INVERSE SEMICLASSICAL SCATTERING AT FIXED ENERGY

FRANÇOIS NICOLEAU

Dedicated to Didier Robert, for his major contributions to semiclassical analysis, scattering theory, and coherent states.

Abstract. We investigate inverse scattering at a fixed energy for semiclassical Schrödinger operators with smooth potentials. For compactly supported potentials, we show that, at a non-trapping energy, if the corresponding semiclassical scattering matrices differ by $o(1)$ in operator norm as $h \to 0$, then the associated classical scattering maps coincide. The proof relies on Ingremeau's description of the action of the scattering matrix on coherent states. Combined with a classical rigidity result of Stefanov–Uhlmann–Vasy, this yields uniqueness of the potential under a natural virial condition, provided the energy lies above the potential. We then consider radial short-range repulsive potentials. Under a monotonicity assumption on the radial force, the classical scattering relation has a single branch. Combining the semiclassical scattering asymptotics of Robert–Tamura with the classical Firsov–Abel inversion formula, we show that the differential cross section at a single fixed energy determines the potential throughout the classically accessible region. Finally, we obtain an analogous rigidity result for simple compactly supported metric perturbations.

1. Introduction

The purpose of this paper is to study the inverse problem for semiclassical scattering at a fixed energy. More precisely, we investigate to what extent the scattering data in the semiclassical limit determine the underlying potential. Let

$$H(h) = -\frac{h^2}{2}\Delta + V(x), \qquad 0 < h \leq 1, \tag{1.1}$$

be the semiclassical Schrödinger operator in $L^2(\mathbb{R}^n)$, $n \geq 3$. Throughout the paper, V is real-valued and smooth. We shall work with short-range potentials satisfying, for some $\mu > 1$,

$$|\partial_x^\alpha V(x)| \leq C_\alpha \langle x\rangle^{-\mu-|\alpha|}, \qquad \alpha \in \mathbb{N}^n, \qquad \langle x\rangle = (1+|x|^2)^{1/2}. \tag{1.2}$$

For such potentials, trajectories escaping to infinity have free asymptotics, while trapped trajectories may still occur at a given energy.

We denote by

$$H_0(h) = -\frac{h^2}{2}\Delta \tag{1.3}$$

the free Hamiltonian associated with $H(h)$, and by

$$R(z;h) = (H(h)-z)^{-1}, \qquad z \in \mathbb{C} \setminus \sigma(H(h)), \tag{1.4}$$

the resolvent of $H(h)$. Let $\mathcal{F}_h$ denote the semiclassical Fourier transform,

$$(\mathcal{F}_h f)(\xi) = (2\pi h)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi/h} f(x)\, dx.$$

For $\lambda > 0$, the restriction of the free spectral transform to the energy λ is given by

$$F_0(\lambda,h)f(\omega) = (2\lambda)^{(n-2)/4} (\mathcal{F}_h f)(\sqrt{2\lambda}\,\omega), \qquad \omega \in \mathbb{S}^{n-1}.$$

The scattering matrix

$$S(\lambda,h) : L^2(\mathbb{S}^{n-1}) \longrightarrow L^2(\mathbb{S}^{n-1})$$

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is unitary and can be written as

$$S(\lambda,h)=\mathsf{Id}-2\pi\,\mathrm{i}\,T(\lambda,h), \tag{1.5}$$

where

$$T(\lambda,h)=F_0(\lambda,h)\big(V-VR(\lambda+\mathrm{i}0;h)V\big)F_0(\lambda,h)^*. \tag{1.6}$$

Here

$$R(\lambda+\mathrm{i}0;h)=s-\lim_{\varepsilon\downarrow 0}\,(H(h)-\lambda-\mathrm{i}\varepsilon)^{-1}$$

denotes the outgoing boundary value of the resolvent, where the strong limit is taken in

$$\mathcal{B}\big(L^2_s(\mathbb{R}^n),L^2_{-s}(\mathbb{R}^n)\big),\qquad s>\frac{1}{2},$$

with

$$L^2_s(\mathbb{R}^n)=\big\{u\in L^2_{\mathrm{loc}}(\mathbb{R}^n);\ \langle x\rangle^s u\in L^2(\mathbb{R}^n)\big\},\qquad \langle x\rangle=(1+|x|^2)^{1/2}. \tag{1.7}$$

Away from the diagonal, $T(\lambda,h)$ has a smooth kernel, denoted by

$$T(\theta,\omega;\lambda,h),\qquad \theta,\omega\in\mathbb{S}^{n-1},\quad \theta\neq\omega.$$

The corresponding scattering amplitude is defined by Kuroda's formula

$$f(\omega\to\theta;\lambda,h)=c(\lambda,h)\,T(\theta,\omega;\lambda,h), \tag{1.8}$$

where

$$c(\lambda,h)=-(2\pi)^{(n+1)/2}(2\lambda)^{-(n-1)/4}h^{(n-1)/2}\exp\left(-\mathrm{i}\frac{(n-3)\pi}{4}\right). \tag{1.9}$$

For the standard scattering theory recalled above, we refer to Isozaki–Kitada [7, 8] and Reed–Simon [18].

Our main inverse problem is to determine to what extent the condition

$$\|S_1(\lambda,h)-S_2(\lambda,h)\|_{\mathcal{B}(L^2(\mathbb{S}^{n-1}))}=o(1),\qquad h\to 0, \tag{1.10}$$

at a fixed energy $\lambda>0$, determines the potential.

There is a natural limitation to such a fixed-energy inverse problem. In [15], it was proved that, for two short-range potentials V_1 and V_2, at a fixed non-trapping energy $\lambda>0$,

$$(\lambda-V_1)_+=(\lambda-V_2)_+,\qquad (a)_+:=\max\{a,0\},\quad\Longrightarrow\quad S_1(\lambda,h)-S_2(\lambda,h)=\mathcal{O}(h^\infty) \tag{1.11}$$

in the operator norm of $\mathcal{B}(L^2(\mathbb{S}^{n-1}))$ as $h\to 0$. Thus, at a fixed energy, one cannot in general expect the semiclassical scattering data to determine the potential in the classically forbidden region $\{V\geq\lambda\}$. The natural inverse problem is therefore to recover V in the classically allowed region $\{V<\lambda\}$, or, equivalently, to recover the function $(\lambda-V)_+$.

The semiclassical direct problem has a precise classical interpretation. Semiclassical asymptotics for scattering amplitudes were first obtained by Protas [16] and Vainberg [24] for compactly supported potentials. Yajima [25] subsequently treated non-compactly supported short-range potentials, obtaining an asymptotic expansion in an L^2 sense with respect to the energy and angular variables. Robert and Tamura [21] then obtained, for short-range potentials, a precise fixed-energy description at a non-trapping energy, expressing the scattering amplitude, for regular outgoing directions, as a finite sum of contributions associated with the underlying classical scattering trajectories.

We recall the classical objects entering the semiclassical scattering asymptotics, using a notation adapted to the present paper. The Hamiltonian associated with (1.1) is

$$p(x,\xi)=\frac{|\xi|^2}{2}+V(x), \tag{1.12}$$

and the corresponding classical trajectories $t\mapsto(q_{\mathrm{cl}}(t),p_{\mathrm{cl}}(t))$ are the integral curves of the Hamiltonian vector field, that is,

$$\dot q_{\mathrm{cl}}(t)=p_{\mathrm{cl}}(t),\qquad \dot p_{\mathrm{cl}}(t)=-\nabla V(q_{\mathrm{cl}}(t)). \tag{1.13}$$

For the standard semiclassical framework and its relation with the underlying classical Hamiltonian dynamics, we refer, for instance, to Robert [19].

At the energy $\lambda > 0$, the energy surface is

$$\Sigma_\lambda = \left\{ (x,\xi) \in T^*\mathbb{R}^n;\ \frac{|\xi|^2}{2} + V(x) = \lambda \right\}. \tag{1.14}$$

We recall that $\lambda > 0$ is a non-trapping energy if, for every $R > 1$ sufficiently large, there exists $T = T(R) > 0$ such that

$$|q_{\mathrm{cl}}(t; y, \eta)| > R, \qquad |t| > T, \tag{1.15}$$

whenever $|y| < R$ and $(y,\eta) \in \Sigma_\lambda$, where $t \mapsto (q_{\mathrm{cl}}(t;y,\eta), p_{\mathrm{cl}}(t;y,\eta))$ denotes the solution of (1.13) with initial data (y,η) at $t = 0$.

Under the non-trapping assumption and (1.2), every classical trajectory $(q_{\mathrm{cl}}(t), p_{\mathrm{cl}}(t))$ on the energy surface Σ_λ is a scattering trajectory and admits asymptotic data $(r_\pm, v_\pm)$ such that

$$q_{\mathrm{cl}}(t) = v_\pm t + r_\pm + o(1), \qquad p_{\mathrm{cl}}(t) = v_\pm + o(1), \qquad t \to \pm\infty. \tag{1.16}$$

Since the energy is conserved and $V(q_{\mathrm{cl}}(t)) \to 0$ as $t \to \pm\infty$, one has $|v_\pm| = \sqrt{2\lambda}$. The map

$$\mathcal{S}_{\mathrm{cl}} : (r_-, v_-) \longmapsto (r_+, v_+) \tag{1.17}$$

is the classical scattering transformation.

For later use, we describe the classical scattering transformation in the coordinates naturally associated with the semiclassical scattering matrix. Fix an incoming direction $\omega \in \mathbb{S}^{n-1}$ and set

$$\Lambda_\omega = \{ z \in \mathbb{R}^n;\ z \cdot \omega = 0 \}. \tag{1.18}$$

For $z \in \Lambda_\omega$, let

$$\big(q_\infty(t; z, \lambda), p_\infty(t; z, \lambda) \big)$$

be the scattering trajectory satisfying

$$q_\infty(t; z, \lambda) = \sqrt{2\lambda}\,\omega t + z + o(1), \qquad p_\infty(t; z, \lambda) = \sqrt{2\lambda}\,\omega + o(1), \qquad t \to -\infty. \tag{1.19}$$

As $t \to +\infty$, there exist smooth functions $\xi_\infty(z;\lambda) \in \mathbb{S}^{n-1}$ and $r_\infty(z;\lambda) \in \mathbb{R}^n$ such that

$$\begin{aligned} q_\infty(t; z, \lambda) &= \sqrt{2\lambda}\,\xi_\infty(z;\lambda) t + r_\infty(z;\lambda) + o(1), \\ p_\infty(t; z, \lambda) &= \sqrt{2\lambda}\,\xi_\infty(z;\lambda) + o(1). \end{aligned} \qquad t \to +\infty. \tag{1.20}$$

Since $T^*_\omega \mathbb{S}^{n-1} \simeq \omega^\perp = \Lambda_\omega$, the space of incoming oriented lines is naturally identified with $T^*\mathbb{S}^{n-1}$. For $\theta \in \mathbb{S}^{n-1}$, let

$$\Pi_\theta x = x - (x \cdot \theta)\theta$$

denote the orthogonal projection onto $\theta^\perp$. The classical scattering map at energy λ is then given by

$$\kappa_\lambda(\omega, z) = \left(\xi_\infty(z;\lambda), \Pi_{\xi_\infty(z;\lambda)} r_\infty(z;\lambda) \right). \tag{1.21}$$

Figure 1 summarizes the geometry. For a genuinely short-range potential, the interaction is not compactly supported; the circle only represents the region where it is significant. The parameter z labels the incoming line, while $\xi_\infty(z;\lambda)$ is the outgoing direction.

