

INVERSE SEMICLASSICAL SCATTERING AT FIXED ENERGY

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Dedicated to Didier Robert, for his major contributions to semiclassical analysis, scattering theory, and coherent states.

ABSTRACT. We investigate inverse scattering at a fixed energy for semiclassical Schrödinger operators with smooth potentials. For compactly supported potentials satisfying a natural virial condition, we prove that $\liminf_{h \rightarrow 0} \|S_1(\lambda, h) - S_2(\lambda, h)\| < \sqrt{2}$ implies $V_1 = V_2$, provided the fixed energy λ lies above both potentials. We then consider radial short-range repulsive potentials. Under a monotonicity assumption on the radial force, we show that the semiclassical differential cross section at a single fixed energy, up to $o(1)$ as $h \rightarrow 0$, determines the potential throughout the classically accessible region. Finally, we obtain an analogous rigidity result for nontrapping compactly supported perturbations of the Euclidean metric under a strict convexity assumption.

1. INTRODUCTION

The purpose of this paper is to study the inverse problem for semiclassical scattering at a fixed energy. More precisely, we investigate to what extent the scattering data in the semiclassical limit determine the underlying potential. Let

$$(1.1) \quad H(h) = -\frac{h^2}{2}\Delta + V(x), \quad 0 < h \leq 1,$$

be the semiclassical Schrödinger operator in $L^2(\mathbb{R}^n)$, $n \geq 3$. Throughout the paper, V is real-valued and smooth. We shall work with short-range potentials satisfying, for some $\mu > 1$,

$$(1.2) \quad |\partial_x^\alpha V(x)| \leq C_\alpha \langle x \rangle^{-\mu-|\alpha|}, \quad \alpha \in \mathbb{N}^n, \quad \langle x \rangle = (1 + |x|^2)^{1/2}.$$

For such potentials, trajectories escaping to infinity have free asymptotics, while trapped trajectories may still occur at a given energy.

We denote by

$$(1.3) \quad H_0(h) = -\frac{h^2}{2}\Delta$$

the free Hamiltonian associated with $H(h)$, and by

$$(1.4) \quad R(z; h) = (H(h) - z)^{-1}, \quad z \in \mathbb{C} \setminus \sigma(H(h)),$$

the resolvent of $H(h)$. Let $\mathcal{F}_h$ denote the semiclassical Fourier transform, defined for $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$(\mathcal{F}_h f)(\xi) = (2\pi h)^{-n/2} \int_{\mathbb{R}^n} e^{-ix \cdot \xi/h} f(x) dx.$$

For $\lambda > 0$, the Fourier restriction operator at energy λ is given by

$$F_0(\lambda, h)f(\omega) = (2\lambda)^{(n-2)/4}(\mathcal{F}_h f)(\sqrt{2\lambda}\omega), \quad \omega \in \mathbb{S}^{n-1}.$$

The scattering matrix

$$S(\lambda, h) : L^2(\mathbb{S}^{n-1}) \longrightarrow L^2(\mathbb{S}^{n-1})$$

is unitary and can be written as

$$(1.5) \quad S(\lambda, h) = \text{Id} - 2\pi i T(\lambda, h),$$

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where

$$(1.6) \quad T(\lambda, h) = F_0(\lambda, h)(V - VR(\lambda + i0; h)V)F_0(\lambda, h)^*.$$

Here

$$R(\lambda + i0; h) = s - \lim_{\varepsilon \downarrow 0} (H(h) - \lambda - i\varepsilon)^{-1}$$

denotes the outgoing boundary value of the resolvent, where the strong limit is taken in

$$\mathcal{B}(L_s^2(\mathbb{R}^n), L_{-s}^2(\mathbb{R}^n)), \quad s > \frac{1}{2},$$

with

$$(1.7) \quad L_s^2(\mathbb{R}^n) = \{u \in L_{\text{loc}}^2(\mathbb{R}^n); \langle x \rangle^s u \in L^2(\mathbb{R}^n)\}, \quad \langle x \rangle = (1 + |x|^2)^{1/2}.$$

Away from the diagonal, $T(\lambda, h)$ has a smooth kernel, denoted by

$$T(\theta, \omega; \lambda, h), \quad \theta, \omega \in \mathbb{S}^{n-1}, \quad \theta \neq \omega.$$

The corresponding scattering amplitude is defined by Kuroda's formula

$$(1.8) \quad f(\omega \rightarrow \theta; \lambda, h) = c(\lambda, h) T(\theta, \omega; \lambda, h),$$

where

$$(1.9) \quad c(\lambda, h) = -(2\pi)^{(n+1)/2} (2\lambda)^{-(n-1)/4} h^{(n-1)/2} \exp\left(-i \frac{(n-3)\pi}{4}\right).$$

For the standard scattering theory recalled above, we refer to Isozaki–Kitada [9, 10], Reed–Simon [21], and Yafaev [29].

Our main inverse problem is to determine to what extent the condition

$$(1.10) \quad \liminf_{h \rightarrow 0} \|S_1(\lambda, h) - S_2(\lambda, h)\|_{\mathcal{B}(L^2(\mathbb{S}^{n-1}))} < \sqrt{2},$$

at a fixed energy $\lambda > 0$, determines the potential.

There is a natural limitation to such a fixed-energy inverse problem. In [17], it was proved that, for two short-range potentials V_1 and V_2 , at a fixed non-trapping energy $\lambda > 0$ (defined below),

$$(1.11) \quad (\lambda - V_1)_+ = (\lambda - V_2)_+ \implies S_1(\lambda, h) - S_2(\lambda, h) = \mathcal{O}(h^\infty)$$

in the operator norm of $\mathcal{B}(L^2(\mathbb{S}^{n-1}))$ as $h \rightarrow 0$, where $(a)_+ := \max\{a, 0\}$. Thus, at a fixed energy, one cannot in general expect the semiclassical scattering data to determine the potential in the classically forbidden region $\{V \geq \lambda\}$. The natural inverse problem is therefore to recover V in the classically allowed region $\{V < \lambda\}$, or, equivalently, to recover the function $(\lambda - V)_+$.

The semiclassical direct problem has a precise classical interpretation. Semiclassical asymptotics for scattering amplitudes were first obtained by Protas [19] and Vainberg [27] for compactly supported potentials. Yajima [30] subsequently treated non-compactly supported short-range potentials, obtaining an asymptotic expansion in an L^2 sense with respect to the energy and angular variables. Robert and Tamura [24] then obtained, for short-range potentials, a precise fixed-energy description at a non-trapping energy, expressing the scattering amplitude, for regular outgoing directions, as a finite sum of contributions associated with the underlying classical scattering trajectories. Further microlocal descriptions of the semiclassical scattering matrix were obtained by Alexandrova [1], as we shall discuss in more detail below.

We recall the classical objects entering the semiclassical scattering asymptotics, using a notation adapted to the present paper. For the standard semiclassical framework and its relation with the underlying classical Hamiltonian dynamics, we refer, for instance, to Robert [22]. The Hamiltonian associated with (1.1) is

$$(1.12) \quad p(x, \xi) = \frac{|\xi|^2}{2} + V(x),$$

and the corresponding classical trajectories $t \mapsto (q_{\text{cl}}(t), p_{\text{cl}}(t))$ are the integral curves of the Hamiltonian vector field, that is,

$$(1.13) \quad \dot{q}_{\text{cl}}(t) = p_{\text{cl}}(t), \quad \dot{p}_{\text{cl}}(t) = -\nabla V(q_{\text{cl}}(t)).$$

At the energy $\lambda > 0$, the energy surface is

$$(1.14) \quad \Sigma_\lambda = \left\{ (x, \xi) \in T^*\mathbb{R}^n; \frac{|\xi|^2}{2} + V(x) = \lambda \right\}.$$

We recall that $\lambda > 0$ is a non-trapping energy if, for every $R > 1$ sufficiently large, there exists $T = T(R) > 0$ such that

$$(1.15) \quad |q_{\text{cl}}(t; y, \eta)| > R, \quad |t| > T,$$

whenever $|y| < R$ and $(y, \eta) \in \Sigma_\lambda$, where $t \mapsto (q_{\text{cl}}(t; y, \eta), p_{\text{cl}}(t; y, \eta))$ denotes the solution of (1.13) with initial data (y, η) at $t = 0$.

Under the non-trapping assumption and (1.2), every classical trajectory $(q_{\text{cl}}(t), p_{\text{cl}}(t))$ on the energy surface Σ_λ is a scattering trajectory and admits asymptotic data $(r_\pm, v_\pm)$ such that

$$(1.16) \quad q_{\text{cl}}(t) = v_\pm t + r_\pm + o(1), \quad p_{\text{cl}}(t) = v_\pm + o(1), \quad t \rightarrow \pm\infty.$$

Since the energy is conserved and $V(q_{\text{cl}}(t)) \rightarrow 0$ as $t \rightarrow \pm\infty$, one has $|v_\pm| = \sqrt{2\lambda}$. The map

$$(1.17) \quad \mathcal{S}_{\text{cl}} : (r_-, v_-) \mapsto (r_+, v_+)$$

is the classical scattering transformation.

For later use, we describe the classical scattering transformation in the coordinates naturally associated with the semiclassical scattering matrix. Fix an incoming direction $\omega \in \mathbb{S}^{n-1}$ and set

$$(1.18) \quad \Lambda_\omega = \{z \in \mathbb{R}^n; z \cdot \omega = 0\}.$$

For $z \in \Lambda_\omega$, let

$$(q_\infty(t; z, \lambda), p_\infty(t; z, \lambda))$$

be the scattering trajectory satisfying

$$(1.19) \quad q_\infty(t; z, \lambda) = \sqrt{2\lambda}\omega t + z + o(1), \quad p_\infty(t; z, \lambda) = \sqrt{2\lambda}\omega + o(1), \quad t \rightarrow -\infty.$$

As $t \rightarrow +\infty$, there exist smooth functions $\xi_\infty(z; \lambda) \in \mathbb{S}^{n-1}$ and $r_\infty(z; \lambda) \in \mathbb{R}^n$ such that

$$(1.20) \quad \begin{aligned} q_\infty(t; z, \lambda) &= \sqrt{2\lambda}\xi_\infty(z; \lambda)t + r_\infty(z; \lambda) + o(1), \\ p_\infty(t; z, \lambda) &= \sqrt{2\lambda}\xi_\infty(z; \lambda) + o(1). \end{aligned} \quad t \rightarrow +\infty.$$

For $\theta \in \mathbb{S}^{n-1}$, let

$$\Pi_\theta x = x - (x \cdot \theta)\theta$$

denote the orthogonal projection onto $\theta^\perp$. The classical scattering map at energy λ is then given by

$$(1.21) \quad \kappa_\lambda(\omega, z) = (\xi_\infty(z; \lambda), \Pi_{\xi_\infty(z; \lambda)} r_\infty(z; \lambda)).$$

Figure 1 summarizes the geometry. For a genuinely short-range potential, the interaction is not compactly supported; the circle only represents the region where it is significant. The parameter z labels the incoming line, while $\xi_\infty(z; \lambda)$ is the outgoing direction.

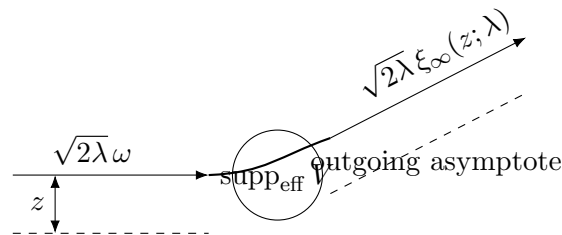


FIGURE 1. A classical scattering trajectory at the fixed energy λ .

