the previous Subsection. The aim is to prove that $y = \zeta'$ is a solution of a linear fourth-order differential equation on $(0,1)$ and satisfies the conditions

$$y\big(\tfrac{1}{2}\big) = y'\big(\tfrac{1}{2}\big) = y''\big(\tfrac{1}{2}\big) = y^{(3)}\big(\tfrac{1}{2}\big) = 0\,.$$

It then follows that y vanishes identically on $(0,1)$, and consequently ζ vanishes almost everywhere on $(0,1)$, since it is an odd function.

Let us set $f := S_1[\zeta]$ and $g = -S_2[f]$, so that $g = T_2[\zeta]$. We now differentiate the expression $A_2(D)\big[g\big]$ four times. Since the application of an even number of derivatives preserves parity in the distributional sense, and since the operator $A_2(D)$ commutes with differentiation, it follows that

$$A_2(D)\big[f^{(4)}(x) + \tfrac{8}{x} f^{(3)}(x)\big]$$

is odd, where we have used Lemma 4.1 (iv) with $\ell = 2$. Using again Lemma 4.1 (iv), now with $\ell = 1$, and substituting into the above expression, we obtain that the function $A_2(D)\big[F\big]$ is odd in the sense of distributions, where for all $x \in (0,1)$,

$$F(x) := \zeta^{(4)}(x) + \frac{12}{x}\,\zeta^{(3)}(x) + \frac{24}{x^2}\zeta''(x) - \frac{24}{x^3}\zeta'(x)\,.$$

As in the cases $\ell = 0$ and $\ell = 1$, we now define $y(x) := \zeta'(x)$. Since ζ is odd with respect to the midpoint $x = \frac{1}{2}$, it follows that y is an even function. Applying the differential operator $4A_2(D) = D^2 - 6D + 12$ to the function F, we define

$$G(x) := 4A_2(D)[F(x)] = \left(\frac{d^2}{dx^2} - 6\frac{d}{dx} + 12\right)\left(y^{(3)} + \frac{12}{x}\,y'' + \frac{24}{x^2}y' - \frac{24}{x^3}y\right)$$

and we find:

$$\begin{aligned} G(x) = {} & \left(-\frac{288}{x^5} - \frac{432}{x^4} - \frac{288}{x^3}\right) y(x) + \left(\frac{288}{x^4} + \frac{432}{x^3} + \frac{288}{x^2}\right) y'(x) \\ & + \left(-\frac{96}{x^3} - \frac{72}{x^2} + \frac{144}{x}\right) y''(x) + \left(12 - \frac{72}{x}\right) y^{(3)}(x) \\ & + \left(-6 + \frac{12}{x}\right) y^{(4)}(x) + y^{(5)}(x)\,. \end{aligned}$$

By the symmetry result, we have that $G(x) + G(1-x) = 0$ in the sense of distributions. This implies that the function y satisfies a linear differential equation of order four, since the fifth derivative $y^{(5)}$ is odd with respect to the midpoint $x = \frac{1}{2}$.

We now prove that $y\big(\frac{1}{2}\big) = y'\big(\frac{1}{2}\big) = y''\big(\frac{1}{2}\big) = y^{(3)}\big(\frac{1}{2}\big) = 0$. First, since y is even, y' and $y^{(3)}$ are both odd and $y'\big(\frac{1}{2}\big) = y^{(3)}\big(\frac{1}{2}\big) = 0$. Recall that (see (3.9) in [27]), for all $x \in (0,1)$,

$$g(x) = T_2[\zeta](x) = -\zeta(x) - 12x\int_x^1 \frac{\zeta(t)}{t^2}\,dt + 24x^3\int_x^1 \frac{\zeta(t)}{t^4}\,dt\,.$$

Applying the differential operator $4A_2(D) = D^2 - 6D + 12$ to g, we obtain for all $x \in (0,1)$,

$$
\begin{aligned}
4A_2(D)[g](x) = &-\zeta''(x) + \left(6 - \frac{12}{x}\right)\zeta'(x) + \left(-\frac{48}{x^2} + \frac{72}{x} - 12\right)\zeta(x) \\
&+ 72(1-2x)\int_x^1 \frac{\zeta(t)}{t^2}\,dt + 144x(x-1)(2x-1)\int_x^1 \frac{\zeta(t)}{t^4}\,dt\,.
\end{aligned}
$$

Evaluating this expression at $x = \frac{1}{2}$ and using that ζ is odd and $y = \zeta'$, we obtain

$$
4A_2(D)\big[g\big]\big(\tfrac{1}{2}\big) = -\zeta''\big(\tfrac{1}{2}\big) - 18\zeta'\big(\tfrac{1}{2}\big) - 60\zeta\big(\tfrac{1}{2}\big) = -18\,y\big(\tfrac{1}{2}\big)\ .
$$

Since $4A_2(D)\big[g\big](x)$ is odd, we therefore conclude that $y\big(\frac{1}{2}\big) = 0$.

Proceeding in the same way, we compute

$$
\begin{aligned}
4D^2A_2(D)[g](x) = &-\zeta^{(4)}(x) + \left(6 - \frac{12}{x}\right)\zeta^{(3)}(x) + \left(-\frac{24}{x^2} + \frac{72}{x} - 12\right)\zeta''(x) \\
&+ \left(\frac{24}{x^3} + \frac{216}{x^2} - \frac{144}{x}\right)\zeta'(x) + \left(\frac{288}{x^3} - \frac{576}{x^2}\right)\zeta(x) \\
&+ 864(2x-1)\int_x^1 \frac{\zeta(t)}{t^4}\,dt\,.
\end{aligned}
$$

Recalling that ζ is odd, $y = \zeta'$ and $y\big(\frac{1}{2}\big) = 0$, we evaluate at $x = \frac{1}{2}$ and obtain

$$
4\,D^2A_2(D)\big[g\big]\big(\tfrac{1}{2}\big) = -\,y^{(3)}\big(\tfrac{1}{2}\big) - 18\,y''\big(\tfrac{1}{2}\big) + 36\,y'\big(\tfrac{1}{2}\big) + 768\,y\big(\tfrac{1}{2}\big) = -18\,y''\big(\tfrac{1}{2}\big)\ .
$$

Since $4\,D^2A_2(D)\big[g\big]$ is also an odd function, we conclude, as above, that $y''\big(\frac{1}{2}\big) = 0$.

In conclusion, y satisfies a fourth–order linear differential equation with the initial conditions

$$
y\big(\tfrac{1}{2}\big) = 0, \qquad y'\big(\tfrac{1}{2}\big) = 0, \qquad y''\big(\tfrac{1}{2}\big) = 0, \qquad y^{(3)}\big(\tfrac{1}{2}\big) = 0\,.
$$

The Cauchy–Lipschitz theorem then implies that $y \equiv 0$. Since $y = \zeta'$ and ζ is odd, this in turn forces $\zeta \equiv 0$. We thus obtain a completeness result in the case $(\ell_1, \ell_2) = (0,2)$, in full analogy with the case $(\ell_1, \ell_2) = (0,1)$:

Theorem 8.1. *Assume that $\zeta \in L^2(0,1)$ satisfies the hypotheses of Proposition 5.1 for $\ell = 0$ and $\ell = 2$. Then $\zeta = 0$ almost everywhere on $(0,1)$.*

In particular, this implies that the differential of the spectral map associated with the pair $(\ell_1, \ell_2) = (0,2)$ is injective.

9 Uniqueness result in the case $(\ell_1, \ell_2) = (1,2)$

In this section, we assume that ζ satisfies the hypotheses of Proposition 5.1 for both $\ell = 1$ and $\ell = 2$. Our goal is to prove that ζ vanishes identically on $(0,1)$.