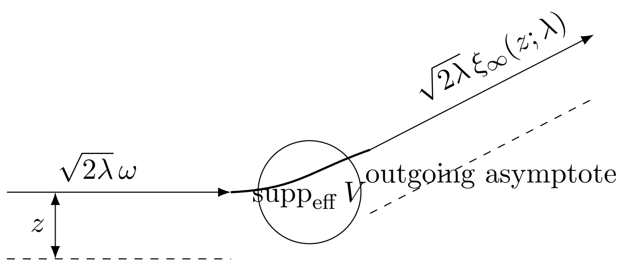


FIGURE 1. A classical scattering trajectory at the fixed energy λ.

We first consider compactly supported potentials. In this case, the semiclassical scattering matrix determines the classical scattering map at a fixed non-trapping energy. Combined with a classical rigidity result, this leads to our first uniqueness result.

Theorem 1.1. *Let $V_1, V_2 \in C_c^\infty(\mathbb{R}^n)$, $n \geq 3$, and let $\lambda > 0$ satisfy*

$$\lambda > \sup_{\mathbb{R}^n} V_j, \qquad j = 1, 2. \tag{1.22}$$

Assume moreover the virial condition

$$2(\lambda - V_j(x)) - x \cdot \nabla V_j(x) > 0, \qquad x \in \mathbb{R}^n, \qquad j = 1, 2. \tag{1.23}$$

If

$$\|S_1(\lambda, h) - S_2(\lambda, h)\|_{\mathcal{B}(L^2(\mathbb{S}^{n-1}))} = o(1), \qquad h \to 0, \tag{1.24}$$

then

$$V_1 = V_2. \tag{1.25}$$

Let us briefly explain the main ingredients of the proof. Alexandrova [1] proved, microlocally away from the diagonal, that the semiclassical scattering matrix is a semiclassical Fourier integral operator associated with the classical scattering relation. Thus, in a precise microlocal sense, the scattering matrix quantizes the underlying classical scattering dynamics. For our inverse problem, a particularly convenient description is due to Ingremeau [6]. For compactly supported potentials at a non-trapping energy, his result describes the action of the scattering matrix on Gaussian states localized at points of $T^*\mathbb{S}^{n-1}$. After a harmless rescaling of the energy, if $\varphi_{\rho,h}$ is a Gaussian state localized near $\rho \in T^*\mathbb{S}^{n-1}$, then, for every $N \in \mathbb{N}$,

$$S(\lambda, h)\phi_{\rho,h} = e^{\,\mathrm{i}\delta/h}\phi^{(N)}_{\kappa_\lambda(\rho),h} + \mathcal{O}(h^N), \tag{1.26}$$

where $\delta \in \mathbb{R}$ and $\phi^{(N)}_{\kappa_\lambda(\rho),h}$ is an outgoing Gaussian state, with an amplitude admitting a finite semiclassical expansion, localized near the phase point $\kappa_\lambda(\rho)$.

The essential point for our purposes is the transport of the center of the Gaussian state from ρ to $\kappa_\lambda(\rho)$. We show that this implies

$$\|S_1(\lambda, h) - S_2(\lambda, h)\|_{\mathcal{B}(L^2(\mathbb{S}^{n-1}))} = o(1) \quad \Longrightarrow \quad \kappa_{1,\lambda} = \kappa_{2,\lambda}. \tag{1.27}$$

Indeed, Gaussian states localized at distinct phase-space points are asymptotically orthogonal. Since the scattering matrices are unitary, applying them to the same incoming Gaussian state yields (1.27).

It remains to pass from the classical scattering map to the potential. Under (1.22), trajectories at energy λ, up to reparametrization, are geodesics of the Jacobi metrics

$$g_j = 2(\lambda - V_j)g_0, \qquad j = 1, 2. \tag{1.28}$$

The virial condition (1.23) implies that λ is a non-trapping energy and that the Euclidean spheres form a strictly convex foliation for the Jacobi metrics (1.28). The former property allows us to apply Ingremeau's result, while the latter allows us to use the rigidity theorem of Stefanov–Uhlmann–Vasy and conclude that $g_1 = g_2$, hence $V_1 = V_2$ We also mention that related fixed-energy scattering rigidity results for simple MP-systems were obtained by Muñoz-Thon [12]. In the purely potential case considered here, simplicity amounts to simplicity of the associated Jacobi metric, and his result yields uniqueness of the potential when the background metric is fixed. By contrast, our result does not require the Jacobi metric to be simple; the simplicity assumption is replaced by the convex foliation condition provided by the virial hypothesis.

We next turn to radial short-range potentials. Here the inverse problem has a different character. In particular, the energy is allowed to lie below the maximum of the potential and, as explained above, one can only expect to recover the potential in the classically accessible region.

Let

$$V_j(x) = v_j(|x|), \qquad j = 1, 2, \tag{1.29}$$

be smooth radial short-range potentials satisfying

$$v_j'(r) < 0, \qquad r > 0, \qquad v_j(0) > \lambda. \tag{1.30}$$

The assumption $v_j(0) > \lambda$, together with the strict monotonicity of v_j, ensures the existence of a unique positive radial turning point $r_{j,\lambda} > 0$ defined by

$$v_j(r_{j,\lambda}) = \lambda. \tag{1.31}$$

The regions $r < r_{j,\lambda}$ and $r > r_{j,\lambda}$ are respectively classically forbidden and classically accessible at the energy λ. We shall further assume the monotonicity condition

$$\big(rv_j'(r)\big)' \geq 0, \qquad r > r_{j,\lambda}, \qquad j = 1, 2. \tag{1.32}$$

In terms of the radial force $F_j(r) = -v_j'(r) > 0$, this means that $rF_j(r)$ is non-increasing in the classically accessible region. Notice also that the virial condition is automatic in this region:

$$2\big(\lambda - v_j(r)\big) - rv_j'(r) > 0, \qquad r > r_{j,\lambda}. \tag{1.33}$$

Since the potentials are radial, the scattering amplitude is invariant under simultaneous rotations of the incoming and outgoing directions. More precisely, for every $R \in SO(n)$,

$$f_j(R\omega \to R\theta; \lambda, h) = f_j(\omega \to \theta; \lambda, h), \qquad j = 1, 2. \tag{1.34}$$

The corresponding semiclassical differential cross section is defined by

$$\frac{d\sigma_{j,h}}{d\Omega}(\omega, \theta; \lambda) = |f_j(\omega \to \theta; \lambda, h)|^2. \tag{1.35}$$

By rotational invariance, both the scattering amplitude and the differential cross section depend only on the scattering angle between ω and θ. We may therefore fix an incoming direction $\omega \in \mathbb{S}^{n-1}$ without loss of information.

Our second result shows that considerably less semiclassical information is sufficient in this radial setting.

Theorem 1.2. *Let $V_j(x) = v_j(|x|)$, $j = 1, 2$, satisfy the short-range condition (1.2) and assumptions (1.30)–(1.32). Fix an incoming direction $\omega \in \mathbb{S}^{n-1}$ and assume that*

$$\frac{d\sigma_{1,h}}{d\Omega}(\omega, \theta; \lambda) - \frac{d\sigma_{2,h}}{d\Omega}(\omega, \theta; \lambda) \longrightarrow 0, \qquad h \longrightarrow 0, \tag{1.36}$$

for every $\theta \in \mathbb{S}^{n-1} \setminus \{\omega, -\omega\}$. Then

$$r_{1,\lambda} = r_{2,\lambda} =: r_\lambda, \tag{1.37}$$

and

$$V_1(x) = V_2(x), \qquad |x| \geq r_\lambda. \tag{1.38}$$

In particular, the semiclassical differential cross section at the single energy λ, for one fixed incoming direction, determines the potential throughout the classically accessible region.

Let us explain the main ingredients of the proof. The starting point is the fixed-energy semiclassical asymptotic formula of Robert and Tamura [21]. For a regular pair of incoming and outgoing directions, the scattering amplitude has the form

$$f(\omega \to \theta; \lambda, h) = \sum_{j=1}^{\ell(\theta,\lambda)} \sigma(w_j; \lambda)^{-1/2} \exp\left(\frac{\mathrm{i}}{h}\mathcal{A}_j(\omega, \theta; \lambda) - \frac{\mathrm{i}\pi}{2}\mu_j\right)\big(1 + \mathcal{O}(h)\big), \tag{1.39}$$

where the sum runs over the classical scattering trajectories connecting the incoming direction ω to the outgoing direction θ. Here $\sigma(w_j; \lambda)$ is the corresponding angular density, $\mathcal{A}_j$ the classical action, and μ_j the Maslov index. The notions of regular direction and classical branch, as well as the precise definitions of the quantities entering (1.39), are recalled in Section 3.

For general potentials, several classical trajectories may connect the same incoming and outgoing directions. The corresponding contributions in (1.39) may then interfere, so that the semiclassical differential cross section does not directly reduce to the classical differential cross section of a single trajectory. In the radial setting considered here, however, the classical dynamics can be described in terms of the deflection function $b \longmapsto \Theta_\lambda(b)$, which associates with each impact parameter $b > 0$ the corresponding scattering angle. Its precise definition is given in Section 3. Under condition (1.32), we shall prove that Θ_λ is strictly decreasing from π to 0.

