Inverse scattering for Stark Hamiltonians with short-range potentials.

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Abstract

We consider the Stark Hamiltonian $H = \frac{1}{2}\ p^2 - x_1 + V(x)$ which describes the scattering of a quantum mechanical particle in $I\!R^n$ by a short-range potential in the presence of a constant electric field. We show that the electric potential V is uniquely determined by the high energy limit of the scattering operator, if the dimension $n \geq 3$. We prove our results using the Enss-Weder's time-dependent method.

1 Introduction.

We study a short-range quantum mechanical scattering in the presence of a constant electric field. For the sake of simplicity, we assume that the electric field acts in the e_1-direction, where $e_1 = (1, 0, ..., 0) \in I\!R^n,\ n \geq 2$.

The corresponding Stark Hamiltonian defined on $L^2(I\!R^n)$ is given by :

$$H_0 \ = \ \frac{1}{2}\ p^2 - x_1 \ , \tag{1.1}$$

where $p = -i\nabla$. It is well-known that H_0 is essentially self-adjoint on $C_0^\infty(I\!R^n)$, (see [11] for example). We denote also by H_0 the self-adjoint realization with domain $D(H_0)$.

Let us recall the Avron-Herbst formula [2] which describes the free time evolution :

$$(1.2) \qquad e^{-itH_0} = e^{-i\frac{t^3}{6}} \ e^{itx_1} \ e^{-i\frac{t^2}{2}p_1} \ e^{-i\frac{t}{2}p^2} \ ,$$

where p_1 is the first component of $p = (p_1, p')$. Equation (1.2) shows that, up to a phase factor, the evolution e^{-itH_0} consists of a translation by $\frac{t^2}{2}$ units to the right in the e_1-direction, followed by the usual free time evolution without electric field.

Now, let H be a second Hamiltonian considered as a perturbation of H_0 : $H = H_0 + V(x)$. We assume that V is a short-range potential, i.e $V \in C^\infty(I\!R^n)$ and it satisfies $\forall \alpha \in I\!N^n$:

$$(H_1) \qquad | \ \partial_x^\alpha V(x) \ | \leq C_\alpha < x >^{-\rho-|\alpha|} \ , \ \rho > \frac{1}{2} \ ,$$

where $< x >= (1+x^2)^{\frac{1}{2}}$. Actually, let us remark that in our paper we only need (H_1) for α with finite order, (for example $| \ \alpha \ | \leq n$). It is well-known that under the asumption (H_1), H is essentially self-adjoint with domain $D(H) = D(H_0)$. Moreover, H has no eigenvalues [2] and $\sigma_{sc} \ (H) = \emptyset$, where $\sigma_{sc} \ (H)$ is the singular continuous spectrum of H, [10].

Under the asumption (H_1) the wave operators :

$$(1.3) \qquad W^\pm \ = \ s - \lim_{t \to \pm\infty} \ e^{itH} \ e^{-itH_0}$$

exist and are complete, (i.e $Ran \ W^\pm = \mathcal{H}^{(ac)}(H)$, the later being the subspace of absolute continuity of H), [2]. Actually, we can prove existence and completeness of $W^\pm$ with weaker asumptions on the potential [2], but we need (H_1) to solve the inverse scattering problem.

We denote $S = S(V) = W^{+*}W^-$ the scattering operator. The inverse scattering problem consists of reconstructing the perturbation V from the scattering operator.

In this paper, we prove that in dimensions $n \geq 3$, the S-operator determines uniquely the potential V. More precisely, it suffices to know the high energy limit of S, (cf Proposition 2). We need $n \geq 3$ in order to use the inversion of the Radon transform (or the Fourier transform) on a hyperplane, (see section 2.3 for details).

Our main result is :

Theorem 1
Let $V_1, \ V_2$ be potentials satisfying (H_1) and assume that $n \geq 3$. Then :

$$S(V_1) = S(V_2) \iff V_1 = V_2 \ .$$

A similar problem has been studied by Weder [12] by a time-dependent method; he obtained the same result for $n \geq 2$ (with weaker conditions on the derivatives of V), but he needed a stronger decay condition on V : he assumed $\rho > \frac{3}{4}$. In the last section, we prove that our method allows to recover this result.

2 Proof of Theorem 1.

In this section, we study the high energy limit of the scattering operator using the Enss-Weder's time-dependent method : see [5] where they study the case of two-body Schrödinger Hamiltonians $H = \frac{1}{2}\, p^2 + V$ on $L^2(I\!R^n)$. This method can be used to study Hamiltonians with electric and magnetic potentials on $L^2(I\!R^n)$ [1], the Dirac equation [7], the N-body case [5] and the Aharonov-Bohm effect [13].