We first consider the case $\ell = 1$. By Theorem 6.6, the distribution $A_1(D)\big[S_1[\zeta]\big]$ is odd, where $A_1(t) = 1 - \frac{t}{2}$ and $D = \frac{d}{dx}$. Thus, if we denote $f := S_1[\zeta]$, this condition is equivalent to requiring that $f' - 2f$ be odd (in the sense of distributions). Decomposing f into its even and odd parts,

$$
f = f_e + f_o\,, \tag{9.1}
$$

where f_e is even and f_o is odd with respect to $x = \frac{1}{2}$, we immediately get

$$f_e = \frac{1}{2} f_o' , \tag{9.2}$$

and therefore

$$f = \frac{1}{2} f_o' + f_o . \tag{9.3}$$

We now exploit the case $\ell = 2$. Since ζ satisfies the assumptions of Proposition 5.1 with $\ell = 2$, Theorem 6.6 yields that $A_2(D)\big[T_2[\zeta]\big]$ is odd in the sense of distributions, where

$$A_2(t) = \tfrac{1}{4}t^2 - \tfrac{3}{2}t + 3 .$$

We define

$$g := S_2[f] = f(x) - 8x^3 \int_x^1 \frac{f(t)}{t^4}\, dt ,$$

so that $g = -T_2[\zeta]$. A straightforward computation yields

$$4A_2(D)[g](x) = f''(x) + \Big(\frac{8}{x} - 6\Big) f'(x) + \Big(\frac{16}{x^2} - \frac{48}{x} + 12\Big) f(x) - 48x(1-x)(1-2x) \int_x^1 \frac{f(t)}{t^4}\, dt . \tag{9.4}$$

Since $4A_2(D)[g]$ is odd with respect to $x = \frac{1}{2}$, evaluating (9.4) at $x = \frac{1}{2}$ yields

$$f''\big(\tfrac{1}{2}\big) + 10\, f'\big(\tfrac{1}{2}\big) - 20\, f\big(\tfrac{1}{2}\big) = 0 .$$

Using the decomposition $f = \frac{1}{2} f_o' + f_o$, where f_o is odd, we obtain

$$f_o^{(3)}\big(\tfrac{1}{2}\big) = 0 . \tag{9.5}$$

Similarly, a straightforward computation gives

$$\begin{aligned} D^2\big(4A_2(D)[g]\big)(x) &= f^{(4)}(x) + \Big(\frac{8}{x} - 6\Big) f^{(3)}(x) + \Big(12 - \frac{48}{x}\Big) f''(x) \\ &\quad + \Big(\frac{96}{x} - \frac{48}{x^2}\Big) f'(x) + \Big(\frac{192}{x^2} - \frac{96}{x^3}\Big) f(x) \\ &\quad + 288(1-2x) \int_x^1 \frac{f(t)}{t^4}\, dt . \end{aligned} \tag{9.6}$$

Since $D^2\big(4A_2(D)[g]\big)$ is odd with respect to $x = \frac{1}{2}$, evaluating at $x = \frac{1}{2}$ gives

$$f_o^{(5)}\big(\tfrac{1}{2}\big) = 64\, f_o^{(3)}\big(\tfrac{1}{2}\big) = 0 . \tag{9.7}$$

Finally, a similar computation yields

$$\begin{aligned} G(x) := D^4\big(4A_2(D)[g]\big)(x) &= f^{(6)}(x) + \Big(\frac{8}{x} - 6\Big) f^{(5)}(x) + \Big(12 - \frac{48}{x} - \frac{16}{x^2}\Big) f^{(4)}(x) \\ &\quad + \Big(\frac{96}{x} + \frac{48}{x^2} + \frac{16}{x^3}\Big) f^{(3)}(x) . \end{aligned} \tag{9.8}$$

Replacing, $f = \frac{1}{2} f_o' + f_o$, we get

$$G(x) = \frac{1}{2}\, f_o^{(7)}(x) + \Big(\frac{4}{x} - 2\Big) f_o^{(6)}(x) - \Big(\frac{16}{x} + \frac{8}{x^2}\Big) f_o^{(5)}(x) + \Big(12 + \frac{8}{x^2} + \frac{8}{x^3}\Big) f_o^{(4)}(x) + \Big(\frac{96}{x} + \frac{48}{x^2} + \frac{16}{x^3}\Big) f_o^{(3)}(x)\,. \tag{9.9}$$

We now use the symmetry condition $G(x)+G(1-x)=0$. This yields the following differential equation.

$$\begin{aligned} 0 = {} & f_o^{(7)} + 4\left(\frac{1}{x} - \frac{1}{1-x}\right) f_o^{(6)} \\ & - \left[16\left(\frac{1}{x} + \frac{1}{1-x}\right) + 8\left(\frac{1}{x^2} + \frac{1}{(1-x)^2}\right)\right] f_o^{(5)} \\ & + 8\left[\left(\frac{1}{x^2} - \frac{1}{(1-x)^2}\right) + \left(\frac{1}{x^3} - \frac{1}{(1-x)^3}\right)\right] f_o^{(4)} \\ & + \left[96\left(\frac{1}{x} + \frac{1}{1-x}\right) + 48\left(\frac{1}{x^2} + \frac{1}{(1-x)^2}\right) + 16\left(\frac{1}{x^3} + \frac{1}{(1-x)^3}\right)\right] f_o^{(3)}\,. \end{aligned} \tag{9.10}$$

Finally, setting

$$y(x) = f_o^{(3)}(x)\,,$$

we obtain a fourth–order differential equation for y. The function y is even with respect to $x = \frac{1}{2}$, and from the previous identities we have

$$y^{(k)}\big(\tfrac{1}{2}\big) = 0\,, \qquad k = 0,1,2,3\,.$$

By the Cauchy–Lipschitz theorem, it follows that $y \equiv 0$. Hence f_o is an odd polynomial of degree at most two, and therefore necessarily of the form

$$f_o(x) = a\,(x - \tfrac{1}{2})\,. \tag{9.11}$$

Using again $f = \frac{1}{2} f_o' + f_o$, we obtain

$$f(x) = \frac{a}{2} + a\,(x - \tfrac{1}{2}) = a\,x\,. \tag{9.12}$$

Since $f = S_1[\zeta]$, applying the left inverse A_1 of S_1 given in Lemma 4.1(iii) yields

$$\zeta(x) = A_1[f](x) = f(x) - \frac{4}{x^3}\int_0^x t^2\, f(t)\,dt = 0\,. \tag{9.13}$$

We have thus proved the following theorem.

Theorem 9.1. *Assume that $\zeta \in L^2(0,1)$ satisfies the hypotheses of Proposition 5.1 for $\ell = 1$ and $\ell = 2$. Then $\zeta = 0$ almost everywhere on $(0,1)$.*

In particular, the differential of the spectral map associated with the pair $(\ell_1, \ell_2) = (1,2)$ is injective.

10 Uniqueness result in the case $(\ell_1, \ell_2) = (0, 3)$

Throughout this section, we assume that ζ satisfies the hypotheses of Proposition 5.1 for both $\ell = 0$ and $\ell = 3$, and we aim to show that ζ vanishes almost everywhere on $(0,1)$. As before, the condition for $\ell = 0$ implies that ζ is odd with respect to the midpoint $x = \frac{1}{2}$. For $\ell = 3$, Theorem 6.6 implies that the function $A_3(D)\big[T_3[\zeta]\big]$ is odd in the sense of distributions.

In order to prove that ζ vanishes almost everywhere on $(0,1)$, we follow the same strategy as in the previous section. The only step where we rely on computer assistance is Step 3, where we provide a computer-assisted proof.

