Stability in the inverse Steklov problem on warped product Riemannian manifolds

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Abstract

In this paper, we study the amount of information contained in the Steklov spectrum of some compact manifolds with connected boundary equipped with a warped product metric. Examples of such manifolds can be thought of as deformed balls in $\mathbb{R}^d$. We first prove that the Steklov spectrum determines uniquely the warping function of the metric. We show in fact that the approximate knowledge (in a given precise sense) of the Steklov spectrum is enough to determine uniquely the warping function in a neighbourhood of the boundary. Second, we provide stability estimates of log-type on the warping function from the Steklov spectrum. The key element of these stability results relies on a formula that, roughly speaking, connects the inverse data (the Steklov spectrum) to the *Laplace transform* of the difference of the two warping factors.

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Contents

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1 The model and statement of the results

In this paper, we shall consider a class of d-dimensional manifolds

$$M = (0, 1] \times S^{d-1}, \tag{1.1}$$

where $d \geq 3$ and with a connected boundary consisting in a copy of S^{d-1}, *i.e.* $\partial M = \{1\} \times S^{d-1}$. We shall assume these manifolds are equipped with Riemannian warped product metrics of the form

$$g = c^4(r)[dr^2 + r^2 g_S]. \tag{1.2}$$

In what follows, g_S denotes a *fixed* smooth Riemannian metric on the sphere S^{d-1} and c is a positive C^m-function, (with $m \geq 2$), of the variable r only. Without loss of generality, we may assume $c(0) = 1$.

We emphasize that under these general assumptions, the metrics (1.2) are not necessarily regular. Actually, one proves that the metrics g are regular if and only if the odd-order derivatives $c^{(2k+1)}(0) = 0$ and $g_S = d\Omega^2$ where $d\Omega^2$ is the round metric on S^{n-1}, (see [24], Section 4.3.4). Nevertheless, in this paper, we consider general metrics of the form (1.2) which could be singular at $r = 0$, but emphasizing that in the regular case, the proofs of our results are much simpler.

It is convenient to use the change of coordinates $x = -\log r \in [0, +\infty[$. In these new coordinates, the metric g has the form

$$g = f^4(x)[dx^2 + g_S], \tag{1.3}$$

where $f(x) = c(e^{-x})e^{-\frac{x}{2}}$. Using the Taylor expansion of $c(r)$ at $r = 0$, we see that the conformal factor $f(x)$ satisfies the following asymptotic expansion:

$$f(x) = e^{-\frac{x}{2}} + \sum_{k=1}^{m} c_k e^{-(k+\frac{1}{2})x} + o(e^{-(m+\frac{1}{2})x}) \ , \ x \to +\infty \ , \tag{1.4}$$

for some suitable real constants c_k.

A typical example of Riemannian manifold belonging to our class is the usual Euclidean metric on a unit ball of $\mathbb{R}^d$. Indeed, simply take $c(r) = 1$, (i.e $f(x) = e^{-\frac{x}{2}}$), and $g_S = d\Omega^2$ to recover it. Hence, under our hypothesis, the Riemannian manifold (M, g) can be viewed topologically as a unit ball of $\mathbb{R}^n$ whose metric is a deformation of the usual Euclidean metric both in the radial direction r through the warping function $c(r)$, but also in the transversal directions through the general metric g_S.

In this paper, we are interested in studying the amount of information contained in the Steklov spectrum associated to the class of Riemannian manifolds (M, g). Recall [16] that the Steklov spectrum is defined as the spectrum of the Dirichlet-to-Neumann map Λ_g (abbreviated later by DN map) associated to (M, g). More precisely, consider the Dirichlet problem

$$\begin{cases} -\triangle_g u = 0, & \text{on } M, \\ u = \psi, & \text{on } \partial M, \end{cases} \tag{1.5}$$

where $\psi \in H^{\frac{1}{2}}(\partial M)$. Using the separation of variables, we shall prove in Section 2 that we can solve uniquely (1.5) even in the case where the metric g is singular. Of course, when the metric is smooth, this result is well-known (see [27, 30]) and the unique solution u of (1.5) belongs to the Sobolev space $H^1(M)$.

In the latter case, we define the DN map Λ_g as the operator Λ_g from $H^{1/2}(\partial M)$ to $H^{-1/2}(\partial M)$ as

$$\Lambda_g \psi = (\partial_\nu u)_{|\partial M}\,, \tag{1.6}$$

where u is the unique solution of (1.5) and $(\partial_\nu u)_{|\partial M}$ is its normal derivative with respect to the unit outer normal vector ν on ∂M. Here $(\partial_\nu u)_{|\partial M}$ is interpreted in the weak sense as an element of $H^{-1/2}(\partial M)$ by

$$\langle \Lambda_g \psi | \phi \rangle = \int_M \langle du, dv \rangle_g \, dVol_g,$$

for any $\psi \in H^{1/2}(\partial M)$ and $\phi \in H^{1/2}(\partial M)$ such that u is the unique solution of (1.5) and v is any element of $H^1(M)$ such that $v_{|\partial M} = \phi$. If ψ is sufficiently smooth, we can check that

$$\Lambda_g \psi = g(\nu, \nabla u)_{|\partial M} = du(\nu)_{|\partial M} = \nu(u)_{|\partial M},$$

where ν denotes the unit outer normal vector to ∂M, so that the expression in local coordinates for the normal derivative is thus given by

$$\partial_\nu u = \nu^i \partial_i u. \tag{1.7}$$

The DN map is a pseudo-differential operator of order 1 and is self-adjont on $L^2(\partial M, dS_g)$ where dS_g denotes the metric induced by g on the boundary ∂M. Therefore, the DN map has a real and discrete spectrum accumulating at infinity. We shall thus denote the Steklov eigenvalues (counted with multiplicity) by

$$0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_k \to \infty. \tag{1.8}$$

We refer the reader to the nice survey [16] and references therein. The main results of this paper are the following. First, we obtain the complete asymptotics of the Steklov spectrum σ_k as $k \to \infty$ in terms of the Taylor series of the effective potential

$$q_f(x) = \frac{(f^{d-2})''(x)}{f^{d-2}(x)} - \frac{(d-2)^2}{4}, \tag{1.9}$$

and of the eigenvalues of the positive Laplace-Beltrami operator of (S^{d-1}, g_S). Note that, in the radial coordinate r, the effective potential is given by:

$$q_f(x) = (d-2)\left((d-3) r^2 \left(\frac{c'(r)}{c(r)} \right)^2 + r^2 \frac{c''(r)}{c(r)} + (d-1) r \frac{c'(r)}{c(r)} \right). \tag{1.10}$$

For later use, we denote the spectrum of $-\triangle_{g_S}$ (counting with multiplicities) by

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \to \infty. \tag{1.11}$$

For each $k \geq 0$, the eigenvalue μ_k is associated to the normalized eigenfunction $Y_k \in L^2(S^{d-1}, dVol_{g_S})$ such that

$$-\triangle_{g_K} Y_k = \mu_k Y_k.$$

Let us finally denote

$$\kappa_k = \sqrt{\mu_k + \frac{(d-2)^2}{4}}, \quad \forall k \geq 0, \tag{1.12}$$

and recall the Weyl asymptotics ([28], Theorem 1.2.1 and Remark 1.2.2)

$$\kappa_k = c_{d-1}\, k^{\frac{1}{d-1}} + O(1), \quad k \to \infty. \tag{1.13}$$

Here $c_{d-1} = 2\pi\, (\omega_{d-1} Vol(S^{d-1}))^{-\frac{1}{d-1}}$ where ω_{d-1} is the volume of the unit ball in $\mathbb{R}^{d-1}$, while $Vol(S^{d-1})$ is the volume of S^{d-1} for the metric g_S.
Our first result is the following:

Theorem 1.1. *Let (M, g) be a Riemannian manifold given by (1.1)-(1.2) and assume that $c \in C^\infty([0,1])$. Then the Steklov spectrum $(\sigma_k)_{k\geq 0}$ satisfies for all $N \in \mathbb{N}$,*

$$\sigma_k = \frac{(d-2)f'(0)}{f^3(0)} + \frac{\kappa_k}{f^2(0)} + \sum_{j=0}^{N} \frac{\beta_j(0)}{f^2(0)} \kappa_k^{-j-1} + O(\kappa_k^{-N-2}), \quad k \to \infty, \tag{1.14}$$

where

$$\begin{cases} \beta_0(x) = \frac{1}{2} q_f(x), \\ \beta_{j+1}(x) = \dfrac{1}{2}\beta_j'(x) + \dfrac{1}{2}\displaystyle\sum_{l=0}^{j} \beta_l(x)\beta_{j-l}(x). \end{cases} \tag{1.15}$$

We make the following observations:

1. According to (1.13), we find that the Steklov spectrum satisfies the usual Weyl law (see [16]):

$$\sigma_k = \frac{c_{d-1}}{f^2(0)}\, k^{\frac{1}{d-1}} + O(1) = 2\pi \left(\frac{k}{\omega_{d-1} Vol(\partial M)} \right)^{\frac{1}{d-1}} + O(1), \quad k \to \infty. \tag{1.16}$$

2. Note that the coefficients $\beta_j(0),\ j \geq 0$ only depend on the derivatives $q_f^{(l)}(0),\ l = 0, \dots, j$ up to order j. Hence the values $f(0), f'(0)$ and the Taylor series at 0 of the effective potential q_f give the leading terms of the asymptotics of the Steklov spectrum in inverse powers of κ_k. But without additional knowledge of the asymptotics of the κ_k, we are not able to determine the warping function f from this information.
3. In the exceptional case where $(S^{d-1}, g_S) = (S^{d-1}, d\Omega^2)$, the spectrum of $-\triangle_{g_K}$ is given by $\{\lambda_k = k(k+d-2),\ k \geq 0\}$ where each eigenvalue λ_k has multiplicity $\frac{2k+d-2}{d-2}\, C_{k+d-3}^{k}$. Hence, if we order the Steklov spectrum σ_k *without counting mutiplicities, i.e.*

$$0 = \sigma_0 < \sigma_1 < \sigma_2 < \cdots < \sigma_k \to \infty,$$

where each σ_k has multiplicity $\frac{2k+d-2}{d-2}\ C_{k+d-3}^k$ and denoting $\kappa_k = \sqrt{\lambda_k + \frac{(d-2)^2}{4}} = k + \frac{d-2}{2}$, we get the more precise asymptotic (without multiplicity):

$$\sigma_k = \frac{(d-2)f'(0)}{f^3(0)} + \frac{k+\frac{d-2}{2}}{f^2(0)} + \sum_{j=0}^{N} \frac{\beta_j(0)}{f^2(0)}(k+\frac{d-2}{2})^{-j-1} + O(k^{-N-2}), \quad k \to \infty. \tag{1.17}$$

As a consequence, the asymptotics of the Steklov spectrum allow to recover inductively the values $f(0), f'(0)$ and the Taylor series at 0 of the effective potential q_f. Using then (1.9), we immediately see that in fact, the Steklov spectrum determines the Taylor serie of f. Hence we have proved:

Corollary 1.2. *If $(S^{d-1}, g_S) = (S^{d-1}, d\Omega^2)$ and if the warping function c is analytic on $[0,1]$, then the Steklov spectrum of (M,g) determines uniquely the warping function c.*

This leads to the question: does the Steklov spectrum determine uniquely the warping function c without any assumption of analyticity on c and on the transversal metric g_S ? The answer is yes. Precisely, we have:

Theorem 1.3. *Let (M,g) be a Riemannian manifolds given by (1.1)-(1.2). Then the Steklov spectrum $(\sigma_k)_{k\geq 0}$ determines uniquely the warping function c.*

In dimension 2, the problem of recovering the metric from the Steklov spectrum on the unit disk was studied in a paper by Jollivet and Sharafutdinov [19]. They show that if the DN maps associated to the metrics g_j, $j = 1,2$ are intertwined (which is a stronger hypothesis than the Steklov isospectrality), then the metrics g_j are equal (up to a diffeomorphism and a conformal factor which is equal to 1 on the boundary of the unit disk).