In [8], [9], a stationary approach is proposed to solve scattering inverse problems for Schrödinger operators with magnetic fields or with the Aharonov-Bohm effect. Unfortunately, for the Stark effect, this method is not easily applicable.

The method proposed below is very close to [5], [12].The main steps in the proof are :

a) we define an auxiliary wave operator $\Omega^\pm$ (a Dollard's modified wave operator) which coincides with $W^\pm$ up to an energy phase.

b) we study the high energy asymptotics of S (by means of $\Omega^\pm$) in a direction orthogonal to e_1.

c) using the high energy asymptotics of S and the fact that $dim\ \Pi_{e_1} \geq 2$, where Π_{e_1} is the orthogonal hyperplane to e_1, we solve the inverse scattering problem.

2.1 Construction of an auxiliary wave operator.

In this section, we construct a modified wave operator $\Omega^\pm$ which is close to the canonical one $W^\pm$. The advantage of using $\Omega^\pm$ unstead of $W^\pm$ is that $\Omega^\pm$ admits sharper estimations when the energy goes to infinity, (see Lemma 3 below, and [12] Lemma 2.2, Corollary 2.3).

First, let us define a free-modified dynamic $U_D(t)$ by :

$$U_D(t) = e^{-itH_0}\ e^{-i \int_0^t\ V(sp' + \frac{1}{2}s^2 e_1)\ ds} \ . \tag{2.1}$$

This dynamic is close to the dynamic introduced in [14] for the long-range case. Note that we take p' unstead of p in the integral since H_0 commutes with p'.

Now, we can define the modified wave operators ("Dollard's modified wave operators") :

$$\Omega^\pm = s - \lim_{t \to \pm\infty}\ e^{itH}\ U_D(t)\ . \tag{2.2}$$

Since $\rho > \frac{1}{2}$, it is clear that this limit exists and we obtain by (1.3) :

$$\Omega^\pm = W^\pm\ e^{-ig^\pm(p')}\ , \tag{2.3}$$

where

$$g^\pm(p')\ =\ \int_0^{\pm\infty}\ V(sp' + \frac{1}{2}s^2 e_1)\ ds\ . \tag{2.4}$$

We set $T = \Omega^{+*}\Omega^{-}$, and by (2.3) we deduce :

$$S = e^{-ig^{+}(p')}\ T\ e^{ig^{-}(p')} \qquad (2.5)$$

2.2 High energy asymptotics of the scattering operator.

In order to formulate the main result of this section, we need additional notation.

- Φ, Ψ are the Fourier transforms of functions in $C_0^\infty(I\!R^n)$.
- $\omega \in S^{n-1} \cap \Pi_{e_1}$ is fixed.
- $\Phi_{\lambda,\omega} = e^{i\sqrt{\lambda}x.\omega}\Phi,\ \Psi_{\lambda,\omega} = e^{i\sqrt{\lambda}x.\omega}\Psi$.

We have the following high energy asymptotics where $<\ ,\ >$ is the usual scalar product in $L^2(I\!R^n)$:

Proposition 2

$$< [S,p]\ \Phi_{\lambda,\omega}\ ,\ \Psi_{\lambda,\omega} > = \lambda^{-\frac{1}{2}}\ < \left(\int_{-\infty}^{+\infty} \nabla V(x+t\omega)\ dt\right)\ \Phi\ ,\ \Psi > + o\ (\lambda^{-\frac{1}{2}})\ .$$

In [12], Weder studied the same asymptotics; he also obtained Proposition 2 but with a stronger decay condition on the potential V : $\rho > \frac{3}{4}$.

The main tool to prove Proposition 2 is the following lemma :

Lemma 3
For $\lambda \gg 1$, we have :

$$(i) \qquad \left|\left| \left(V(x) - V(tp' + \frac{1}{2}t^2 e_1)\right)\ U_D(t)\ e^{ig^{\pm}(p')}\Phi_{\lambda,\omega}\ \right|\right| \leq\ C\ (1+ \mid t\sqrt{\lambda} \mid)^{-\frac{1}{2}-\rho}\ .$$

$$(ii) \qquad ||\ (e^{-itH}\Omega^{\pm} - U_D(t))e^{ig^{\pm}(p')}\Phi_{\lambda,\omega}\ || =\ O\ (\lambda^{-\frac{1}{2}})\ ,\ \text{uniformly for } t \in I\!R\ .$$

Proof of Lemma 3 (the case +).
(i) First, we need some notation :

Let $\chi \in C^\infty(I\!R^{n-1})$ such that $\chi(x') = 1$ if $\mid x' \mid \geq \frac{1}{2}$, $\chi(x') = 0$ if $\mid x' \mid \leq \frac{1}{4}$. Denote $V_{\lambda,t}(x) = V(x)\ \chi(\frac{x'}{t\sqrt{\lambda}})$ where $x = (x_1, x') \in I\!R^n$, and choose $\theta \in C_0^\infty(I\!R^{n-1})$ such that $\theta(p')\ \Phi = \Phi$, $f \in C_0^\infty(I\!R)$ such that $f(p_1)\ \Phi = \Phi$.