- **Step 1.** We show that

$$\Big(\forall\, k \in \{0,1,2\},\ D^{2k} A_3(D)\big[T_3[\zeta]\big]\big(\tfrac{1}{2}\big) = 0\Big) \quad \Longrightarrow \quad (S)\,,$$

  where (S) denotes a triangular linear system involving the even-order derivatives of $y = \zeta'$ at $x = \frac{1}{2}$, uniquely determined by the single value $y(\frac{1}{2})$.

- **Step 2.** We show that y satisfies a linear differential equation of order 8 on the interval $(0,1)$. The associated indicial equation at $x = 0$ has the characteristic roots

$$\rho = -2,\ 1,\ 3,\ 5,\ 6,\ 7,\ -2 \pm 2i\sqrt{11}\,.$$

- **Step 3.** A numerical investigation shows that the unique solution of this differential equation satisfying $y(\frac{1}{2}) = 1$ and whose initial data at $x = \frac{1}{2}$ are prescribed by the system (S) exhibits a pronounced blow-up as $x \to 0^+$ and $x \to 1^-$. The numerical solution $y(x)$ displays oscillatory growth compatible with the Frobenius exponents $\alpha = -2 \pm 2i\sqrt{11}$. As a consequence, the corresponding function ζ fails to be square-integrable on $(0,1)$.

- **Step 4.** We infer that all derivatives of y at $x = \frac{1}{2}$ must vanish. It then follows that $y \equiv 0$ on $(0,1)$, and consequently $\zeta = 0$ almost everywhere on $(0,1)$, since ζ is odd with respect to the midpoint $x = \frac{1}{2}$.

Step 1.a: for all odd $k \in \{0, \dots 7\}$**, i.e.,** $k \in \{1,3,5,7\}$ **,** $y^{(k)}\big(\frac{1}{2}\big) = 0$**.** This is an immediate consequence of the fact that $y^{(k)}$ is odd whenever k is odd.

Step 1.b. We derive a triangular linear system (S) for the even-order jets of y at $x = \frac{1}{2}$. We follow the strategy used in cases $(0,1)$ and $(0,2)$ but as we will see, due to the increasing degree of the polynomial A_ℓ, we need to work a little more.

Step 1.b.i: using $A_3(D)\big[T_3[\zeta]\big]\big(\frac{1}{2}\big) = 0$. Recall that (see (3.9) in [27]), for all $x \in (0,1)$,

$$h(x) = T_3[\zeta](x) = \zeta(x) - 24x \int_x^1 \frac{\zeta(t)}{t^2}\, dt + 120x^3 \int_x^1 \frac{\zeta(t)}{t^4}\, dt - 120x^5 \int_x^1 \frac{\zeta(t)}{t^6}\, dt\,.$$

Applying the differential operator $8A_3(D) = -D^3 + 12D^2 - 60D + 120$ to h, we obtain for all $x \in (0,1)$,

$$\begin{aligned} 8A_3(D)[h](x) = &- \zeta'''(x) + \left(12 - \frac{24}{x}\right)\zeta''(x) + \left(-60 + \frac{288}{x} - \frac{216}{x^2}\right)\zeta'(x) \\ &+ \left(120 - \frac{1440}{x} + \frac{2880}{x^2} - \frac{1200}{x^3}\right)\zeta(x) + 1440(-2x+1)\int_x^1 \frac{\zeta(t)}{t^2}\,dt \\ &+ 720(20x^3 - 30x^2 + 12x - 1)\int_x^1 \frac{\zeta(t)}{t^4}\,dt \\ &+ 7200(-2x^5 + 5x^4 - 4x^3 + x^2)\int_x^1 \frac{\zeta(t)}{t^6}\,dt\,. \end{aligned}$$

Evaluating this expression at $x = \frac{1}{2}$ and using that ζ is odd and $y = \zeta'$, we obtain

$$8A_3(D)\big[h\big]\left(\tfrac{1}{2}\right) = -\zeta'''\left(\tfrac{1}{2}\right) - 36\zeta''\left(\tfrac{1}{2}\right) - 348\zeta'\left(\tfrac{1}{2}\right) - 840\zeta\left(\tfrac{1}{2}\right) = -y''\left(\tfrac{1}{2}\right) - 348\,y\left(\tfrac{1}{2}\right)\,.$$

Since $8A_3(D)[h]$ is odd, we therefore conclude that

$$-y''\left(\tfrac{1}{2}\right) - 348\,y\left(\tfrac{1}{2}\right) = 0\,.$$

Step 1.b.ii: using $D^2A_3(D)\big[T_3[\zeta]\big]\left(\frac{1}{2}\right) = 0$. After some calculations we obtain :

$$8D^2A_3(D)\big[h\big]\left(\tfrac{1}{2}\right) = -\zeta^{(5)}\left(\tfrac{1}{2}\right) - 36\zeta^{(4)}\left(\tfrac{1}{2}\right) - 156\zeta'''\left(\tfrac{1}{2}\right) + 3384\zeta''\left(\tfrac{1}{2}\right) + 18432\zeta'\left(\tfrac{1}{2}\right)\,.$$

Since $8D^2A_3(D)[h]$ is odd, we therefore conclude that

$$-y^{(4)}\left(\tfrac{1}{2}\right) - 156y''\left(\tfrac{1}{2}\right) + 18432\,y\left(\tfrac{1}{2}\right) = 0\,.$$

Step 1.b.iii: using $D^4A_3(D)\big[T_3[\zeta]\big]\left(\frac{1}{2}\right) = 0$. After some calculations we obtain :

$$8D^4A_3(D)\big[h\big]\left(\tfrac{1}{2}\right) = -\zeta^{(7)}\left(\tfrac{1}{2}\right) - 36\zeta^{(6)}\left(\tfrac{1}{2}\right) + 36\zeta^{(5)}\left(\tfrac{1}{2}\right) + 6072\zeta^{(4)}\left(\tfrac{1}{2}\right) - 18432\zeta'''\left(\tfrac{1}{2}\right) - 294912\zeta''\left(\tfrac{1}{2}\right) + 589824\zeta'\left(\tfrac{1}{2}\right)\,.$$

Since $8D^4A_3(D)[h]$ is odd, we therefore conclude that

$$-y^{(6)}\left(\tfrac{1}{2}\right) + 36y^{(4)}\left(\tfrac{1}{2}\right) - 18432y''\left(\tfrac{1}{2}\right) + 589824y\left(\tfrac{1}{2}\right) = 0\,.$$

Thus, in contrast with the case $(0,2)$, where for $k \in \{0,1\}$ one has

$$D^{2k}A_2(D)\big[T_2[\zeta]\big]\left(\tfrac{1}{2}\right) = 0 \quad\Longrightarrow\quad y^{(2k)}\left(\tfrac{1}{2}\right) = 0\,,$$

we obtain here

$$\left(\forall\, k \in \{0,1,2\}\,,\ D^{2k}A_3(D)\big[T_3[\zeta]\big]\left(\tfrac{1}{2}\right) = 0\right) \quad\Longrightarrow\quad (S)\,,$$

where (S) denotes the following triangular linear system:

$$\begin{cases} y''\left(\tfrac{1}{2}\right) + 348\,y\left(\tfrac{1}{2}\right) = 0, \\ y^{(4)}\left(\tfrac{1}{2}\right) + 156\,y''\left(\tfrac{1}{2}\right) - 18432\,y\left(\tfrac{1}{2}\right) = 0, \\ y^{(6)}\left(\tfrac{1}{2}\right) - 36\,y^{(4)}\left(\tfrac{1}{2}\right) + 18432\,y''\left(\tfrac{1}{2}\right) - 589824\,y\left(\tfrac{1}{2}\right) = 0\,. \end{cases}$$

This system is uniquely determined by the single value $y(\frac{1}{2})$.