In fact, we obtain an even better uniqueness result than Theorem 1.3 from the Steklov spectrum : the subsequence $(\sigma_{k^{d-1}})$ determines uniquely the warping function c. The proof is based on the following *local* uniqueness property:

Theorem 1.4. *Let (M,g) and $(M,\tilde{g})$ be Riemannian manifolds given by (1.1)-(1.2). Then, for a positive constant $a > 0$, the two following assertions are equivalent:*

$$\sigma_{k^{d-1}} - \tilde{\sigma}_{k^{d-1}} \;=\; O(e^{-2a\kappa_{k^{d-1}}}), \quad k \to \infty. \tag{1.18}$$

$$c(r) \;=\; \tilde{c}(r), \quad \forall r \in [e^{-a}, 1]. \tag{1.19}$$

This theorem asserts that the knowledge of a subsequence of the Steklov spectrum up to an error that decays exponentially as $k \to \infty$ (see (1.18)) determines uniquely the warping function c in a neighbourhood of the boundary $r = 1$ provided that (S^{d-1}, g_S) is known. Of course, Theorem 1.3 is an immediate consequence of Theorem 1.4. The question of whether the Steklov spectrum determines uniquely the warping function without the additional knowledge of the transversal Riemannian manifold (S^{n-1}, g_S) remains open.

Notice that the above result can be viewed as a weak form of a stability result. It stresses the important fact that the asymptotic behaviour of the Steklov spectrum allows one to determine the warping function c in a neighbourhood of the boundary $r = 1$. In contrast, we shall show that the first Steklov eigenvalues determine in a stable way the warping function c on $[0,1]$. Before stating our main result, let us define our set $\mathcal{C}(A)$ of admissible warping functions $c(r)$, where A is any positive constant, $m \geq 3$ and $2 \leq p \leq m-1$:

$$\mathcal{C}(A) = \{c \in C^m([0,1]) \text{ s. t. } ||c||_{C^m([0,1])} + ||\frac{1}{c}||_{C^m([0,1])} \leq A \text{ and for } k = 1, ..., p-1,\ c^{(k)}(0) = 0\}. \tag{1.20}$$

Roughly speaking, the conditions on the derivatives of $c(r)$ at $r = 0$ tell us that the metric g is relatively flat in a neighborhood of the origin $r = 0$. These assumptions are only introduced to take into account the regularity of the warping function in the stability estimates obtained by integration by parts. We only assume $m \geq 3$ for the simplicity of the proofs. In particular, using (1.10) we see that, if $c \in \mathcal{C}(A)$, the effective potential q_f satisfies the uniform estimates:

$$|q_f^{(k)}(x)| \leq C_A \ e^{-px} \ , \forall x \geq 0, \ \forall k = 0, ..., m-2, \tag{1.21}$$

where the constant C_A depends only on A.

In Section 4, we shall prove two log-type estimates results. The first one concerns the case where the metric g is smooth, i.e we assume that the odd order derivatives of the warping functions $c^{(2k+1)}(0) = 0$, and we assume that the transversal metric $g_S = d\Omega^2$ is the usual Euclidean metric on S^{n-1}. In this regular case, we have the following estimate:

Theorem 1.5. *Let (M, g) and $(M, \tilde{g})$ be smooth Riemannian manifolds given by (1.1)-(1.2) with c, $\tilde{c} \in \mathcal{C}(A)$ where $A > 0$ is fixed. Assume that for $\epsilon > 0$ small enough, one has:*

$$\sup_{k \geq 0} \ |\sigma_k - \tilde{\sigma}_k| \leq \epsilon. \tag{1.22}$$

Then, there exists a positive constant C_A, depending only on A such that,

$$||c - \tilde{c}||_{L^\infty(0,1)} \leq C_A \ \left(\frac{1}{\log(\frac{1}{\epsilon})} \right)^{p-1} \tag{1.23}$$

This result is relatively close in spirit to the logarithmic stability estimates obtained by Alessandrini [1], and then improved by Novikov [23], for the Schrödinger equation on a bounded domain M in $\mathbb{R}^d$. Indeed, it is well-known there is a close connection between the DN map which we have defined in the introduction for the metric $g = c^4(r) \ g_e$, (g_e being the Euclidean metric on $\mathbb{R}^d$), and the DN map associated with the Schrödinger equation $(-\Delta_{g_e} + V)u = 0$ where the potential V is defined below. It is induced by the transformation law for the Laplace-Beltrami operator under conformal changes of metric:

$$-\Delta_{c^4 g_e} u = c^{-(d+2)} \left(-\Delta_{g_e} + V\right) \left(c^{d-2} u\right), \tag{1.24}$$

where

$$V = c^{-d+2} \Delta_{g_e} c^{d-2}. \tag{1.25}$$

We recall that if 0 is not an eigenvalue of $-\Delta_{g_e} + V$, then for all $\psi \in H^{1/2}(\partial M)$, there exists a unique u solution of

$$\begin{cases} (-\Delta_{g_e} + V)u = 0, & \text{on } M, \\ u = \psi, & \text{on } \partial M. \end{cases} \tag{1.26}$$

The DN map for this Schrödinger equation is then given by $\Lambda_V(\psi) = \partial_\nu u_{|\partial M}$, where $(\partial_\nu u)_{|\partial M}$ is its normal derivative with respect to the unit outer normal vector ν on ∂M.

If the warping function $c(r)$ satisfies $c_{|\partial M} = 1$ and $\partial_\nu c_{|\partial M} = 0$, (or equivalently $c(1) = 1$ and $c'(1) = 0$ in our case), one can easily show (see for instance [27]) that:

$$\Lambda_{g_e} = \Lambda_V. \tag{1.27}$$

For the Schrödinger equation, Alessandrini [1] has obtained the following stability estimate : assume that, for $j = 1, 2$, the potentials V_j belong to the Sobolev space $W^{m,1}(M)$ with $||V_j||_{W^{m,1}(M)} \leq A$. Then there exists a constant $C_A > 0$ such that

$$||V_1 - V_2||_{L^\infty(M)} \leq C_A \left(\log(3 + |||\Lambda_{V_1} - \Lambda_{V_2}|||^{-1}\right)^{-\alpha}, \tag{1.28}$$

with $\alpha = 1 - \frac{d}{m}$ and where $|||B|||$ is the norm of the operator $B : L^\infty(\partial M) \to L^\infty(\partial M)$. This stability estimate was then improved by Novikov in [23], where he proved one can take $\alpha = m - d$.

In Theorem 1.5, under some additional assumptions on the derivatives of the warping function at $r = 0$, we obtain a rather similar stability estimate with, roughly speaking, α replaced by $p - 1$. Note that our potential V is only C^{m-2} and that $p \leq m - 1$. We emphasize we do not assume that $c(1) = 1$ and $c'(1) = 0$, thus the connection between Λ_g and Λ_V is not so clear and depends implicitly of these unknown values $c(1)$ and $c'(1)$ which we are trying to estimate.

In the singular case, we also obtain a log-type estimate result, but which is much less precise. Indeed, in this case, we do not have explicit formula for the angular eigenvalues κ_k, so we need to use the Weyl's law to reasonably approximate the κ_k. We can prove the following theorem:

Theorem 1.6. *Let (M, g) and $(M, \tilde{g})$ be singular Riemannian manifolds given by (1.1)-(1.2) with c, $\tilde{c} \in \mathcal{C}(A)$ where $A > 0$ is fixed. Assume that for $\epsilon > 0$ small enough, one has:*

$$\sup_{k \geq 0} |\sigma_k - \tilde{\sigma}_k| \leq \epsilon. \tag{1.29}$$

Then, there exists $\theta \in (0, 1)$ and a positive constant C_A, depending only on A such that,

$$||c - \tilde{c}||_{L^\infty(0,1)} \leq C_A \left(\frac{1}{\log(\frac{1}{\epsilon})}\right)^{(p-1)\theta} \tag{1.30}$$

Remark 1.7. *In the definition of our admissible warping functions set $\mathcal{C}(A)$, we impose that the integer $p \geq 2$. In particular, the first derivative at $r = 0$ of the warping function $c(r)$ must vanish. When the metric g is smooth, this condition is automatically satisfied. When the metric g is singular, we only make this assumption for simplicity of exposition in order to use the same strategy as for the regular case. This technical assumption could be certainly relaxed with a little effort, and one would obtain a logarithmic stabilily estimate close to the one stated in Theorem 1.6.*

The rest of our paper is organized as follows. In Section 2, using the separation of variables, we recall the standard construction of the Dirichlet to Neumann map (see [9], [10], [11] for details), and some basics facts on the Steklov spectrum. In Section 3, we prove that the warping function $c(r)$ is uniquely determined by the knowledge of the Steklov spectrum. We also obtain a local uniqueness result from the approximate knowledge of the Steklov spectrum. In Section 4, we prove our stability results: the proofs are based on the Müntz-Jacskon approximation, starting from the knowledge of the Hausdorff moments.

2 The Steklov spectrum

In this section, we construct explicitly the Steklov spectrum by using the warped product structure of the manifold (M, g). More precisely, we use the underlying symmetry of the warped product in order to diagonalize the DN map onto the Hilbert basis of harmonics $\{Y_k\}_{k\geq 0}$, *i.e.* the normalized eigenfunctions, of $-\triangle_{g_S}$. On each harmonic, the DN map acts as an operator of multiplication by essentially the Weyl-Titchmarsh function associated to the countable family of Schrödinger operators arising from the separation of variables procedure. The Weyl-Titchmarsh theory will then allow us to prove the asymptotic of theorem 1.1 as well the (local) uniqueness results of Theorems 1.3 and 1.4 in Section 3.