At last , denote $U^{\pm}(t,p') = e^{i\int_t^{\pm\infty}\ V(sp' + \frac{1}{2}s^2 e_1)\ ds}$.

Our strategy is close to Lemma 3.3 [5]. We have :

$$\begin{aligned} A(t,\lambda) &:= \| \ \Big(V(x) - V(tp' + \frac{1}{2}t^2 e_1)\Big) \ U_D(t) \ e^{ig^+(p')}\Phi_{\lambda,\omega} \ \| \\ &= \| \ \Big(V(x) - V(tp' + \frac{1}{2}t^2 e_1)\Big) \ e^{-itH_0} \ U^+(t,p')\Phi_{\lambda,\omega} \ \| \\ &= \| \ \Big(V(x) - V(tp' + \frac{1}{2}t^2 e_1)\Big) \ e^{-itH_0} \ \theta(p' - \sqrt{\lambda}\omega) \ U^+(t,p')\Phi_{\lambda,\omega} \ \| \ , \end{aligned}$$

since $\omega \in \Pi_{e_1}$ and $\theta(p') \ \Phi = \Phi$. Thus,

$$\begin{aligned} A(t,\lambda) &\leq \| \ (V(x) - V_{\lambda,t}(x)) \ e^{-itH_0} \ \theta(p' - \sqrt{\lambda}\omega) \ U^+(t,p')\Phi_{\lambda,\omega} \ \| \\ &+ \| \ \Big(V_{\lambda,t}(x) - V(tp' + \frac{1}{2}t^2 e_1)\Big) \ e^{-itH_0} \ \theta(p' - \sqrt{\lambda}\omega) \ U^+(t,p')\Phi_{\lambda,\omega} \ \| \\ &:= (1) + (2). \end{aligned}$$

Step 1 :

We remark that $Supp \ (V - V_{\lambda,t}) \subset \{ \ x \in I\!R^n \ : | \ x' \ | \leq \ \frac{|t\sqrt{\lambda}|}{2} \ \}$, so we have :

$$(1) \ \leq \ C \ \| \ \ F(| \ x' - t\sqrt{\lambda}\omega \ | \geq \frac{| \ t\sqrt{\lambda} \ |}{2}) \ e^{-itH_0} \ \theta(p' - \sqrt{\lambda}\omega) \ U^+(t,p')\Phi_{\lambda,\omega} \ \| \ ,$$

where $F(x \in O)$ denotes the multiplication operator with the characteristic function on the set O. Thus,

$$\begin{aligned} (1) \ &\leq \ C \ \| \ \ F(| \ x' - t\sqrt{\lambda}\omega \ | \geq \frac{| \ t\sqrt{\lambda} \ |}{2}) \ e^{-itH_0} \ \theta(p' - \sqrt{\lambda}\omega) \ F(| \ x' \ | < \frac{| \ t\sqrt{\lambda} \ |}{8}) \ \| \\ &+ C \ \| \ \ F(| \ x' \ | \geq \frac{| \ t\sqrt{\lambda} \ |}{8}) \ U^+(t,p')\Phi_{\lambda,\omega} \ \| \\ &:= (a) + (b). \end{aligned}$$

First, we estimate (a). Using the Avron-Herbst formula (1.2) we obtain :

$$(a) \ \leq \| \ \ F(| \ x' - t\sqrt{\lambda}\omega \ | \geq \frac{| \ t\sqrt{\lambda} \ |}{2}) \ e^{-i\frac{t}{2}p'^2} \ \theta(p' - \sqrt{\lambda}\omega) \ F(| \ x' \ | < \frac{| \ t\sqrt{\lambda} \ |}{8}) \ \| \ \ .$$