Step 2 : y is a solution of a linear 8^{th}-order differential equation. Let us set $f := S_1[\zeta]$, $g := S_2[f]$, and $h := S_3[g]$, so that $h = T_3[\zeta] = S_3 S_2 S_1[\zeta]$. Using Lemma 4.1 (iv) with $\ell = 3$, $\ell = 2$ and $\ell = 1$, we have

$$D^6[h] = D^6[g] + \frac{12}{x} D^5[g] \quad ; \quad D^4[g] = D^4[f] + \frac{8}{x} D^3[f] \quad ; \quad D^2[f] = D^2[\zeta] + \frac{4}{x} D[\zeta].$$

Then, we can write $D^6[h]$ as

$$D^6[h] = D^6[\zeta] + \sum_{k=1}^{5} a_k\big(\tfrac{1}{x}\big)\, D^k[\zeta] \tag{10.1}$$

where for all $k \in \{1, \dots, 5\}$, a_k is a polynomial function. We now differentiate the expression $A_3(D)\big[h\big]$ six times. Since applying an even number of derivatives preserves parity in the distributional sense, and since the operator $A_3(D)$ commutes with differentiation, it follows that the function

$$G := 8D^6 A_3(D)\big[h\big] = 8A_3(D)\big[D^6[h]\big]$$

is odd. Using the expression of $A_3(D)$ given in Remark 6.7 and the form of $D^6[h]$ given in (10.1), we obtain that the odd function G can be written as

$$G(x) = (-D^3 + 12D^2 - 60D + 120)\big[D^6[\zeta]\big](x) + \sum_{k=1}^{5} a_k\big(\tfrac{1}{x}\big)\, D^k[\zeta](x) = -D_9[\zeta](x) + \sum_{k=1}^{8} b_k\big(\tfrac{1}{x}\big)\, D^k[\zeta](x)\,,$$

where for all $k \in \{1, \dots, 8\}$, b_k is a polynomial function. Thus, since G is odd on $(0,1)$, it satisfies $G(x) + G(1-x) = 0$ in the sense of distributions. It follows that $y = \zeta' = D[\zeta]$ satisfies the following linear 8^{th}-order differential equation on $(0,1)$:

Lemma 10.1. *Assume that $\zeta \in L^2(0,1)$ satisfies the hypotheses of Lemma 5.1 for $\ell = 0$ and $\ell = 3$, and set $y(x) := \zeta'(x)$. Then y satisfies the following differential equation*

$$
\begin{aligned}
&- y^{(8)}(x) + 12\left(\frac{1}{1-x} - \frac{1}{x}\right) y^{(7)}(x) \\
&+ \left[-60 + 144\left(\frac{1}{x} + \frac{1}{1-x}\right) - 36\left(\frac{1}{x^2} + \frac{1}{(1-x)^2}\right)\right] y^{(6)}(x) \\
&+ \left[-720\left(\frac{1}{x} - \frac{1}{1-x}\right) + 576\left(\frac{1}{x^2} - \frac{1}{(1-x)^2}\right) + 336\left(\frac{1}{x^3} - \frac{1}{(1-x)^3}\right)\right] y^{(5)}(x) \\
&+ \left[-720\left(\frac{1}{x^4} + \frac{1}{(1-x)^4}\right) - 2880\left(\frac{1}{x^3} + \frac{1}{(1-x)^3}\right) - 3600\left(\frac{1}{x^2} + \frac{1}{(1-x)^2}\right) + 1440\left(\frac{1}{x} + \frac{1}{1-x}\right)\right] y^{(4)}(x) \\
&+ \left[-2880\left(\frac{1}{x^5} - \frac{1}{(1-x)^5}\right) + 7200\left(\frac{1}{x^3} - \frac{1}{(1-x)^3}\right) + 8640\left(\frac{1}{x^2} - \frac{1}{(1-x)^2}\right)\right] y^{(3)}(x) \\
&+ \left[23040\left(\frac{1}{x^6} + \frac{1}{(1-x)^6}\right) + 34560\left(\frac{1}{x^5} + \frac{1}{(1-x)^5}\right) + 21600\left(\frac{1}{x^4} + \frac{1}{(1-x)^4}\right) + 2880\left(\frac{1}{x^3} + \frac{1}{(1-x)^3}\right)\right] y''(x) \\
&- \left[60480\left(\frac{1}{x^7} - \frac{1}{(1-x)^7}\right) + 103680\left(\frac{1}{x^6} - \frac{1}{(1-x)^6}\right) + 86400\left(\frac{1}{x^5} - \frac{1}{(1-x)^5}\right) + 34560\left(\frac{1}{x^4} - \frac{1}{(1-x)^4}\right)\right] y'(x)
\end{aligned}
$$

$$+\left[60480\left(\frac{1}{x^8}+\frac{1}{(1-x)^8}\right)+103680\left(\frac{1}{x^7}+\frac{1}{(1-x)^7}\right)+86400\left(\frac{1}{x^6}+\frac{1}{(1-x)^6}\right)+34560\left(\frac{1}{x^5}+\frac{1}{(1-x)^5}\right)\right]y(x) \;=\; 0\,,$$

in the sense of distributions on $(0,1)$.

The indicial equation at $x=0$ associated with the differential equation of Lemma 10.1 takes the form

$$-\,(\rho-7)(\rho-6)(\rho-5)(\rho-3)(\rho-1)(\rho+2)\left(\rho^2+4\rho+48\right)=0\,.$$

Consequently, the corresponding indicial roots are

$$\rho \;=\; -2,\ 1,\ 3,\ 5,\ 6,\ 7,\ -2\pm 2i\sqrt{11}\,.$$

Step 3: Numerical study of the solution. We numerically integrate the eighth–order differential equation given in Lemma 10.1 with initial conditions prescribed at $x=\frac{1}{2}$, namely $y(\frac{1}{2})=1$ together with the even derivatives $y^{(2k)}(\frac{1}{2})$ computed in Step 1, the odd derivatives being zero by symmetry. The resulting solution y, which is even with respect to $x=\frac{1}{2}$, blows up at both endpoints $x=0$ and $x=1$.

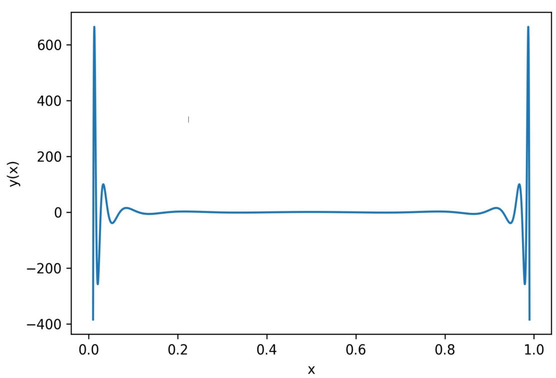

Figure 1: Numerical solution $y(x)$ on $[10^{-2},\,1-10^{-2}]$. The oscillations visible on a logarithmic scale are compatible with the complex Frobenius exponents $\alpha=-2\pm 2i\sqrt{11}$ at the endpoints.