We first consider compactly supported potentials. At a fixed non-trapping energy, the condition $\liminf_{h \rightarrow 0} \|S_1(\lambda, h) - S_2(\lambda, h)\|_{\mathcal{B}(L^2(\mathbb{S}^{n-1}))} < \sqrt{2}$ already implies equality of the classical scattering maps. Combined with a classical rigidity result, this yields our first uniqueness theorem.

Theorem 1.1. *Let $V_1, V_2 \in C_c^\infty(\mathbb{R}^n)$, $n \geq 3$, and let $\lambda > 0$ satisfy*

$$(1.22) \quad \lambda > \sup_{\mathbb{R}^n} V_j, \quad j = 1, 2.$$

Assume moreover the virial condition

$$(1.23) \quad 2(\lambda - V_j(x)) - x \cdot \nabla V_j(x) > 0, \quad x \in \mathbb{R}^n, \quad j = 1, 2.$$

If

$$(1.24) \quad \liminf_{h \rightarrow 0} \|S_1(\lambda, h) - S_2(\lambda, h)\|_{\mathcal{B}(L^2(\mathbb{S}^{n-1}))} < \sqrt{2},$$

then

$$(1.25) \quad V_1 = V_2.$$

Let us briefly explain the main ingredients of the proof. A fundamental result of Alexandrova [1] shows, for short-range potentials and microlocally away from the diagonal, that the semiclassical scattering matrix is a semiclassical Fourier integral operator associated with the classical scattering relation. In particular, her work provides a precise microlocal link between the quantum scattering matrix and the underlying classical scattering dynamics.

Here we use instead a particularly convenient description due to Ingremau [8]. For compactly supported potentials at a non-trapping energy, his result describes explicitly the action of the scattering matrix on Gaussian states localized at points of $T^*\mathbb{S}^{n-1}$. The asymptotic orthogonality of Gaussian states localized at distinct phase-space points then yields a quantitative threshold: it is enough that

$$(1.26) \quad \liminf_{h \rightarrow 0} \|S_1(\lambda, h) - S_2(\lambda, h)\| < \sqrt{2}$$

to recover the classical scattering map.

We now recall the precise form of Ingremau's result that we shall use. His result is stated at the fixed energy $1/2$. By a harmless rescaling, we may reduce our problem at energy $\lambda > 0$ to this setting. If $\phi_{\rho, h}$ is a Gaussian state localized near $\rho \in T^*\mathbb{S}^{n-1}$, then, for every $N \in \mathbb{N}$,

$$(1.27) \quad S(\lambda, h)\phi_{\rho, h} = e^{i\delta/h} \phi_{\kappa_\lambda(\rho), h}^{(N)} + \mathcal{O}(h^N),$$

where $\delta \in \mathbb{R}$ and $\phi_{\kappa_\lambda(\rho), h}^{(N)}$ is an outgoing Gaussian state, with an amplitude admitting a finite semiclassical expansion, localized near the phase point $\kappa_\lambda(\rho)$.

The essential point for our purposes is the transport of the center of the Gaussian state from ρ to $\kappa_\lambda(\rho)$. We show that this implies

$$(1.28) \quad \liminf_{h \rightarrow 0} \|S_1(\lambda, h) - S_2(\lambda, h)\|_{\mathcal{B}(L^2(\mathbb{S}^{n-1}))} < \sqrt{2} \implies \kappa_{1, \lambda} = \kappa_{2, \lambda}.$$

Indeed, Gaussian states localized at distinct phase-space points are asymptotically orthogonal. Since the scattering matrices are unitary, their images of the same normalized incoming Gaussian state would then have distance asymptotic to $\sqrt{2}$, which yields (1.28).

It remains to pass from the classical scattering map to the potential. Following the approach used by Alexandrova [1], we interpret the classical trajectories in terms of the associated Jacobi metric. Under (1.22), trajectories at energy λ , up to reparametrization, are geodesics of the Jacobi metrics

$$(1.29) \quad g_j = 2(\lambda - V_j)g_0, \quad j = 1, 2.$$

The virial condition (1.23) implies that λ is a non-trapping energy and that the Euclidean spheres form a strictly convex foliation for the Jacobi metrics (1.29). The former property allows us to apply Ingremau's result [8], while the latter allows us to use the rigidity theorem

of Stefanov–Uhlmann–Vasy [25] and conclude that $g_1 = g_2$, hence $V_1 = V_2$. We also mention that related fixed-energy scattering rigidity results for simple MP-systems were obtained by Muñoz-Thon [14]. In the purely potential case considered here, simplicity amounts to simplicity of the associated Jacobi metric, and his result yields uniqueness of the potential when the background metric is fixed. By contrast, our result does not require the Jacobi metric to be simple; the simplicity assumption is replaced by the convex foliation condition provided by the virial hypothesis.

We next turn to radial short-range potentials. Here the inverse problem has a different character. In particular, the energy is allowed to lie below the maximum of the potential and, as explained above, one can only expect to recover the potential in the classically accessible region.

Let

$$(1.30) \quad V_j(x) = v_j(|x|), \quad j = 1, 2,$$

be smooth radial short-range potentials satisfying

$$(1.31) \quad v'_j(r) < 0, \quad r > 0, \quad v_j(0) > \lambda.$$

The assumption $v_j(0) > \lambda$, together with the strict monotonicity of v_j , ensures the existence of a unique positive radial turning point $r_{j,\lambda} > 0$ defined by

$$(1.32) \quad v_j(r_{j,\lambda}) = \lambda.$$

The regions $r < r_{j,\lambda}$ and $r > r_{j,\lambda}$ are respectively classically forbidden and classically accessible at the energy λ . We shall further assume the monotonicity condition

$$(1.33) \quad (rv'_j(r))' \geq 0, \quad r > r_{j,\lambda}, \quad j = 1, 2.$$

In terms of the radial force $F_j(r) = -v'_j(r) > 0$, this means that $rF_j(r)$ is non-increasing in the classically accessible region. Notice also that the virial condition is automatic in this region:

$$(1.34) \quad 2(\lambda - v_j(r)) - rv'_j(r) > 0, \quad r > r_{j,\lambda}.$$

Since the potentials are radial, the scattering amplitude is invariant under simultaneous rotations of the incoming and outgoing directions. More precisely, for every $R \in SO(n)$, the scattering amplitudes satisfy

$$(1.35) \quad f_j(R\omega \rightarrow R\theta; \lambda, h) = f_j(\omega \rightarrow \theta; \lambda, h), \quad j = 1, 2.$$

The corresponding semiclassical differential cross section is defined by

$$(1.36) \quad \frac{d\sigma_{j,h}}{d\Omega}(\omega, \theta; \lambda) = |f_j(\omega \rightarrow \theta; \lambda, h)|^2.$$

By rotational invariance, both the scattering amplitude and the differential cross section depend only on the scattering angle between ω and θ . We may therefore fix an incoming direction $\omega \in \mathbb{S}^{n-1}$ without loss of information.

Our second result shows that, in the radial setting, the differential cross section alone is sufficient to recover the potential in the classically accessible region.

Theorem 1.2. *Let $V_j(x) = v_j(|x|)$, $j = 1, 2$, satisfy the short-range condition (1.2) and assumptions (1.31)–(1.33). Fix an incoming direction $\omega \in \mathbb{S}^{n-1}$ and assume that*

$$(1.37) \quad \frac{d\sigma_{1,h}}{d\Omega}(\omega, \theta; \lambda) - \frac{d\sigma_{2,h}}{d\Omega}(\omega, \theta; \lambda) \longrightarrow 0, \quad h \longrightarrow 0,$$

for every $\theta \in \mathbb{S}^{n-1} \setminus \{\omega, -\omega\}$. Then

$$(1.38) \quad r_{1,\lambda} = r_{2,\lambda} =: r_\lambda,$$

and

$$(1.39) \quad V_1(x) = V_2(x), \quad |x| \geq r_\lambda.$$

Let us explain the main ingredients of the proof. The starting point is the fixed-energy semiclassical asymptotic formula of Robert and Tamura [24]. For a regular pair of incoming and outgoing directions, the scattering amplitude has the form

$$(1.40) \quad f(\omega \rightarrow \theta; \lambda, h) = \sum_{j=1}^{\ell(\theta, \lambda)} \sigma(w_j; \lambda)^{-1/2} \exp \left(\frac{i}{h} \mathcal{A}_j(\omega, \theta; \lambda) - \frac{i\pi}{2} \mu_j \right) (1 + \mathcal{O}(h)),$$

where the sum runs over the classical scattering trajectories connecting the incoming direction ω to the outgoing direction θ . Here $\sigma(w_j; \lambda)$ is the corresponding angular density, $\mathcal{A}_j$ the classical action, and μ_j the Maslov index. The notions of regular direction and classical branch, as well as the precise definitions of the quantities entering (1.40), are recalled in Section 3.

For general potentials, several classical trajectories may connect the same incoming and outgoing directions. The corresponding contributions in (1.40) may then interfere, so that the semiclassical differential cross section does not directly reduce to the classical differential cross section of a single trajectory. In the radial setting considered here, however, the classical dynamics can be described in terms of the deflection function $b \mapsto \Theta_\lambda(b)$, which associates with each impact parameter $b > 0$ the corresponding scattering angle. Its precise definition is given in Section 3. Under condition (1.33), we shall prove that Θ_λ is strictly decreasing from π to 0.

For incoming and outgoing directions $\omega, \theta \in \mathbb{S}^{n-1}$, we denote their scattering angle by

$$(1.41) \quad \Theta = \arccos(\omega \cdot \theta) \in [0, \pi].$$

Thus a trajectory with impact parameter b and outgoing direction θ satisfies

$$\Theta = \Theta_\lambda(b).$$

The strict monotonicity of Θ_λ therefore implies that, for every $\Theta \in (0, \pi)$, there is a unique impact parameter b . For fixed incoming and outgoing directions ω and θ with this scattering angle, the direction of the impact vector is also uniquely determined, yielding a unique classical trajectory. Moreover, as shown in Section 3, the corresponding outgoing directions are regular. The Robert–Tamura formula thus applies with $\ell(\theta, \lambda) = 1$. In particular, there are no interference terms, and since $\sigma(w; \lambda)^{-1}$ is precisely the classical differential cross section associated with the unique trajectory, we obtain

$$(1.42) \quad |f(\omega \rightarrow \theta; \lambda, h)|^2 \longrightarrow \frac{d\sigma_{\text{cl}}}{d\Omega}(\Theta; \lambda), \quad 0 < \Theta < \pi.$$

The classical differential cross section determines the inverse function $b = b(\Theta)$ of the deflection function $b \mapsto \Theta_\lambda(b)$, and hence determines the deflection function itself. A classical Firsov–Abel inversion [4, 12] then recovers the radial potential throughout the classically accessible region. This proves Theorem 1.2.

We also show that the same strategy extends to nontrapping compactly supported metric perturbations of the Euclidean Laplacian. The same Gaussian-state argument shows that the classical scattering relation is determined under the threshold condition introduced above. Since the metrics are Euclidean near the boundary of a sufficiently large ball, a first variation argument also gives the lengths of the corresponding geodesics. Combining these lens data with the rigidity theorem of Stefanov–Uhlmann–Vasy [26, Theorem 8.1], we obtain uniqueness of the metric up to a boundary-fixing isometry, assuming that one of the metrics admits a smooth strictly convex function whose zero set is the boundary of the ball. No simplicity assumption is needed. Related scattering rigidity results for simple Riemannian metrics, including metrics in a fixed conformal class and real-analytic metrics, were obtained by Muñoz-Thon in the more general framework of magnetic-potential systems [14].