This term describes the free propagation for the free Hamiltonian $\frac{1}{2} \ p'^2$ into the classical forbidden region. So, we have for all N and $\lambda \gg 1$ (see [3]) :

$$(a) \ \leq C_N \ (1 + | \ t\sqrt{\lambda} \ |)^{-N} \ . \tag{2.6}$$

Now, we estimate (b) :

$$(b) \ = \ C \ \| < x' >^{-N} \ \ F(| \ x' \ | \geq \frac{| \ t\sqrt{\lambda} \ |}{8}) < x' >^N \ U^+(t,p')\Phi_{\lambda,\omega} \ \|$$

$$\leq C\ (1+\mid t\sqrt{\lambda}\mid)^{-N}\ \ || < x' >^N U^+(t,p') < x' >^{-N} ||\ .$$

Writing $< x' >^N U^+(t,p') < x' >^{-N} = U^+(t,p') + [< x' >^N, U^+(t,p')] < x' >^{-N}$ and using standard pseudo-differential calculus, it is easy to show that

$$|| < x' >^N U^+(t,p') < x' >^{-N} || \leq\ C_N \text{ uniformly for } t \in I\!R\ .$$

So, we obtain :

$$(b) \leq C_N\ (1+\mid t\sqrt{\lambda}\mid)^{-N}\ . \tag{2.7}$$

Step 2 :

Since $\theta(p' - \sqrt{\lambda}\omega)$ commutes with H_0, we have for $\lambda \gg 1$,

$$(2)\ = ||\ \left(V_{\lambda,t}(x) - V_{\lambda,t}(tp' + \frac{1}{2}t^2 e_1)\right)\ e^{-itH_0}\ \theta(p' - \sqrt{\lambda}\omega)\ U^+(t,p')\Phi_{\lambda,\omega}\ \ ||\ \ .$$

The Avron-Herst formula (1.2) implies :

$$(2)\ \ = ||\ \ \left(V_{\lambda,t}(x + tp + \frac{1}{2}t^2 e_1) - V_{\lambda,t}(tp' + \frac{1}{2}t^2 e_1)\right)\ \theta(p' - \sqrt{\lambda}\omega)\ U^+(t,p')\Phi_{\lambda,\omega}\ \ ||\ ,$$

where by definition $g(x+tp)$ is equal to $e^{i\frac{t}{2}p^2}\ g(x)\ e^{-i\frac{t}{2}p^2}$ for any borelian function g .

Thus,

$$\begin{aligned}(2)\ \leq ||\ &\left(V_{\lambda,t}(x + tp + \frac{1}{2}t^2 e_1) - V_{\lambda,t}(tp + \frac{1}{2}t^2 e_1)\right)\ \theta(p' - \sqrt{\lambda}\omega)\ U^+(t,p')\Phi_{\lambda,\omega}\ \ || \\ +\ ||\ &\left(V_{\lambda,t}(tp + \frac{1}{2}t^2 e_1) - V_{\lambda,t}(tp' + \frac{1}{2}t^2 e_1)\right)\ \theta(p' - \sqrt{\lambda}\omega)\ U^+(t,p')\Phi_{\lambda,\omega}\ \ ||\ \ . \\ := (a) + (b).\end{aligned}$$

First, we estimate (a). Using the following formula given in [4] :

$$\begin{aligned}V_{\lambda,t}(x + tp + \frac{1}{2}t^2 e_1) - V_{\lambda,t}(tp + \frac{1}{2}t^2 e_1)\ \ =\ \ &\int_0^1\ (\nabla V_{\lambda,t})(sx + tp + \frac{1}{2}t^2 e_1).x \\ &-\frac{it}{2}\ (\Delta V_{\lambda,t})(sx + tp + \frac{1}{2}t^2 e_1)\ ds\ ,\end{aligned} \tag{2.8}$$

as well as the estimate $\mid \partial_x^\alpha\ V_{\lambda,t}(x) \mid \leq C_\alpha\ (1+\mid t\sqrt{\lambda}\mid)^{-\rho-|\alpha|}$, we obtain :

$$(a)\ \ \leq\ \ C\ (1 + t\sqrt{\lambda})^{-\rho-1}\ \ ||< x > U^+(t,p') < x >^{-1}||$$

Thus, since $||< x > U^+(t,p') < x >^{-1}|| = O(1)$ uniformly for $t \in I\!R$ by using the standard pseudo-differential calculus (see Step 1), we have :

$$(a)\ \ \leq\ \ C\ (1+\mid t\sqrt{\lambda}\mid)^{-\rho-1}. \tag{2.9}$$

Now, we estimate (b). We use again that ω is orthononal to e_1. First, we see that :

$$V_{\lambda,t}(tp+\frac{1}{2}t^2e_1)-V_{\lambda,t}(tp'+\frac{1}{2}t^2e_1) \ = \ \left(\int_0^1 \ t\ (\partial_1 V_{\lambda,t})\ ((stp_1+\frac{1}{2}t^2)e_1+tp')\ ds\right)\ p_1\ .$$