Guided by these indicial roots, we extract the boundary asymptotics at $x=0$ by fitting $x^2y(x)$ on shrinking windows near the boundary. We obtain numerically

$$y(x)=\frac{C_0}{x^2}\,\cos\!\big(2\sqrt{11}\,\ln x+\phi\big)\;+\;o(x^{-2})\,,\qquad x\to 0^+\,,$$

with fitted constants

$$C_0\approx 0.11325\,,\qquad \phi\approx -2.79\,.$$

Refining the fit, we construct an explicit boundary approximation $y_{\rm app}$ such that

$$y-y_{\rm app}=o(1)\,,\qquad x\to 0^+.$$

Based on the Frobenius analysis below, we introduce the following approximation near $x = 0$, retaining all terms up to order $o(1)$:

$$\begin{aligned} y_{\text{app}}(x) = \frac{1}{x^2} \Big[&\big(A_0 + A_1 x + A_2 x^2\big) \cos\!\big(2\sqrt{11}\, \ln x\big) \\ &+ \big(B_0 + B_1 x + B_2 x^2\big) \sin\!\big(2\sqrt{11}\, \ln x\big) \Big], \qquad x \to 0^+ . \end{aligned} \tag{10.2}$$

The fitted numerical coefficients are

$$\begin{aligned} A_0 &\approx -0.10617307\,, & B_0 &\approx 0.03940251\,, \\ |A_1| &\lesssim 10^{-6}\,, & |B_1| &\lesssim 10^{-6}\,, \\ A_2 &\approx -0.0923567\,, & B_2 &\approx 0.3818868\,. \end{aligned}$$

Theoretical Frobenius asymptotics. We recall that the indicial equation at $x = 0$ associated with the differential equation in Lemma 10.1 has the following roots:

$$\rho = -2,\ 1,\ 3,\ 5,\ 6,\ 7\,, \qquad \rho = -2 \pm 2i\sqrt{11}\,.$$

Using Mathematica, we compute the Frobenius expansions associated with each indicial root. Truncating the resulting expressions up to terms $o(1)$ as $x \to 0^+$, we obtain the following local expansion:

$$\begin{aligned} y(x) = {} & A\Big(\frac{120}{x^2} - 120\Big) \\ &+ B\, x^{-2-2i\sqrt{11}} \Big(9375i + 1215\sqrt{11} + \big(6542i + 10474\sqrt{11}\big)x^2\Big) \\ &+ C\, x^{-2+2i\sqrt{11}} \Big(-9375i + 1215\sqrt{11} + \big(-6542i + 10474\sqrt{11}\big)x^2\Big) \\ &+\ o(1)\,, \qquad x \to 0^+ . \end{aligned} \tag{10.3}$$

for some complex constants A, B, C. Note that the remaining Frobenius solutions associated with the real roots $\rho = 1, 3, 5, 6, 7$ behave like $O(x)$ or higher powers of x, and are therefore absorbed into the $o(1)$ remainder.

Thus, taking either the real or the imaginary part of the previous complex solution $y(x)$ yields a real solution admitting the asymptotic expansion

$$\begin{aligned} y(x) = {} & C_{-2}\Big(\frac{120}{x^2} - 120\Big) \\ &+ \frac{1}{x^2}\Big[\big(C_c + \widetilde{C}_c\, x^2\big) \cos\!\big(2\sqrt{11}\, \ln x\big) + \big(C_s + \widetilde{C}_s\, x^2\big) \sin\!\big(2\sqrt{11}\, \ln x\big)\Big] \\ &+\ o(1)\,, \qquad x \to 0^+ , \end{aligned} \tag{10.4}$$

for some real constants $C_{-2}, C_c, C_s, \widetilde{C}_c, \widetilde{C}_s$. These constants are not independent: (C_c, C_s) uniquely determines $(\widetilde{C}_c, \widetilde{C}_s)$ through the linear relation

$$\begin{pmatrix} \widetilde{C}_c \\ \widetilde{C}_s \end{pmatrix} = \begin{pmatrix} \frac{29}{15} & \frac{13\sqrt{11}}{15} \\ -\frac{13\sqrt{11}}{15} & \frac{29}{15} \end{pmatrix} \begin{pmatrix} C_c \\ C_s \end{pmatrix}.$$

We also observe that the oscillatory modes do not contain any term of order x^{-1}.

Comparison with numerics. Comparing with the numerical ansatz (10.2), we obtain the identifications

$$A_0 = C_c\,, \qquad B_0 = C_s\,, \qquad A_2 = \widetilde{C}_c\,, \qquad B_2 = \widetilde{C}_s\,.$$

Moreover, the Frobenius structure predicts that

$$A_1 = B_1 = 0\,.$$

The fitted numerical values satisfy $|A_1|, |B_1| \lesssim 10^{-6}$, which is fully compatible with this theoretical prediction. Any nonzero values obtained at lower precision should therefore be interpreted as numerical artefacts.

The numerical fit shows no evidence of a non-oscillatory x^{-2} contribution, which suggests

$$C_{-2} = 0\,.$$

Identifying the leading oscillatory coefficients yields

$$C_c \approx A_0 \approx -0.106173\,, \qquad C_s \approx B_0 \approx \quad 0.039403\,.$$

Equivalently, writing

$$C_c \cos\!\big(2\sqrt{11}\,\ln x\big) + C_s \sin\!\big(2\sqrt{11}\,\ln x\big) = C_0 \cos\!\big(2\sqrt{11}\,\ln x + \phi\big)\,,$$

we recover

$$C_0 = \sqrt{C_c^2 + C_s^2} \approx 0.11325\,, \qquad \phi = \arctan\!\Big(-\frac{C_s}{C_c}\Big) \approx -2.79\,.$$

Finally, the Frobenius expansion implies that the constant-order oscillatory coefficients $\widetilde{C}_c, \widetilde{C}_s$ are not independent. Using the fitted values of C_c and C_s, this relation predicts

$$\widetilde{C}_c^{\text{pred}} \approx -0.0917\,, \qquad \widetilde{C}_s^{\text{pred}} \approx \quad 0.3819\,.$$

These values are in excellent agreement with the numerical coefficients

$$\widetilde{C}_c \approx A_2 \approx -0.09236\,, \qquad \widetilde{C}_s \approx B_2 \approx \quad 0.38189\,,$$

with relative discrepancies of approximately $0.7\,\%$ for $\widetilde{C}_c$ and $3 \times 10^{-4}\,\%$ for $\widetilde{C}_s$. This quantitative agreement provides strong numerical evidence for the presence of the quadratic correction in the oscillatory Frobenius modes.

Role of the numerical analysis. At this stage, it is important to clarify the role played by the numerical computations. They are used solely to detect that the solution y of the eighth–order equation is not bounded near $x = 0$ and $x = 1$. This observation implies that at least one of the singular Frobenius modes associated with the indicial roots

$$\rho = -2 \qquad \text{or} \qquad \rho = -2 \pm 2i\sqrt{11}$$

is present in the expansion of y near the endpoints. No numerical information on the corresponding coefficients is required for this conclusion.

In particular, the proof that the primitive ζ does not belong to $L^2(0,1)$ relies only on the qualitative oscillatory behaviour induced by these Frobenius modes, and not on the explicit values of the constants appearing in the asymptotic expansion. The numerical computation of the coefficients $(C_c, C_s, \widetilde{C}_c, \widetilde{C}_s)$ should therefore be viewed as an additional quantitative illustration, confirming the Frobenius structure predicted by the theory.