Let us conclude by mentioning a possible extension of the radial result to short-range potentials which are radial only near infinity. At sufficiently large energy, classical inverse scattering results of Jollivet [11] provide uniqueness from suitable classical scattering data. Extending the semiclassical recovery of the classical scattering data to this setting would therefore lead to a corresponding semiclassical inverse result. We do not pursue this question here.

The paper is organized as follows. In Section 2, we recover the classical scattering map from the semiclassical scattering matrix for compactly supported potentials, using Ingremeau's propagation of Gaussian states, and then deduce uniqueness of the potential from the rigidity theorem of Stefanov–Uhlmann–Vasy. Section 3 is devoted to radial short-range potentials. After studying the classical radial dynamics, we use the Robert–Tamura asymptotics and the Firsov–Abel inversion formula to recover the potential in the classically accessible region from the semiclassical differential cross section. Finally, Section 4 extends the argument to nontrapping compactly supported perturbations of the Euclidean metric under a strict convexity assumption, without requiring simplicity.

2. COMPACTLY SUPPORTED POTENTIALS

2.1. Recovery of the scattering map. By a simple scaling argument, we may assume without loss of generality that the fixed non-trapping energy is $\lambda = 1/2$. We therefore recall Ingremeau's result [8] in this normalized setting, in the form needed below. For

$$(2.1) \quad \rho = (\omega, z) \in T^*\mathbb{S}^{n-1}, \quad z \in \omega^\perp,$$

we denote by $\kappa = \kappa_{1/2}$ the classical scattering map introduced in (1.21). Thus, if

$$(2.2) \quad \rho_1 = (\omega_1, z_1) = \kappa(\rho),$$

then ρ_1 is the outgoing phase-space point associated with the incoming datum ρ .

Following Ingremeau [8], let Γ_0 be a symmetric complex matrix with positive definite real part, let Q_0 be a polynomial, and choose $\chi \in C^\infty(\mathbb{R}; [0, 1])$ such that

$$\chi(r) = 1 \quad \text{for } r \leq \frac{1}{2}, \quad \chi(r) = 0 \quad \text{for } r \geq \frac{3}{4}.$$

Associated with $\rho = (\omega, z) \in T^*\mathbb{S}^{n-1}$, we define the Gaussian state

$$(2.3) \quad \varphi_{\rho, \Gamma_0, Q_0}(\hat{x}; h) = \chi\left(\frac{|\hat{x} - \omega|}{h^{1/3}}\right) Q_0\left(\frac{\hat{x} - \omega}{\sqrt{h}}\right) e^{-\frac{i}{h} z \cdot \hat{x}} e^{-\frac{1}{2h} (\hat{x} - \omega) \cdot \Gamma_0 (\hat{x} - \omega)},$$

where $\hat{x} \in \mathbb{S}^{n-1}$. The Gaussian factor localizes the state at the scale $\sqrt{h}$ around ω , while the oscillatory factor localizes it microlocally at the cotangent variable z . Thus $\varphi_{\rho, \Gamma_0, Q_0}$ is microlocally concentrated near the phase-space point $\rho = (\omega, z)$.

The Gaussian states used by Ingremeau may be viewed as the scattering counterpart of the coherent states familiar from semiclassical propagation. In particular, their localization in phase space and their propagation along the classical scattering map are analogous to the semiclassical propagation of Gaussian coherent states; see, for instance, Combescure and Robert [23].

We now recall the main result of [8], which describes the action of the scattering matrix on these Gaussian states at the non-trapping energy $\lambda = 1/2$. With the conventions adopted in Section 1, Ingremeau's scattering matrix coincides with $S(\lambda, h)$, $\lambda = \frac{1}{2}$. Let

$$(2.4) \quad (\omega_1, z_1) = \kappa(\omega, z).$$

Then there exist a real number δ_1 , a symmetric complex matrix Γ_1 with positive definite real part, and polynomials Q_1^k , $k \in \mathbb{N}$, such that the following holds. For every $N \in \mathbb{N}$, setting

$$(2.5) \quad \tilde{Q}_1^N = \sum_{k=0}^N h^{k/2} Q_1^k,$$

one has

$$(2.6) \quad S(\lambda, h) \varphi_{\omega, z, \Gamma_0, Q_0} = e^{i\delta_1/h} \varphi_{\omega_1, z_1, \Gamma_1, \tilde{Q}_1^N} + R_N,$$

where

$$(2.7) \quad \|R_N\|_{C^0(\mathbb{S}^{n-1})} = O\left(h^{(N+1)/2}\right).$$

Thus, to arbitrary order in the semiclassical expansion, the scattering matrix maps a Gaussian state localized near the incoming phase point (ω, z) to a Gaussian state localized near the outgoing phase point $\kappa(\omega, z)$. This direct transport of phase-space centers is the feature of Ingremau's approach that will be used below to recover the classical scattering map from the semiclassical scattering matrix.

For the inverse argument below, we shall only need the particular choice $\Gamma_0 = I$ and $Q_0 = 1$. We therefore set

$$(2.8) \quad \phi_{\rho, h}(\widehat{x}) = c_h \chi \left(\frac{|\widehat{x} - \omega|}{h^{1/3}} \right) e^{-\frac{i}{h} z \cdot \widehat{x}} e^{-\frac{|\widehat{x} - \omega|^2}{2h}},$$

where $c_h > 0$ is chosen so that

$$(2.9) \quad \|\phi_{\rho, h}\|_{L^2(\mathbb{S}^{n-1})} = 1.$$

Alexandrova's Fourier integral description shows that the semiclassical scattering matrix microlocally encodes the classical scattering relation. Here, using Ingremau's propagation result, we show that the quantitative condition

$$(2.10) \quad \liminf_{h \rightarrow 0} \|S_1(\lambda, h) - S_2(\lambda, h)\|_{\mathcal{B}(L^2(\mathbb{S}^{n-1}))} < \sqrt{2}$$

is sufficient to ensure that the corresponding classical scattering maps coincide. We state the result for an arbitrary fixed energy $\lambda > 0$, since by a simple scaling one can reduce to the case $\lambda = 1/2$.

Lemma 2.1. *Let $V_1, V_2 \in C_c^\infty(\mathbb{R}^n)$ and assume that $\lambda > 0$ is a non-trapping energy for both potentials. Let $S_j(\lambda, h)$ and $\kappa_{j, \lambda}$ denote respectively the semiclassical scattering matrix and the classical scattering map associated with V_j , $j = 1, 2$. If*

$$(2.11) \quad \liminf_{h \rightarrow 0} \|S_1(\lambda, h) - S_2(\lambda, h)\|_{\mathcal{B}(L^2(\mathbb{S}^{n-1}))} < \sqrt{2},$$

then

$$(2.12) \quad \kappa_{1, \lambda} = \kappa_{2, \lambda}.$$

Proof. Fix $\rho = (\omega, z) \in T^*\mathbb{S}^{n-1}$ and suppose, by contradiction, that

$$(2.13) \quad \kappa_{1, \lambda}(\rho) \neq \kappa_{2, \lambda}(\rho).$$

Write

$$\kappa_{j, \lambda}(\rho) = (\omega_j, z_j), \quad j = 1, 2.$$

By Ingremau's theorem, for every $M > 0$,

$$(2.14) \quad S_j(\lambda, h)\phi_{\rho, h} = e^{i\delta_j/h}\Phi_{j, h} + O(h^M) \quad \text{in } L^2(\mathbb{S}^{n-1}),$$

where $\Phi_{j, h}$ is a Gaussian state centered at (ω_j, z_j) . Since the expansion in (2.6) is available to arbitrary order, M can be chosen arbitrarily large. We claim that (2.13) implies

$$(2.15) \quad \langle S_1(\lambda, h)\phi_{\rho, h}, S_2(\lambda, h)\phi_{\rho, h} \rangle = O(h^\infty).$$

Indeed, if $\omega_1 \neq \omega_2$, the cutoffs in the two outgoing Gaussian states have disjoint supports for h sufficiently small, since they are supported in balls of radius $O(h^{1/3})$ around the distinct points ω_1 and ω_2 . It remains to consider the case

$$(2.16) \quad \omega_1 = \omega_2 =: \omega_+, \quad z_1 \neq z_2.$$

The two outgoing states are then localized near the same point ω_+ , but the product entering their scalar product contains the oscillatory factor

$$(2.17) \quad \exp \left(\frac{i}{h} (z_1 - z_2) \cdot \widehat{x} \right).$$

Since $z_1 - z_2 \in \omega_+^\perp$ and $z_1 - z_2 \neq 0$, the phase in (2.17) has no critical point on the support of the Gaussian states for h sufficiently small. Hence, by non-stationary phase,

$$(2.18) \quad \langle \Phi_{1, h}, \Phi_{2, h} \rangle = O(h^\infty),$$

and (2.15) follows. Since the scattering matrices are unitary, by (2.9) we have

$$(2.19) \quad \|S_j(\lambda, h)\phi_{\rho, h}\|_{L^2} = 1, \quad j = 1, 2.$$

Hence, using (2.15),

$$(2.20) \quad \begin{aligned} \|(S_1(\lambda, h) - S_2(\lambda, h))\phi_{\rho, h}\|_{L^2}^2 &= 2 - 2 \operatorname{Re} \langle S_1(\lambda, h)\phi_{\rho, h}, S_2(\lambda, h)\phi_{\rho, h} \rangle \\ &= 2 + O(h^\infty). \end{aligned}$$

Since $\|\phi_{\rho, h}\|_{L^2} = 1$, it follows that

$$\|S_1(\lambda, h) - S_2(\lambda, h)\|_{\mathcal{B}(L^2(\mathbb{S}^{n-1}))} \geq \|(S_1(\lambda, h) - S_2(\lambda, h))\phi_{\rho, h}\|_{L^2} = \sqrt{2} + O(h^\infty).$$

Consequently,

$$\liminf_{h \rightarrow 0} \|S_1(\lambda, h) - S_2(\lambda, h)\|_{\mathcal{B}(L^2(\mathbb{S}^{n-1}))} \geq \sqrt{2},$$

which contradicts (2.11). Therefore

$$(2.21) \quad \kappa_{1, \lambda}(\rho) = \kappa_{2, \lambda}(\rho).$$

Since $\rho \in T^*\mathbb{S}^{n-1}$ was arbitrary, (2.12) follows. $\square$

Remark 2.2. The constant $\sqrt{2}$ in Lemma 2.1 has a simple interpretation in the radial case. Indeed, let $\{Y_{\ell, m}\}$ be an orthonormal basis of spherical harmonics on $\mathbb{S}^{n-1}$. If $\delta_{j, \ell}(\lambda, h)$ denotes the phase shift corresponding to the angular momentum ℓ , then

$$S_j(\lambda, h)Y_{\ell, m} = e^{2i\delta_{j, \ell}(\lambda, h)}Y_{\ell, m},$$

and therefore

$$\|S_1(\lambda, h) - S_2(\lambda, h)\|_{\mathcal{B}(L^2(\mathbb{S}^{n-1}))} = 2 \sup_{\ell \geq 0} |\sin(\delta_{1, \ell}(\lambda, h) - \delta_{2, \ell}(\lambda, h))|.$$

Thus the assumption of Lemma 2.1 is equivalent to

$$\liminf_{h \rightarrow 0} \sup_{\ell \geq 0} \operatorname{dist}(\delta_{1, \ell}(\lambda, h) - \delta_{2, \ell}(\lambda, h), \pi\mathbb{Z}) < \frac{\pi}{4}.$$

The critical value $\sqrt{2}$ corresponds to a phase-shift difference at distance $\pi/4$ from $\pi\mathbb{Z}$, that is, equal to $\pm\pi/4$ modulo π .