So, since $\omega\in\Pi_{e_1}$,

$$(b)\ \ =||\left(\int_0^1 \ t\ (\partial_1 V_{\lambda,t})\ ((stp_1+\frac{1}{2}t^2)e_1+tp')\ ds\right)\ p_1 f(p_1)\ \theta(p'-\sqrt{\lambda}\omega)\ U^+(t,p')\Phi_{\lambda,\omega}\ \ ||\ \ .$$

On the other hand $(\partial_1 V_{\lambda,t})\ (x)=\partial_1 V(x)\ \chi(\frac{x'}{t\sqrt{\lambda}})$, hence $|\ (\partial_1 V_{\lambda,t})\ (x)\ |\leq C\ \ <x>^{-1-\rho}$. Using that, in Fourier representation, $|\ (stp_1+\frac{1}{2}t^2)e_1+tp'\ |\geq C(t^2+\mid t\sqrt{\lambda}\mid)$, we obtain :

$$(b)\leq C\ \ |\ t\ |\ \ (t^2+\mid t\sqrt{\lambda}\mid+1)^{-1-\rho}\leq C\ \ |\ t\ |\ \ (t^2+\mid t\sqrt{\lambda}\mid+1)^{-\frac{1}{2}}\ (t^2+\mid t\sqrt{\lambda}\mid+1)^{-\frac{1}{2}-\rho}$$

thus

$$(b)\ \ \leq C\ (\mid t\sqrt{\lambda}\mid+1)^{-\frac{1}{2}-\rho}\ . \tag{2.10}$$

Then, (i) follows from $(2.6)-(2.10)$.

(ii) Arguing as in Corollary 3.4, [5], we have :

$$\begin{aligned} e^{-itH}\ \Omega^+ \ &= \ s-\lim_{\tau\to+\infty}\ e^{-itH}\ e^{i\tau H}\ U_D(\tau) \\ &= \ s-\lim_{T\to+\infty}\ \ e^{iTH}\ U_D(t+T)\ . \end{aligned}$$

So,

$$e^{-itH}\ \Omega^+-U_D(t)\ \ =\ i\ \int_0^{+\infty}\ \ e^{iTH}\ \left(V(x)-V((T+t)p'+\frac{1}{2}((T+t)^2e_1)\right)\ U_D(T+t)\ dt\ .$$

Thus,

$$||\ (e^{-itH}\ \Omega^+-U_D(t))\ e^{ig^+(p')}\ \Phi_{\lambda,\omega}\ ||\leq\ \int_{-\infty}^{+\infty}\ \ ||\left(V(x)-V(sp'+\frac{1}{2}s^2e_1)\right)\ U_D(s)\ e^{ig^+(p')}\ \Phi_{\lambda,\omega}\ ||\ \ ds\ .$$

Using (i), we obtain :

$$||\ (e^{-itH}\ \Omega^+-U_D(t))\ e^{ig^+(p')}\ \Phi_{\lambda,\omega}\ ||=\ O(\lambda^{-\frac{1}{2}})\ \text{uniformly for}\ t\in I\!R\ .\ \ \square$$

Proof of Proposition 2.
We denote $F(\lambda,\omega)=<[S,p]\ \Phi_{\lambda,\omega}\ ,\ \Psi_{\lambda,\omega}>$. Using (2.5), we have :

$$F(\lambda,\omega)\ \ =<[e^{-ig^+(p')}\ T\ e^{ig^-(p')},p]\ \Phi_{\lambda,\omega}\ ,\ \Psi_{\lambda,\omega}>$$

$$= < [T,p]\ e^{ig^-(p')}\Phi_{\lambda,\omega}\ ,\ e^{ig^+(p')}\Psi_{\lambda,\omega} >$$
$$= < [T-1, p-\sqrt{\lambda}\omega]\ e^{ig^-(p')}\Phi_{\lambda,\omega}\ ,\ e^{ig^+(p')}\Psi_{\lambda,\omega} >$$
$$= < (T-1)\ e^{ig^-(p')}(p\Phi)_{\lambda,\omega}\ ,\ e^{ig^+(p')}\Psi_{\lambda,\omega} >$$
$$-\ < (T-1)\ e^{ig^-(p')}\Phi_{\lambda,\omega}\ ,\ e^{ig^+(p')}(p\Psi)_{\lambda,\omega} >$$
$$:= F_1(\lambda,\omega) - F_2(\lambda,\omega).$$