Non–square integrability of the primitive. Let ζ be a primitive of y, namely

$$\zeta(x) := \int_{1/2}^{x} y(s)\,ds\,, \qquad x \in (0,1).$$

At this stage, and for the sake of generality, we do not assume that $C_{-2} = 0$. Integrating the leading terms in the asymptotic expansion of y as $x \to 0^+$ yields

$$\zeta(x) = \frac{-120\,C_{-2}}{x} + \frac{C_0}{45}\,\frac{-\cos\!\big(2\sqrt{11}\,\ln x + \phi\big) + 2\sqrt{11}\,\sin\!\big(2\sqrt{11}\,\ln x + \phi\big)}{x} + O(1)\,, \qquad x \to 0^+\,.$$

Setting $t = -\ln x$, we can rewrite this behaviour as

$$\zeta(e^{-t}) = e^t\,A(t) + O(1)\,, \qquad t \to +\infty,$$

where the function A is bounded, periodic, and explicitly given by

$$A(t) := -120\,C_{-2} + \frac{C_0}{45}\Big(-\cos\!\big(2\sqrt{11}\,t - \phi\big) - 2\sqrt{11}\,\sin\!\big(2\sqrt{11}\,t - \phi\big)\Big)\,.$$

Since $A \not\equiv 0$ and A is continuous and periodic, there exist constants $\delta > 0$ and $c_0 > 0$, and a sequence $t_n \to +\infty$, such that

$$|A(t)| \geq c_0 \qquad \text{for all } t \in I_n := [t_n, t_n + \delta]\,.$$

Consequently,

$$\int_0^1 |\zeta(x)|^2\,dx = \int_0^{+\infty} |\zeta(e^{-t})|^2\,e^{-t}\,dt = \int_0^{+\infty} e^t\,|A(t)|^2\,dt = +\infty\,.$$

because

$$\int_0^{+\infty} e^t\,|A(t)|^2\,dt \;\geq\; \sum_n \int_{I_n} e^t\,|A(t)|^2\,dt \;\geq\; c_0^2 \sum_n \int_{t_n}^{t_n+\delta} e^t\,dt = +\infty\,.$$

This proves that $\zeta \notin L^2(0,1)$.

Step 4: Conclusion. We therefore conclude that one must take $y(\frac{1}{2}) = 0$. By the triangular system (S), all derivatives of y at $x = \frac{1}{2}$ up to order 7 vanish. The Cauchy–Lipschitz theorem then implies that $y \equiv 0$ on $(0,1)$. Consequently, $\zeta \equiv 0$ on $(0,1)$.

In particular, this implies that the differential of the spectral map associated with the pair $(\ell_1, \ell_2) = (0,3)$ is injective.

11 Conclusion

The techniques developed in this paper rely on the Kneser–Sommerfeld formula. In the case $(\ell_1, \ell_2) = (0,3)$, the analysis required the use of computer-assisted numerical computations. Even with the assistance of computer algebra, the study of the resulting differential equations already becomes rapidly cumbersome in the case $(\ell_1, \ell_2) = (1,3)$. Larger values of (ℓ_1, ℓ_2) currently seem out of reach, which suggests that a more conceptual approach is needed.

We are nevertheless led to the following natural conjecture, originally formulated by Rundell and Sacks: *local uniqueness holds near the zero potential as soon as the Dirichlet spectra are known for two distinct values of ℓ.*

A Proof of Theorem 1.1

In this appendix, we briefly recall the scattering-theoretic framework underlying Theorem 1.1. Throughout this section, we fix the energy $\lambda \in \mathbb{C}$ and consider the radial Schrödinger equation

$$-u''(r) + \left(\frac{\ell(\ell+1)}{r^2} + \hat{q}(r)\right) u(r) = \lambda\, u(r)\,, \qquad r > 0, \tag{A.1}$$

where $\hat{q}$ denotes the extension of $q \in L^2(0,1)$ by zero for $r \geq 1$.[1]

For each $\nu = \ell + \frac{1}{2}$, we denote by $\varphi(r,\nu,\lambda)$ the *regular solution* of (A.1), characterized by the boundary condition

$$\varphi(r,\nu,\lambda) \sim r^{\nu+\frac{1}{2}}\,, \qquad r \to 0\,. \tag{A.2}$$

By definition, a solution of (B.1) is a C^1-function whose derivative with respect to r is locally absolutely continuous on $(0,+\infty)$. For instance, when $q \equiv 0$, the regular solution is explicitly given by

$$\varphi_0(r,\nu,\lambda) = 2^\nu\, \lambda^{-\nu/2}\, \Gamma(\nu+1)\, \sqrt{r}\, J_\nu(\sqrt{\lambda}\, r)\,, \tag{A.3}$$

where $\sqrt{\lambda}$ is defined by the principal branch.[2]

The existence and uniqueness of a regular solution are ensured by the following theorem, proved in [15, Lemma 2.2] (see also [16, Lemma B.2]) for locally integrable potentials q on $(0,+\infty)$ under the additional assumption $\int_0^1 r\,|q(r)|\,dr < \infty$.

Theorem A.1. *There exists a unique regular solution* $\varphi(r,\nu,\lambda)$ *of* (A.1) *satisfying:*

(i) The map $\lambda \mapsto \varphi(r,\nu,\lambda)$ *is an entire function of order* $\frac{1}{2}$.

(ii) There exists $C > 0$ *such that for all* $r > 0$ *and* $\lambda \in \mathbb{C}$,

$$|\varphi(r,\nu,\lambda)| \;\leq\; C \left(\frac{r}{1+|\sqrt{\lambda}|\,r}\right)^{\ell+1} e^{|\Im\sqrt{\lambda}|\,r}\,. \tag{A.4}$$

(iii) Moreover, there exists $C > 0$ *such that for all* $r > 0$ *and* $\lambda \in \mathbb{C}$,

$$\big|\varphi(r,\nu,\lambda) - \varphi_0(r,\nu,\lambda)\big| \;\leq\; C \left(\frac{r}{1+|\sqrt{\lambda}|\,r}\right)^{\ell+1} e^{|\Im\sqrt{\lambda}|\,r} \int_0^r \frac{t\,|q(t)|}{1+|\sqrt{\lambda}|\,t}\,dt\,. \tag{A.5}$$

The *Dirichlet spectrum* associated with a fixed angular momentum ℓ (or equivalently $\nu = \ell + \frac{1}{2}$) is defined as the set of values $\{\lambda_{\ell,n}(q)\}_{n\geq 1}$ such that the regular solution satisfies the boundary condition

$$\varphi(1,\nu,\lambda_{\ell,n}(q)) = 0\,. \tag{A.6}$$

Equivalently, the Dirichlet spectrum consists of the zeros of the entire function $\lambda \mapsto \varphi(1,\nu,\lambda)$.

[1] In the remainder of this appendix, we omit the hat notation and use the same notation q for the potential in $L^2(0,1)$ and its extension by 0 in $L^2(0,+\infty)$.

[2] By convention, if $\lambda = |\lambda| e^{i\theta}$ with $\theta \in (-\pi,\pi]$, then $\sqrt{\lambda} = \sqrt{|\lambda|}\, e^{i\theta/2}$.

It is well known that, for such potentials q and for each fixed angular momentum $\ell \in \mathbb{N}$, the associated Dirichlet spectrum forms a countable sequence of simple, real eigenvalues $\{\lambda_{\ell,n}(q)\}_{n\geq 1}$ diverging to infinity. These facts are classical and can be found in [5, 6].

For instance, when $q \equiv 0$, the Dirichlet eigenvalues are explicitly given by

$$\lambda_{\ell,n}(0) = j_{\nu,n}^2 \,, \tag{A.7}$$

where $j_{\nu,n}$ denotes the n-th positive zero of $J_\nu(z)$. Recall that the Bessel function $J_\nu(z)$ admits the classical Hadamard factorization

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{j_{\nu,n}^2}\right) . \tag{A.8}$$

As a consequence, the free regular solution evaluated at $r = 1$ can be written as

$$\varphi_0(1,\nu,\lambda) = \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{j_{\nu,n}^2}\right) . \tag{A.9}$$

Similarly, by applying Hadamard's factorization theorem, $\varphi(1,\nu,\lambda)$ admits the following canonical product representation:

Lemma A.2. *For each fixed $\nu = \ell + \frac{1}{2}$ and $q \in L^2(0,1)$, the regular solution satisfies*

$$\varphi(1,\nu,\lambda) = C_\nu(q)\ \lambda^m \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_{\ell,n}(q)}\right) ,$$

where $C_\nu(q) \neq 0$ is a constant depending on ν and q, and $m \in \{0,1\}$ denotes the order of the zero at $\lambda = 0$.