2.2. From the scattering map to the potential. We now turn to the classical inverse problem. We assume first that the energy lies above the potentials,

$$(2.22) \quad \lambda > \sup_{\mathbb{R}^n} V_j, \quad j = 1, 2,$$

and that the virial condition

$$(2.23) \quad 2(\lambda - V_j(x)) - x \cdot \nabla V_j(x) > 0, \quad x \in \mathbb{R}^n, \quad j = 1, 2,$$

holds. The condition (2.23) implies that the Hamiltonian flow at energy λ is non-trapping.

By the Maupertuis principle, under (2.22), the trajectories at energy λ , up to a reparametrization, are the geodesics of the Jacobi metrics

$$(2.24) \quad g_j = 2(\lambda - V_j)g_0, \quad j = 1, 2,$$

where g_0 denotes the Euclidean metric on $\mathbb{R}^n$. Equivalently, we may write

$$(2.25) \quad g_j = c_j^{-2}g_0, \quad c_j = (2(\lambda - V_j))^{-1/2}, \quad j = 1, 2.$$

This reduction was already used by Alexandrova [1] under the assumption that the Jacobi metric is simple. In that setting, she used an argument implicit in Michel [13] to recover the boundary distance function from the scattering relation, and then applied the boundary rigidity results of Mukhometov [16] and Mukhometov–Romanov [15] to determine the conformal factor. More recently, in a purely classical setting, Muñoz-Thon [14] obtained scattering rigidity results at a single energy for simple magnetic-potential systems. In particular, in the non-magnetic case, his result recovers the classical rigidity statement used by Alexandrova: the classical scattering relation determines the potential when the associated Jacobi metric is simple.

The geometric assumption used here is different. We do not require the Jacobi metrics to be simple, but instead use the rigidity theorem of Stefanov–Uhlmann–Vasy [25], which applies when the manifold admits a strictly convex foliation. Our virial condition provides precisely such a foliation for the Jacobi metrics. Since this condition does not exclude conjugate points, it applies to geometries which need not be simple. We note that Muñoz-Thon [14] explicitly raised the question of extending scattering rigidity for magnetic-potential systems from the simple setting to the strictly convex foliation framework.

Definition 2.3. Let (M, g) be a compact Riemannian manifold with boundary. We say that M admits a strictly convex foliation if there exist $T > 0$ and a smooth function

$$\rho : M \longrightarrow [0, +\infty)$$

such that, for every $t \in [0, T)$, the level set

$$\Sigma_t = \rho^{-1}(t)$$

is a strictly convex hypersurface, $d\rho \neq 0$ on Σ_t , $\Sigma_0 = \partial M$, and

$$M \setminus \bigcup_{0 \leq t < T} \Sigma_t$$

has empty interior.

In our setting, the virial condition provides precisely such a foliation. Indeed, on $M = \overline{B(0, R)}$, we take $\rho(x) = R^2 - |x|^2$. Then, for $0 \leq t < R^2$,

$$(2.26) \quad \Sigma_t = \rho^{-1}(t) = \{|x| = \sqrt{R^2 - t}\}.$$

Thus the leaves of the foliation are precisely the Euclidean spheres centered at the origin.

The following lemma shows that the virial condition indeed guarantees the strict convexity of these leaves with respect to the Jacobi metrics. This observation is likely known, but we have been unable to find an explicit statement of it in the literature.

Lemma 2.4. *Under (2.22) and (2.23), the Euclidean spheres form a strictly convex foliation for the Jacobi metrics (2.24).*

Proof. It is enough to consider one potential V . Writing the Jacobi metric in the form

$$(2.27) \quad g = c^{-2}g_0, \quad c = (2(\lambda - V))^{-1/2},$$

we have

$$(2.28) \quad \frac{\partial}{\partial r} \left(\frac{r}{c} \right) = \frac{2(\lambda - V(x)) - x \cdot \nabla V(x)}{\sqrt{2(\lambda - V(x))}}.$$

Hence (2.23) implies the Herglotz–Wiechert–Zoeppritz condition

$$(2.29) \quad \frac{\partial}{\partial r} \left(\frac{r}{c} \right) > 0.$$

By [25, Proposition 6.1], this is equivalent to the strict convexity of the Euclidean spheres for the Jacobi metric. As the concentric spheres foliate $\mathbb{R}^n \setminus \{0\}$, the conclusion follows. $\square$

We briefly recall the notion of scattering relation used in Stefanov–Uhlmann–Vasy [25]. Let (M, g) be a compact Riemannian manifold with boundary. In our application below, M will simply be the closed Euclidean ball $\overline{B(0, R)} \subset \mathbb{R}^n$, equipped with one of the Jacobi metrics (2.24). We denote by $\partial_- SM$ and $\partial_+ SM$ the sets of unit tangent vectors based at ∂M and pointing respectively into and out of M . For $(x_-, v_-) \in \partial_- SM$, let γ_{x_-, v_-} be the unique unit speed geodesic issued from (x_-, v_-) . In our setting, the non-trapping assumption ensures that this geodesic has a finite exit time, defining an outgoing point and direction $(x_+, v_+) \in \partial_+ SM$. The scattering relation is defined by

$$(2.30) \quad L(x_-, v_-) = (x_+, v_+).$$

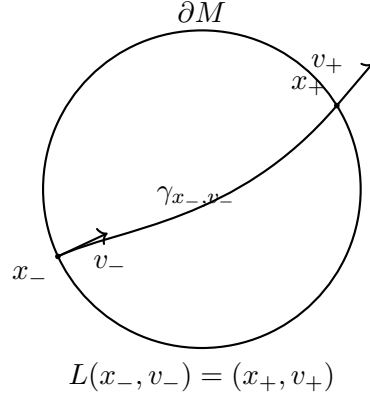


FIGURE 2. The scattering relation of a Riemannian metric: an incoming point and direction (x_-, v_-) are mapped to the corresponding exit point and direction (x_+, v_+) .

By Lemma 2.1, the assumption

$$(2.31) \quad \liminf_{h \rightarrow 0} \|S_1(\lambda, h) - S_2(\lambda, h)\|_{\mathcal{B}(L^2(\mathbb{S}^{n-1}))} < \sqrt{2},$$

implies

$$(2.32) \quad \kappa_{1,\lambda} = \kappa_{2,\lambda}.$$

Since V_1 and V_2 are compactly supported, we may choose $R > 0$ such that

$$(2.33) \quad \text{supp } V_1 \cup \text{supp } V_2 \subset B(0, R).$$

Outside $B(0, R)$ the Hamiltonian trajectories are straight lines. Hence the incoming and outgoing asymptotic data defining $\kappa_{j,\lambda}$ determine uniquely the corresponding entry and exit points and directions on $\partial B(0, R)$. It follows from (2.32) that the scattering relations L_j of the Jacobi metrics on $B(0, R)$ satisfy

$$(2.34) \quad L_1 = L_2.$$

Notice that no travel-time information is needed here, since the rigidity result [25, Theorem 1.4] only requires equality of the scattering relations. Finally, since $V_1 = V_2 = 0$ near $\partial B(0, R)$, the Jacobi metrics coincide there:

$$(2.35) \quad g_1 = g_2 = 2\lambda g_0 \quad \text{near } \partial B(0, R).$$

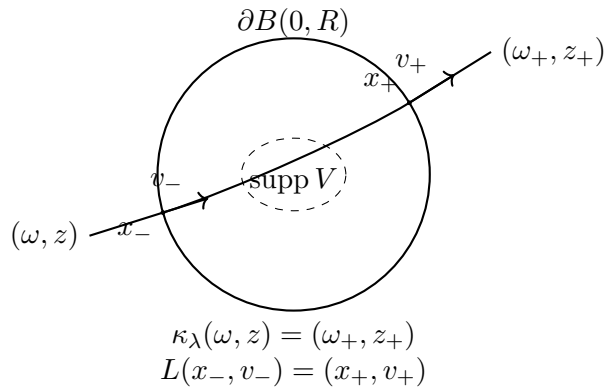


FIGURE 3. Relation between the classical scattering map κ_λ and the scattering relation L on $\partial B(0, R)$. Outside the support of V , the trajectories are straight lines, so that the asymptotic data determine uniquely the corresponding entry and exit data on the boundary.

We may therefore apply the global rigidity theorem of Stefanov–Uhlmann–Vasy [25, Theorem 1.4]. Indeed, Lemma 2.4 provides the required strictly convex foliation. It follows that $g_1 = g_2$. In view of (2.24), we conclude that $V_1 = V_2$.

3. RADIAL SHORT-RANGE POTENTIALS

We now consider smooth radial repulsive short-range potentials

$$(3.1) \quad V(x) = v(|x|), \quad v \in C^\infty([0, +\infty)), \quad |\partial_r^k v(r)| \leq C_k \langle r \rangle^{-\mu-k}, \quad \mu > 1.$$

Following the classical setting of Keller–Kay–Shmoys [12], we assume that

$$(3.2) \quad v(r) > 0, \quad v'(r) < 0, \quad r > 0,$$

and that

$$(3.3) \quad v(0) > \lambda.$$

Since $v(r) \rightarrow 0$ as $r \rightarrow +\infty$, assumptions (3.2)–(3.3) imply that there exists a unique $r_\lambda > 0$ such that

$$(3.4) \quad v(r_\lambda) = \lambda.$$

The region $0 < r < r_\lambda$ is classically forbidden at energy λ , whereas the classically accessible region is $r > r_\lambda$. Thus, (3.3) ensures the presence of a nonempty classically forbidden core. This assumption also plays an important role in the scattering geometry. Indeed, if $v(0) < \lambda$, one can find that the deflection angle $\Theta_\lambda(b)$, defined below in (3.24), tends to zero both as the impact parameter b tends to zero and as b tends to infinity. Hence $\Theta_\lambda(b)$ cannot be strictly monotone in this case, and several impact parameters may correspond to the same scattering angle. The presence of the forbidden core, together with the monotonicity assumption introduced below, will instead imply that Θ_λ is strictly monotone.

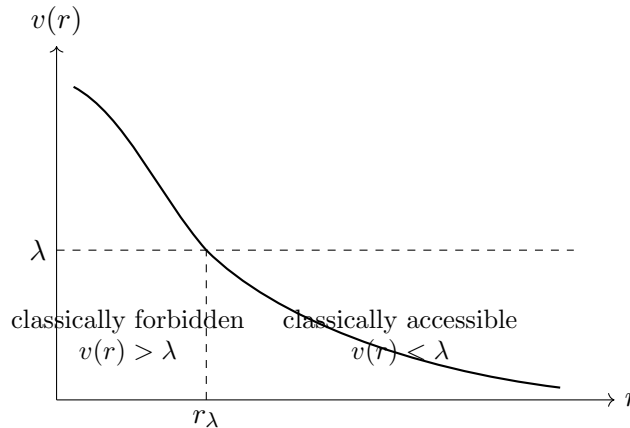


FIGURE 4. A radial repulsive potential in the setting of Keller–Kay–Shmoys. The unique radius r_λ is defined by $v(r_\lambda) = \lambda$. At the fixed energy λ , the region $0 < r < r_\lambda$ is classically forbidden, whereas $r > r_\lambda$ is classically accessible.

In the radial setting, the virial condition (2.23) takes the form

$$(3.5) \quad 2(\lambda - v(r)) - rv'(r) > 0.$$

On the energy surface

$$\frac{1}{2}|\xi|^2 + v(r) = \lambda,$$

one necessarily has $v(r) \leq \lambda$, and hence $r \geq r_\lambda$. Since $v'(r) < 0$, (3.5) is therefore satisfied on the whole energy surface, including at $r = r_\lambda$. In particular, the classical dynamics at energy λ is non-trapping, so that the semiclassical scattering results of Robert–Tamura apply.