We study $F_1(\lambda,\omega)$ at first. Writing $T-1 = (\Omega^+ - \Omega^-)^*\Omega^-$ and using

$$\Omega^+ - \Omega^- \ =\ i\ \int_{-\infty}^{+\infty}\ e^{itH}\ \left(V(x) - V(tp' + \frac{1}{2}t^2 e_1)\right)\ U_D(t)\ dt\ , \tag{2.11}$$

we obtain :

$$T-1 = -i\ \int_{-\infty}^{+\infty}\ U_D(t)^*\ \left(V(x) - V(tp' + \frac{1}{2}t^2 e_1)\right)\ e^{-itH}\ \Omega^-\ dt\ . \tag{2.12}$$

Thus,

$$F_1(\lambda,\omega) = -i\ \int_{-\infty}^{+\infty}\ < e^{-itH}\ \Omega^-\ e^{ig^-(p')}(p\Phi)_{\lambda,\omega}\ ,\ \left(V(x) - V(tp' + \frac{1}{2}t^2 e_1)\right)\ U_D(t)\ e^{ig^+(p')}\Psi_{\lambda,\omega} >\ dt$$
$$= -i\ \int_{-\infty}^{+\infty}\ < U_D(t)\ e^{ig^-(p')}(p\Phi)_{\lambda,\omega}\ ,\ \left(V(x) - V(tp' + \frac{1}{2}t^2 e_1)\right)\ U_D(t)\ e^{ig^+(p')}\Psi_{\lambda,\omega} >\ dt$$
$$+\ R_1(\lambda,\omega)\ ,$$

where

$$R_1(\lambda,\omega)\ =\ -i\ \int_{-\infty}^{+\infty}\ < (e^{-itH}\ \Omega^- - U_D(t))\ e^{ig^-(p')}(p\Phi)_{\lambda,\omega}\ , \left(V(x) - V(tp' + \frac{1}{2}t^2 e_1)\right)\ U_D(t)\ e^{ig^+(p')}\Psi_{\lambda,\omega} >\ dt\ . \tag{2.13}$$

Now, by Lemma 3, we have easily $R_1(\lambda,\omega) = O\ (\lambda^{-1})$. Thus,

$$F_1(\lambda,\omega)\ = -\frac{i}{\sqrt{\lambda}}\ \int_{-\infty}^{+\infty}\ < U_D(\frac{t}{\sqrt{\lambda}})\ e^{ig^-(p')}\ (p\Phi)_{\lambda,\omega}\ , \left(V(x) - V(\frac{t}{\sqrt{\lambda}}p' + \frac{1}{2\lambda}t^2 e_1)\right)\ U_D(\frac{t}{\sqrt{\lambda}})\ e^{ig^+(p')}\Psi_{\lambda,\omega} >\ dt + O\ (\lambda^{-1})\ . \tag{2.14}$$

Denote by $f_1(t,\lambda,\omega)$ the integrand of the (R.H.S) of (2.14). By Lemma 3 (i),

$$|\ f_1(t,\lambda,\omega)\ | \leq C\ (1+ |\ t\ |)^{-\frac{1}{2}-\rho}\ .$$

So, by Lebesgue's theorem, to obtain the asymptotics of $F_1(\lambda,\omega)$, it suffices to determine $\lim_{\lambda\to+\infty}\ f_1(t,\lambda,\omega)\ ,\ \forall t \in I\!R.$

We have :

$$(2.15) \qquad f_1(t,\lambda,\omega) \ = < e^{-i\frac{t}{\sqrt{\lambda}}H_0} \ U^-(\frac{t}{\sqrt{\lambda}}, p') \ (p\Phi)_{\lambda,\omega} \ ,$$

$$\left(V(x) - V(\frac{t}{\sqrt{\lambda}}p' + \frac{1}{2\lambda}t^2 e_1) \right) \ e^{-i\frac{t}{\sqrt{\lambda}}H_0} \ U^+(\frac{t}{\sqrt{\lambda}}, p') \ \Psi_{\lambda,\omega} > \ .$$

Using the Avron-Herbst formula (1.2), we deduce that :

$$(2.16) \qquad f_1(t,\lambda,\omega) \ = < e^{-i\frac{t}{2\sqrt{\lambda}}p^2} \ U^-(\frac{t}{\sqrt{\lambda}}, p') \ (p\Phi)_{\lambda,\omega} \ ,$$

$$\left(V(x + \frac{1}{2\lambda}t^2 e_1) - V(\frac{t}{\sqrt{\lambda}}p' + \frac{1}{2\lambda}t^2 e_1) \right) \ e^{-i\frac{t}{2\sqrt{\lambda}}p^2} \ U^+(\frac{t}{\sqrt{\lambda}}, p') \ \Psi_{\lambda,\omega} > \ .$$