We next show that equality of the Dirichlet spectra implies equality of the regular solutions evaluated at $r = 1$.

Lemma A.3. *Let $\nu = \ell + \frac{1}{2}$ and let $q\,, \tilde{q} \in L^2(0,1)$ be two potentials such that their Dirichlet spectra coincide:*

$$\{\lambda_{\ell,n}(q)\}_{n\geq 1} = \{\lambda_{\ell,n}(\tilde{q})\}_{n\geq 1} \,. \tag{A.10}$$

Then the corresponding regular solutions satisfy

$$\varphi(1,\nu,\lambda;q) = \varphi(1,\nu,\lambda;\tilde{q})\,, \qquad \forall\,\lambda \in \mathbb{C}\,. \tag{A.11}$$

Proof. We denote by a tilde all quantities associated with the potential $\tilde{q}$, and, for brevity, we write $\varphi(1,\nu,\lambda;q)$ simply as $\varphi(1,\nu,\lambda)$. Since the Dirichlet spectra coincide, we have $m = \tilde{m}$. By Theorem A.1 (with $r = 1$), for real $\lambda = k^2$ and $|k| \to \infty$,

$$\big|\varphi(1,\nu,\lambda) - \varphi_0(1,\nu,\lambda)\big| = \mathcal{O}(|k|^{-\ell-2})\,. \tag{A.12}$$

From the standard asymptotics of Bessel functions, one has as $\lambda \to +\infty$,

$$\varphi_0(1,\nu,\lambda) \sim c_\nu\, k^{-\ell-1} \cos\!\Big(k - \tfrac{\pi\nu}{2} - \tfrac{\pi}{4}\Big)\,, \qquad k = \sqrt{\lambda} \in \mathbb{R}\,. \tag{A.13}$$

Choosing

$$k_n = \tfrac{\pi}{2}\Big(\nu + \tfrac{1}{2}\Big) + 2\pi n\,, \qquad \lambda_n = k_n^2\,, \tag{A.14}$$

with $n \in \mathbb{N}$, so that $\cos(\cdot) = 1$, we obtain as $n \to +\infty$,

$$\varphi(1,\nu,\lambda_n) \sim \varphi_0(1,\nu,\lambda_n) \sim \tilde{\varphi}(1,\nu,\lambda_n)\,. \tag{A.15}$$

Therefore, using Lemma A.2, the leading coefficients coincide, that is, $C_\nu(q) = C_\nu(\tilde{q})$, which completes the proof. □

One can also define the *Jost solutions* $f^{\pm}(r,\nu,\lambda)$ as the unique C^1 solutions with respect to r of (A.1) satisfying the oscillatory asymptotics

$$f^{\pm}(r,\nu,\lambda) \sim e^{\pm i\sqrt{\lambda}\, r}\,, \qquad r \to +\infty\,. \tag{A.16}$$

The proof of this result is given in [16, Lemma B.4] under the assumptions $\int_0^1 r\,|q(r)|\,dr < \infty$ and $\int_1^{+\infty} |q(r)|\,dr < \infty$. In particular, for $\lambda = 1$ (which is the case considered in [7] and [24]), and for such potentials supported in $[0,1]$, one has, for all $r \geq 1$ (see [16, Lemma B.3]),

$$f^{+}(r,\nu,1) = e^{i(\nu+\frac{1}{2})\frac{\pi}{2}} \sqrt{\frac{\pi r}{2}}\, H_\nu^{(1)}(r) \tag{A.17}$$

and

$$f^{-}(r,\nu,1) = e^{-i(\nu+\frac{1}{2})\frac{\pi}{2}} \sqrt{\frac{\pi r}{2}}\, H_\nu^{(2)}(r)\,, \tag{A.18}$$

where $H_\nu^{(j)}(r)$ denotes the Hankel function of order ν. It is a fundamental observation that the Jost solutions are *entirely independent of the potential* for all $r \geq 1$.

The pair of Jost solutions $\{f^{+}(r,\nu,1),\ f^{-}(r,\nu,1)\}$ forms a fundamental system of solutions of (A.1). Hence, at the fixed energy $\lambda = 1$, the regular solution can be written as

$$\varphi(r,\nu,1) = \alpha(\nu)\, f^{+}(r,\nu,1) + \beta(\nu)\, f^{-}(r,\nu,1)\,, \tag{A.19}$$

where $\alpha(\nu),\ \beta(\nu) \in \mathbb{C}$ are called the *Jost functions*.

We now recall some basic properties of the Jost functions (see [7] for details). They satisfy the conjugation relation $\overline{\alpha(\nu)} = \beta(\nu)$, and neither $\alpha(\nu)$ nor $\beta(\nu)$ vanishes. This allows us to introduce the *Regge interpolation*:

$$\sigma(\nu) = e^{i\pi(\nu+\frac{1}{2})}\, \frac{\alpha(\nu)}{\beta(\nu)}\,. \tag{A.20}$$

Note that, following [25], the so-called *phase shifts* $\delta(\nu)$ are defined as a continuous function of $\nu \in (0,+\infty)$ through the relation

$$\sigma(\nu) = e^{2i\,\delta(\nu)}\,.$$

The phase shifts become uniquely determined by imposing the condition $\delta(\nu) \to 0$ as $\nu \to +\infty$, and they can be meromorphically continued to complex values of ν.

Consider now a sequence $(\ell_k)_{k\geq 1}$ of positive integers satisfying

$$\sum_k \frac{1}{\ell_k} = +\infty\,, \tag{A.21}$$

that is, the *Müntz condition*. Then the following uniqueness result holds (see Theorem 2.1 in [24]): if $q, \tilde{q} \in L^2(0,1)$ are two potentials whose Regge interpolation functions coincide for $\nu_k = \ell_k + \frac{1}{2}$, then

$$q = \tilde{q} \quad \text{a.e. on } (0,1) . \tag{A.22}$$

Assume now that q and $\tilde{q}$ share the same Dirichlet spectra $\{\lambda_{\ell_k,n}(q)\}_{k,n\geq 1}$. Since the Jost solutions $f^{\pm}(r,\nu,1)$ never vanish (see [7, Lemma 4.2]) and are *independent of the potential at* $r = 1$, it follows from (A.19) and Lemma A.3 that the corresponding Regge interpolation functions coincide for $\nu_k = \ell_k + \frac{1}{2}$. This concludes the proof of Theorem 1.1.

B Proof of Theorem 2.5

In this section, we prove that the range of the differential of the spectral map at the zero potential is closed in the cases $(\ell_1, \ell_2) = (0,1)$, $(1,2)$ and $(0,3)$. Moreover, in the case $(0,1)$, we give an explicit description of this range.