For incoming and outgoing directions $\omega, \theta \in \mathbb{S}^{n-1}$, we denote their scattering angle by

$$\Theta = \arccos(\omega \cdot \theta) \in [0, \pi]. \tag{1.40}$$

Thus a trajectory with impact parameter b and outgoing direction θ satisfies

$$\Theta = \Theta_\lambda(b).$$

The strict monotonicity of Θ_λ therefore implies that, for every $\Theta \in (0, \pi)$, there is a unique impact parameter b and hence a unique classical trajectory with this scattering angle. Moreover, as shown in Section 3, the corresponding outgoing directions are regular. The Robert–Tamura formula thus applies with $\ell(\theta, \lambda) = 1$. In particular, there are no interference terms, and since $\sigma(w; \lambda)^{-1}$ is precisely the classical differential cross section associated with the unique trajectory, we obtain

$$|f(\omega \to \theta; \lambda, h)|^2 \longrightarrow \frac{d\sigma_{\rm cl}}{d\Omega}(\Theta; \lambda), \qquad 0 < \Theta < \pi. \tag{1.41}$$

The classical differential cross section determines the inverse function $b = b(\Theta)$ of the deflection function $b \mapsto \Theta_\lambda(b)$, and hence determines the deflection function itself. A classical Firsov–Abel inversion [4, 10] then recovers the radial potential throughout the classically accessible region. This proves Theorem 1.2.

We also show that the same strategy extends to compactly supported metric perturbations of the Euclidean Laplacian. In the simple case, the semiclassical scattering matrix determines the classical scattering relation, and hence the boundary distance function. Combining this observation with the boundary rigidity results of Stefanov–Uhlmann–Vasy [23], we obtain uniqueness of the metric up to a boundary-fixing isometry under the geometric assumptions of their rigidity theorem. Related scattering rigidity results for simple Riemannian metrics, including metrics in a fixed conformal class and real-analytic metrics, were obtained by Muñoz-Thon in the more general framework of magnetic-potential systems [12].

Let us conclude by mentioning a possible extension of the radial result to short-range potentials which are radial only near infinity. At sufficiently large energy, classical inverse scattering results of Jollivet [9] provide uniqueness from suitable classical scattering data. Extending the semiclassical recovery of the classical scattering data to this setting would therefore lead to a corresponding semiclassical inverse result. We do not pursue this question here.

The paper is organized as follows. In Section 2, we recover the classical scattering map from the semiclassical scattering matrix for compactly supported potentials, using Ingremeau's propagation of Gaussian states, and then deduce uniqueness of the potential from the rigidity theorem of Stefanov–Uhlmann–Vasy. Section 3 is devoted to radial short-range potentials. After studying the classical radial dynamics, we use the Robert–Tamura asymptotics and the Firsov–Abel inversion formula to recover the potential in the classically accessible region from the semiclassical differential cross section. Finally, Section 4 extends the argument to simple compactly supported perturbations of the Euclidean metric.

2. Compactly supported potentials: recovery of the scattering map

By a simple scaling argument, we may assume without loss of generality that the fixed non-trapping energy is $\lambda = 1/2$. We therefore recall Ingremeau's result [6] in this normalized setting, in the form needed below. For

$$\rho = (\omega, z) \in T^*\mathbb{S}^{n-1}, \qquad z \in \omega^\perp, \tag{2.1}$$

we denote by $\kappa = \kappa_{1/2}$ the classical scattering map introduced in (1.21). Thus, if

$$\rho_1 = (\omega_1, z_1) = \kappa(\rho), \tag{2.2}$$

then ρ_1 is the outgoing phase-space point associated with the incoming datum ρ.

Following Ingremeau [6], let Γ_0 be a symmetric complex matrix with positive definite real part, let Q_0 be a polynomial, and choose $\chi \in C^\infty(\mathbb{R};[0,1])$ such that

$$\chi(r) = 1 \quad \text{for } r \le \frac{1}{2}, \qquad \chi(r) = 0 \quad \text{for } r \ge \frac{3}{4}.$$

Associated with $\rho = (\omega, z) \in T^*\mathbb{S}^{n-1}$, we define the Gaussian state

$$\varphi_{\rho,\Gamma_0,Q_0}(\widehat{x};h) = \chi\left(\frac{|\widehat{x}-\omega|}{h^{1/3}}\right) Q_0\left(\frac{\widehat{x}-\omega}{\sqrt{h}}\right) e^{-\frac{\mathrm{i}}{h} z\cdot\widehat{x}} e^{-\frac{1}{2h}(\widehat{x}-\omega)\cdot\Gamma_0(\widehat{x}-\omega)}. \tag{2.3}$$

The Gaussian factor localizes the state at the scale $\sqrt{h}$ around ω, while the oscillatory factor localizes it microlocally at the cotangent variable z. Thus $\varphi_{\rho,\Gamma_0,Q_0}$ is microlocally concentrated near the phase-space point $\rho = (\omega, z)$.

The Gaussian states used by Ingremeau may be viewed as the scattering counterpart of the coherent states familiar from semiclassical propagation. In particular, their localization in phase space and their propagation along the classical scattering map are analogous to the semiclassical propagation of Gaussian coherent states; see, for instance, Combescure and Robert [20].

We now recall Ingremeau's propagation result. His convention for the scattering matrix differs from the standard one by conjugation with the antipodal map. More precisely, if

$$(Tf)(\widehat{x}) = f(-\widehat{x}), \tag{2.4}$$

then Ingremeau's scattering matrix S_h is related to the standard semiclassical scattering matrix by

$$S_h = TS(1/2,h)T. \tag{2.5}$$

Since this fixed reflection plays no role in the argument below, we use Ingremeau's notation S_h in the statement of his result. The main result of [6] describes the action of the scattering matrix on these Gaussian states. Let

$$(\omega_1, z_1) = \kappa(\omega, z). \tag{2.6}$$

Then there exist a real number δ_1, a symmetric complex matrix Γ_1 with positive definite real part, and polynomials Q_1^k, $k \in \mathbb{N}$, such that the following holds. For every $N \in \mathbb{N}$, setting

$$\widetilde{Q}_1^N = \sum_{k=0}^{N} h^{k/2} Q_1^k, \tag{2.7}$$

one has

$$S_h \varphi_{\omega,z,\Gamma_0,Q_0} = e^{i\delta_1/h} \varphi_{\omega_1,z_1,\Gamma_1,\widetilde{Q}_1^N} + R_N, \tag{2.8}$$

where

$$\|R_N\|_{C^0(\mathbb{S}^{n-1})} = O\left(h^{(N+1)/2}\right). \tag{2.9}$$

Thus, to arbitrary order in the semiclassical expansion, the scattering matrix maps a Gaussian state localized near the incoming phase point (ω, z) to a Gaussian state localized near the outgoing phase point $\kappa(\omega, z)$. This direct transport of phase-space centers is the feature of Ingremeau's approach that will be used below to recover the classical scattering map from the semiclassical scattering matrix.

For the inverse argument below, we shall only need the particular choice $\Gamma_0 = I$ and $Q_0 = 1$. We therefore set

$$\phi_{\rho,h}(\widehat{x}) = c_h \, \chi\left(\frac{|\widehat{x}-\omega|}{h^{1/3}}\right) e^{-\frac{\mathrm{i}}{h} z\cdot\widehat{x}} e^{-\frac{|\widehat{x}-\omega|^2}{2h}}, \tag{2.10}$$

where $c_h > 0$ is chosen so that

$$\|\phi_{\rho,h}\|_{L^2(\mathbb{S}^{n-1})} = 1. \tag{2.11}$$

We can now use Ingremeau's propagation result to recover the classical scattering map from the semiclassical scattering matrix. We state the result at an arbitrary fixed energy $\lambda > 0$, although, in view of the scaling discussed above, it is enough to prove it for $\lambda = 1/2$.

Lemma 2.1. *Let $V_1, V_2 \in C_c^\infty(\mathbb{R}^n)$ and assume that $\lambda > 0$ is a non-trapping energy for both potentials. Let $S_{j,h}(\lambda)$ and $\kappa_{j,\lambda}$ denote respectively the semiclassical scattering matrix and the classical scattering map associated with V_j, $j = 1, 2$. If*

$$\|S_{1,h}(\lambda) - S_{2,h}(\lambda)\|_{\mathcal{B}(L^2(\mathbb{S}^{n-1}))} = o(1), \qquad h \to 0, \tag{2.12}$$

then

$$\kappa_{1,\lambda} = \kappa_{2,\lambda}. \tag{2.13}$$

Proof. Fix $\rho = (\omega, z) \in T^*\mathbb{S}^{n-1}$ and suppose, by contradiction, that

$$\kappa_{1,\lambda}(\rho) \neq \kappa_{2,\lambda}(\rho). \tag{2.14}$$

Write

$$\kappa_{j,\lambda}(\rho) = (\omega_j, z_j), \qquad j = 1, 2.$$

By Ingremeau's theorem, for every $M > 0$,

$$S_{j,h}(\lambda)\phi_{\rho,h} = e^{i\delta_j/h}\Phi_{j,h} + O(h^M) \qquad \text{in } L^2(\mathbb{S}^{n-1}), \tag{2.15}$$

where $\Phi_{j,h}$ is a Gaussian state centered at (ω_j, z_j). Since the expansion in (2.8) is available to arbitrary order, M can be chosen arbitrarily large. We claim that (2.14) implies

$$\langle S_{1,h}(\lambda)\phi_{\rho,h}, S_{2,h}(\lambda)\phi_{\rho,h}\rangle = O(h^\infty). \tag{2.16}$$

Indeed, if $\omega_1 \neq \omega_2$, the cutoffs in the two outgoing Gaussian states have disjoint supports for h sufficiently small, since they are supported in balls of radius $O(h^{1/3})$ around the distinct points ω_1 and ω_2. It remains to consider the case

$$\omega_1 = \omega_2 =: \omega_+, \qquad z_1 \neq z_2. \tag{2.17}$$

The two outgoing states are then localized near the same point ω_+, but the product entering their scalar product contains the oscillatory factor

$$\exp\left(\frac{i}{h}(z_1 - z_2)\cdot\widehat{x}\right). \tag{2.18}$$

Since $z_1 - z_2 \in \omega_+^\perp$ and $z_1 - z_2 \neq 0$, the phase in (2.18) has no critical point on the support of the Gaussian states for h sufficiently small. Hence, by non-stationary phase,

$$\langle \Phi_{1,h}, \Phi_{2,h}\rangle = O(h^\infty), \tag{2.19}$$

and (2.16) follows. Since the scattering matrices are unitary, by (2.11) we have

$$\|S_{j,h}(\lambda)\phi_{\rho,h}\|_{L^2} = 1, \qquad j = 1, 2. \tag{2.20}$$

Hence, using (2.16),

$$\begin{aligned}\left\|(S_{1,h}(\lambda) - S_{2,h}(\lambda))\phi_{\rho,h}\right\|_{L^2}^2 &= 2 - 2\,\mathrm{Re}\,\langle S_{1,h}(\lambda)\phi_{\rho,h}, S_{2,h}(\lambda)\phi_{\rho,h}\rangle \\ &= 2 + O(h^\infty).\end{aligned} \tag{2.21}$$

On the other hand, since $\|\phi_{\rho,h}\|_{L^2} = 1$, (2.12) gives

$$\left\|(S_{1,h}(\lambda) - S_{2,h}(\lambda))\phi_{\rho,h}\right\|_{L^2} = o(1). \tag{2.22}$$

This contradicts (2.21). Therefore

$$\kappa_{1,\lambda}(\rho) = \kappa_{2,\lambda}(\rho). \tag{2.23}$$

Since $\rho \in T^*\mathbb{S}^{n-1}$ was arbitrary, (2.13) follows. $\square$

2.1. **From the scattering map to the potential.** We now turn to the classical inverse problem. We assume first that the energy lies above the potentials,

$$\lambda > \sup_{\mathbb{R}^n} V_j, \qquad j = 1, 2, \tag{2.24}$$

and that the virial condition

$$2(\lambda - V_j(x)) - x \cdot \nabla V_j(x) > 0, \qquad x \in \mathbb{R}^n, \qquad j = 1, 2, \tag{2.25}$$

holds. The condition (2.25) implies that the Hamiltonian flow at energy λ is non-trapping.