Let us first solve the Dirichlet problem (1.5). In the coordinate system (x, ω), the Laplace equation $-\triangle_g u = 0$ reads

$$[-\partial_x^2 - \triangle_{g_S} + q_f(x)]v = -\frac{(d-2)^2}{4} v, \tag{2.1}$$

where $v = f^{d-2} u$ and q_f is given by (1.9). Observe that under the hypothesis (1.4), the effective potential q_f is a smooth function on $[0, +\infty)$ that satisfies the asymptotics

$$q_f(x) = O(e^{-px}), \quad x \to \infty. \tag{2.2}$$

We now use the warped structure to separate variables. We thus look for solutions of (2.1) of the form

$$v = \sum_{k=0}^{\infty} v_k(x) Y_k. \tag{2.3}$$

Hence, for each $k \geq 0$, the functions v_k satisfy the Schrödinger equation on the half-line

$$-v_k'' + q_f(x) v_k = -\kappa_k^2 v_k, \quad x \in [0, \infty), \tag{2.4}$$

where κ_k is given by (1.12) for all $k \geq 0$. Actually, it is very useful to consider complex spectral parameter $z \in \mathbb{C}$ and we are interested by some special solutions of the Sturm-Liouville equation:

$$-v'' + q_f(x) v = zv. \tag{2.5}$$

We denote by $\{C_0(x,z), S_0(x,z)\}$ the fundamental system of solutions of (2.5) with a spectral parameter $z \in \mathbb{C}$ that satisfy Neumann and Dirichlet conditions at $x = 0$ respectively, given by

$$C_0(0,z) = 1, \; C_0'(0,z) = 0, \; S_0(0,z) = 0, \; S_0'(0,z) = 1. \tag{2.6}$$

Note that

$$W(C_0(x,z), S_0(x,z)) = 1, \tag{2.7}$$

where the Wronskian is defined by $W(u,v) = uv' - u'v$. Moreover, the functions $z \mapsto C_0(x,z), S_0(x,z)$ are entire in z. Note also that under the hypothesis (2.2), the Schrödinger operator $H = -\frac{d^2}{dx^2} + q_f$ is in the limit point case at $x = \infty$. In consequence, for all $z \in \mathbb{C}$, there exists a unique (up to constant factor) solution $S_\infty(x,z)$ of (2.5) that is L^2 in a neighbourhood of $x = \infty$, (see [25], Theorem XI.57 where our spectral parameter $z = k^2$). We write this function as

$$S_\infty(x,z) = A(z) \left(C_0(x,z) - M(z) S_0(x,z) \right). \tag{2.8}$$

Using (2.7), we thus get the following expressions for the function $A(z)$ and the Weyl-Titchmarsh function $M(z)$

$$A(z) = W(S_\infty(x,z), S_0(x,z)), \qquad M(z) = -\frac{W(C_0(x,z), S_\infty(x,z))}{W(S_0(x,z), S_\infty(x,z))} = \frac{S_\infty'(0,z)}{S_\infty(0,z)}. \tag{2.9}$$

For later use, we also introduce the characteristic functions

$$\Delta(z) = W(S_0(x,z), S_\infty(x,z)) = -A(z), \quad d(z) = W(C_0(x,z), S_\infty(x,z)). \tag{2.10}$$

We thus have:

$$M(z) = -\frac{d(z)}{\Delta(z)}. \tag{2.11}$$

Notice that the characteristic functions $\Delta(z)$ and $d(z)$ are analytic on $\mathbb{C}$, and the WT function M is analytic on $\mathbb{C}\backslash[\beta, +\infty[$ with $-\beta$ sufficiently large. The zeros $(-\alpha_j^2)_{j\geq 0}$ of the function $z \mapsto \Delta(z)$ are precisely the Dirichlet eigenvalues of the self-adjoint operator H (and thus are real) whereas the zeros $(-\gamma_j^2)_{j\geq 0}$ of the function $z \mapsto d(z)$ are the Neumann eigenvalues of the self-adjont operator H (and thus are real too). Moreover, we know that $\sigma_{ess}(H) = [0, +\infty)$ and that the essential spectrum contains no embedded eigenvalues ([26], Thm XIII.56). In consequence of the spectral theorem, the eigenvalues $(-\alpha_j^2)_{j\geq 0}$ and $(-\gamma_j^2)_{j\geq 0}$ must then satisfy

$$-\alpha_j^2, -\gamma_j^2 \in [\min(q_f), 0] \quad , \quad \forall j \geq 0, \tag{2.12}$$

with the usual convention that $[\min(q_f), 0] = \emptyset$ if $\min(q_f) > 0$.

We have a more precise result on the location of the eigenvalues $-\alpha_j^2$ of H thanks to

Lemma 2.1. *The discrete Dirichlet spectrum of $H = -\frac{d^2}{dx^2} + q_f$ is finite and contained in $(-\frac{(d-2)^2}{4}, 0)$.*

Proof. Introduce the operators $L = H + \frac{(d-2)^2}{4}$ and $K = f^{-d+2} L f^{d-2}$. An easy calculation shows that

$$K = -\frac{1}{f^{d-2}} \frac{d}{dx} \left(f^{d-2} \frac{d}{dx} \right).$$

We remark that K is selfadjoint on $L^2(\mathbb{R}^+; f^{2d-4}(x)dx)$ if we impose Dirichlet boundary condition at $x = 0$. Moreover, the operator K is clearly positive. Hence we have $\sigma_{pp}(K) \subset (0, +\infty)$. But λ is an eigenvalue of K if and only if $\lambda - \frac{(d-2)^2}{4}$ is an eigenvalue of H. In particular, we obtain that an eigenvalue $-\alpha_j^2$ of H always satisfy $-\alpha_j^2 > -\frac{(d-2)^2}{4}$. Together with (2.12), this proves the result. $\square$

Corollary 2.2. 0 *does not belong to the Dirichlet spectrum of $-\Delta_g$.*

Proof. Assume the converse. Then there exists $u \neq 0$ such that $-\Delta_g u = 0$ on M and $u = 0$ on ∂M. Using separation of variables, this means that there exists $k \geq 0$, $v_k \neq 0$ and $v_k(0) = 0$ such that $H v_k = -\kappa_k^2 v_k$. In other words, $-\kappa_k^2$ is an eigenvalue of H with Dirichlet boundary condition. According to Lemma 2.1, we then must have $-\kappa_k^2 > -\frac{(d-2)^2}{4}$. But since, $\mu_k \geq 0$, we always have $-\kappa_k^2 \leq -\frac{(d-2)^2}{4}$. Contradiction. $\square$

Finally, the DN map can be clearly diagonalized onto the Hilbert basis of harmonics $\{Y_k\}_{k\geq 0}$. If we represent the Dirichlet data as $\psi = \sum_{k\geq 0} \psi_k Y_k$, then the global DN map has the expression

$$\Lambda_g \psi = \sum_{k=0}^{\infty} (\Lambda_g^k \psi_k) Y_k, \tag{2.13}$$

where the diagonalized DN map are defined by

$$\Lambda_g^k \psi_k = \frac{(d-2)f'(0)}{f^{d+1}(0)} v_k(0) - \frac{v_k'(0)}{f^d(0)}. \tag{2.14}$$

Using that $v_k(x) = \alpha_k C_0(x, \kappa_k) + \beta_k S_0(x, \kappa_k)$, a straightforward calculation shows that the partial DN map Λ_g^k acts as an operator of multiplication, precisely

$$\Lambda_g^k \psi_k = \left(\frac{(d-2)f'(0)}{f^3(0)} - \frac{M(-\kappa_k^2)}{f^2(0)} \right) \psi_k. \tag{2.15}$$

We see immediately from (2.15) that the partial DN map Λ_g^k acts essentially by an operator of multiplication by the WT function $M(-\kappa_k^2)$ associated to (2.4) up to some boundary values of the warping function f ans its first derivative f'. In consequence, we infer that the Steklov spectrum of (M, g), that is the set of eigenvalues of Λ_g, is precisely given by

$$\{\sigma_k, \ k \geq 0\} = \left\{ \frac{(d-2)f'(0)}{f^3(0)} - \frac{M(-\kappa_j^2)}{f^2(0)}, \ j \geq 0 \right\}. \tag{2.16}$$

In fact, we have the following exact identification:

Lemma 2.3. *1. The function $y \in \mathbb{R} \to M(y)$ is strictly increasing on $\mathbb{R} \setminus \{-\alpha_j^2, \ j \geq 0\}$.*
2. Then $\forall k \geq 0$, $\sigma_k = \frac{(d-2)f'(0)}{f^3(0)} - \frac{M(-\kappa_k^2)}{f^2(0)}$.

Proof. 1. Let $y \in \mathbb{R} \backslash \{-{\alpha_j}^2\}$. Notice then that $C_0(x, y), S_0(x, y), S_\infty(x, y)$ are real and thus $\Delta(y)$ and $d(y)$ are real too. Let $y^* \in \mathbb{R}$. Since $(S_\infty(x, y)S'_\infty(x, y^*) - S'_\infty(x, y)S_\infty(x, y^*))' = (y - y^*)S_\infty(x, y)S_\infty(x, y^*)$, and using that for $k = 0, 1$, $\lim\limits_{x \to \infty} S_\infty^{(k)}(x, y) = 0$ and (2.10), we have:

$$(y - y^*) \int_0^\infty S_\infty(x, y)S_\infty(x, y^*)dx = \Delta(y)d(y^*) - \Delta(y^*)d(y).$$

Hence letting $y^* \to y$, we get

$$\int_0^\infty S_\infty(x, y)^2 dx = d(y)\dot{\Delta}(y) - \Delta(y)\dot{d}(y) = \Delta^2(y)\dot{M}(y),$$

where $\dot{}$ denotes the derivative with respect to y. We conclude from this that 1. holds. Observe that between two Dirichlet eigenvalues $-\alpha_j^2 < -\alpha_{j+1}^2$, the WT function $M(y)$ is strictly increasing and goes from from $-\infty$ to $+\infty$.

2. From Lemma 2.1, we know that the Dirichlet eigenvalues of H satisfy $-\frac{(d-2)^2}{4} < -\alpha_j^2 \leq 0$ for all $j \geq 0$. In other words, we have

$$\forall j \geq 0, \quad 0 \leq \alpha_j < \frac{d-2}{2}.$$

Since $\kappa_k \geq \frac{d-2}{2}$ for all $k \geq 0$ (see (1.12)), we deduce from 1. that the function $\kappa_k \mapsto M(-\kappa_k^2)$ is strictly decreasing for $k \geq 0$. Hence 2. follows from (2.16) and the ordering of the Steklov spectrum $(\sigma_k)_{k \geq 0}$. □

Having obtained the exact expression of the Steklov spectrum in terms of the WT functions (2.9) evaluated at the $-\kappa_k^2$, we can prove easily Theorem 1.1 by using the well-known asymptotic of a WT function due to Danielyan and Levitan [8]. We refer for instance to [29], Section 4 and the references therein for a proof and more asymptotic results.