We further assume that

$$(3.6) \quad (rv'(r))' \geq 0, \quad r > r_\lambda.$$

This condition will be used below to prove the strict monotonicity of the deflection function $b \mapsto \Theta_\lambda(b)$, and hence the uniqueness of the impact parameter associated with a given scattering angle. Related sufficient conditions for the monotonicity of the deflection function were obtained, in the setting of compactly supported radial potentials, by Pulvirenti–Saffirio–Simonella [20, Appendix A, condition (A.8)] and Gallagher–Saint-Raymond–Texier [5, Lemma 8.3.1]. Their condition is

$$rv''(r) + 2v'(r) \geq 0,$$

whereas (3.6) reads

$$rv''(r) + v'(r) \geq 0.$$

Since $v' < 0$, their condition is stronger than (3.6). Moreover, the potentials considered here are short-range and need not be compactly supported. Although the argument below is similar in spirit to these previous monotonicity arguments, their results do not apply directly in the present setting. We therefore give the complete argument below.

Remark 3.1. Condition (3.6) has a natural interpretation in terms of the radial force

$$F(r) := -v'(r) > 0.$$

It is equivalent to requiring that the weighted radial force $rF(r) = -rv'(r)$ be non-increasing on the classically accessible region. For instance, for the inverse-power profile

$$(3.7) \quad v(r) = Cr^{-\rho}, \quad r \geq R > 0, \quad C > 0, \quad \rho > 0,$$

one has

$$(3.8) \quad (rv'(r))' = \rho^2 Cr^{-\rho-1} > 0.$$

3.1. Review of classical radial dynamics. We briefly recall some standard facts about the classical dynamics of radial potentials; see, for instance, Goldstein [6], Keller–Kay–Shmoys [12], Jollivet [11], Pulvirenti–Saffirio–Simonella [20], and Gallagher–Saint-Raymond–Texier [5]. These facts are classical, and we recall them here only to fix the notation and sign conventions used below.

Under the assumptions (3.2)–(3.3), the motion at the fixed energy λ takes place in the classically accessible region $r \geq r_\lambda$. Together with the additional assumption (3.6), these hypotheses will imply that each scattering angle in $(0, \pi)$ corresponds to a unique impact parameter $b > 0$.

Let $b = |z| > 0$ be the impact parameter. Since the force is central, the trajectory lies in the scattering plane

$$(3.9) \quad \Pi_{\omega, z} := \text{span}\{\omega, z\},$$

and the angular momentum $q_\infty \wedge p_\infty$ is conserved; see, for instance, [6, Sections 3.2 and 3.10]. Its magnitude L is determined by the incoming asymptotics:

$$(3.10) \quad L = |q_\infty \wedge p_\infty| = \sqrt{2\lambda} b.$$

Let $(r(t), \varphi(t))$ be polar coordinates in the scattering plane $\Pi_{\omega, z}$, with $r(t) = |q_\infty(t; z, \lambda)|$. Conservation of angular momentum gives

$$(3.11) \quad r(t)^2 |\dot{\varphi}(t)| = L = \sqrt{2\lambda} b.$$

Using conservation of energy, we obtain the radial equation

$$(3.12) \quad \frac{1}{2} \dot{r}(t)^2 + \frac{L^2}{2r(t)^2} + v(r(t)) = \frac{1}{2} \dot{r}(t)^2 + \lambda \frac{b^2}{r(t)^2} + v(r(t)) = \lambda.$$

It is useful to introduce the effective radial potential

$$(3.13) \quad v_{\text{eff}}(r; b) := v(r) + \lambda \frac{b^2}{r^2}.$$

Under the assumption $v'(r) < 0$, one has

$$(3.14) \quad \partial_r v_{\text{eff}}(r; b) = v'(r) - 2\lambda \frac{b^2}{r^3} < 0, \quad r > 0.$$

Thus $v_{\text{eff}}(\cdot; b)$ is strictly decreasing. Moreover, for $b > 0$,

$$(3.15) \quad v_{\text{eff}}(r; b) \longrightarrow +\infty \quad \text{as } r \rightarrow 0^+, \quad v_{\text{eff}}(r; b) \longrightarrow 0 \quad \text{as } r \rightarrow +\infty.$$

Consequently, for every $b > 0$, there exists a unique turning point $r_0(b)$ characterized by

$$(3.16) \quad v(r_0(b)) + \lambda \frac{b^2}{r_0(b)^2} = \lambda.$$

Since (3.16) yields $v(r_0(b)) < \lambda = v(r_\lambda)$ and v is strictly decreasing, one necessarily has $r_0(b) > r_\lambda$. Geometrically, $r_0(b)$ is the distance of closest approach of the trajectory to the origin. More precisely, if t_0 denotes the time at which the trajectory reaches the turning point, then $\dot{r}(t_0) = 0$ and

$$(3.17) \quad r_0(b) = r(t_0) = \min_{t \in \mathbb{R}} r(t).$$

By rotational invariance, the distance of closest approach depends on the impact vector z only through $b = |z|$. Rewriting (3.16), we obtain

$$(3.18) \quad b = \rho_\lambda(r_0(b)), \quad \rho_\lambda(r) := r \sqrt{1 - \frac{v(r)}{\lambda}}, \quad r \geq r_\lambda.$$

Thus, the function ρ_λ relates the radius $r_0(b)$ of the turning point to the corresponding impact parameter b . It will play a central role in the inversion argument below.

By (3.2), for $r > r_\lambda$,

$$(3.19) \quad \rho'_\lambda(r) = \sqrt{1 - \frac{v(r)}{\lambda}} \left(1 - \frac{rv'(r)}{2(\lambda - v(r))} \right) > 0,$$

since $\lambda - v(r) > 0$ and $v'(r) < 0$. Hence ρ_λ is strictly increasing from 0 to $+\infty$ on $[r_\lambda, +\infty)$, and

$$(3.20) \quad b = \rho_\lambda(r_0(b))$$

provides a one-to-one parametrization of the radial turning points by the impact parameter $b \geq 0$. On the outgoing part of the trajectory, $r(t)$ increases from $r_0(b)$ to $+\infty$. From (3.12), we have

$$(3.21) \quad \dot{r}(t) = \sqrt{2\lambda} \sqrt{1 - \frac{v(r(t))}{\lambda} - \frac{b^2}{r(t)^2}}.$$

For $r > r_0(b)$, we may therefore use r as a parameter along the trajectory. Combining (3.21) with (3.11) and using the chain rule, we obtain

$$(3.22) \quad \left| \frac{d\varphi}{dr} \right| = \frac{|\dot{\varphi}(t)|}{\dot{r}(t)} = \frac{b}{r^2 \sqrt{1 - \frac{v(r)}{\lambda} - \frac{b^2}{r^2}}}.$$

Then, the angular variation along the outgoing part of the trajectory is

$$(3.23) \quad \Phi_\lambda(b) := b \int_{r_0(b)}^{+\infty} \frac{dr}{r^2 \sqrt{1 - \frac{v(r)}{\lambda} - \frac{b^2}{r^2}}}.$$

By the symmetry of the central-force trajectory with respect to the line through the origin and the point of closest approach, the angular variation from the incoming asymptote to the turning point equals that from the turning point to the outgoing asymptote. Hence the total angular

variation between the two asymptotes is $2\Phi_\lambda(b)$. We therefore define the signed deflection function by

$$(3.24) \quad \Theta_\lambda(b) := \pi - 2\Phi_\lambda(b) = \pi - 2b \int_{r_0(b)}^{+\infty} \frac{dr}{r^2 \sqrt{1 - \frac{v(r)}{\lambda} - \frac{b^2}{r^2}}}.$$

This is the standard formula for the deflection angle in a central potential; see Goldstein [6, p. 108, Eq. (3.96)], with the notation adapted to the present setting.

Remark 3.2. Let us check the above convention in the free case $V = 0$. Then $v(r) = 0$ and the turning point equation gives $r_0(b) = b$. Consequently,

$$(3.25) \quad \Phi_\lambda(b) = b \int_b^{+\infty} \frac{dr}{r^2 \sqrt{1 - \frac{b^2}{r^2}}} = \frac{\pi}{2}.$$

Therefore, $\Theta_\lambda(b) = \pi - 2\Phi_\lambda(b) = 0$, as expected, since in the absence of a potential the incoming and outgoing directions coincide.

The geometry of the resulting radial scattering trajectory is illustrated in Figure 5.

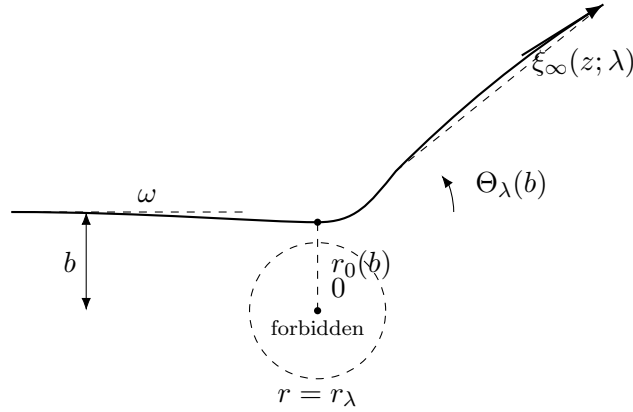


FIGURE 5. Radial scattering at the fixed energy λ . The dashed circle $r = r_\lambda$, defined by $v(r_\lambda) = \lambda$, bounds the classically forbidden region. For $b > 0$, the distance of closest approach satisfies $r_0(b) > r_\lambda$. The trajectory is asymptotic to the incoming and outgoing free lines as $t \rightarrow -\infty$ and $t \rightarrow +\infty$, respectively.

Since ρ_λ is strictly increasing on $[r_\lambda, +\infty)$, it can be used as a new radial variable. We denote its inverse by

$$(3.26) \quad r = r_\lambda(\rho), \quad \rho = \rho_\lambda(r),$$

and introduce

$$(3.27) \quad q_\lambda(\rho) := \frac{d}{d\rho} \log \frac{r_\lambda(\rho)}{\rho}.$$

The additional assumption (3.6), which is not needed for the Firsov–Abel inversion itself, provides a convenient sufficient condition ensuring that the radial scattering geometry has only one classical branch. More precisely, we have the following result.