By the Maupertuis principle, under (2.24), the trajectories at energy λ, up to a reparametrization, are the geodesics of the Jacobi metrics

$$g_j = 2(\lambda - V_j)g_0, \qquad j = 1, 2, \tag{2.26}$$

where g_0 denotes the Euclidean metric on $\mathbb{R}^n$. Equivalently, we may write

$$g_j = c_j^{-2} g_0, \qquad c_j = \big(2(\lambda - V_j)\big)^{-1/2}, \qquad j = 1, 2. \tag{2.27}$$

This reduction was already used by Alexandrova [1] under the stronger assumption that the Jacobi metric is simple. In that setting, she used an argument implicit in Michel [11] to recover the boundary distance function from the scattering relation, and then applied the boundary rigidity results of Mukhometov [14] and Mukhometov–Romanov [13] to determine the conformal factor. More recently, Muñoz-Thon [12] obtained scattering rigidity results at a single energy for simple magnetic-potential systems. In particular, in the non-magnetic setting considered here, his results yield uniqueness of the potential from the classical scattering relation when the associated Jacobi metric is simple.

The geometric assumption used here is different. We do not require the Jacobi metrics to be simple, but instead use the rigidity theorem of Stefanov–Uhlmann–Vasy [22], which applies when the manifold admits a strictly convex foliation. Our virial condition provides precisely such a foliation for the Jacobi metrics. Since this condition does not exclude conjugate points, it applies to geometries which need not be simple. We note that Muñoz-Thon [12] explicitly raised the question of extending scattering rigidity for magnetic-potential systems from the simple setting to the strictly convex foliation framework.

Definition 2.2. Let (M, g) be a compact Riemannian manifold with boundary. We say that M admits a strictly convex foliation if there exist $T > 0$ and a smooth function

$$\rho : M \longrightarrow [0, +\infty)$$

such that, for every $t \in [0, T)$, the level set

$$\Sigma_t = \rho^{-1}(t)$$

is a strictly convex hypersurface, $d\rho \neq 0$ on Σ_t, $\Sigma_0 = \partial M$, and

$$M \setminus \bigcup_{0 \le t < T} \Sigma_t$$

has empty interior.

In our setting, the virial condition provides precisely such a foliation. Indeed, on $M = \overline{B(0, R)}$, we take $\rho(x) = R^2 - |x|^2$. Then, for $0 \le t < R^2$,

$$\Sigma_t = \rho^{-1}(t) = \{|x| = \sqrt{R^2 - t}\}. \tag{2.28}$$

Thus the leaves of the foliation are precisely the Euclidean spheres centered at the origin.

The following lemma shows that the virial condition indeed guarantees the strict convexity of these leaves with respect to the Jacobi metrics.

Lemma 2.3. *Under* (2.24) *and* (2.25)*, the Euclidean spheres form a strictly convex foliation for the Jacobi metrics* (2.26)*.*

Proof. It is enough to consider one potential V. Writing the Jacobi metric in the form

$$g = c^{-2} g_0, \qquad c = \big(2(\lambda - V)\big)^{-1/2}, \tag{2.29}$$

we have

$$\frac{\partial}{\partial r}\left(\frac{r}{c}\right) = \frac{2(\lambda - V(x)) - x \cdot \nabla V(x)}{\sqrt{2(\lambda - V(x))}}. \tag{2.30}$$

Hence (2.25) implies the Herglotz–Wiechert–Zoeppritz condition

$$\frac{\partial}{\partial r}\left(\frac{r}{c}\right) > 0. \tag{2.31}$$

By [22, Proposition 6.1], this is equivalent to the strict convexity of the Euclidean spheres for the Jacobi metric. As the concentric spheres foliate $\mathbb{R}^n \setminus \{0\}$, the conclusion follows. $\square$

We briefly recall the notion of scattering relation used in Stefanov–Uhlmann–Vasy [22]. Let (M, g) be a compact Riemannian manifold with boundary. In our application below, M will simply be the closed Euclidean ball $\overline{B(0,R)} \subset \mathbb{R}^n$, equipped with one of the Jacobi metrics (2.26). We denote by $\partial_- SM$ and $\partial_+ SM$ the sets of unit tangent vectors based at ∂M and pointing respectively into and out of M. For $(x_-, v_-) \in \partial_- SM$, let γ_{x_-, v_-} be the unique unit speed geodesic issued from (x_-, v_-). In our setting, the non-trapping assumption ensures that this geodesic exits M at a unique point and direction $(x_+, v_+) \in \partial_+ SM$. The scattering relation is defined by

$$L(x_-, v_-) = (x_+, v_+). \tag{2.32}$$

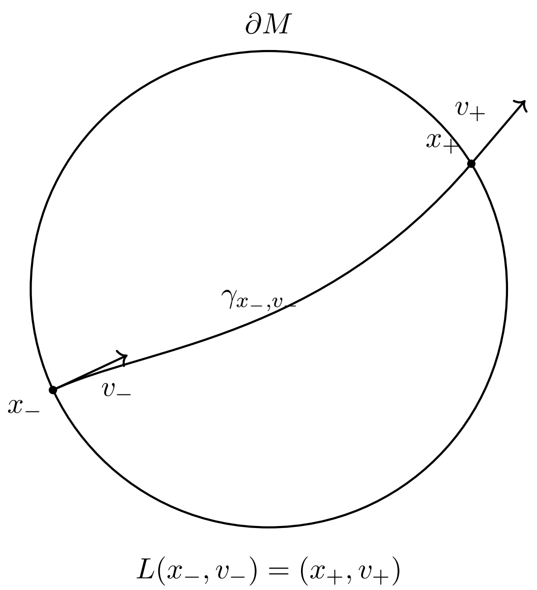


FIGURE 2. The scattering relation of a Riemannian metric: an incoming point and direction (x_-, v_-) are mapped to the corresponding exit point and direction (x_+, v_+).

By Lemma 2.1, the assumption

$$\|S_{1,h}(\lambda) - S_{2,h}(\lambda)\|_{\mathcal{B}(L^2(\mathbb{S}^{n-1}))} = o(1), \qquad h \to 0, \tag{2.33}$$

implies

$$\kappa_{1,\lambda} = \kappa_{2,\lambda}. \tag{2.34}$$

Since V_1 and V_2 are compactly supported, we may choose $R > 0$ such that

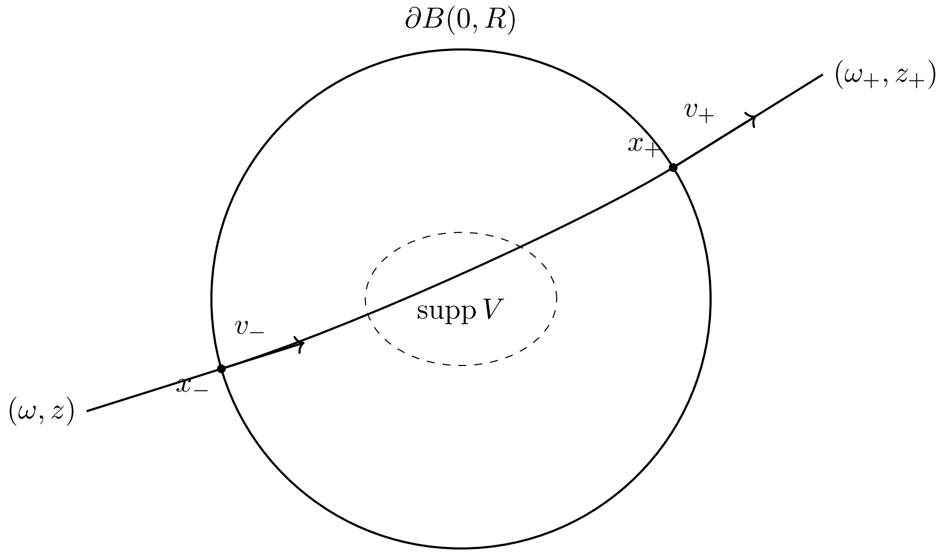
$$\operatorname{supp} V_1 \cup \operatorname{supp} V_2 \subset B(0,R). \tag{2.35}$$

Outside $B(0,R)$ the Hamiltonian trajectories are straight lines. Hence the incoming and outgoing asymptotic data defining $\kappa_{j,\lambda}$ determine uniquely the corresponding entry and exit points and directions on $\partial B(0,R)$. It follows from (2.34) that the scattering relations L_j of the Jacobi metrics on $B(0,R)$ satisfy

$$L_1 = L_2. \tag{2.36}$$

Notice that no travel-time information is needed here, since the rigidity result [22, Theorem 1.4] only requires equality of the scattering relations. Finally, since $V_1 = V_2 = 0$ near $\partial B(0,R)$, the Jacobi metrics coincide there:

$$g_1 = g_2 = 2\lambda g_0 \qquad \text{near } \partial B(0,R). \tag{2.37}$$



$$\kappa_\lambda(\omega, z) = (\omega_+, z_+)$$
$$L(x_-, v_-) = (x_+, v_+)$$

FIGURE 3. Relation between the classical scattering map κ_λ and the scattering relation L on $\partial B(0,R)$. Outside the support of V, the trajectories are straight lines, so that the asymptotic data determine uniquely the corresponding entry and exit data on the boundary.

We may therefore apply the global rigidity theorem of Stefanov-Uhlmann-Vasy [22, Theorem 1.4]. Indeed, Lemma 2.3 provides the required strictly convex foliation. It follows that $g_1 = g_2$. In view of (2.26), we conclude that $V_1 = V_2$.