Proof of Theorem 1.1. Recall the asymptotic of the WT function from [29], Thm 4.5. If $q_f \in C^N([0,\delta))$ with $\delta > 0$, then as $\kappa \to \infty$, we have

$$M(-\kappa^2) = -\kappa - \sum_{j=0}^{N} \beta_j(0)\kappa^{-j-1} + O(\kappa^{-N-2}), \tag{2.17}$$

where the constants $\beta_j(0)$ can be calculated inductively by (1.15). Since the potential q_f is assumed to be smooth, the asymptotic (2.17) together with lemma 2.3 lead to the result. □

3 Uniqueness and local uniqueness

In this section, we prove local uniqueness, that is Theorem 1.4, under the assumption that the transversal Riemanniann manifold (S^{n-1}, g_S) is known. The general idea of local uniqueness inverse results arises in the different versions of the local Borg-Marchenko Theorem stated first by Simon in [29] and proved differently or extended to singular settings in [3, 13, 14, 15, 20]. We shall follow here the initial version due to Simon that makes an intensive use of a representation of the WT function M as the Laplace transform of what Simon called the A-function. Precisely, Simon showed in [29], Thm 2.1, that there exists a function A on $[0,\infty)$ with $A - q_f$ continuous, obeying

$$|A(\alpha) - q_f(\alpha)| \leq Q(\alpha)^2 e^{\alpha Q(\alpha)}, \quad Q(\alpha) = \int_0^\alpha |q_f(s)|ds, \tag{3.1}$$

such that, if $\kappa > \frac{1}{2}\|q_f\|_{L^1}$, then

$$M(-\kappa^2) = -\kappa - \int_0^\infty A(\alpha)e^{-2\kappa\alpha}d\alpha. \tag{3.2}$$

We also have

$$|A(\alpha, q_f) - A(\alpha, \tilde{q_f})| \leq \|q_f - \tilde{q_f}\|_{L^1}[Q(\alpha) + \tilde{Q}(\alpha)]e^{\alpha[Q(\alpha)+\tilde{Q}(\alpha)]}. \tag{3.3}$$

Finally, Simon also proved the local uniqueness result

Theorem 3.1 ([29], Thm 1.5)**.** *The potential q_f on $[0,a]$ is a function of A on $[0,a]$. Explicitly, if q_f and $\tilde{q_f}$ are two potentials, let A and $\tilde{A}$ be their A-functions. Then*

$$A(\alpha) = \tilde{A}(\alpha),\ \forall \alpha \in [0,a] \iff q_f(x) = \tilde{q_f}(x),\ \forall x \in [0,a].$$

We shall use these results as follows.

Proof of theorem 1.4. Suppose that the assumption (1.18) is satisfied. Using Theorem 1.1 and Lemma 2.3, we see immediately that

$$f(0) = \tilde{f}(0), \quad f'(0) = \tilde{f}'(0). \tag{3.4}$$

Hence the assumption (1.18) can be equivalently read as

$$M(-\kappa_{k^{d-1}}^2) - \tilde{M}(-\kappa_{k^{d-1}}^2) = O(e^{-2a\kappa_{k^{d-1}}}), \quad k \to \infty. \tag{3.5}$$

We use then the representation of the WT functions (3.2), the behavior (2.2) of the potentials q_f, $q_{\tilde{f}}$ and the estimate (3.3) to show that (3.5) entails

$$\int_0^a \left[A(\alpha) - \tilde{A}(\alpha)\right] e^{-2\kappa_{k^{d-1}}\alpha}d\alpha = O(e^{-2a\kappa_{k^{d-1}}}), \quad k \to \infty. \tag{3.6}$$

Now, we need the following proposition which is a slight generalization to the case of noninteger $(\kappa_k)_{k\geq 0}$ of Proposition 2.4 in [17].

Proposition 3.2. *Let $f \in L^1(0,a)$. Assume that*

$$\int_0^a \ e^{-\kappa_{k^{d-1}} t} f(t) \ dt \ = \ O(e^{-a\kappa_{k^{d-1}}}) \ , \quad k \to +\infty. \tag{3.7}$$

Then, $f = 0$ almost everywhere.

Proof. Setting $\lambda_k = \frac{1}{c_{d-1}} \kappa_{k^{d-1}}$, we deduce from (1.13) there exists $C > 0$ such that $|\lambda_k - k| \leq C$.

For a fixed $N \in \mathbb{N}$ large enough, we set $\nu_k = \frac{\lambda_{kN}}{N}$, and we have $|\nu_k - k| \leq \frac{C}{N} < \frac{1}{4}$. Using (3.7), a straightforward calculation gives:

$$\int_0^b g(y) e^{-\nu_k y} \ dy = O(e^{-b\nu_k}), \ k \to +\infty, \tag{3.8}$$

where we have set $b = c_{d-1} N a$, and $g(y) = f(\dfrac{y}{c_{d-1}N})$. Now, let us define for $z \in \mathbb{C}$,

$$F(z) = e^{bz} \int_0^b g(y) e^{-zy} \ dy. \tag{3.9}$$

Clearly, $F(z)$ is an entire function which obeys

$$|F(z)| \leq ||g||_1 \ e^{bRe_+(z)}, \tag{3.10}$$

where $Re_+(z)$ is the positive part of $Re\ z$. Moreover, from (3.8), we get $F(\nu_k) = O(1)$. It follows from a theorem of Duffin and Schaeffer ([5], Theorem 10.5.1) that $F(x)$ is bounded for $x > 0$, or equivalently

$$\int_0^b g(y) e^{-xy} \ dy = O(e^{-bx}) \ , \ x \to +\infty. \tag{3.11}$$

Then, using ([29], Lemma A.2.1), we have $g = 0$ almost everywhere in $(0, b)$ which concludes the proof of this Proposition.

□

Thus, Proposition 3.2 implies in particular that

$$A(\alpha) = \tilde{A}(\alpha), \quad \forall \alpha \in [0, a],$$

from which we infer using Theorem 3.1 that

$$q_f(x) = q_{\tilde{f}}(x), \quad \forall x \in [0, a].$$

Recalling the definition (1.9) of q_f, we thus see that

$$(f^{d-2})''(x) = \frac{(\tilde{f}^{d-2})''(x)}{\tilde{f}^{d-2}(x)} f^{d-2}(x), \quad \forall x \in [0, a] \tag{3.12}$$

We finish the proof seeing that (3.12) can be viewed as a linear second-order ODE for f^{d-2}. Recalling from (3.4) that the Cauchy data (3.4) of f^{d-2} are equal to those of $\tilde{f}^{d-2}$, we conclude that the unique solution of (3.12) on $[0,a]$ is $f^{d-2} = \tilde{f}^{d-2}$. Whence the asserted result.

Conversely, assume that (1.19) holds. In particular, $q_f(x) = q_{\tilde{f}}(x)$ for all $x \in (0,a)$. So, using Theorem 3.1, we get $A(\alpha) = \tilde{A}(\alpha)$ for all $\alpha \in (0,a)$. Then, using the same arguments as in the first part of the proof, we obtain

$$M(-\kappa^2) = \tilde{M}(-\kappa^2) + O(e^{-2a\kappa}) \ \ , \ \ \kappa \to +\infty. \tag{3.13}$$

Then, using Lemma 2.3 and noting that $f(0), f'(0)$ are also known, we get immediately (1.18).

□

4 Proof of Theorems 1.5 and 1.6

4.1 A Volterra type integral operator

Let us begin by an elementary lemma:

Lemma 4.1. *For any κ large enough, we have:*

$$S_\infty(0,-\kappa^2)\tilde{S}_\infty(0,-\kappa^2)\left(M(-\kappa^2) - \tilde{M}(-\kappa^2)\right) = \int_0^{+\infty} (q_{\tilde{f}}(x) - q_f(x))\ S_\infty(x,-\kappa^2)\tilde{S}_\infty(x,-\kappa^2)\ dx.$$

Proof. To simplify the notation, we set $z = -\kappa^2$ and we integrate over the interval $(0,\infty)$ the obvious equality:

$$\left(S_\infty(x,z)\tilde{S}'_\infty(x,z) - S'_\infty(x,z)\tilde{S}_\infty(x,z)\right)' = (q_f(x) - q_{\tilde{f}}(x))S_\infty(x,z)\tilde{S}_\infty(x,z). \tag{4.1}$$

For $k=0$ or 1, $S_\infty^{(k)}(x,z) \to 0$ as $x \to +\infty$, (see for instance [4]), so we get immediately

$$S'_\infty(0,z)\tilde{S}_\infty(0,z) - S_\infty(0,z)\tilde{S}'_\infty(0,z) = \int_0^{+\infty} (q_f(x) - q_{\tilde{f}}(x))S_\infty(x,z)\tilde{S}_\infty(x,z)\ dx, \tag{4.2}$$

which implies the Lemma, thanks to (2.9). □

In the following lemma, we express the Weyl solution $S_\infty(x,z)$ with the help of the well-known Marchenko's representation, (we refer to [22], Chapter III for details):

Lemma 4.2. *Assume that $c \in \mathcal{C}(A)$. Then, there exists a C^{m-1} function $K(x,t)$ for $0 \le x \le t < \infty$, satisfying the properties:*

$$S_\infty(x,-\kappa^2) \ = \ e^{-\kappa x} + \int_x^{+\infty} K(x,t)e^{-\kappa t}\ dt \ ,\ \kappa > 0. \tag{4.3}$$

$$K(x,x) \ = \ \frac{1}{2}\int_x^{+\infty} q_f(t)\ dt. \tag{4.4}$$

Moreover, there exists a constant $C_A > 0$ depending only on A such that,

$$|\partial_x^k \partial_t^l K(x,t)| \le C_A\ e^{-\frac{p}{2}(x+t)} \ ,\ \forall k,l \le m-1. \tag{4.5}$$

Proof. The existence of the Marchenko's kernel $K(x,t)$ is proved in ([22], Lemma 3.1.1), and we have the following estimate:

$$|K(x,t)| \leq \frac{1}{2}\ \sigma(\frac{x+t}{2})\ e^{\sigma_1(x)-\sigma_1(\frac{x+t}{2})}, \tag{4.6}$$

where

$$\sigma(x) = \int_x^{+\infty} |q_f(s)|\ ds\ ,\ \sigma_1(x) = \int_x^{+\infty} \sigma(s)\ ds. \tag{4.7}$$

Thus, using (1.21), we see that $|K(x,t)| \leq C_A e^{-\frac{p}{2}(x+t)}$, i.e we have proved (4.5) in the case $k = l = 0$. Now, let us define $H(u,v) = K(u-v, u+v)$ for $0 \leq v \leq u$. Thanks to ([22], Lemma 3.1.2), H obeys:

$$\frac{\partial H}{\partial u}(u,v) \quad = \quad -\frac{1}{2}q_f(u) - \int_0^v q_f(u-s)H(u,s)\ ds, \tag{4.8}$$

$$\frac{\partial H}{\partial v}(u,v) \quad = \quad \int_u^{+\infty} q_f(s-v)H(s,v)\ ds. \tag{4.9}$$