Lemma 3.3. *Assume (3.1), (3.2), (3.3), and (3.6). Then the deflection function $b \mapsto \Theta_\lambda(b)$ is strictly decreasing:*

$$(3.28) \quad \Theta'_\lambda(b) < 0, \quad b > 0.$$

Moreover,

$$(3.29) \quad \lim_{b \rightarrow 0^+} \Theta_\lambda(b) = \pi, \quad \lim_{b \rightarrow +\infty} \Theta_\lambda(b) = 0.$$

In particular, for every scattering angle $\Theta \in (0, \pi)$, there exists a unique impact parameter $b > 0$ such that

$$(3.30) \quad \Theta_\lambda(b) = \Theta.$$

Proof. Using the change of variables $\rho = \rho_\lambda(r)$ in (3.24), and recalling that $\rho_\lambda(r_0(b)) = b$, we have

$$(3.31) \quad \sqrt{1 - \frac{v(r)}{\lambda} - \frac{b^2}{r^2}} = \frac{\sqrt{\rho^2 - b^2}}{r},$$

and

$$(3.32) \quad \Phi_\lambda(b) = b \int_b^{+\infty} \frac{1}{r_\lambda(\rho)} \frac{dr_\lambda}{d\rho}(\rho) \frac{d\rho}{\sqrt{\rho^2 - b^2}}.$$

By (3.27),

$$(3.33) \quad \frac{1}{r_\lambda(\rho)} \frac{dr_\lambda}{d\rho}(\rho) = \frac{1}{\rho} + q_\lambda(\rho).$$

Since

$$(3.34) \quad b \int_b^{+\infty} \frac{d\rho}{\rho \sqrt{\rho^2 - b^2}} = \frac{\pi}{2},$$

substitution into $\Theta_\lambda(b) = \pi - 2\Phi_\lambda(b)$ yields

$$(3.35) \quad \Theta_\lambda(b) = -2b \int_b^{+\infty} \frac{q_\lambda(\rho)}{\sqrt{\rho^2 - b^2}} d\rho.$$

After the change of variables $\rho = bt$, we obtain

$$(3.36) \quad \Theta_\lambda(b) = -2b \int_1^{+\infty} \frac{q_\lambda(bt)}{\sqrt{t^2 - 1}} dt.$$

Differentiating with respect to b gives

$$(3.37) \quad \Theta'_\lambda(b) = -2 \int_1^{+\infty} \frac{\frac{d}{d\rho}(\rho q_\lambda(\rho))|_{\rho=bt}}{\sqrt{t^2 - 1}} dt.$$

To determine the sign of the integrand, set

$$(3.38) \quad Q_\lambda(r) := -\frac{rv'(r)}{\lambda - v(r)}, \quad r > r_\lambda.$$

A direct computation gives

$$(3.39) \quad Q'_\lambda(r) = -\frac{(\lambda - v(r))(rv'(r))' + r(v'(r))^2}{(\lambda - v(r))^2}.$$

By (3.6), since $\lambda - v(r) > 0$ and $v'(r) < 0$, the numerator in (3.39) is strictly positive. Hence

$$(3.40) \quad Q'_\lambda(r) < 0, \quad r > r_\lambda.$$

On the other hand, from the definitions of ρ_λ and q_λ , one obtains

$$(3.41) \quad \rho q_\lambda(\rho) = \frac{1}{1 + \frac{1}{2}Q_\lambda(r_\lambda(\rho))} - 1.$$

Since $r'_\lambda(\rho) > 0$ and $Q'_\lambda < 0$, we conclude that

$$(3.42) \quad \frac{d}{d\rho}(\rho q_\lambda(\rho)) > 0.$$

Inserting (3.42) into (3.37) proves (3.28). Finally, when $b \rightarrow 0^+$, the turning point $r_0(b)$ tends to r_λ and the trajectory approaches the radial trajectory. Indeed, since $L = \sqrt{2\lambda}b$, the limit $b \rightarrow 0^+$ corresponds to vanishing angular momentum, and hence to a radial trajectory which reaches the turning radius r_λ and is backscattered along the same radial line; hence

$$(3.43) \quad \Theta_\lambda(b) \longrightarrow \pi, \quad b \rightarrow 0^+.$$

On the other hand, the short-range assumption implies that the scattering becomes asymptotically free as $b \rightarrow +\infty$, and thus

$$(3.44) \quad \Theta_\lambda(b) \longrightarrow 0, \quad b \rightarrow +\infty.$$

This proves (3.29). Therefore $b \mapsto \Theta_\lambda(b)$ is one-to-one from $(0, +\infty)$ onto $(0, \pi)$. $\square$

3.2. The Robert–Tamura asymptotics. We now recall the semiclassical asymptotics of the scattering amplitude obtained by Robert and Tamura [24]. This provides the link between the quantum differential cross section and the classical scattering data introduced in the previous subsection.

Fix an incoming direction $\omega \in \mathbb{S}^{n-1}$ and choose oriented orthonormal coordinates $(z_1, \dots, z_{n-1})$ on the impact plane

$$\Lambda_\omega = \omega^\perp.$$

For $z \in \Lambda_\omega$, let

$$(q_\infty(t; z, \lambda), p_\infty(t; z, \lambda))$$

be the scattering trajectory defined by the incoming asymptotics

$$q_\infty(t; z, \lambda) = \sqrt{2\lambda}\omega t + z + o(1), \quad p_\infty(t; z, \lambda) = \sqrt{2\lambda}\omega + o(1), \quad t \rightarrow -\infty.$$

Assume that $\lambda > 0$ is a non-trapping energy. Then there exist $\xi_\infty(z; \lambda) \in \mathbb{S}^{n-1}$ and $r_\infty(z; \lambda) \in \mathbb{R}^n$ such that

$$(3.45) \quad \begin{aligned} q_\infty(t; z, \lambda) &= \sqrt{2\lambda}\xi_\infty(z; \lambda)t + r_\infty(z; \lambda) + o(1), \\ p_\infty(t; z, \lambda) &= \sqrt{2\lambda}\xi_\infty(z; \lambda) + o(1), \end{aligned} \quad t \rightarrow +\infty.$$

For fixed $\omega \in \mathbb{S}^{n-1}$, consider the smooth map

$$(3.46) \quad F_\omega : \Lambda_\omega \longrightarrow \mathbb{S}^{n-1}, \quad F_\omega(z) = \xi_\infty(z; \lambda).$$

The angular density associated with the trajectory issued from $z \in \Lambda_\omega$ is defined by

$$(3.47) \quad \sigma(z; \lambda) = |\det(\xi_\infty(z; \lambda), \partial_{z_1}\xi_\infty(z; \lambda), \dots, \partial_{z_{n-1}}\xi_\infty(z; \lambda))|.$$

Thus $\sigma(z; \lambda)$ is the absolute value of the Jacobian of F_ω between Λ_ω and $\mathbb{S}^{n-1}$.

An outgoing direction $\theta \in \mathbb{S}^{n-1}$, $\theta \neq \omega$, is said to be regular for the incoming direction ω if

$$(3.48) \quad \sigma(z; \lambda) \neq 0 \quad \text{for all } z \in \Lambda_\omega \text{ such that } F_\omega(z) = \theta.$$

Assume that θ is regular. Since $\theta \neq \omega$ and $\xi_\infty(z; \lambda) \rightarrow \omega$ as $|z| \rightarrow +\infty$, the set

$$F_\omega^{-1}(\theta) = \{z \in \Lambda_\omega; \xi_\infty(z; \lambda) = \theta\}$$

is compact. By the inverse function theorem, its points are isolated, and hence this set is finite. We denote its cardinality by $\ell(\theta, \lambda)$ and its elements by

$$(3.49) \quad F_\omega^{-1}(\theta) = \{w_1(\theta; \lambda), \dots, w_{\ell(\theta, \lambda)}(\theta; \lambda)\}.$$

Each $w_j(\theta; \lambda)$ determines a classical scattering trajectory with incoming direction ω and outgoing direction θ . For each such trajectory, Robert and Tamura associate the action

$$(3.50) \quad \begin{aligned} \mathcal{A}_j(\omega, \theta; \lambda) &= \int_{-\infty}^{+\infty} \left(\frac{|p_\infty(t; w_j, \lambda)|^2}{2} - V(q_\infty(t; w_j, \lambda)) - \lambda \right) dt \\ &\quad - \left\langle r_\infty(w_j; \lambda), \sqrt{2\lambda}\theta \right\rangle, \end{aligned}$$

where $w_j = w_j(\theta; \lambda)$. The integral in (3.50) represents the difference between the interacting and free actions along trajectories which are asymptotic as $t \rightarrow -\infty$. Let $\mu_j \in \mathbb{Z}$ denote the Keller–Maslov–Morse index of the corresponding trajectory.

With the normalization of the scattering amplitude introduced above, the Robert–Tamura formula [24] reads

$$(3.51) \quad f(\omega \rightarrow \theta; \lambda, h) = \sum_{j=1}^{\ell(\theta, \lambda)} \sigma(w_j; \lambda)^{-1/2} \exp \left(\frac{i}{h} \mathcal{A}_j(\omega, \theta; \lambda) - \frac{i\pi}{2} \mu_j \right) + O(h), \quad h \rightarrow 0.$$

More precisely, under the same assumptions, the scattering amplitude admits a complete asymptotic expansion in powers of h .

In the radial setting, the monotonicity of the deflection function also implies that all non-forward and non-backward scattering directions are regular.

Lemma 3.4. *Under the assumptions of Lemma 3.3, every outgoing direction*

$$\theta \in \mathbb{S}^{n-1} \setminus \{\omega, -\omega\}$$

is regular for the incoming direction ω . Moreover, there is exactly one classical trajectory connecting ω to θ , that is,

$$\ell(\theta, \lambda) = 1.$$

Proof. Fix

$$(3.52) \quad \theta \in \mathbb{S}^{n-1} \setminus \{\omega, -\omega\},$$

and set

$$(3.53) \quad \Theta = \arccos(\omega \cdot \theta) \in (0, \pi).$$

By Lemma 3.3, there exists a unique $b > 0$ such that

$$(3.54) \quad \Theta_\lambda(b) = \Theta.$$

Define

$$(3.55) \quad \eta = \frac{\theta - (\omega \cdot \theta)\omega}{\sin \Theta} \in \mathbb{S}^{n-2} \subset \omega^\perp,$$

and set

$$(3.56) \quad z = b\eta.$$

By rotational invariance and the definition of the deflection function,

$$(3.57) \quad F_\omega(z) = \cos \Theta_\lambda(b) \omega + \sin \Theta_\lambda(b) \eta = \theta.$$

Thus

$$(3.58) \quad F_\omega^{-1}(\theta) \neq \emptyset.$$

We now prove that θ is regular. Let $z = b\eta \in F_\omega^{-1}(\theta)$ be arbitrary, with $b > 0$ and $\eta \in \mathbb{S}^{n-2} \subset \omega^\perp$. Let $h \in T_z \Lambda_\omega = \omega^\perp$, and write

$$h = a\eta + h_\perp, \quad h_\perp \in T_\eta \mathbb{S}^{n-2}.$$

Since

$$b(z) = |z|, \quad \eta(z) = \frac{z}{|z|},$$

we have

$$Db(z)[h] = a, \quad D\eta(z)[h] = \frac{1}{b} h_\perp.$$

Hence

$$(3.59) \quad D_z F_\omega[h] = a\Theta'_\lambda(b)(-\sin \Theta_\lambda(b) \omega + \cos \Theta_\lambda(b) \eta) + \frac{\sin \Theta_\lambda(b)}{b} h_\perp.$$

By Lemma 3.3,

$$(3.60) \quad 0 < \Theta_\lambda(b) < \pi, \quad \Theta'_\lambda(b) < 0.$$

In particular,

$$(3.61) \quad \Theta'_\lambda(b) \neq 0, \quad \frac{\sin \Theta_\lambda(b)}{b} \neq 0.$$

Moreover,

$$(3.62) \quad -\sin \Theta_\lambda(b) \omega + \cos \Theta_\lambda(b) \eta \perp T_\eta \mathbb{S}^{n-2}.$$

It follows from (3.59) that

$$(3.63) \quad D_z F_\omega[h] = 0 \implies a = 0, \quad h_\perp = 0.$$

Hence $D_z F_\omega$ is injective. Since

$$\dim T_z \Lambda_\omega = \dim T_\theta \mathbb{S}^{n-1} = n - 1,$$

it follows that $D_z F_\omega$ is invertible and therefore

$$\sigma(z; \lambda) \neq 0.$$

Since $z \in F_\omega^{-1}(\theta)$ was arbitrary, θ is regular.