3. Radial short-range potentials

We now consider smooth radial short-range potentials

$$V(x) = v(|x|), \qquad v \in C^\infty([0,+\infty)), \qquad |\partial_r^k v(r)| \le C_k \langle r \rangle^{-\mu-k}, \qquad \mu > 1. \tag{3.1}$$

Following the classical setting of Keller–Kay–Shmoys [10], we assume that

$$v(r) > 0, \qquad v'(r) < 0, \qquad r > 0, \tag{3.2}$$

and that

$$v(0) > \lambda. \tag{3.3}$$

Since $v(r) \to 0$ as $r \to +\infty$, assumptions (3.2)–(3.3) imply that there exists a unique $r_\lambda > 0$ such that $v(r_\lambda) = \lambda$. The region $0 < r < r_\lambda$ is classically forbidden at energy λ, whereas the classically accessible region is $r > r_\lambda$. The assumption (3.3) is thus precisely what ensures the presence of a nonempty classically forbidden core; it is also part of the classical monotonicity framework used below for the Firsov–Abel inversion.

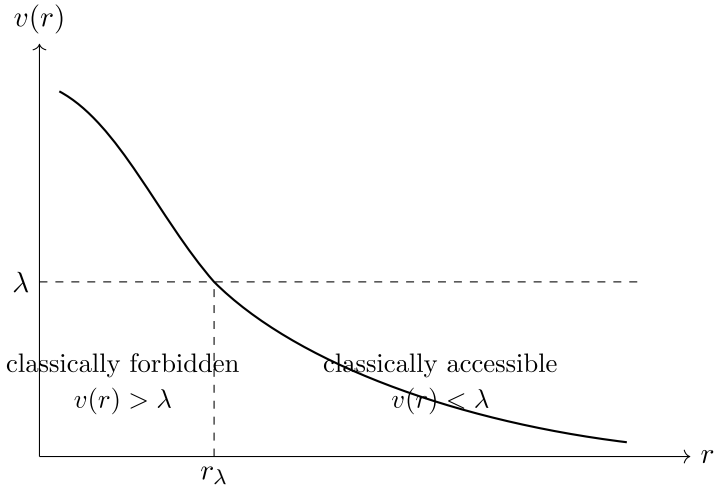


FIGURE 4. A radial repulsive potential in the setting of Keller–Kay–Shmoys. The unique radius r_λ is defined by $v(r_\lambda) = \lambda$. At the fixed energy λ, the region $0 < r < r_\lambda$ is classically forbidden, whereas $r > r_\lambda$ is classically accessible.

In the radial setting, the virial condition (2.25) takes the form

$$2\big(\lambda - v(r)\big) - rv'(r) > 0. \tag{3.4}$$

On the energy surface

$$\frac{1}{2}|\xi|^2 + v(r) = \lambda,$$

one necessarily has $v(r) \le \lambda$, and hence $r \ge r_\lambda$. Since $v'(r) < 0$, (3.4) is therefore satisfied on the whole energy surface, including at $r = r_\lambda$. In particular, the classical dynamics at energy λ is non-trapping, so that the semiclassical scattering results of Robert–Tamura apply.

We further assume that

$$\big(rv'(r)\big)' \ge 0, \qquad r > r_\lambda. \tag{3.5}$$

Remark 3.1. Condition (3.5) has a natural interpretation in terms of the radial force. Indeed, since

$$F(x) = -\nabla V(x) = -v'(r)\frac{x}{r}, \qquad r = |x|,$$

it is equivalent to requiring that the weighted radial force $-rv'(r)$ be non-increasing on the classically accessible region. For instance, for the inverse-power profile

$$v(r) = Cr^{-\rho}, \qquad r \ge r_0 > 0, \qquad C > 0, \quad \rho > 0, \tag{3.6}$$

one has

$$\big(rv'(r)\big)' = \rho^2 C r^{-\rho-1} > 0. \tag{3.7}$$

3.1. **Classical radial dynamics.** We first recall some standard facts about the classical dynamics of a radial potential; see, for instance, Keller–Kay–Shmoys [10] and Jollivet [9]. We include the details needed below in order to fix our notation and sign conventions. Under the assumptions (3.2)–(3.3), the motion at the fixed energy λ takes place in the classically accessible region $r \ge r_\lambda$. Together with the additional assumption (3.5), these hypotheses will imply that each scattering angle corresponds to a unique classical trajectory.

Let $b = |z|$ be the impact parameter. Since V is radial, the force is central and the classical motion takes place in the scattering plane

$$\Pi_{\omega,z} := \mathrm{span}\{\omega, z\}; \tag{3.8}$$

see, for instance, [5, Sections 3.2 and 3.10]. The angular momentum is also conserved. More precisely, the angular momentum bivector is

$$\mathcal{L}(t) = q_\infty(t;z,\lambda) \wedge p_\infty(t;z,\lambda), \tag{3.9}$$

where, for $x, \xi \in \mathbb{R}^n$, the bivector $x \wedge \xi$ has coordinates

$$(x \wedge \xi)_{ij} = x_i\xi_j - x_j\xi_i, \qquad 1 \le i < j \le n. \tag{3.10}$$

Its magnitude, with respect to the Euclidean norm, will be denoted by $L = |\mathcal{L}(t)|$. One has

$$\frac{d}{dt}\mathcal{L}(t) = p_\infty \wedge p_\infty - q_\infty \wedge \nabla V(q_\infty) = 0, \tag{3.11}$$

since

$$\nabla V(q_\infty) = v'(|q_\infty|)\frac{q_\infty}{|q_\infty|} \tag{3.12}$$

is parallel to q_∞. Thus L is constant along the trajectory. Using the incoming asymptotics

$$q_\infty(t;z,\lambda) = \sqrt{2\lambda}\,\omega t + z + o(1), \qquad p_\infty(t;z,\lambda) = \sqrt{2\lambda}\,\omega + o(1), \qquad t \to -\infty, \tag{3.13}$$

together with $z \perp \omega$ and $|z| = b$, we obtain

$$L = \sqrt{2\lambda}\,|z| = \sqrt{2\lambda}\,b. \tag{3.14}$$

We now introduce polar coordinates in the scattering plane $\Pi_{\omega,z}$. After choosing an oriented orthonormal basis of this plane, we write

$$q_\infty(t;z,\lambda) = r(t)e_r(t), \qquad r(t) = |q_\infty(t;z,\lambda)|, \tag{3.15}$$

where, if $\varphi(t)$ denotes the polar angle,

$$e_r(t) = \big(\cos\varphi(t), \sin\varphi(t)\big), \qquad e_\varphi(t) = \big(-\sin\varphi(t), \cos\varphi(t)\big). \tag{3.16}$$

The vectors $e_r(t)$ and $e_\varphi(t)$ form an orthonormal basis of the scattering plane and satisfy

$$\dot e_r(t) = \dot\varphi(t)e_\varphi(t). \tag{3.17}$$

It follows that

$$p_\infty(t;z,\lambda) = \dot q_\infty(t;z,\lambda) = \dot r(t)e_r(t) + r(t)\dot\varphi(t)e_\varphi(t), \tag{3.18}$$

and therefore

$$|p_\infty(t;z,\lambda)|^2 = \dot r(t)^2 + r(t)^2\dot\varphi(t)^2. \tag{3.19}$$

From (3.14) and (3.18), we obtain

$$L = r(t)^2|\dot\varphi(t)| = \sqrt{2\lambda}\,b. \tag{3.20}$$

Hence

$$\frac{1}{2}r(t)^2\dot\varphi(t)^2 = \frac{L^2}{2r(t)^2} = \lambda\frac{b^2}{r(t)^2}. \tag{3.21}$$

Thus, using conservation of energy,

$$\frac{1}{2}|p_\infty(t;z,\lambda)|^2 + v(r(t)) = \lambda, \tag{3.22}$$

we obtain the radial energy equation

$$\frac{1}{2}\dot r(t)^2 + \lambda\frac{b^2}{r(t)^2} + v(r(t)) = \lambda. \tag{3.23}$$

It is useful to introduce the effective radial potential

$$v_{\mathrm{eff}}(r;b) := v(r) + \lambda\frac{b^2}{r^2}. \tag{3.24}$$

Under the assumption $v'(r) < 0$, one has

$$\partial_r v_{\mathrm{eff}}(r;b) = v'(r) - 2\lambda\frac{b^2}{r^3} < 0, \qquad r > 0. \tag{3.25}$$

Thus $v_{\rm eff}(\,\cdot\,;b)$ is strictly decreasing. Moreover, for $b>0$,

$$v_{\rm eff}(r;b) \longrightarrow +\infty \quad \text{as } r\to 0^+, \qquad v_{\rm eff}(r;b) \longrightarrow 0 \quad \text{as } r\to +\infty. \tag{3.26}$$

Consequently, for every $b>0$, there exists a unique turning point $r_0(b)$ characterized by

$$v(r_0(b)) + \lambda\frac{b^2}{r_0(b)^2} = \lambda. \tag{3.27}$$

Since (3.27) yields $v(r_0(b)) < \lambda = v(r_\lambda)$ and v is strictly decreasing, one necessarily has $r_0(b) > r_\lambda$. Geometrically, $r_0(b)$ is the distance of closest approach of the trajectory to the origin: if $\dot r(t_0) = 0$ at the turning point, then

$$r_0(b) = r(t_0) = \min_{t\in\mathbb{R}} r(t). \tag{3.28}$$

By rotational invariance, the distance of closest approach depends on the impact vector z only through $b=|z|$. Rewriting (3.27), we obtain

$$b = \rho_\lambda(r_0(b)), \qquad \rho_\lambda(r) := r\sqrt{1-\frac{v(r)}{\lambda}}, \qquad r\ge r_\lambda. \tag{3.29}$$

The function ρ_λ is the optical radius associated with the fixed energy λ; see, for instance, Keller–Kay–Shmoys [10]. It relates the physical radius of a turning point to the corresponding impact parameter and plays a central role in the classical radial inverse problem.