Then, (4.5) follows from a straightforward calculation. □

As a by-product, we get:

Corollary 4.3. *Let $c \in \mathcal{C}(A)$ be a warping function for the metric (1.2). Then, there exists a constant C_A such that,*

$$|S_\infty(0,-\kappa^2) - 1| \leq \frac{C_A}{\kappa+1}\ ,\ \text{for all } \kappa \geq 0.$$

Proof. Using (4.5) for $k = p = 0$, we get:

$$S_\infty(0,-\kappa^2) - 1 = \int_0^{+\infty} K(0,t)\ e^{-\kappa t}\ dt \leq \frac{C_A}{\kappa+1}.$$

□

Now, let us introduce a new kernel $K_1(x,t)$ for $0 \leq t \leq x < \infty$, by the formula:

$$K_1(x,t) = 2K(t, 2x-t) + 2\tilde{K}(t, 2x-t) + 2\int_t^{2x-t} K(t,u)\tilde{K}(t, 2x-u)\ du, \tag{4.10}$$

where $\tilde{K}(x,t)$ is the Marchenko's kernel associated with the potential $q_{\tilde{f}}$, (see Lemma 4.2). We have the following estimate which follows immediately from (4.5):

Lemma 4.4. *Assume that $c \in \mathcal{C}(A)$. Then, for all $\alpha < p$, there exists a constant $C_{A,\alpha} > 0$ depending only on A and α such that,*

$$|\partial_x^k \partial_t^l K_1(x,t)| \leq C_{A,\alpha}\ e^{-\alpha x}\ ,\ \forall k,l \leq m-1. \tag{4.11}$$

Finally, we consider the corresponding Volterra type integral operator B given by:

$$Bh(x) = h(x) + \int_0^x K_1(x,t)h(t)\ dt. \tag{4.12}$$

This operator B is crucial to our analysis because it links the Steklov spectrum to the difference of the potentials q_f and $q_{\tilde{f}}$. More precisely, one has the following result:

Lemma 4.5. *For κ sufficiently large, we have:*

$$S_\infty(0,-\kappa^2)\tilde{S}_\infty(0,-\kappa^2)\left(M(-\kappa^2)-\tilde{M}(-\kappa^2)\right)=\int_0^{+\infty} e^{-2\kappa x} B[q_{\tilde{f}}-q_f](x)\ dx. \tag{4.13}$$

Proof. Thanks to ([18], Lemma 2.5), we have:

$$\int_0^{+\infty}(q_f(x)-q_{\tilde{f}}(x))S_\infty(x,-\kappa^2)\tilde{S}_\infty(x,-\kappa^2)\ dx=\int_0^{+\infty} e^{-2\kappa x} B[q_{\tilde{f}}-q_f](x)\ dx. \tag{4.14}$$

Then, using Lemma 4.1 and (4.14), we get (4.13). □

This Volterra operator B also possesses good properties on some L^2-spaces equipped with exponential weights, and which are defined by:

$$\mathcal{H}_\delta=\{q:||q||^2_{\mathcal{H}_\delta}:=\int_0^{+\infty}|q(x)|^2\ e^{\delta x}\ dx<\infty\}. \tag{4.15}$$

If $c\in\mathcal{C}(A)$, then it results from (1.21) that $q_f\in\mathcal{H}_\delta$ for any $\delta<2p$. Moreover, there exists a constant $C_{A,\delta}$ depending only on A and δ such that

$$||q_f||_{\mathcal{H}_\delta}\leq C_{A,\delta}\ \text{for all}\ c\in\mathcal{C}(A). \tag{4.16}$$

In the following Proposition, which is close to [18], Lemmas 2.5 - 2.6, we give a uniform estimate on the norm of $B:\mathcal{H}_\delta\to\mathcal{H}_\delta$, and its inverse, when the warping functions belong to the admissible set $\mathcal{C}(A)$. This result will be very useful to estimate the difference of the potentials $q_{\tilde{f}}-q_f$, (for the topology of $\mathcal{H}_\delta$).

Proposition 4.6. *Let $c,\tilde{c}$ be warping functions belonging to $\mathcal{C}(A)$. Then, for any $0<\delta<p$, we have:*

$$B:\mathcal{H}_\delta\to\mathcal{H}_\delta\ \text{is an isomorphism,}$$

and there exists a constant $C_{A,\delta}$ depending only on A and δ such that

$$||B||+||B^{-1}||\leq C_{A,\delta}. \tag{4.17}$$

Proof. By convention, in what follows, C_A or $C_{A,\delta}$ denote constants depending only on A, (or only on A and δ), which can differ from one line to the other. Let $\alpha\in]\delta,p[$ be fixed. We split the operator B as $B=Id+C$ where C is the Volterra operator given by

$$Ch(x)=\int_0^x K_1(x,t)\ h(t)\ dt \tag{4.18}$$

For $h\in\mathcal{H}_\delta$, using (4.11), we get immediately

$$|Ch(x)|\leq C_{A,\delta}\ e^{-\alpha x}||h||_{\mathcal{H}_\delta}, \tag{4.19}$$

which clearly implies $||Ch||_{\mathcal{H}_\delta}\leq C_{A,\delta}\ ||h||_{\mathcal{H}_\delta}$. Then, $B:\mathcal{H}_\delta\to\mathcal{H}_\delta$ is bounded and we have $||B||\leq C_{A,\delta}$.

Now, let us prove that $C : \mathcal{H}_\delta \to \mathcal{H}_\delta$ is a Hilbert-Schmidt operator. To this end, we calculate:

$$\begin{aligned}
||C||_{HS}^2 &:= \int_0^{+\infty} \int_0^{+\infty} |K_1(x,t)\mathbf{1}_{\{\mathbf{t}\leq\mathbf{x}\}}|^2 \; e^{\delta(x+t)} \; dx \; dt \\
&= \int_0^{+\infty} \int_0^{x} |K_1(x,t)|^2 \; e^{\delta(x+t)} \; dt \; dx \\
&\leq C_{A,\delta} \int_0^{+\infty} \int_0^{x} e^{-2\alpha x} \; e^{\delta(x+t)} \; dt \; dx \\
&\leq C_{A,\delta} \int_0^{+\infty} x e^{-2\alpha x} \; e^{2\delta x} \; dt \; dx \\
&\leq C_{A,\delta},
\end{aligned}$$

since $\alpha \in]\delta, p[$. It follows that C is a Hilbert-Schmidt operator, and a fortiori C is a compact operator. So, if B is not an isomomorphism, then -1 must be an eigenvalue of C. But this is impossible for a Volterra operator with continuous kernel.

Now, let us estimate the norm of the inverse operator $B : \mathcal{H}_\delta \to \mathcal{H}_\delta$. We denote by $K_n(x,t)$ the integral kernel of the operator C^n, $n \geq 1$. Clearly, these kernels satisfy the relation

$$K_{n+1}(x,t) = \int_t^x K_1(x,s) K_n(s,t) \; ds \;, \; t \leq x, \tag{4.20}$$

and we have the following estimates which can be easily proved by induction: for $\delta < 2$,

$$|K_n(x,t)| \leq \left(C_{A,\delta} e^{-\delta x}\right)^n \; \frac{(x-t)^{n-1}}{(n-1)!} \;, \; t \leq x. \tag{4.21}$$

For $n \geq 2$, using the rough bound $||C^n|| \leq ||C^n||_{HS}$, we get:

$$\begin{aligned}
||C^n|| &\leq \int_0^{+\infty} \int_0^{x} \left(C_{A,\delta} e^{-\delta x}\right)^{2n} \; \left(\frac{(x-t)^{n-1}}{(n-1)!}\right)^2 \; e^{\delta(t+x)} \; dt \; dx \\
&\leq \left(\frac{(C_{A,\delta})^n}{(n-1)!}\right)^2 \int_0^{+\infty} x^{2n-1} \; e^{-2(n-1)\delta x} \; dx \\
&\leq \left(\frac{(C_{A,\delta})^n}{(n-1)!}\right)^2 \left(\frac{1}{2(n-1)\delta}\right)^{2n} \; \int_0^{+\infty} y^{2n-1} e^{-y} \; dy \\
&\leq \left(\frac{C_{A,\delta}}{2(n-1)\delta}\right)^{2n} \; \frac{\Gamma(2n)}{\Gamma(n)^2}.
\end{aligned}$$

So, thanks to Stirling's formula, we get

$$\sum_{n=0}^{\infty} ||C^n|| \leq 1 + ||C|| + \sum_{n=2}^{+\infty} \left(\frac{C_{A,\delta}}{2(n-1)\delta}\right)^{2n} \; \frac{\Gamma(2n)}{\Gamma(n)^2} \leq C_{A,\delta}. \tag{4.22}$$

This means that the Neumann series

$$B^{-1} = \sum_{n=0}^{+\infty} (-1)^n C^n \tag{4.23}$$

is convergent in the operator norm and $||B^{-1}|| \leq C_{A,\delta}$. □

Now, let us assume again that the warping functions $c, \tilde{c} \in \mathcal{C}(A)$ and that for all $k \geq 0$,

$$|\sigma_k - \tilde{\sigma}_k| \leq \epsilon. \tag{4.24}$$

By making $k \to +\infty$ in (4.24), and thanks to Theorem 1.1, we deduce that $f(0) = \tilde{f}(0)$, and also

$$\left| \frac{(d-2)f'(0)}{f^3(0)} - \frac{(d-2)\tilde{f}'(0)}{\tilde{f}^3(0)} \right| \leq \epsilon. \tag{4.25}$$

Thus, recalling that for sufficiently large k, we have

$$\sigma_k = \frac{(d-2)f'(0)}{f^3(0)} - \frac{M(-\kappa_k^2)}{f^2(0)}, \tag{4.26}$$

we obtain:

$$|\tilde{M}(-\kappa^2) - M(-\kappa^2)| \leq 2f^2(0)\ \epsilon \leq 2A^2\ \epsilon, \tag{4.27}$$

since $c \in \mathcal{C}(A)$. So, plugging (4.27) into (4.13) and using Corollary 4.3, we easily get the following result:

Lemma 4.7. *Let $c, \tilde{c} \in \mathcal{C}(A)$. Assume that for all $k \geq 0$, $|\sigma_k - \tilde{\sigma}_k| \leq \epsilon$. Then, there exists a positive constant C_A which does not depend on ϵ such that*

$$\left| \int_0^{+\infty} e^{-2\kappa_k x} B[q_{\tilde{f}} - q_f](x)\ dx \right| \leq C_A\ \epsilon. \tag{4.28}$$

Note that this kind of estimates fits into the so-called *moment theory*, (see for instance [2] for a nice exposition and references therein), and this is the object of the next sections.