Then, we obtain :

$$(2.17) \qquad f_1(t,\lambda,\omega) \ = < e^{-i\frac{t}{2\sqrt{\lambda}}(p+\sqrt{\lambda}\omega)^2} \ U^-(\frac{t}{\sqrt{\lambda}}, p' + \sqrt{\lambda}\omega) \ p\Phi \ ,$$

$$\left(V(x + \frac{1}{2\lambda}t^2 e_1) - V(\frac{t}{\sqrt{\lambda}}(p' + \sqrt{\lambda}\omega) + \frac{1}{2\lambda}t^2 e_1) \right) \ e^{-i\frac{t}{2\sqrt{\lambda}}(p+\sqrt{\lambda}\omega)^2} \ U^+(\frac{t}{\sqrt{\lambda}}, p' + \sqrt{\lambda}\omega) \ \Psi > \ .$$

Since

$$(2.18) \qquad e^{-i\frac{t}{2\sqrt{\lambda}}(p+\sqrt{\lambda}\omega)^2} \ = \ e^{-i\frac{t\sqrt{\lambda}}{2}} \ e^{-it\omega.p} \ e^{-i\frac{t}{2\sqrt{\lambda}}p^2} \ ,$$

we have

$$(2.19) \qquad f_1(t,\lambda,\omega) \ = < e^{-i\frac{t}{2\sqrt{\lambda}}p^2} \ U^-(\frac{t}{\sqrt{\lambda}}, p' + \sqrt{\lambda}\omega) \ \ p\Phi \ ,$$

$$\left(V(x + t\omega + \frac{1}{2\lambda}t^2 e_1) - V(t\omega + \frac{t}{\sqrt{\lambda}}p' + \frac{1}{2\lambda}t^2 e_1) \right) \ e^{-i\frac{t}{2\sqrt{\lambda}}p^2} \ U^+(\frac{t}{\sqrt{\lambda}}, p' + \sqrt{\lambda}\omega) \ \Psi > \ .$$

Since $| \ V(s(p' + \sqrt{\lambda}\omega) + \frac{1}{2}s^2 e_1)) \ | \leq \ C \ (s^2+1)^{-\rho} \in L^1(I\!R, \ ds)$, it is easy to show by Lebesgue's theorem that :

$$(2.20) \qquad s - \lim_{\lambda \to +\infty} \ U^\pm(\frac{t}{\sqrt{\lambda}}, p' + \sqrt{\lambda}\omega) \ = 1 \ .$$

Then,

$$(2.21) \qquad \lim_{\lambda \to +\infty} \ f_1(t,\lambda,\omega) \ = < p\Phi \ , \ (V(x+t\omega) - V(t\omega)) \ \Psi > \ .$$

So, we have obtained

$$(2.22) \qquad F_1(\lambda,\omega) \ = \ -\frac{i}{\sqrt{\lambda}} \ < p\Phi, \ \left(\int_{-\infty}^{+\infty} \ (V(x+t\omega) - V(t\omega)) \ dt \right) \ \Psi > + o \ (\frac{1}{\sqrt{\lambda}}) \ ,$$

In the same way, we obtain

$$F_2(\lambda,\omega) \ = \ -\frac{i}{\sqrt{\lambda}} \ < \Phi, \ \left(\int_{-\infty}^{+\infty} (V(x+t\omega) - V(t\omega)) \ dt\right) \ p\Psi > + o\,(\frac{1}{\sqrt{\lambda}}) \ , \tag{2.23}$$

so

$$\begin{aligned} F(\lambda,\omega) \ &= F_1(\lambda,\omega) \ - F_2(\lambda,\omega) \\ &= \frac{1}{\sqrt{\lambda}} \ < \Phi, \ \left(\int_{-\infty}^{+\infty} \nabla V(x+t\omega) \ dt\right)) \ \Psi > + o\,(\frac{1}{\sqrt{\lambda}}) \ . \quad \square \end{aligned} \tag{2.24}$$

2.3 Uniqueness of the potential.

In this section, we use Proposition 2 to prove Theorem 1.