By (3.2), for $r>r_\lambda$,

$$\rho_\lambda'(r) = \sqrt{1-\frac{v(r)}{\lambda}}\left(1-\frac{rv'(r)}{2(\lambda - v(r))}\right) > 0, \tag{3.30}$$

since $\lambda - v(r) > 0$ and $v'(r)<0$. Hence ρ_λ is strictly increasing from 0 to $+\infty$ on $[r_\lambda, +\infty)$, and

$$b = \rho_\lambda(r_0(b)) \tag{3.31}$$

provides a one-to-one parametrization of the radial turning points by the impact parameter $b\ge 0$. On the outgoing part of the trajectory, $r(t)$ increases from $r_0(b)$ to $+\infty$. From (3.23), we have

$$\dot r(t) = \sqrt{2\lambda}\sqrt{1-\frac{v(r(t))}{\lambda} - \frac{b^2}{r(t)^2}}. \tag{3.32}$$

For $r>r_0(b)$, we may therefore use r as a parameter along the trajectory. Combining (3.32) with (3.20) and using the chain rule, we obtain

$$\left|\frac{d\varphi}{dr}\right| = \frac{|\dot\varphi(t)|}{\dot r(t)} = \frac{b}{r^2\sqrt{1-\frac{v(r)}{\lambda}-\frac{b^2}{r^2}}}. \tag{3.33}$$

The angular variation along the outgoing part of the trajectory is

$$\Phi_\lambda(b) := b\int_{r_0(b)}^{+\infty} \frac{dr}{r^2\sqrt{1-\frac{v(r)}{\lambda}-\frac{b^2}{r^2}}}. \tag{3.34}$$

By the symmetry of the central-force trajectory with respect to the line through the origin and the point of closest approach, the angular variation from the incoming asymptote to the turning point equals that from the turning point to the outgoing asymptote. Hence the total angular variation between the two asymptotes is $2\Phi_\lambda(b)$. We therefore define the signed deflection function by

$$\Theta_\lambda(b) := \pi - 2\Phi_\lambda(b) = \pi - 2b\int_{r_0(b)}^{+\infty} \frac{dr}{r^2\sqrt{1-\frac{v(r)}{\lambda}-\frac{b^2}{r^2}}}. \tag{3.35}$$

Remark 3.2. Let us check the above convention in the free case $V = 0$. Then $v(r) = 0$ and the turning point equation gives $r_0(b) = b$. Consequently,

$$\Phi_\lambda(b) = b \int_b^{+\infty} \frac{dr}{r^2\sqrt{1 - \frac{b^2}{r^2}}} = \frac{\pi}{2}. \tag{3.36}$$

Therefore, $\Theta_\lambda(b) = \pi - 2\Phi_\lambda(b) = 0$, as expected, since in the absence of a potential the incoming and outgoing directions coincide.

The geometry of the resulting radial scattering trajectory is illustrated in Figure 5.

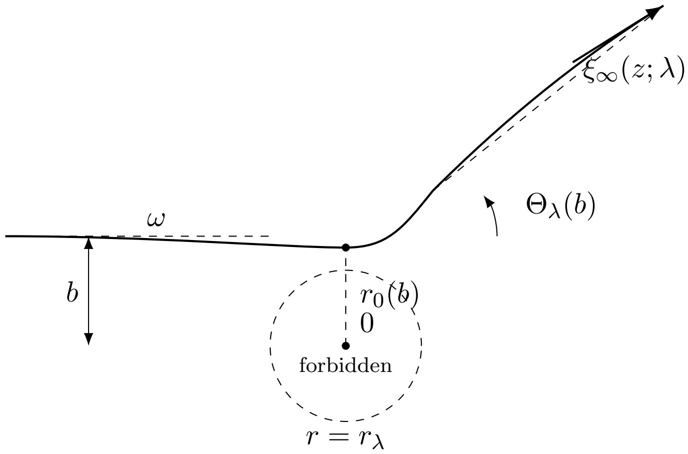


FIGURE 5. Radial scattering at the fixed energy λ. The dashed circle $r = r_\lambda$, defined by $v(r_\lambda) = \lambda$, bounds the classically forbidden region. For $b > 0$, the distance of closest approach satisfies $r_0(b) > r_\lambda$. The trajectory is asymptotic to the incoming and outgoing free lines as $t \to -\infty$ and $t \to +\infty$, respectively.

Since ρ_λ is strictly increasing on $[r_\lambda, +\infty)$, it can be used as a new radial variable. We denote its inverse by

$$r = r_\lambda(\rho), \qquad \rho = \rho_\lambda(r), \tag{3.37}$$

and introduce

$$q_\lambda(\rho) := \frac{d}{d\rho} \log \frac{r_\lambda(\rho)}{\rho}. \tag{3.38}$$

The additional assumption (3.5), which is not needed for the Firsov–Abel inversion itself, provides a convenient sufficient condition ensuring that the radial scattering geometry has only one classical branch. More precisely, we have the following result.

Lemma 3.3. *Assume* (3.1), (3.2), (3.3), *and* (3.5). *Then the deflection function $b \mapsto \Theta_\lambda(b)$ is strictly decreasing:*

$$\Theta'_\lambda(b) < 0, \qquad b > 0. \tag{3.39}$$

Moreover,

$$\lim_{b\to 0^+} \Theta_\lambda(b) = \pi, \qquad \lim_{b\to+\infty} \Theta_\lambda(b) = 0. \tag{3.40}$$

In particular, for every scattering angle $\Theta \in (0, \pi)$, there exists a unique impact parameter $b > 0$ such that

$$\Theta_\lambda(b) = \Theta. \tag{3.41}$$

Proof. Using the change of variables $\rho = \rho_\lambda(r)$ in (3.35), and recalling that $\rho_\lambda(r_0(b)) = b$, we have

$$\sqrt{1 - \frac{v(r)}{\lambda} - \frac{b^2}{r^2}} = \frac{\sqrt{\rho^2 - b^2}}{r}, \tag{3.42}$$

and

$$\Phi_\lambda(b) = b \int_b^{+\infty} \frac{1}{r_\lambda(\rho)} \frac{dr_\lambda}{d\rho}(\rho) \frac{d\rho}{\sqrt{\rho^2 - b^2}}. \tag{3.43}$$

By (3.38),

$$\frac{1}{r_\lambda(\rho)} \frac{dr_\lambda}{d\rho}(\rho) = \frac{1}{\rho} + q_\lambda(\rho). \tag{3.44}$$

Since

$$b \int_b^{+\infty} \frac{d\rho}{\rho\sqrt{\rho^2 - b^2}} = \frac{\pi}{2}, \tag{3.45}$$

substitution into $\Theta_\lambda(b) = \pi - 2\Phi_\lambda(b)$ yields

$$\Theta_\lambda(b) = -2b \int_b^{+\infty} \frac{q_\lambda(\rho)}{\sqrt{\rho^2 - b^2}}\, d\rho. \tag{3.46}$$

After the change of variables $\rho = bt$, we obtain

$$\Theta_\lambda(b) = -2b \int_1^{+\infty} \frac{q_\lambda(bt)}{\sqrt{t^2 - 1}}\, dt. \tag{3.47}$$

Differentiating with respect to b gives

$$\Theta'_\lambda(b) = -2 \int_1^{+\infty} \frac{\frac{d}{d\rho}\big(\rho q_\lambda(\rho)\big)\big|_{\rho = bt}}{\sqrt{t^2 - 1}}\, dt. \tag{3.48}$$

To determine the sign of the integrand, set

$$Q_\lambda(r) := -\frac{r v'(r)}{\lambda - v(r)}, \qquad r > r_\lambda. \tag{3.49}$$

A direct computation gives

$$Q'_\lambda(r) = -\frac{(\lambda - v(r))(r v'(r))' + r(v'(r))^2}{(\lambda - v(r))^2}. \tag{3.50}$$

By (3.5), since $\lambda - v(r) > 0$ and $v'(r) < 0$, the numerator in (3.50) is strictly positive. Hence

$$Q'_\lambda(r) < 0, \qquad r > r_\lambda. \tag{3.51}$$

On the other hand, from the definitions of ρ_λ and q_λ, one obtains

$$\rho q_\lambda(\rho) = \frac{1}{1 + \frac{1}{2} Q_\lambda(r_\lambda(\rho))} - 1. \tag{3.52}$$

Since $r'_\lambda(\rho) > 0$ and $Q'_\lambda < 0$, we conclude that

$$\frac{d}{d\rho}\big(\rho q_\lambda(\rho)\big) > 0. \tag{3.53}$$

Inserting (3.53) into (3.48) proves (3.39). Finally, when $b \to 0^+$, the turning point $r_0(b)$ tends to r_λ and the trajectory approaches the radial trajectory. Indeed, since $L = \sqrt{2\lambda}\, b$, the limit $b \to 0^+$ corresponds to vanishing angular momentum, and hence to a radial trajectory which reaches the turning radius r_λ and is backscattered along the same radial line; hence

$$\Theta_\lambda(b) \longrightarrow \pi, \qquad b \to 0^+. \tag{3.54}$$

On the other hand, the short-range assumption implies that the scattering becomes asymptotically free as $b \to +\infty$, and thus

$$\Theta_\lambda(b) \longrightarrow 0, \qquad b \to +\infty. \tag{3.55}$$

This proves (3.40). Therefore $b \mapsto \Theta_\lambda(b)$ is one-to-one from $(0,+\infty)$ onto $(0,\pi)$. $\square$

3.2. **The Robert–Tamura asymptotics.** We now recall the semiclassical asymptotics of the scattering amplitude obtained by Robert and Tamura [21]. This provides the link between the quantum differential cross section and the classical scattering data introduced in the previous subsection.

Fix an incoming direction $\omega \in \mathbb{S}^{n-1}$ and choose oriented orthonormal coordinates $(z_1,\dots,z_{n-1})$ on the impact plane

$$\Lambda_\omega = \omega^\perp.$$

For $z \in \Lambda_\omega$, let

$$(q_\infty(t;z,\lambda), p_\infty(t;z,\lambda))$$

be the scattering trajectory defined by the incoming asymptotics

$$q_\infty(t;z,\lambda) = \sqrt{2\lambda}\,\omega t + z + o(1), \qquad p_\infty(t;z,\lambda) = \sqrt{2\lambda}\,\omega + o(1), \qquad t \to -\infty.$$

Assume that $\lambda > 0$ is a non-trapping energy. Then there exist $\xi_\infty(z;\lambda) \in \mathbb{S}^{n-1}$ and $r_\infty(z;\lambda) \in \mathbb{R}^n$ such that

$$\begin{aligned} q_\infty(t;z,\lambda) &= \sqrt{2\lambda}\,\xi_\infty(z;\lambda)t + r_\infty(z;\lambda) + o(1), \\ p_\infty(t;z,\lambda) &= \sqrt{2\lambda}\,\xi_\infty(z;\lambda) + o(1), \end{aligned} \qquad t \to +\infty. \tag{3.56}$$

For fixed $\omega \in \mathbb{S}^{n-1}$, consider the smooth map

$$F_\omega : \Lambda_\omega \longrightarrow \mathbb{S}^{n-1}, \qquad F_\omega(z) = \xi_\infty(z;\lambda). \tag{3.57}$$

The angular density associated with the trajectory issued from $z \in \Lambda_\omega$ is defined by

$$\sigma(z;\lambda) = \left|\det\left(\xi_\infty(z;\lambda), \partial_{z_1}\xi_\infty(z;\lambda), \dots, \partial_{z_{n-1}}\xi_\infty(z;\lambda)\right)\right|. \tag{3.58}$$

Thus $\sigma(z;\lambda)$ is the absolute value of the Jacobian of F_ω between Λ_ω and $\mathbb{S}^{n-1}$.