4.2 A Müntz-Jackson's theorem.

Let $\Lambda_\infty = \{0 \leq \lambda_0 < \lambda_1 < ... < \lambda_n < ...\}$, $\lambda_n \to +\infty$ be a sequence of positive real numbers. The classical Müntz-Szász's Theorem characterizes the sequences (λ_k) for which all functions in $C^0([0,1])$, or in $L^2([0,1])$, can be approximated by "Müntz polynomials" of the form:

$$P(x) = \sum_{k=0}^{n} a_k\ x^{\lambda_k}, \tag{4.29}$$

with real coefficients a_k. More precisely, one has:

Theorem 4.8. *Let Λ_∞ be a sequence of positive real numbers as above. Then, span $\{x^{\lambda_0}, x^{\lambda_1}, ...\}$ is dense in $L^2([0,1])$ if and only if*

$$\sum_{k=1}^{+\infty} \frac{1}{\lambda_k} = \infty. \tag{4.30}$$

Moreover, if $\lambda_0 = 0$, the denseness of the Müntz polynomials in $C^0([0,1])$ in the sup norm is also characterized by (4.30).

Now, for $n \geq 1$, let us consider the finite sequence

$$\Lambda := \Lambda_n : 0 \leq \lambda_0 < \lambda_1 < ... < \lambda_n. \tag{4.31}$$

We define the subspace of the "Müntz polynomials of degree λ_n" as:

$$\mathcal{M}(\Lambda) = \{P : \; P(x) = \sum_{k=0}^{n} a_k \; x^{\lambda_k}\}. \tag{4.32}$$

The error of approximation from $\mathcal{M}(\Lambda)$ of a function f in $C^0([0,1])$ or in $L^2([0,1])$ is given by:

$$E(f, \Lambda)_p := \inf_{P \in \mathcal{M}(\Lambda)} ||f - P||_p = ||f - P_0||_p, \tag{4.33}$$

for some $P_0 \in \mathcal{M}(\Lambda)$ depending on whether $p = 2$ or $p = \infty$. Clearly, one has $E(f,\Lambda)_2 \leq E(f,\Lambda)_\infty$ and $E(f,\Lambda)_2 = ||f - \pi_n(f)||_2$ where $\pi_n(f)$ is the orthogonal projection of f on the subspace $\mathcal{M}(\Lambda)$. An estimation from above of $E(f,\Lambda)_p$ in terms of the smoothness of f is called a *Müntz-Jackson's theorem.*

To estimate this error of approximation for the uniform norm, let us consider the so-called Blaschke product $B(z)$, $z \in \mathbb{C}$,

$$B(z) := B(z, \Lambda) = \prod_{k=0}^{n} \frac{z - \lambda_k}{z + \lambda_k}. \tag{4.34}$$

The *index of approximation* of Λ in $C^0([0,1])$ is defined as :

$$\epsilon_\infty(\Lambda) = \max_{y \geq 0} \left| \frac{B(1 + iy)}{1 + iy} \right|. \tag{4.35}$$

Its relevance is justified by the following result ([21], Theorem 2.6, Chapter 11) when $\lambda_0 = 0$:

Proposition 4.9. *Let $\Lambda : 0 = \lambda_0 < \lambda_1 < ... < \lambda_n$ be a finite sequence. Then, for each $f \in C^1([0,1])$,*

$$E(f, \Lambda)_\infty \leq 20 \; \epsilon_\infty(\Lambda) \; ||f'||_\infty. \tag{4.36}$$

As a by-product, we can easily prove:

Corollary 4.10. *Let $\Lambda^* : 0 < \lambda_1 < ... < \lambda_n$ be a finite sequence. Then, for each $f \in C^1([0,1])$ with $f(0) = 0$, one has:*

$$E(f, \Lambda^*)_\infty \leq 40 \; \epsilon_\infty(\Lambda^*) \; ||f'||_\infty. \tag{4.37}$$

Proof. Consider the finite sequence $\Lambda : 0 = \lambda_0 < \lambda_1 < ... < \lambda_n$. Thanks to Proposition 4.9, there exists $P \in \mathcal{M}(\Lambda)$ such that

$$E(f, \Lambda)_\infty = ||f - P||_\infty \leq 20 \; \epsilon_\infty(\Lambda) \; ||f'||_\infty. \tag{4.38}$$

Since $f(0) = 0$, we deduce that $|P(0)| \leq 20 \; \epsilon_\infty(\Lambda) \; ||f'||_\infty$. Thus, setting $Q = P - P(0) \in \mathcal{M}(\Lambda^*)$, we obtain:

$$||f - Q||_\infty \leq 40 \; \epsilon_\infty(\Lambda) \; ||f'||_\infty, \tag{4.39}$$

which concludes the proof since $\epsilon_\infty(\Lambda) = \epsilon_\infty(\Lambda^*)$. □

In the same way, we can recursively approximate C^r-differentiable functions, $(r \geq 1)$. To this end, let $\Lambda^* : \; r - 1 < \lambda_1 < ... < \lambda_n$ be a finite sequence. For $0 \leq k \leq r - 1$, we set:

$$\Lambda^{(k)} : \; \lambda_1^{(k)} = \lambda_1 - k, \; ... \; , \; \lambda_n^{(k)} = \lambda_n - k, \tag{4.40}$$

and the indices of approximation $\epsilon_\infty^{(k)} = \epsilon_\infty(\Lambda^{(k)}), \; k = 0, 2, ..., r - 1$.

We have the following result:

Corollary 4.11. *For each $f \in C^r([0,1])$ such that $f^{(k)}(0) = 0$ for all $k = 0, ..., r-1$, one has:*

$$E(f, \Lambda^*)_\infty \le 40^r \prod_{k=0}^{r-1} \epsilon_\infty^{(k)} \quad ||f^{(r)}||_\infty. \tag{4.41}$$

Proof. For $r = 1$, this estimate is nothing but Corollary 4.10. Now, assume that $r = 2$ and consider $f \in C^2([0,1])$ with $f(0) = f'(0) = 0$. Using Corollary 4.10 for the function f' and the finite sequence $\Lambda^{(1)}$, we see there exists $P(x) = \sum_{k=1}^{n} a_k \ x^{\lambda_k - 1} \in \mathcal{M}(\Lambda^{(1)})$ such that

$$E(f', \Lambda^{(1)})_\infty = ||f' - P||_\infty \le 40 \ \epsilon_\infty^{(1)} \ ||f''||_\infty. \tag{4.42}$$

We set $F(x) = f(x) - \int_0^x P(t) \ dt = f(x) - \sum_{k=1}^{n} \frac{a_k}{\lambda_k} \ x^{\lambda_k}$. Since $F(0) = 0$, using again Corollary 4.10 for the finite sequence Λ^*, we see there exists $Q \in \mathcal{M}(\Lambda^*)$ such that

$$E(F, \Lambda^*)_\infty = ||F - Q||_\infty \le 40 \ \epsilon_\infty \ ||F'||_\infty, \tag{4.43}$$

or equivalently

$$||f - (\int_0^x P(t) \ dt + Q)|| \le 40 \ \epsilon_\infty \ ||f' - P||_\infty. \tag{4.44}$$

Observing that $\int_0^x P(t) \ dt + Q(x) = \sum_{k=1}^{n} \frac{a_k}{\lambda_k} x^{\lambda_k} + Q(x) \in \mathcal{M}(\Lambda^*)$, and using (4.42), we obtain:

$$E(f, \Lambda^*)_\infty \le ||f - (\int_0^x P(t) \ dt + Q)|| \le 40^2 \ \epsilon_\infty \ \epsilon_\infty^{(1)} \ ||f''||_\infty, \tag{4.45}$$

which proves Corollary 4.11 in the case $r = 2$. For $r \ge 3$, the proof is identical. □

For special finite sequences Λ, the index of approximation $\epsilon_\infty(\Lambda)$ can be replaced by a much simpler expression. For instance, we have the following result, ([21], Theorem 4.1, Chapter 11):

Theorem 4.12. *Let $\Lambda : 0 = \lambda_0 < \lambda_1 < \lambda_2 < ... < \lambda_n$ be a finite sequence. Assume that $\lambda_{k+1} - \lambda_k \ge 2$ for $k \ge 0$. Then,*

$$\epsilon_\infty(\Lambda) = |B(1, \Lambda)| = \prod_{k=1}^{n} \frac{\lambda_k - 1}{\lambda_k + 1} \tag{4.46}$$

In particular, if $\lambda_k = 2k + b$ for $k = 1, ..., n$ where $b > 0$, one has

$$\epsilon_\infty(\Lambda) = \frac{b+1}{2n + b + 1}. \tag{4.47}$$

4.3 A Hausdorff moment problem with non-integral powers.

As an application of the previous section, we shall give an approximation of the L^2-norm of a function f from the approximately knowledge of a finite number of these moments:

$$m_k = \int_0^1 t^{\lambda_k} \ f(t) \ dt \ , \ k \in X \subset \mathbb{N}. \tag{4.48}$$

Thanks to Theorem 4.8, if $\Lambda_\infty = \{0 \le \lambda_0 < \lambda_1 < ... < \lambda_n < ...\}$, $\lambda_n \to +\infty$ is a sequence of positive real numbers such that

$$\sum_{k=1}^{+\infty} \frac{1}{\lambda_k} = \infty, \tag{4.49}$$

the system $\{x^{\lambda_0},\ x^{\lambda_1}, ...\}$ is complete in $L^2([0,1])$. Thus, the knowledge of the complete sequence $(m_k)_{k\ge 0}$, uniquely determines the function f. But, in practice, one has available only a finite set $m_0, ..., m_n$ of moments, and furthermore these moments are usually corrupted with noise. It is well-known that this problem is severely ill-posed, (see for instance [2] and references therein, for further details).