Let V_1 and V_2 be potentials satisfying (H_1) such that $S(V_1) = S(V_2)$. By proposition 2, we have :

$$\int_{-\infty}^{+\infty} \nabla V(x+t\omega) \ dt \ = \ 0 \ , \ \forall x \in I\!R^n \ , \ \forall \omega \in \Pi_{e_1} \cap S^{n-1} \ , \tag{2.25}$$

where $V = V_1 - V_2$. Now, fix $a \in I\!R$ and define for $x' \in \Pi_{e_1}$, $V_a(x') = V(ae_1 + x')$. Using (2.25), we have $\forall \alpha \in I\!N^{n-1}$ with $\mid \alpha \mid \geq 1$:

$$\int_{-\infty}^{+\infty} \partial_{x'}^{\alpha} V_a(x'+t\omega) \ dt \ = \ 0 \ , \ \forall x' \in \ \Pi_{e_1} \ , \ \forall \omega \in \Pi_{e_1} \cap S^{n-1} \ . \tag{2.26}$$

Now, we can easily prove that $V_a = 0$.

First, remark that for $n \geq 3$, $dim \ \Pi_{e_1} \cap \Pi_\omega = n-2 \geq 1$. Let $\xi' \in \Pi_{e_1} \cap \Pi_\omega$ and consider for $\mid \alpha \mid \geq n-1$, the Fourier transform in $L^1(\Pi_{e_1})$:

$$\widehat{\partial_{x'}^{\alpha} V_a}(\xi') \ = \ \int_{\Pi_{e_1}} e^{-ix'.\xi'} \ \partial_{x'}^{\alpha'} V_a(x') \ dx' \ . \tag{2.27}$$

Writing $x' = y + t\omega$ where $y \in \Pi_{e_1} \cap \Pi_\omega$, we have by (2.26)

$$\widehat{\partial_{x'}^{\alpha'} V_a}(\xi') \ = \ \int_{\Pi_{e_1} \cap \Pi_\omega} e^{-iy.\xi'} \ \left(\int_{-\infty}^{+\infty} \partial_{x'}^{\alpha'} V_a(y+t\omega) \ dt\right) \ dy \ = \ 0 \ . \tag{2.28}$$

Varying ω, by the uniqueness of the Fourier transform in $L^1(\Pi_{e_1})$, we have $\partial_{x'}^{\alpha} V_a = 0$. So V_a is polynomial and goes to zero at infinity. We deduce that $V_a = 0$, $\forall a \in I\!R$. Then $V = 0$. $\square$

Remark.
In [12], Weder proved uniqueness of the potential by using the inversion for the Radon transform (see [6]).

3 Comments.

With the method proposed above, we can recover Weder's result [12]. Actually, in [12], he showed this result with weaker conditions on the derivatives of the potentials.

Theorem 4 ([12], Theorem 2.4)
Let $V_1,\ V_2$ be potentials satisfying (H_1) with $\rho > \frac{3}{4}$ and assume that $n \geq 2$. Then :

$$S(V_1) = S(V_2) \iff V_1 = V_2 \ .$$

Sketch of proof.
Let $\omega \in S^{n-1}$ fixed such that $\mid \omega.e_1 \mid < 1$. We easily show that if $\mid p - \sqrt{\lambda}\omega \mid \leq C$, (in Fourier representation) , there exists $C_1 > 0$ such that :

$$(3.1) \qquad \forall t \in I\!R \ , \ \forall s \in [0,1] \ , \ \forall \lambda \gg 1 \ , \ \mid (stp_1 + \frac{1}{2}t^2)e_1 + tp' \mid \geq C_1 \ (t^2 + \mid t\sqrt{\lambda} \mid) \ .$$

Now, following the proof of Lemma 3 and using (3.1) in the Step 2 (b), we obtain :

$$(3.2) \qquad || \ \left(V(x) - V(tp' + \frac{1}{2}t^2 e_1)\right) \ U_D(t) \ e^{ig^{\pm}(p')}\Phi_{\lambda,\omega} \ || \leq \ C \ (1 + \mid t\sqrt{\lambda} \mid)^{-1-\rho}$$

$$+C \ \mid t\sqrt{\lambda} \mid \ (1 + \mid t\sqrt{\lambda} \mid + t^2)^{-1-\rho}.$$

Then, using the same arguments as in Lemma 3 (ii), we have for $\delta < min \ (\rho, 1)$:

$$(3.3) \qquad || \ (e^{-itH}\Omega^{\pm} - U_D(t))e^{ig^{\pm}(p')}\Phi_{\lambda,\omega} \ || = \ O \ (\lambda^{\frac{1}{2}-\delta}) \ , \ \text{uniformly for } t \in I\!R \ .$$

With the notation of the proof of Proposition 2, we obtain easily :

$$(3.4) \qquad R_1(\lambda, \omega) = O(\lambda^{1-2\delta}) \ .$$

So, if $\rho > \frac{3}{4}$, Proposition 2 is valid. By a standard continuity argument, we have (2.25) for all $\omega \in S^{n-1}$ and using the same arguments as in the section 2.3, we obtain Theorem 4. □

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