An outgoing direction $\theta \in \mathbb{S}^{n-1}$, $\theta \neq \omega$, is said to be regular for the incoming direction ω if

$$\sigma(z;\lambda) \neq 0 \quad \text{for all } z \in \Lambda_\omega \text{ such that } F_\omega(z) = \theta. \tag{3.59}$$

Assume that θ is regular. Since $\theta \neq \omega$ and $\xi_\infty(z;\lambda) \to \omega$ as $|z| \to +\infty$, the set

$$F_\omega^{-1}(\theta) = \{z \in \Lambda_\omega;\ \xi_\infty(z;\lambda) = \theta\}$$

is compact. By the inverse function theorem, its points are isolated, and hence this set is finite. We denote its cardinality by $\ell(\theta,\lambda)$ and its elements by

$$F_\omega^{-1}(\theta) = \{w_1(\theta;\lambda), \dots, w_{\ell(\theta,\lambda)}(\theta;\lambda)\}. \tag{3.60}$$

Each $w_j(\theta;\lambda)$ determines a classical scattering trajectory with incoming direction ω and outgoing direction θ. For each such trajectory, Robert and Tamura associate the action

$$\begin{aligned} \mathcal{A}_j(\omega,\theta;\lambda) = &\int_{-\infty}^{+\infty} \left(\frac{|p_\infty(t;w_j,\lambda)|^2}{2} - V(q_\infty(t;w_j,\lambda)) - \lambda\right) \mathrm{d}t \\ &- \left\langle r_\infty(w_j;\lambda), \sqrt{2\lambda}\,\theta\right\rangle, \end{aligned} \tag{3.61}$$

where $w_j = w_j(\theta;\lambda)$. The integral in (3.61) represents the difference between the interacting and free actions along trajectories which are asymptotic as $t \to -\infty$. Let $\mu_j \in \mathbb{Z}$ denote the Keller–Maslov–Morse index of the corresponding trajectory.

With the normalization of the scattering amplitude introduced above, the Robert–Tamura formula [21] reads

$$f(\omega \to \theta;\lambda,h) = \sum_{j=1}^{\ell(\theta,\lambda)} \sigma(w_j;\lambda)^{-1/2} \exp\left(\frac{\mathrm{i}}{h}\mathcal{A}_j(\omega,\theta;\lambda) - \frac{\mathrm{i}\pi}{2}\mu_j\right) + O(h), \qquad h \to 0. \tag{3.62}$$

More precisely, under the same assumptions, the scattering amplitude admits a complete asymptotic expansion in powers of h.

In the radial setting, the monotonicity of the deflection function also implies that all non-forward and non-backward scattering directions are regular.

Lemma 3.4. *Under the assumptions of Lemma 3.3, every outgoing direction*

$$\theta \in \mathbb{S}^{n-1} \setminus \{\omega, -\omega\}$$

is regular for the incoming direction ω. Moreover, there is exactly one classical trajectory connecting ω to θ, that is,

$$\ell(\theta, \lambda) = 1.$$

Proof. Fix

$$\theta \in \mathbb{S}^{n-1} \setminus \{\omega, -\omega\}, \tag{3.63}$$

and set

$$\Theta = \arccos(\omega \cdot \theta) \in (0, \pi). \tag{3.64}$$

By Lemma 3.3, there exists a unique $b > 0$ such that

$$\Theta_\lambda(b) = \Theta. \tag{3.65}$$

Define

$$\eta = \frac{\theta - (\omega \cdot \theta)\omega}{\sin \Theta} \in \mathbb{S}^{n-2} \subset \omega^\perp, \tag{3.66}$$

and set

$$z = b\eta. \tag{3.67}$$

By rotational invariance and the definition of the deflection function,

$$F_\omega(z) = \cos \Theta_\lambda(b)\, \omega + \sin \Theta_\lambda(b)\, \eta = \theta. \tag{3.68}$$

Thus

$$F_\omega^{-1}(\theta) \neq \varnothing. \tag{3.69}$$

We now prove that θ is regular. Let $z = b\eta \in F_\omega^{-1}(\theta)$ be arbitrary, with $b > 0$ and $\eta \in \mathbb{S}^{n-2} \subset \omega^\perp$. Let $h \in T_z\Lambda_\omega = \omega^\perp$, and write

$$h = a\eta + h_\perp, \qquad h_\perp \in T_\eta \mathbb{S}^{n-2}.$$

Since

$$b(z) = |z|, \qquad \eta(z) = \frac{z}{|z|},$$

we have

$$Db(z)[h] = a, \qquad D\eta(z)[h] = \frac{1}{b}\, h_\perp.$$

Hence

$$D_z F_\omega[h] = a\Theta_\lambda'(b)\big(-\sin \Theta_\lambda(b)\, \omega + \cos \Theta_\lambda(b)\, \eta\big) + \frac{\sin \Theta_\lambda(b)}{b}\, h_\perp. \tag{3.70}$$

By Lemma 3.3,

$$0 < \Theta_\lambda(b) < \pi, \qquad \Theta_\lambda'(b) < 0. \tag{3.71}$$

In particular,

$$\Theta_\lambda'(b) \neq 0, \qquad \frac{\sin \Theta_\lambda(b)}{b} \neq 0. \tag{3.72}$$

Moreover,

$$-\sin \Theta_\lambda(b)\, \omega + \cos \Theta_\lambda(b)\, \eta \perp T_\eta \mathbb{S}^{n-2}. \tag{3.73}$$

It follows from (3.70) that

$$D_z F_\omega[h] = 0 \quad \Longrightarrow \quad a = 0, \qquad h_\perp = 0. \tag{3.74}$$

Hence D_zF_ω is injective. Since

$$\dim T_z\Lambda_\omega = \dim T_\theta\mathbb{S}^{n-1} = n-1,$$

it follows that D_zF_ω is invertible and therefore

$$\sigma(z;\lambda) \neq 0.$$

Since $z \in F_\omega^{-1}(\theta)$ was arbitrary, θ is regular.

It remains to prove that the preimage is unique. Let $z = b\eta \in F_\omega^{-1}(\theta)$. Taking the scalar product of (3.68) with ω, we obtain

$$\cos\Theta_\lambda(b) = \cos\Theta. \tag{3.75}$$

Since both angles belong to $(0,\pi)$,

$$\Theta_\lambda(b) = \Theta.$$

By Lemma 3.3, the impact parameter b is uniquely determined. Moreover, from (3.68),

$$\eta = \frac{\theta - (\omega\cdot\theta)\omega}{\sin\Theta},$$

so η is uniquely determined as well. Hence $z = b\eta$ is the unique preimage of θ under F_ω. Therefore $\ell(\theta,\lambda) = 1$. □

For $\theta \in \mathbb{S}^{n-1}\setminus\{\omega,-\omega\}$, let $w(\theta;\lambda)$ denote the unique element of $F_\omega^{-1}(\theta)$. By Lemma 3.4 and the Robert–Tamura formula,

$$|f(\omega\to\theta;\lambda,h)|^2 = \sigma(w(\theta;\lambda);\lambda)^{-1} + O(h), \qquad h\to 0. \tag{3.76}$$

Since, in the one-branch case, $\sigma(w(\theta;\lambda);\lambda)^{-1}$ is precisely the classical differential cross section, we obtain

$$|f(\omega\to\theta;\lambda,h)|^2 \longrightarrow \frac{d\sigma_{\mathrm{cl}}}{d\Omega}(\theta;\lambda), \qquad h\to 0. \tag{3.77}$$

This is the semiclassical information that will be used below to recover the radial potential.

3.3. **Uniqueness from the differential cross section.** Let $V_j(x) = v_j(|x|)$, $j = 1,2$, be two radial short-range potentials satisfying the assumptions above, and let $r_{j,\lambda}$ be the unique radius defined by

$$v_j(r_{j,\lambda}) = \lambda. \tag{3.78}$$

We denote by $f_{j,h}(\omega,\theta;\lambda)$ the corresponding semiclassical scattering amplitudes. Assume that

$$|f_{1,h}(\omega,\theta;\lambda)|^2 - |f_{2,h}(\omega,\theta;\lambda)|^2 \longrightarrow 0, \qquad h\longrightarrow 0, \tag{3.79}$$

for every regular scattering angle.

By the Robert–Tamura asymptotics and the one-trajectory property established above,

$$|f_{j,h}(\omega,\theta;\lambda)|^2 \longrightarrow \frac{d\sigma_{\mathrm{cl},j}}{d\Omega}(\theta;\lambda), \qquad j = 1,2. \tag{3.80}$$

Consequently, (3.79) implies

$$\frac{d\sigma_{\mathrm{cl},1}}{d\Omega}(\theta;\lambda) = \frac{d\sigma_{\mathrm{cl},2}}{d\Omega}(\theta;\lambda) \tag{3.81}$$

for every regular scattering angle.