We first recall that in both cases $(0,1)$, $(1,2)$ and $(0,3)$, the differential of the spectral map at $q = 0$ is injective. Equivalently, the intersection of the tangent spaces to the corresponding isospectral manifolds at $q = 0$ is trivial. Following the terminology of Shubin Christ [29] and Carlson–Shubin [5], we introduce

$$W := T_0\mathcal{M}_{\ell_1} \cap T_0\mathcal{M}_{\ell_2} ,$$

where $\mathcal{M}_\ell$ denotes the isospectral set associated with the Dirichlet spectrum for angular momentum ℓ. Injectivity of the differential is precisely the statement that

$$W = \{0\} .$$

Thus, in the case $(\ell_1, \ell_2) = (0,1)$, the first part of Theorem 2.5 follows from ([29], Lemma 5), while for $(\ell_1, \ell_2) = (1,2)$ or $(0,3)$, it follows from the analysis carried out in the proof of ([5], Theorem 5.6).

We have thus shown that in the cases $(\ell_1, \ell_2) = (0,1)$, $(\ell_1, \ell_2) = (1,2)$ and $(\ell_1, \ell_2) = (0,3)$, the differential of the spectral map at $q = 0$ is injective and has closed range. In other words, the operator $d_0\mathcal{S}_{\ell_1,\ell_2}$ is semi-Fredholm (see [12, p. 230]). Moreover, if an operator B is Fredholm or semi-Fredholm, then any compact perturbation remains Fredholm or semi-Fredholm and the Fredholm index is preserved (see [12, Theorem 5.26]). We now use these theoretical results to refine the analysis in the case $(0,1)$. To this end, we set

$$\Phi_\ell(x) := \frac{\pi}{2}\, x\, J_{\ell+\frac{1}{2}}(x)^2 = (j_\ell(x))^2 .$$

First, we decompose the differential $d_0\mathcal{S}_{\ell_1,\ell_2}$ into two parts,

$$d_0\mathcal{S}_{\ell_1,\ell_2} = A_{\ell_1,\ell_2} + K_{\ell_1,\ell_2} ,$$

where we define

$$A_{\ell_1,\ell_2}(\zeta) = \Big(\langle \zeta, 1\rangle, \big(\langle \zeta,\ 2\Phi_{\ell_1}\big((n+\tfrac{\ell_1}{2})\pi\,\cdot\big) - 1\rangle\big)_{n\geq 1} , \\ \big(\langle \zeta,\ 2\Phi_{\ell_2}\big((n+\tfrac{\ell_2}{2})\pi\,\cdot\big) - 1\rangle\big)_{n\geq 1} \Big) , \tag{B.1}$$

and

$$K_{\ell_1,\ell_2}(\zeta) = \Bigg(0,\ \big(\langle\zeta,\ g^2_{\ell_1,n} - 2\Phi_{\ell_1}\big((n+\tfrac{\ell_1}{2})\pi\,\cdot\big)\rangle\big)_{n\geq 1}\,,$$

$$\big(\langle\zeta,\ g^2_{\ell_2,n} - 2\Phi_{\ell_2}\big((n+\tfrac{\ell_2}{2})\pi\,\cdot\big)\rangle\big)_{n\geq 1}\Bigg)\,. \tag{B.2}$$

According to Theorem 2.1 and Corollary 2.2 in Serier [28], one has

$$\|g^2_{\ell,n} - 2\Phi_\ell(j_{\nu,n}\,\cdot)\|_{L^2(0,1)} = \mathcal{O}\Big(\frac{1}{n}\Big)\,.$$

Since $|\Phi'_\ell(x)| \leq C_\ell$ uniformly on $\mathbb{R}$ and

$$\sum_{n\geq 1}\big|j_{\ell+\frac{1}{2},n} - (n+\tfrac{\ell}{2})\pi\big|^2 < \infty \qquad \text{for any integer } \ell\,,$$

it follows that

$$\sum_{n\geq 1}\big\|g^2_{\ell,n} - 2\Phi_\ell\big((n+\tfrac{\ell}{2})\pi\,\cdot\big)\big\|^2_{L^2(0,1)} < \infty\,.$$

Therefore, by the Cauchy–Schwarz inequality, the operator K_{ℓ_1,ℓ_2} is Hilbert-Schmidt, and hence compact, from $L^2(0,1)$ into $\mathbb{R}\times\ell^2(\mathbb{N})\times\ell^2(\mathbb{N})$. As a consequence of the stability of semi-Fredholm operators under compact perturbations, it follows that, for $(\ell_1,\ell_2)=(0,1)$, $(\ell_1,\ell_2)=(1,2)$ or $(0,3)$ the range of A_{ℓ_1,ℓ_2}, which is a compact perturbation of $d_0\mathcal{S}_{\ell_1,\ell_2}$, is also closed. Now, according to [27, Corollary 5.2], in the case $(\ell_1,\ell_2)=(0,1)$, the family

$$\Big(1,\ \big(\Phi_{\ell_1}((n+\tfrac{\ell_1}{2})\pi\,\cdot)\big)_{n\geq 1},\ \big(\Phi_{\ell_2}((n+\tfrac{\ell_2}{2})\pi\,\cdot)\big)_{n\geq 1}\Big)$$

is referred to as a *basis* of $L^2(0,1)$[3]. We define

$$\mathcal{F}_{\ell_1,\ell_2} = \Big(f_0,\ (f_{1,n})_{n\geq 1}\,,\ (f_{2,n})_{n\geq 1}\Big) \subset L^2(0,1)\,,$$

where

$$f_0(r) = 1, \qquad f_{1,n}(r) = 2\Phi_{\ell_1}\big((n+\tfrac{\ell_1}{2})\pi r\big) - 1\,, \qquad f_{2,n}(r) = 2\Phi_{\ell_2}\big((n+\tfrac{\ell_2}{2})\pi r\big) - 1\,.$$

The family $\mathcal{F}_{\ell_1,\ell_2}$ remains complete and algebraically linearly independent in $L^2(0,1)$. With this notation, the operator A_{ℓ_1,ℓ_2} can be written as

$$A_{\ell_1,\ell_2}(\zeta) = \Big(\langle\zeta,f_0\rangle,\ (\langle\zeta,f_{1,n}\rangle)_{n\geq 1},\ (\langle\zeta,f_{2,n}\rangle)_{n\geq 1}\Big)\,,$$

and is therefore injective. Recall that c_{00} denotes the space of finitely supported sequences. Since

$$\mathbb{R}\times c_{00}\times c_{00} \quad \text{is dense in} \quad \mathbb{R}\times\ell^2_{\mathbb{R}}(\mathbb{N})\times\ell^2_{\mathbb{R}}(\mathbb{N})\,,$$

[3] This statement should be interpreted with care. In the sense of Gohberg-Krein [10], this only means that the above family is complete and ω-linearly independent in $L^2(0,1)$ This property does not, in general, imply that this family is a Krein basis or a Riesz basis.

it follows from a straightforward algebraic argument that the range of A_{ℓ_1,ℓ_2} is also dense in $\mathbb{R} \times \ell^2_{\mathbb{R}}(\mathbb{N}) \times \ell^2_{\mathbb{R}}(\mathbb{N})$. Since this range is closed, as shown above, we conclude that

$$\operatorname{Ran} A_{\ell_1,\ell_2} = \mathbb{R} \times \ell^2_{\mathbb{R}}(\mathbb{N}) \times \ell^2_{\mathbb{R}}(\mathbb{N})\,.$$

In particular, the operator A_{ℓ_1,ℓ_2} is Fredholm of index zero.

Finally, since $d_0\mathcal{S}_{0,1}$ is a compact perturbation of $A_{0,1}$, it follows that $d_0\mathcal{S}_{0,1}$ is also Fredholm with the same index. Because it is injective, it must therefore be an isomorphism from $L^2(0,1)$ onto $\mathbb{R} \times \ell^2_{\mathbb{R}}(\mathbb{N}) \times \ell^2_{\mathbb{R}}(\mathbb{N})$. This completes the proof of Theorem 2.5.

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