Let us briefly explain the approach given in [2]. Using the Gram-Schmidt process, we define the polynomials $(L_m(x))$ as $L_0(x) = 1$, and for $m \ge 1$,

$$L_m(x) = \sum_{j=0}^{m} C_{mj} x^{\lambda_j}, \tag{4.50}$$

where we have set

$$C_{mj} = \sqrt{2\lambda_m + 1}\ \frac{\prod_{r=0}^{m-1}(\lambda_j + \lambda_r + 1)}{\prod_{r=0, r\ne j}^{m}(\lambda_j - \lambda_r)}. \tag{4.51}$$

The family $(L_m(x))$ defines an orthonormal Hilbert basis of $L^2([0,1])$. For instance, if $\lambda_k = k$, the polynomials $(L_m(x))$ are the Legendre polynomials, and in this case, the latter coefficients C_{mj} are given by:

$$C^0_{mj} := \sqrt{2m+1}\ (-1)^{m-j}\ \frac{(m+j)!}{(m-j)!\ j!^2} \tag{4.52}$$

Note that, the generalized binomial theorem gives easily the upper bound:

$$|C^0_{mj}| \le \sqrt{2m+1}\ 3^{m+j}. \tag{4.53}$$

Now, let us consider the finite sequence

$$\Lambda := \Lambda_n : 0 \le \lambda_0 < \lambda_1 < ... < \lambda_n. \tag{4.54}$$

Assume that the $(n+1)$ first moments of a function $f \in L^2([0,1])$ are equal to zero up to noise, i.e there exists $\epsilon > 0$ such that

$$\left|\int_0^1 f(t)\ t^{\lambda_k}\ dt\right| \le \epsilon\ ,\ \ \forall k = 0, ..., n. \tag{4.55}$$

We denote $\pi_n(f)$ the orthogonal projection on on the subspace $\mathcal{M}(\Lambda)$:

$$\pi_n(f) = \sum_{k=0}^{n} < f, L_k > L_k. \tag{4.56}$$

Thus, we deduce that:

$$\begin{aligned} ||\pi_n(f)||_2^2 &= \sum_{k=0}^{n} \left| < f, \sum_{p=0}^{k} C_{kp}\ x^{\lambda_p} > \right|^2 \\ &\le \epsilon^2 \sum_{k=0}^{n} \left(\sum_{p=0}^{k} |C_{kp}| \right)^2, \end{aligned} \tag{4.57}$$

thanks to our hypothesis on the moments (4.55). So, we get immediately:

$$\begin{aligned} ||f||_2^2 &= ||\pi_n(f)||_2^2 + ||f - \pi_n(f)||_2^2 \\ &= ||\pi_n(f)||_2^2 + E(f,\Lambda)_2^2 \\ &\leq \epsilon^2 \sum_{k=0}^{n} \left(\sum_{p=0}^{k} |C_{kp}| \right)^2 + E(f,\Lambda)_\infty^2 \\ &\leq \epsilon^2 \sum_{k=0}^{n} \left(\sum_{p=0}^{k} |C_{kp}| \right)^2 + E(f,\Lambda^*)_\infty^2, \end{aligned}$$

where $\Lambda^* : 0 < \lambda_1 < ... < \lambda_n$. In particular, if $f \in C^r([0,1]$ with $f^{(k)}(0) = 0$ for $k = 0, ..., r-1$, and if $r - 1 < \lambda_1$, we get using Corollary 4.11:

$$||f||_2^2 \leq \epsilon^2 \sum_{k=0}^{n} \left(\sum_{p=0}^{k} |C_{kp}| \right)^2 + \left(40^r \prod_{k=0}^{r-1} \epsilon_\infty^{(k)} \ \ ||f^{(r)}||_\infty \right)^2. \tag{4.58}$$

At this stage, it is important to make the following remark: on the one hand, the double sum appearing in the (RHS) of (4.58) can be very large with respect to n. Indeed, in the case $\lambda_k = k$, the coefficients C_{k0}^0 are equal to $\sqrt{2k+1}$ thanks to (4.52). On the other hand, if $\lambda_k = 2k$, Theorem 4.12 suggests that the second term in the (RHS) of (4.58) is equal to $O(n^{-r})$, $n \to +\infty$. Thus, if we want to control reasonably (with respect to ϵ) the L^2-norm of the function f, we have to choose a suitable $n = n(\epsilon)$ in the equation (4.58). Of course, this choice will depend heavily of the behaviour of the coefficients C_{kp}. This will be done in the next two sections where we separate the simpler case where the metric in regular, and the case where the metric is singular at $r = 0$.

4.4 The regular case.

4.4.1 Stability estimates for the potentials.

In this case, we recall that necessarily, the metric $g_S = d\Omega^2$ and $\kappa_k = k + \frac{d-2}{2}$, (we order the Steklov spectrum without counting multiplicity). So, making the change of variables $t = e^{-x}$ in (4.28), we obtain:

$$| \int_0^1 t^{2k+d-3} \ B[q_{\tilde{f}} - q_f](-\log t) \ dt| \leq C_A \ \epsilon, \ \forall k \geq 0. \tag{4.59}$$

Thus, introducing $\delta \in]0,1[$, we get:

$$| \int_0^1 t^{\lambda_k} \ h(t) \ dt| \leq C_A \ \epsilon, \ \forall k \geq 0, \tag{4.60}$$

where we have set $h(t) = t^{-\frac{\delta+1}{2}} B[q_{\tilde{f}} - q_f](-\log t)$ and $\lambda_k = 2k + d - 3 + \frac{\delta+1}{2}$. Note that

$$||h||_{L^2(0,1)} = ||B[q_{\tilde{f}} - q_f]||_{\mathcal{H}_\delta}. \tag{4.61}$$

Using Lemma 4.4, we see that $h(t) \in C^{m-2}((0,1])$ and we have the following result:

Lemma 4.13. *The function $t \to h(t)$ extends in a C^{p-1}-differentiable function on $[0,1]$ with $h^{(k)}(0) = 0$ for $k = 0, ..., p-1$. Moreover, there exists a constant $C_A > 0$ such that, for all $k \leq p-1$,*

$$||h^{(k)}||_\infty \leq C_A. \tag{4.62}$$

Proof. We only sketch the proof since the arguments are straightforward. Recalling that $B = Id + C$, we write :

$$\begin{aligned} h(t) &= t^{-\frac{\delta+1}{2}}[q_{\tilde{f}} - q_f](-\log t) + t^{-\frac{\delta+1}{2}} C[q_{\tilde{f}} - q_f](-\log t) \\ &:= h_1(t) + h_2(t). \end{aligned}$$

First, let us estimate the derivatives of $h_1(t)$. Setting $Q(x) := [q_{\tilde{f}} - q_f](x)$, we deduce immediately from (1.21) that:

$$|Q^{(k)}(x)| \leq C_A \ e^{-px}, \forall x \geq 0, \ \forall k = 0, ..., m-2. \tag{4.63}$$

Then, we get easily:

$$|h_1^{(k)}(t)| \leq C_A \ t^{p-k-\frac{\delta+1}{2}} \ , \ \forall k = 0, ..., m-2, \ \ \forall t \in]0,1]. \tag{4.64}$$

In the same way, using again Lemma 4.4, a tedious calculation shows that for all $\alpha < p$, there exists a constant $C_{A,\alpha} > 0$ such that

$$|h_2^{(k)}(t)| \leq C_{A,\alpha} \ t^{\alpha-k-\frac{\delta+1}{2}} \ , \ \forall k = 0, ..., m-2. \tag{4.65}$$

Then, it follows from (4.64) and (4.65) that the estimate (4.62) is satisfied. Moreover, for all $k = 0, ..., p-1$, we see that $h^{(k)}(t) \to 0$ as $t \to 0$. This concludes the proof. □

Now, let us estimate the Müntz coefficients C_{mj} associated with $\lambda_k = 2k + d - 3 + \frac{\delta+1}{2}$. To simplify the notation we set $b = d - 3 + \frac{\delta+1}{2}$. Using (4.51), we get:

$$C_{mj} = \sqrt{4m + 2b + 1} \ \frac{\prod_{r=0}^{m-1}(2j + 2r + 4b + 1)}{\prod_{r=0, r\neq j}^{m}(2j - 2r)}. \tag{4.66}$$

Setting $M = \max(2, 4b+1)$, we get easily:

$$|C_{mj}| \leq \sqrt{4m + \frac{M+1}{2}} \ \left(\frac{M}{2}\right)^m \ \left| \frac{\prod_{r=0}^{m-1}(j + r + 1)}{\prod_{r=0, r\neq j}^{m}(j - r)} \right|, \tag{4.67}$$

or equivalently

$$|C_{mj}| \leq \sqrt{\frac{4m + \frac{M+1}{2}}{2m+1}} \ \left(\frac{M}{2}\right)^m \ |C_{mj}^0|. \tag{4.68}$$

Thus, there exists a universal constant $B > 0$ such that

$$|C_{mj}| \leq B \ \left(\frac{M}{2}\right)^m \ |C_{mj}^0|. \tag{4.69}$$

Thanks to (4.69), we can estimate the L^2-norm of the ortogonal projection $\pi_n(h)$ on the subspace $\mathcal{M}(\Lambda)$ where $\Lambda = \Lambda_n : 0 < \lambda_0 < \lambda_1 < ... < \lambda_n$, where $\lambda_k = 2k + d - 3 + \frac{\delta+1}{2}$:

$$\begin{aligned}
||\pi_n(h)||_2^2 &\leq \epsilon^2 \sum_{k=0}^{n} \left(\sum_{p=0}^{k} |C_{kp}| \right)^2 \\
&\leq \epsilon^2 \sum_{k=0}^{n} \left(\sum_{p=0}^{k} \left(\frac{M}{2}\right)^k |C^0_{kp}| \right)^2 \\
&\leq B^2 \epsilon^2 \sum_{k=0}^{n} \left(\frac{M}{2}\right)^k \left(\sum_{p=0}^{k} |C^0_{kp}| \right)^2 .
\end{aligned}$$

Using (4.53), we see that

$$\sum_{p=0}^{k} |C^0_{kp}| \leq \sqrt{2k+1} \sum_{p=0}^{k} 3^{k+p} \leq \frac{3}{2} \sqrt{2k+1}\ 3^{2k}. \tag{4.70}$$

We deduce that

$$\begin{aligned}
\sum_{k=0}^{n} \left(\frac{M}{2}\right)^k \left(\sum_{p=0}^{k} |C^0_{kp}| \right)^2 &\leq \frac{9}{4} \sum_{k=0}^{n} \left(\frac{M}{2}\right)^{2k} (2k+1) 3^{4k} \\
&\leq \frac{9}{4} (2n+1) \sum_{k=0}^{n} \left(\frac{9M}{2}\right)^{2k} \\
&\leq \frac{9}{4} (2n+1) \frac{\left(\frac{9M}{2}\right)^{2n+2}}{\left(\frac{9M}{2}\right)^2 - 1} := g(n)^2,
\end{aligned}$$

where $g : [0, +\infty[$ is the strictly increasing function defined for $t \in [0, +\infty[$ as

$$g(t) = \frac{3}{2} \frac{1}{\sqrt{\left(\frac{9M}{2}\right)^2 - 1}} \sqrt{2t+1} \left(\frac{9M}{2}\right)^{t+1}. \tag{4.71}$$