We first recover the deflection function from the classical differential cross section. By Lemma 3.3, the deflection function $b\mapsto\Theta_\lambda(b)$ is strictly decreasing from π to 0. Its inverse will be denoted by

$$b = b(\Theta), \qquad 0<\Theta<\pi. \tag{3.82}$$

The classical differential cross section is given by (see, for instance, Keller–Kay–Shmoys [10] for $n = 3$),

$$\frac{d\sigma_{\rm cl}}{d\Omega}(\Theta;\lambda) = \frac{b(\Theta)^{n-2}}{\sin^{n-2}\Theta}\left|\frac{db}{d\Theta}(\Theta)\right|. \tag{3.83}$$

Since $b(\Theta)$ is strictly decreasing, (3.83) yields

$$\frac{d}{d\Theta}b(\Theta)^{n-1} = -(n-1)\sin^{n-2}\Theta\,\frac{d\sigma_{\rm cl}}{d\Omega}(\Theta;\lambda). \tag{3.84}$$

Moreover, Lemma 3.3 gives

$$b(\Theta) \longrightarrow 0 \qquad \text{as } \Theta \to \pi^-. \tag{3.85}$$

Integrating (3.84) from Θ to π, we obtain

$$b(\Theta)^{n-1} = (n-1)\int_\Theta^\pi \sin^{n-2}\alpha\,\frac{d\sigma_{\rm cl}}{d\Omega}(\alpha;\lambda)\,d\alpha. \tag{3.86}$$

Thus the classical differential cross section uniquely determines $b(\Theta)$, and hence its inverse $\Theta_\lambda(b)$. It follows from (3.81) that

$$\Theta_{1,\lambda}(b) = \Theta_{2,\lambda}(b), \qquad b > 0. \tag{3.87}$$

It remains to recover the potential from the deflection function. For this purpose, we use the classical Firsov–Abel inversion procedure of Firsov [4] and Keller–Kay–Shmoys [10]; see also Jollivet [9] for a related treatment of short-range radial potentials. We briefly recall the argument in our notation. By (3.46),

$$\Theta_\lambda(b) = -2b\int_b^{+\infty}\frac{q_\lambda(\rho)}{\sqrt{\rho^2-b^2}}\,d\rho, \tag{3.88}$$

where

$$q_\lambda(\rho) = \frac{d}{d\rho}\log\frac{r_\lambda(\rho)}{\rho}. \tag{3.89}$$

Set

$$F_\lambda(\rho) = \log\frac{r_\lambda(\rho)}{\rho}. \tag{3.90}$$

Then $q_\lambda = F_\lambda'$, and (3.88) becomes

$$\frac{\Theta_\lambda(b)}{b} = -2\int_b^{+\infty}\frac{F_\lambda'(\rho)}{\sqrt{\rho^2-b^2}}\,d\rho. \tag{3.91}$$

After the quadratic change of variables $x = b^2$, $y = \rho^2$, this reduces to the standard Abel transform. Applying the classical Abel inversion formula and using

$$F_\lambda(\rho) = O(\rho^{-\mu}), \qquad \rho \to +\infty, \tag{3.92}$$

we obtain

$$F_\lambda(\rho) = \frac{1}{\pi}\int_\rho^{+\infty}\frac{\Theta_\lambda(b)}{\sqrt{b^2-\rho^2}}\,db, \qquad \rho > 0. \tag{3.93}$$

Recalling the definition of F_λ, we obtain

$$r_\lambda(\rho) = \rho\exp\left(\frac{1}{\pi}\int_\rho^{+\infty}\frac{\Theta_\lambda(b)}{\sqrt{b^2-\rho^2}}\,db\right), \qquad \rho > 0. \tag{3.94}$$

Applying (3.94) to V_1 and V_2 and using (3.87), we obtain

$$r_{1,\lambda}(\rho) = r_{2,\lambda}(\rho), \qquad \rho > 0. \tag{3.95}$$

Since the functions $r_{1,\lambda}$ and $r_{2,\lambda}$ coincide on $(0,+\infty)$, their ranges coincide. Since

$$r_{j,\lambda}\big((0,+\infty)\big) = (r_{j,\lambda},+\infty), \tag{3.96}$$

their left endpoints coincide, and hence

$$r_{1,\lambda} = r_{2,\lambda} =: r_\lambda. \tag{3.97}$$

Taking inverse functions in (3.95), we then obtain

$$\rho_{1,\lambda}(r) = \rho_{2,\lambda}(r), \qquad r > r_\lambda. \tag{3.98}$$

By continuity, the equality also holds at $r = r_\lambda$. Finally, from

$$v_j(r) = \lambda\left(1 - \frac{\rho_{j,\lambda}(r)^2}{r^2}\right), \qquad r \geq r_\lambda, \tag{3.99}$$

we conclude that

$$v_1(r) = v_2(r), \qquad r \geq r_\lambda. \tag{3.100}$$

Equivalently,

$$V_1(x) = V_2(x), \qquad |x| \geq r_\lambda. \tag{3.101}$$

Thus the semiclassical differential cross section at the fixed energy λ determines the radial potential throughout the classically accessible region.

4. An extension to simple metric perturbations

We conclude with an extension of the preceding argument to compactly supported perturbations of the Euclidean metric. The proof follows the same general strategy as in the potential case, and we therefore only give its main steps.

Let g be a smooth Riemannian metric on $\mathbb{R}^n$, $n \geq 3$, such that

$$g = g_0 \qquad \text{outside a compact set,} \tag{4.1}$$

where g_0 denotes the Euclidean metric. We consider the semiclassical Laplace–Beltrami operator

$$P_g(h) = -\frac{h^2}{2}\Delta_g \tag{4.2}$$

and denote by $S_{g,h}(\lambda)$ its scattering matrix at the fixed energy $\lambda > 0$, with the Euclidean Laplacian as the free reference operator.

Assume that the geodesic flow of g is non-trapping. Since g is Euclidean outside a compact set, every geodesic has incoming and outgoing Euclidean asymptotic data. As in the potential case, these data define the classical scattering map

$$\kappa_{g,\lambda} : T^*\mathbb{S}^{n-1} \longrightarrow T^*\mathbb{S}^{n-1}. \tag{4.3}$$

Although Ingremeau formulates his Gaussian-state propagation result for potential scattering,

he points out that the same result holds for compactly supported metric perturbations of the Laplacian, with a similar proof [6]. Consequently, the proof of Lemma 2.1 applies in the metric setting as well.

Lemma 4.1. *Let g_1 and g_2 be smooth Riemannian metrics on $\mathbb{R}^n$ which coincide with the Euclidean metric outside a compact set, and assume that their geodesic flows are non-trapping. If*

$$\|S_{g_1,h}(\lambda) - S_{g_2,h}(\lambda)\|_{\mathcal{B}(L^2(\mathbb{S}^{n-1}))} = o(1), \qquad h \to 0, \tag{4.4}$$

then

$$\kappa_{g_1,\lambda} = \kappa_{g_2,\lambda}. \tag{4.5}$$

We now restrict ourselves to the simple case. Recall that a compact Riemannian manifold (M, g) with boundary is called simple if its boundary is strictly convex, it is non-trapping, and it has no conjugate points; see, for instance, [17, p. xi]. In this case, any two points of M are joined by a unique geodesic, which is minimizing. Let $R > 0$ be sufficiently large so that

$$\operatorname{supp}(g_j - g_0) \Subset B(0, R), \qquad j = 1, 2,$$

and assume that $(B(0, R), g_j)$ is simple for $j = 1, 2$. Since the geodesics are straight lines outside $B(0, R)$, their Euclidean asymptotic data determine their entry and exit points and directions on $\partial B(0, R)$. It follows that equality of the classical scattering maps implies equality of the corresponding scattering relations,

$$L_{g_1} = L_{g_2} \qquad \text{on } \partial_- SB(0, R). \tag{4.6}$$

For simple manifolds, the scattering relation determines the boundary distance function by the classical argument of Michel [11]. Consequently, (4.6) implies

$$d_{g_1}(x, y) = d_{g_2}(x, y), \qquad x, y \in \partial B(0, R). \tag{4.7}$$

More generally, scattering rigidity for simple Riemannian metrics is also covered by the framework of simple magnetic-potential systems developed by Muñoz-Thon [12]. In the purely Riemannian case, his results include scattering rigidity for metrics in a fixed conformal class, as well as for real-analytic metrics. His approach relies on the reduction to simple magnetic systems and on the corresponding rigidity results of Dairbekov, Paternain, Stefanov and Uhlmann [3]. Here we follow a different route and use the boundary rigidity theorem of Stefanov–Uhlmann–Vasy [23]. This allows us to treat smooth simple metrics, without conformality or analyticity assumptions, under the geometric conditions stated below.

Theorem 4.2. *Let g_1 and g_2 be smooth Riemannian metrics on $\mathbb{R}^n$, $n \geq 3$. Assume that there exists $R > 0$ such that*

$$\operatorname{supp}(g_j - g_0) \Subset B(0, R), \qquad j = 1, 2, \tag{4.8}$$

and that $(B(0, R), g_j)$ is simple for $j = 1, 2$. Assume moreover that $(B(0, R), g_1)$ satisfies one of the following conditions:

(1) the sectional curvature is non-positive;
(2) there are no focal points;
(3) the sectional curvature is non-negative.

If

$$\|S_{g_1,h}(\lambda) - S_{g_2,h}(\lambda)\|_{\mathcal{B}(L^2(\mathbb{S}^{n-1}))} = o(1), \qquad h \to 0, \tag{4.9}$$

then there exists a diffeomorphism

$$\psi : B(0, R) \longrightarrow B(0, R), \qquad \psi|_{\partial B(0,R)} = \mathrm{Id}, \tag{4.10}$$

such that

$$g_1 = \psi^* g_2. \tag{4.11}$$

Proof. Since $(B(0, R), g_j)$ is simple and $g_j = g_0$ outside $B(0, R)$, the geodesic flow of g_j is non-trapping, $j = 1, 2$. Hence Lemma 4.1 applies. Together with the discussion above, (4.9) implies

$$d_{g_1}(x, y) = d_{g_2}(x, y), \qquad x, y \in \partial B(0, R).$$

The conclusion then follows directly from Corollary 1.2 of Stefanov–Uhlmann–Vasy [23]. □

In the particular case where one of the metrics is Euclidean, the additional geometric assumptions in Theorem 4.2 are no longer needed, since the Euclidean metric satisfies the hypotheses of Corollary 1.2 in [23]. We thus obtain the following rigidity result.

Corollary 4.3. *Let g be a smooth Riemannian metric on $\mathbb{R}^n$, $n \geq 3$. Assume that there exists $R > 0$ such that*

$$\operatorname{supp}(g - g_0) \Subset B(0, R), \tag{4.12}$$

and that $(B(0, R), g)$ is simple. If

$$\|S_{g,h}(\lambda) - S_{g_0,h}(\lambda)\|_{\mathcal{B}(L^2(\mathbb{S}^{n-1}))} = o(1), \qquad h \to 0, \tag{4.13}$$

then there exists a diffeomorphism

$$\psi : B(0, R) \longrightarrow B(0, R), \qquad \psi|_{\partial B(0,R)} = \mathrm{Id}, \tag{4.14}$$

such that

$$g = \psi^* g_0. \tag{4.15}$$

Proof. This is an immediate consequence of the preceding theorem, applied with $g_1 = g$ and $g_2 = g_0$. □

Remark 4.4. The simplicity assumption is used above precisely to recover the boundary distance function from the scattering relation. For non-simple metrics admitting a strictly convex foliation, the global rigidity theorem of Stefanov–Uhlmann–Vasy [23] is formulated in terms of the full lens data. Whether the semiclassical scattering matrix alone determines the additional travel-time information in this more general setting remains an interesting question, which we plan to investigate in future work.

AI usage disclosure

The author used ChatGPT (OpenAI, GPT-5.6) to improve the English and the presentation of the manuscript. The author takes full responsibility for its mathematical content.

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(F. Nicoleau) Laboratoire de Mathématiques Jean Leray, CNRS, Nantes Université, 44000 Nantes, France.
Email address: francois.nicoleau@univ-nantes.fr