It remains to prove that the preimage is unique. Let $z = b\eta \in F_\omega^{-1}(\theta)$. Taking the scalar product of (3.57) with ω , we obtain

$$(3.64) \quad \cos \Theta_\lambda(b) = \cos \Theta.$$

Since both angles belong to $(0, \pi)$,

$$\Theta_\lambda(b) = \Theta.$$

By Lemma 3.3, the impact parameter b is uniquely determined. Moreover, from (3.57),

$$\eta = \frac{\theta - (\omega \cdot \theta)\omega}{\sin \Theta},$$

so η is uniquely determined as well. Hence $z = b\eta$ is the unique preimage of θ under F_ω . Therefore $\ell(\theta, \lambda) = 1$. $\square$

For $\theta \in \mathbb{S}^{n-1} \setminus \{\omega, -\omega\}$, let $w(\theta; \lambda)$ denote the unique element of $F_\omega^{-1}(\theta)$. By Lemma 3.4 and the Robert–Tamura formula,

$$(3.65) \quad |f(\omega \rightarrow \theta; \lambda, h)|^2 = \sigma(w(\theta; \lambda); \lambda)^{-1} + O(h), \quad h \rightarrow 0.$$

Since, in the one-branch case, $\sigma(w(\theta; \lambda); \lambda)^{-1}$ is precisely the classical differential cross section, we obtain

$$(3.66) \quad |f(\omega \rightarrow \theta; \lambda, h)|^2 \longrightarrow \frac{d\sigma_{\text{cl}}}{d\Omega}(\theta; \lambda), \quad h \rightarrow 0.$$

This is the semiclassical information that will be used below to recover the radial potential.

3.3. Uniqueness from the differential cross section. Let $V_j(x) = v_j(|x|)$, $j = 1, 2$, be two radial short-range potentials satisfying the assumptions above, and let $r_{j,\lambda}$ be the unique radius defined by

$$(3.67) \quad v_j(r_{j,\lambda}) = \lambda.$$

We denote by $f_{j,h}(\omega, \theta; \lambda)$ the corresponding semiclassical scattering amplitudes. Assume that

$$(3.68) \quad |f_{1,h}(\omega, \theta; \lambda)|^2 - |f_{2,h}(\omega, \theta; \lambda)|^2 \longrightarrow 0, \quad h \longrightarrow 0,$$

for every regular scattering angle.

By the Robert–Tamura asymptotics and the one-trajectory property established above,

$$(3.69) \quad |f_{j,h}(\omega, \theta; \lambda)|^2 \longrightarrow \frac{d\sigma_{\text{cl},j}}{d\Omega}(\theta; \lambda), \quad j = 1, 2.$$

Consequently, (3.68) implies

$$(3.70) \quad \frac{d\sigma_{\text{cl},1}}{d\Omega}(\theta; \lambda) = \frac{d\sigma_{\text{cl},2}}{d\Omega}(\theta; \lambda)$$

for every $\theta \in \mathbb{S}^{n-1} \setminus \{\omega, -\omega\}$.

We first recover the deflection function from the classical differential cross section. By Lemma 3.3, the deflection function $b \mapsto \Theta_\lambda(b)$ is strictly decreasing from π to 0. Its inverse will be denoted by

$$(3.71) \quad b = b(\Theta), \quad 0 < \Theta < \pi.$$

The classical differential cross section is given by (see, for instance, Keller–Kay–Shmoys [12] for $n = 3$),

$$(3.72) \quad \frac{d\sigma_{\text{cl}}}{d\Omega}(\Theta; \lambda) = \frac{b(\Theta)^{n-2}}{\sin^{n-2} \Theta} \left| \frac{db}{d\Theta}(\Theta) \right|.$$

Since $b(\Theta)$ is strictly decreasing, (3.72) yields

$$(3.73) \quad \frac{d}{d\Theta} (b(\Theta)^{n-1}) = -(n-1) \sin^{n-2} \Theta \frac{d\sigma_{\text{cl}}}{d\Omega}(\Theta; \lambda).$$

Moreover, Lemma 3.3 gives

$$(3.74) \quad b(\Theta) \longrightarrow 0 \quad \text{as } \Theta \rightarrow \pi^-.$$

Integrating (3.73) from Θ to π , we obtain

$$(3.75) \quad b(\Theta)^{n-1} = (n-1) \int_{\Theta}^{\pi} \sin^{n-2} \alpha \frac{d\sigma_{\text{cl}}}{d\Omega}(\alpha; \lambda) d\alpha.$$

Thus the classical differential cross section uniquely determines $b(\Theta)$, and hence its inverse $\Theta_\lambda(b)$. It follows from (3.70) that

$$(3.76) \quad \Theta_{1,\lambda}(b) = \Theta_{2,\lambda}(b), \quad b > 0.$$

It remains to recover the potential from the deflection function. For this purpose, we use the classical Firsov–Abel inversion procedure of Firsov [4] and Keller–Kay–Shmoys [12]; see also Jollivet [11] for a related treatment of short-range radial potentials. We briefly recall the argument in our notation. By (3.35),

$$(3.77) \quad \Theta_\lambda(b) = -2b \int_b^{+\infty} \frac{q_\lambda(\rho)}{\sqrt{\rho^2 - b^2}} d\rho,$$

where

$$(3.78) \quad q_\lambda(\rho) = \frac{d}{d\rho} \log \frac{r_\lambda(\rho)}{\rho}.$$

Set

$$(3.79) \quad F_\lambda(\rho) = \log \frac{r_\lambda(\rho)}{\rho}.$$

Then $q_\lambda = F'_\lambda$, and (3.77) becomes

$$(3.80) \quad \frac{\Theta_\lambda(b)}{b} = -2 \int_b^{+\infty} \frac{F'_\lambda(\rho)}{\sqrt{\rho^2 - b^2}} d\rho.$$

After the quadratic change of variables $x = b^2$, $y = \rho^2$, this reduces to the standard Abel transform. Applying the classical Abel inversion formula and using

$$(3.81) \quad F_\lambda(\rho) = O(\rho^{-\mu}), \quad \rho \rightarrow +\infty,$$

we obtain

$$(3.82) \quad F_\lambda(\rho) = \frac{1}{\pi} \int_{\rho}^{+\infty} \frac{\Theta_\lambda(b)}{\sqrt{b^2 - \rho^2}} db, \quad \rho > 0.$$

Recalling the definition of F_λ , we obtain

$$(3.83) \quad r_\lambda(\rho) = \rho \exp \left(\frac{1}{\pi} \int_{\rho}^{+\infty} \frac{\Theta_\lambda(b)}{\sqrt{b^2 - \rho^2}} db \right), \quad \rho > 0.$$

Applying (3.83) to V_1 and V_2 and using (3.76), we obtain

$$(3.84) \quad r_{1,\lambda}(\rho) = r_{2,\lambda}(\rho), \quad \rho > 0.$$

Since the functions $r_{1,\lambda}$ and $r_{2,\lambda}$ coincide on $(0, +\infty)$, their ranges coincide. Since

$$(3.85) \quad r_{j,\lambda}((0, +\infty)) = (r_{j,\lambda}, +\infty),$$

their left endpoints coincide, and hence

$$(3.86) \quad r_{1,\lambda} = r_{2,\lambda} =: r_\lambda.$$

Taking inverse functions in (3.84), we then obtain

$$(3.87) \quad \rho_{1,\lambda}(r) = \rho_{2,\lambda}(r), \quad r > r_\lambda.$$

By continuity, the equality also holds at $r = r_\lambda$. Finally, from

$$(3.88) \quad v_j(r) = \lambda \left(1 - \frac{\rho_{j,\lambda}(r)^2}{r^2} \right), \quad r \geq r_\lambda,$$

we conclude that

$$(3.89) \quad v_1(r) = v_2(r), \quad r \geq r_\lambda.$$

Equivalently,

$$(3.90) \quad V_1(x) = V_2(x), \quad |x| \geq r_\lambda.$$

Thus the semiclassical differential cross section at the fixed energy λ determines the radial potential throughout the classically accessible region.

4. AN EXTENSION TO METRIC PERTURBATIONS

We conclude with an extension of the preceding argument to compactly supported perturbations of the Euclidean metric. The proof follows the same general strategy as in the potential case, and we therefore only give its main steps.

Let g be a smooth Riemannian metric on $\mathbb{R}^n$, $n \geq 3$, such that

$$(4.1) \quad g = g_0 \quad \text{outside a compact set,}$$

where g_0 denotes the Euclidean metric. We consider the semiclassical Laplace–Beltrami operator

$$(4.2) \quad P_g(h) = -\frac{h^2}{2} \Delta_g$$

and denote by $S_{g,h}(\lambda)$ its scattering matrix at the fixed energy $\lambda > 0$, with the Euclidean Laplacian as the free reference operator.

Assume that the geodesic flow of g is non-trapping. Since g is Euclidean outside a compact set, every geodesic has incoming and outgoing Euclidean asymptotic data. As in the potential case, these data define the classical scattering map

$$(4.3) \quad \kappa_{g,\lambda} : T^*\mathbb{S}^{n-1} \longrightarrow T^*\mathbb{S}^{n-1}.$$

Although Ingremau formulates his Gaussian-state propagation result for potential scattering,

he points out that the same result holds for compactly supported metric perturbations of the Laplacian, with a similar proof [8]. Consequently, the proof of Lemma 2.1 applies in the metric setting as well. The relation between the semiclassical scattering matrix and the classical scattering relation has also been studied by Hassell and Wunsch [7] for nontrapping scattering metrics.

Lemma 4.1. *Let g_1 and g_2 be smooth Riemannian metrics on $\mathbb{R}^n$ which coincide with the Euclidean metric outside a compact set, and assume that their geodesic flows are non-trapping. If*

$$(4.4) \quad \liminf_{h \rightarrow 0} \|S_{g_1, h}(\lambda) - S_{g_2, h}(\lambda)\|_{\mathcal{B}(L^2(\mathbb{S}^{n-1}))} < \sqrt{2},$$

then

$$(4.5) \quad \kappa_{g_1, \lambda} = \kappa_{g_2, \lambda}.$$

Choose $R > 0$ so that both metrics are Euclidean near and outside $\partial B(0, R)$. Since geodesics are straight lines outside this ball, their asymptotic data determine their entry and exit points and directions at the boundary. Thus, equality of the scattering maps in Lemma 4.1 implies equality of the boundary scattering relations. To apply the lens rigidity theorem of Stefanov–Uhlmann–Vasy, we also need equality of the lengths of the corresponding geodesics inside the ball. This follows from the next lemma.

Lemma 4.2. *Let $n \geq 2$, and let g_1 and g_2 be two smooth nontrapping Riemannian metrics on $M = \overline{B(0, R)}$, equal to the Euclidean metric near ∂M . If their scattering relations coincide, then their lens data coincide.*

Proof. We use the first variation argument recalled in [28, Section 1.3]. Let $\partial_- SM$ denote the common set of strictly inward-pointing unit vectors at ∂M . For $(x, v) \in \partial_- SM$, let $\ell_j(x, v)$ be the length of the g_j -geodesic starting from x with initial velocity v , up to its first exit from M . The functions ℓ_j are smooth, since both metrics are nontrapping and the boundary is strictly convex.

Let $s \mapsto (x(s), v(s))$ be any smooth curve in $\partial_- SM$. Here s parametrizes a family of initial data. Since the scattering relations are equal, the two geodesics corresponding to each s have the same exit point $y(s)$ and exit direction $w(s)$. The first variation of length gives

$$(4.6) \quad \frac{d}{ds} \ell_j(x(s), v(s)) = \langle w(s), y'(s) \rangle - \langle v(s), x'(s) \rangle, \quad j = 1, 2.$$

Only the endpoints contribute to this formula, since the curves are geodesics. Both endpoints lie on ∂M , where the two metrics are Euclidean. The right-hand side is the same for $j = 1, 2$. Since the curve of initial data is arbitrary, we obtain $d(\ell_1 - \ell_2) = 0$. Moreover, $\partial_- SM$ is connected for $n \geq 2$. Thus $\ell_1 - \ell_2$ is constant.