At this stage, we have obtained

$$||\Pi_n h||_2^2 \leq B^2 \epsilon^2\ g(n)^2. \tag{4.72}$$

Now, let us choose a suitable integer n to control properly the norm of the projection $||\pi_n h||_2^2$. We set $n(\epsilon) := E\ (g^{-1}(\dfrac{1}{\sqrt{\epsilon}}))$. Clearly, since g is an increasing function, one has $g(n(\epsilon)) \leq \frac{1}{\sqrt{\epsilon}}$, and thanks to (4.72) we get immediately:

$$||\pi_{n(\epsilon)} h||_2^2 \leq B^2 \epsilon. \tag{4.73}$$

It remains to estimate $||h - \pi_{n(\epsilon)} h||_2 = E(h, \Lambda_{n(\epsilon)})_2$. First, let us introduce for ϵ small enough,

$$\tilde{\Lambda}_{n(\epsilon)} : \lambda_{k_0} < \lambda_{k_0+1} < ... < \lambda_{n(\epsilon)}, \tag{4.74}$$

where $\lambda_{k_0} > p-2$. Obviously, $E(h,\Lambda_{n(\epsilon)})_2 \leq E(h,\tilde{\Lambda}_{n(\epsilon)})_2 \leq E(h,\tilde{\Lambda}_{n(\epsilon)}))_\infty$. So using Corollary 4.11 with $r = p-1$, we obtain:

$$E(h,\tilde{\Lambda}_{n(\epsilon)})_\infty \leq 40^{p-1} \prod_{k=0}^{p-2} \epsilon_\infty^{(k)} \ \ ||h^{(p-1)}||_\infty, \tag{4.75}$$

where $\epsilon_\infty^{(k)} = \epsilon_\infty(\tilde{\Lambda}_{n(\epsilon)}^{(k)})$. Since $\lambda_{j+1}^{(k)} - \lambda_j^{(k)} = 2$, we can use Theorem 4.12, and thanks to (4.62), we get easily:

$$E(h,\tilde{\Lambda}_{n(\epsilon)})_\infty \leq C_A \ \left(\frac{1}{n(\epsilon)}\right)^{p-1}. \tag{4.76}$$

Now, a straightforward calculation shows that

$$n(\epsilon) \sim C\log(\frac{1}{\epsilon}),\ \ \epsilon \to 0, \tag{4.77}$$

for a suitable constant $C > 0$. Thus, we have proved

$$E(h,\tilde{\Lambda}_{n(\epsilon)})_\infty \leq C_A \ \left(\frac{1}{\log(\frac{1}{\epsilon})}\right)^{p-1}. \tag{4.78}$$

As a conclusion, thanks to (4.61), (4.73) and (4.78), we have obtained for ϵ small enough:

$$||B[q_{\tilde{f}} - q_f]||_{\mathcal{H}_\delta} \leq C_A \left(\frac{1}{\log(\frac{1}{\epsilon})}\right)^{p-1}. \tag{4.79}$$

By Proposition 4.6, $B : \mathcal{H}_\delta \to \mathcal{H}_\delta$ is an isomorphism and $||B^{-1}|| \leq C_A$, then we get a stability estimate between the two potentials $q_{\tilde{f}}$ and q_f for the topology of $\mathcal{H}_\delta$:

$$||q_{\tilde{f}} - q_f||_{\mathcal{H}_\delta} \leq C_A \left(\frac{1}{\log(\frac{1}{\epsilon})}\right)^{p-1}. \tag{4.80}$$

4.4.2 Stability estimates for the warping functions.

First, let us prove a stability estimate for the warping functions $\tilde{f}$ and f in the variable $x \in (0,+\infty)$. In order to simplify the notation, we set $F = f^{d-2},\ \tilde{F} = \tilde{f}^{d-2}$. We we write:

$$\begin{aligned}(\tilde{F}'F - F'\tilde{F})'(y) &= F\tilde{F}\ (q_{\tilde{f}} - q_f)(y) \\ &\leq C_A\ e^{-(d-2)y}\ |(q_{\tilde{f}} - q_f)(y)|,\end{aligned}$$

since for instance, $f(y) = c(e^{-y})e^{-\frac{y}{2}}$ and $c \in \mathcal{C}(A)$. Integrating this last inequality on the interval (x,∞), and setting $G = \tilde{F}'F - F'\tilde{F}$, we get:

$$\begin{aligned}|G(x)| &\leq C_A\ e^{-(d-2)x} \int_x^{+\infty} |(q_{\tilde{f}} - q_f)(y)|\ dy \\ &\leq C_A\ e^{-(d-2)x} \int_x^{+\infty} e^{-\frac{\delta}{2}y}\ e^{\frac{\delta}{2}y}|(q_{\tilde{f}} - q_f)(y)|\ dy \\ &\leq C_A\ e^{-(d-2+\frac{\delta}{2})x}\ ||q_{\tilde{f}} - q_f||_{\mathcal{H}_\delta} \\ &\leq C_A\ e^{-(d-2+\frac{\delta}{2})x}\ \left(\frac{1}{\log(\frac{1}{\epsilon})}\right)^{p-1},\end{aligned}$$

where we have used (4.80) and the Cauchy-Schwarz inequality. Writing $\left(\frac{\tilde{F}}{F}\right)' = \frac{G}{F^2}$, and since $c \in \mathcal{C}(A)$, we deduce from (4.81) that

$$\begin{aligned} \left(\frac{\tilde{F}}{F}\right)'(y) &\leq C_A \ e^{(d-2)y} \ e^{-(d-2+\frac{\delta}{2})y} \left(\frac{1}{\log(\frac{1}{\epsilon})}\right)^{p-1} \\ &\leq C_A \ e^{-\frac{\delta}{2}y} \left(\frac{1}{\log(\frac{1}{\epsilon})}\right)^{p-1}. \end{aligned}$$

We integrate again this equality on (x, ∞) and we get:

$$|1 - \frac{\tilde{F}}{F}(x)| \leq C_A \ e^{-\frac{\delta}{2}x} \ \left(\frac{1}{\log(\frac{1}{\epsilon})}\right)^{p-1} \tag{4.81}$$

Then, we deduce easily that, for all $x \geq 0$,

$$|\tilde{f}(x) - f(x)| \leq C_A \ e^{-\frac{\delta+1}{2}x} \ \left(\frac{1}{\log(\frac{1}{\epsilon})}\right)^{p-1}. \tag{4.82}$$

As a consequence, in the variable $r = e^{-x} \in (0, 1]$, we get:

$$|\tilde{c}(r) - c(r)| \leq C_A \ r^{\frac{\delta}{2}} \ \left(\frac{1}{\log(\frac{1}{\epsilon})}\right)^{p-1}, \tag{4.83}$$

and the proof of Theorem 1.5 is now complete.

4.5 The singular case.

We only sketch the proof since it is very similar to the previous one in the regular case. The real difference lies in the fact that we have no explicit formula for the angular eigenvalues κ_k, and thus we have to use the Weyl asymptotics (1.13). First, let us consider the sub-sequence $\nu_k = \kappa_{k^{d-1}}$. Thanks to Lemma 4.7, we get:

$$\left| \int_0^{+\infty} e^{-2\nu_k x} B[q_{\tilde{f}} - q_f](x) \ dx \right| \leq C_A \ \epsilon \ , \ \forall k \geq 0. \tag{4.84}$$

Thus, making the change of variables $t = e^{-x}$, we obtain immediately:

$$\left| \int_0^{1} t^{2\nu_k - 1} B[q_{\tilde{f}} - q_f](-\log t) \ dt \right| \leq C_A \ \epsilon \ , \ \forall k \geq 0. \tag{4.85}$$

Now, let us choose $\alpha \in]\frac{1}{2}, \frac{3}{2}[$ and let N be an integer large enough which we shall specify below. We deduce easily from (4.85) that:

$$\left| \int_0^{1} t^{\lambda_k} h(t) \ dt \right| \leq C_A \ \epsilon \ , \ \forall k \geq 0, \tag{4.86}$$

where we have set $h(t) = t^{-\alpha} B[q_{\tilde{f}} - q_f](-\log t)$ and $\lambda_k = 2\nu_{Nk} + \alpha - 1$. Note that $||h||_{L^2(0,1)} = ||B[q_{\tilde{f}} - q_f]||_{\mathcal{H}_\delta}$ with $\delta = (2\alpha - 1) \in (0, 1[$.

First, let us verify that we are in the separate case in the framework of the Müntz-Jackson approximation, i.e $\lambda_{k+1} - \lambda_k \geq 2$. It follows from the Weyl's law (1.13) that there exists $C \geq 1$ such that:

$$|\nu_k - c_{d-1}k| \ \leq \ C \ , \ \forall k \geq 0. \tag{4.87}$$

Now, let us fix an integer N large enough such that $B := c_{d-1}N > 3C$. One deduces that

$$\begin{aligned} \lambda_{k+1} - \lambda_k &= 2[\nu_{(k+1)N} - \nu_{kN}] \\ &\geq 2[(B(k+1) - C) - (Bk + C)] \\ &\geq 2B - 4C \geq 2C \geq 2. \end{aligned}$$

Second, let us estimate the index of approximation $\epsilon_\infty(\Lambda)$. We recall that

$$\epsilon_\infty(\Lambda) = \prod_{k=1}^{n} \frac{\lambda_k - 1}{\lambda_k + 1} = \prod_{k=1}^{n} \left(1 - \frac{2}{\lambda_k + 1}\right). \tag{4.88}$$

Since $\lambda_k > 1$, we easily get:

$$\begin{aligned} \log \epsilon_\infty(\Lambda) &= \sum_{k=1}^{n} \log \ (1 - \frac{2}{\lambda_k + 1}) \\ &\leq -2 \sum_{k=1}^{n} \frac{1}{\lambda_k + 1} \\ &\leq -2 \sum_{k=1}^{n} \frac{1}{2(Bk + C) + \alpha}. \end{aligned}$$

Then, we deduce the following estimate :

$$\epsilon_\infty(\Lambda) = O\left(\frac{1}{n^{\frac{1}{B}}}\right). \tag{4.89}$$

Finally, we have to estimate the Müntz coefficients C_{mj} associated with our sequence λ_k as in the previous section in the regular case. We recall that

$$C_{mj} = \sqrt{2\lambda_m + 1} \ \frac{\prod_{r=0}^{m-1}(\lambda_j + \lambda_r + 1)}{\prod_{r=0, r\neq j}^{m}(\lambda_j - \lambda_r)}. \tag{4.90}$$

Clearly, for $j > r$ (for instance), one has $\lambda_j - \lambda_r = 2(\nu_{kN} - \nu_{jN}) \geq 2[B(j-r) - 2C]$. Thus, choosing $b \in (0, B - 2C]$, we get immediately $\lambda_j - \lambda_r \geq b(j - r)$. We deduce that:

$$|\prod_{r=0, r\neq j}^{m} (\lambda_j - \lambda_r)| \ \geq \ (2b)^m \prod_{r=0, r\neq j}^{m} |j - r| \tag{4.91}$$

In the same way, one has:

$$\begin{aligned} |\lambda_j + \lambda_r| &= |\nu_{kj} + \nu_{kr} + 2\alpha - 1| \\ &\geq |B(j + r) + 2C + 2\alpha - 1| \\ &\geq M \ |j + r + 1|, \end{aligned}$$

where $M = max\{B, 2C+2\alpha-1\}$. It follows there exists $D>0$ such that

$$|C_{mj}| \leq D \left(\frac{M}{2b}\right)^m |C_{mj}^0|. \tag{4.92}$$

Now, following exactly the same approach as in the regular case, and taking $\theta = \frac{1}{B} \in (0,1[$, we get Theorem 1.6. The details are left to the reader.

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