To determine this constant, choose an inward direction sufficiently close to a tangent direction at the boundary. The corresponding geodesics are then the same short Euclidean chord, contained in the neighborhood of ∂M where both metrics are Euclidean. Their lengths are equal, so the constant is zero. Therefore $\ell_1 = \ell_2$, which proves the lemma. $\square$

We now apply the lens rigidity theorem of Stefanov–Uhlmann–Vasy [26, Theorem 8.1]. We assume that one of the metrics admits a smooth strictly convex function. Here, strict convexity means that the Riemannian Hessian of this function is positive definite. In the global Cartesian coordinates on $\mathbb{R}^n$, this Hessian is given by

$$(4.7) \quad (\text{Hess}_g f)_{ij} = \partial_i \partial_j f - \sum_{k=1}^n \Gamma_{ij}^k(g) \partial_k f, \quad 1 \leq i, j \leq n,$$

where

$$(4.8) \quad \Gamma_{ij}^k(g) = \frac{1}{2} \sum_{\ell=1}^n g^{k\ell} (\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij})$$

are the Christoffel symbols of g , and $(g^{k\ell})$ is the inverse matrix of $(g_{k\ell})$. For the Euclidean metric, the Christoffel symbols vanish, and (4.7) reduces to the usual Hessian.

Theorem 4.3. *Let g_1 and g_2 be smooth Riemannian metrics on $\mathbb{R}^n$, $n \geq 3$, with nontrapping geodesic flows. Assume that there exists $R > 0$ such that*

$$(4.9) \quad \text{supp}(g_j - g_0) \Subset B(0, R), \quad j = 1, 2.$$

Set $M = \overline{B(0, R)}$. Assume also that there exists $f \in C^\infty(M)$ such that

$$(4.10) \quad \text{Hess}_{g_1} f > 0 \quad \text{on } M, \quad f^{-1}(0) = \partial M.$$

If, for some fixed $\lambda > 0$,

$$(4.11) \quad \liminf_{h \rightarrow 0} \|S_{g_1, h}(\lambda) - S_{g_2, h}(\lambda)\|_{\mathcal{B}(L^2(\mathbb{S}^{n-1}))} < \sqrt{2},$$

then there exists a smooth diffeomorphism

$$(4.12) \quad \psi : M \longrightarrow M, \quad \psi|_{\partial M} = \text{Id},$$

such that

$$(4.13) \quad g_1 = \psi^* g_2.$$

The condition on f is imposed only on g_1 . The function f has a unique critical point in the interior of M , where it attains its minimum. This is allowed in [26, Theorem 8.1]. No assumption on conjugate points is needed. The relation between strictly convex functions and convex foliations is explained in [26, Section 8].

We give some examples before proving the theorem. The Hessian computations for Examples 2 and 3 are given in Appendix A.

Example 1 (Euclidean metric and small perturbations). For $g_1 = g_0$, take

$$f(x) = |x|^2 - R^2.$$

Then

$$\text{Hess}_{g_0} f = 2g_0 > 0, \quad f^{-1}(0) = \partial M.$$

The same function works for any metric g_1 sufficiently close to g_0 in $C^1(M)$. Indeed, in Euclidean coordinates,

$$(\text{Hess}_{g_1} f)_{ij} = 2\delta_{ij} - 2 \sum_{k=1}^n \Gamma_{ij}^k(g_1) x_k,$$

which remains positive definite when g_1 is sufficiently close to g_0 in $C^1(M)$. This example includes perturbations which are not radial.

Example 2 (Radial conformal metrics). Let

$$g_1 = c(r)^{-2} g_0, \quad r = |x|,$$

where c is smooth and positive, constant near $r = 0$, and equal to 1 near and outside $r = R$. Assume the Herglotz condition

$$(4.14) \quad \frac{d}{dr} \left(\frac{r}{c(r)} \right) > 0, \quad 0 < r \leq R.$$

This condition expresses strict convexity of the Euclidean spheres for the metric g_1 ; see [25, Proposition 6.1] and [26, Section 8]. In this radial case, we can give the function explicitly:

$$f(x) = - \int_{|x|}^R \frac{s}{c(s)^2} ds.$$

It is smooth at the origin, since c is constant there, and $f^{-1}(0) = \partial M$. A direct computation gives

$$\text{Hess}_{g_1} f = \left(1 - \frac{rc'(r)}{c(r)} \right) g_1.$$

Thus (4.14) implies (4.10). In particular, this example does not require the metric to be close to the Euclidean one.

Example 3 (Radial warped metrics). In polar coordinates, consider

$$g_1 = dr^2 + a(r)^2 g_{\mathbb{S}^{n-1}},$$

where $g_{\mathbb{S}^{n-1}}$ is the standard sphere metric. Assume that a is smooth, that $a(r) = r$ near $r = 0$ and near and outside $r = R$, and that

$$a(r) > 0, \quad a'(r) > 0, \quad r > 0.$$

The metric is then smooth at the origin and Euclidean outside a compact subset of $B(0, R)$. For $f(x) = |x|^2 - R^2$, direct computation gives, for $r > 0$,

$$\text{Hess}_{g_1} f = 2 dr^2 + 2r a(r) a'(r) g_{\mathbb{S}^{n-1}} > 0.$$

Near the origin, this is simply $2g_0$. For instance, one may take

$$a(r) = r + \varepsilon \chi(r), \quad \chi \in C_c^\infty((0, R)),$$

provided

$$|\varepsilon| \|\chi'\|_\infty < 1.$$

There are also geometric conditions which ensure the existence of a suitable function. By [26, Corollary 1.1 and Section 8], this is the case for a compact manifold with strictly convex boundary if it is simply connected and has non-positive sectional curvature, if it is simply connected and has no focal points, or if it has non-negative sectional curvature. Thus the geometric conditions used in the simple case are covered by the same lens rigidity result.

For completeness, the convexity assumption already implies that g_1 is nontrapping in M . Indeed, compactness gives a constant $c_0 > 0$ such that, along any unit-speed g_1 -geodesic contained in M ,

$$\frac{d^2}{dt^2} f(\gamma(t)) = \text{Hess}_{g_1} f(\dot{\gamma}(t), \dot{\gamma}(t)) \geq c_0.$$

Such a geodesic cannot stay in M for all positive time, since f is bounded on M . Outside the ball, geodesics are straight lines. We have kept the nontrapping assumption for both metrics in the statement to make the use of Lemma 4.1 explicit.

Proof of Theorem 4.3. By Lemma 4.1, (4.11) gives

$$\kappa_{g_1, \lambda} = \kappa_{g_2, \lambda}.$$

Since both metrics are Euclidean outside M , the asymptotic lines determine the entry and exit points and directions at ∂M . After normalizing the velocities to unit speed, we obtain

$$(4.15) \quad L_{g_1} = L_{g_2} \quad \text{on } \partial_- SM.$$

Lemma 4.2 then gives

$$\ell_{g_1} = \ell_{g_2}.$$

Hence the two metrics have the same lens data.

The boundary of M is strictly convex for both metrics, since they are Euclidean near ∂M . Together with (4.10), these are the assumptions of [26, Theorem 8.1]. That theorem gives a diffeomorphism $\psi : M \rightarrow M$, equal to the identity on ∂M , such that $g_1 = \psi^* g_2$. $\square$

Taking the Euclidean metric as the reference metric gives the following consequence.

Corollary 4.4. *Let g be a smooth Riemannian metric on $\mathbb{R}^n$, $n \geq 3$, with nontrapping geodesic flow. Assume that, for some $R > 0$,*

$$(4.16) \quad \text{supp}(g - g_0) \subseteq B(0, R).$$

If, for some fixed $\lambda > 0$,

$$(4.17) \quad \liminf_{h \rightarrow 0} \|S_{g, h}(\lambda) - S_{g_0, h}(\lambda)\|_{\mathcal{B}(L^2(\mathbb{S}^{n-1}))} < \sqrt{2},$$

then there exists a smooth diffeomorphism

$$(4.18) \quad \psi : \overline{B(0, R)} \longrightarrow \overline{B(0, R)}, \quad \psi|_{\partial B(0, R)} = \text{Id},$$

such that

$$(4.19) \quad g = \psi^* g_0.$$

Proof. Apply Theorem 4.3 with $g_1 = g_0$, $g_2 = g$, and $f(x) = |x|^2 - R^2$. We obtain a diffeomorphism ϕ fixing the boundary such that $g_0 = \phi^* g$. Taking $\psi = \phi^{-1}$ gives the stated conclusion. $\square$

Remark 4.5. The argument uses the full lens data, rather than the boundary distance function. The common Euclidean neighborhood of the boundary allows us to recover the geodesic lengths from the scattering relation by Lemma 4.2. Thus simplicity is not needed in the theorem or in the Euclidean corollary.

Remark 4.6. Muñoz-Thon [14] studies scattering rigidity in the more general setting of simple magnetic-potential systems. In the purely Riemannian case, his results include metrics in a fixed conformal class and real-analytic metrics. Here we consider only Riemannian metrics, but we do not assume simplicity. We use instead a strictly convex function and the lens rigidity theorem of Stefanov–Uhlmann–Vasy. Moreover, our assumption on the semiclassical scattering matrices is quantitative: we only require $\liminf_{h \rightarrow 0} \|S_{g_1, h}(\lambda) - S_{g_2, h}(\lambda)\|_{\mathcal{B}(L^2(\mathbb{S}^{n-1}))} < \sqrt{2}$.

APPENDIX A. HESSIAN COMPUTATIONS FOR THE RADIAL EXAMPLES

We use the coordinate formula

$$(\text{Hess}_g f)_{ij} = \partial_i \partial_j f - \sum_k \Gamma_{ij}^k \partial_k f.$$

In Example 2, $g_1 = c(r)^{-2} g_0$ and $f'(r) = r/c(r)^2$. For $r > 0$, the Christoffel symbols in Cartesian coordinates are

$$\Gamma_{ij}^k = -\frac{c'(r)}{rc(r)} (x_i \delta_{jk} + x_j \delta_{ik} - x_k \delta_{ij}).$$

Since

$$\partial_i f = \frac{x_i}{c(r)^2}, \quad \partial_i \partial_j f = \frac{\delta_{ij}}{c(r)^2} - \frac{2c'(r)}{rc(r)^3} x_i x_j,$$

substitution gives

$$\text{Hess}_{g_1} f = \left(1 - \frac{rc'(r)}{c(r)}\right) g_1.$$

The coefficient is positive by the Herglotz condition, since $1 - rc'/c = c(r)(r/c(r))'$.

In Example 3, write $g_1 = dr^2 + a(r)^2 \gamma$, where $\gamma = g_{\mathbb{S}^{n-1}}$, and let $f = r^2 - R^2$. With angular indices α, β , the relevant Christoffel symbols are

$$\Gamma_{rr}^r = \Gamma_{r\alpha}^r = 0, \quad \Gamma_{\alpha\beta}^r = -a(r)a'(r)\gamma_{\alpha\beta}.$$

Since $\partial_r f = 2r$ and the angular derivatives of f vanish,

$$\text{Hess}_{g_1} f = 2dr^2 + 2ra(r)a'(r)\gamma,$$

which is positive definite for $r > 0$, since $a(r) > 0$ and $a'(r) > 0$.

These computations extend smoothly to the origin: since c is constant there in Example 2 and $a(r) = r$ there in Example 3, the Hessians near the origin are respectively g_1 and $2g_0$.

AI USAGE DISCLOSURE

The author used AI tools for assistance with language editing, graphical preparation, and checking the computations in Section 3 and Appendix A. The author takes full responsibility for the mathematical content of the paper.

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