An inverse scattering problem with the Aharonov-Bohm effect

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A direct and an inverse scattering method is developed for Schrödinger operators with electromagnetic fields in the case of obstacles in order to study the well-known Aharonov-Bohm effect. In dimension greater or equal to three, we show that the electric potential and the magnetic field are uniquely determined by the S-operator. In the two-dimensional case, some obstruction appears based on a quantification of the magnetic flux.

I. INTRODUCTION.

In classical mechanics, the electromagnetic field strength describes completely the electromagnetic effects (Lorentz force). In quantum mechanics, this is not the case. In 1959, Aharonov and Bohm[1], using the Schrödinger equation, considered the scattering of an electron off an impenetrable solenoid and found an effect (interferences). This showed that in quantum mechanics, the electromagnetic field acts on charged particles even in a region where this field is zero.

The Aharonov-Bohm phenomenon has attracted a great deal of interest : it has been studied from a spectral viewpoint[2], and in a quantum scattering process[3] (radial magnetic interaction in the two-dimensional case).

In this note, we study Schrödinger operators with an electric potential and a magnetic field in the presence of obstacles. In section III, we will hide the singularity of a magnetic potential inside the obstacle, a singularity which accounts for the "Aharonov-Bohm effect" : when magnetic potentials have singularities, Hamiltonians associated with the same magnetic fields need not be gauge equivalent.

The goals of this paper are the following :

a - we show existence and asymptotic completeness of the Moeller wave operators in order to define the scattering operator.

b - we study the inverse scattering problem in the presence of the Aharonov-Bohm effect.

In the case of two-body Schrödinger Hamiltonians $H = -\Delta + V$ on $L^2(I\!R^n)$, $n \geq 2$, V short-range, such a problem has been studied with high-frequency asymptotic methods[4]. Recently, for short or long range potentials, Enss and Weder[5] have used a geometrical method. They show that the potential is uniquely determined by the high velocity limit of the S-operator.

This method can be used to study Hamiltonians with electric and magnetic potentials[6] on $L^2(I\!R^n)$, the Dirac equation[7], and the N-body case[5] (see also Ref. 8,9,10 for similar problems with different approachs).
In Ref. 11, we used a stationary method to study Hamiltonians with smooth electric and magnetic potentials, based on the construction of suitable modified wave operators. This approach gives the complete asymptotic expansion of the S-operator at high energies. In this paper, we will see that the problem with obstacles can be treated in the same way by determining a class of test functions which have negligible interaction with the obstacle (see Lemma 10). To our knowledge, the Enss-Weder method has not yet been used for such a problem.

First, let us begin by some notation :

$H_0 = -\frac{1}{2}\Delta$ is the free Schrödinger operator on $L^2(I\!R^n)$, $n \geq 2$, with domain $H^2(I\!R^n)$.
We consider a compact obstacle K in $I\!R^n$, $0 \in K$, with smooth boundary and we denote $\Omega = I\!R^n \backslash K$.
We define the Hamiltonian H as the following differential operator on Ω :

$$H \ = H(A,V) \ = \ \frac{1}{2} \ (D - A(x))^2 + V(x) \ , \ D = -i\nabla \ . \tag{1.1}$$

We suppose that the electrostatic potential $V \in C^\infty(\overline{\Omega})$, with the short-range condition :

$$\mid \partial_x^\alpha V(x) \mid \leq \ C_\alpha \ < x >^{-\delta-|\alpha|} \ , \ \delta > 1 \ , \tag{H_1}$$

where $< x >= (1+x^2)^{\frac{1}{2}}$.

Let B be the two-form magnetic field, $B \in C^\infty(I\!R^n)$, (identified with an antisymmetric matrix (b_{jk})), with the decay condition :

$$\mid \partial_x^\alpha B(x) \mid \leq \ C_\alpha \ < x >^{-\mu-|\alpha|} \ , \ \mu > \frac{3}{2} \ . \tag{H_2}$$

We consider the magnetic potential, (transversality gauge), $A \in C^\infty(I\!R^n)$, $(B = dA)$, given by :

$$A(x) \ = \ -\int_0^1 \ sB(sx).x \ ds \ . \tag{1.2}$$

It is easy to see that A satisfies the following estimations :

$$\mid \partial_x^\alpha A(x) \mid \leq \ C_\alpha \ < x >^{-\rho-|\alpha|} \ , \ \rho = min \ (1, \mu - 1) > \frac{1}{2} \ , \tag{1.3}$$

and the geometrical condition,

$$\forall x \in I\!R^n \ , \ A(x).x \ = \ 0 \ . \tag{1.4}$$

We denote also by H the Dirichlet realization, with domain $D(H) = H^2(\Omega) \cap H_0^1(\Omega)$, and with essential spectrum $\sigma_{ess}(H) = [0, +\infty[$.

Finally , let $\mathcal{I}$ be the restriction operator, $\mathcal{I} : L^2(I\!R^n) \to L^2(\Omega)$ defined by $\mathcal{I}\Phi = \Phi_{|\Omega}$.

II. EXISTENCE AND COMPLETENESS OF THE WAVE OPERATORS.

A. Existence of the wave operators.

In order to define the wave operators $W^{\pm}$, we use the two-Hilbert spaces setting[12,13,14].

Lemma 1
The wave operators $W^{\pm} : L^2(\mathbb{R}^n) \to L^2(\Omega)$ *defined by*

$$W^{\pm} \ = \ s - \lim_{t \to \pm\infty} \ e^{itH} \ \mathcal{I} \ e^{-itH_0} \ , \tag{2.1}$$

exist and are isometric.

Proof
Let $\chi \in C^{\infty}(\mathbb{R}^n)$ such that $\chi = 0$ in a neighborhood of K, $\chi = 1$ in a neighborhood of infinity. For Φ such that $\hat{\Phi} \in C_0^{\infty}(\mathbb{R}^n)$, we have :

$$e^{itH} \ \mathcal{I} \ e^{-itH_0} \ \Phi \ = \ e^{itH} \ \mathcal{I} \ \chi \ \ e^{-itH_0} \ \Phi + o(1) \ , \ t \to \pm\infty \ .$$

Using Cook's argument, it is easy to show[15,16] that the right-hand side admits a limit when $t \to \pm\infty$.
To see that the wave operators are isometric, it suffices to show :

$$\lim_{t \to \pm\infty} \ || \, \mathcal{I} e^{-itH_0} \Phi \, ||_{L^2(\Omega)} = || \, \Phi \, ||_{L^2(\mathbb{R}^n)} \ .$$

We have :

$$|| \, \mathcal{I} e^{-itH_0} \Phi \, ||^2_{L^2(\Omega)} \ = < \mathcal{I} e^{-itH_0} \Phi , \mathcal{I} e^{-itH_0} \Phi >_{L^2(\Omega)} \ ,$$

$$=< (\mathcal{I}^* \mathcal{I} - 1) e^{-itH_0} \Phi , e^{-itH_0} \Phi >_{L^2(\mathbb{R}^n)} + || \, \Phi \, ||^2_{L^2(\mathbb{R}^n)} \ .$$

Since $\mathcal{I}^* \mathcal{I} - 1$=multiplication by $\mathbf{1}_K$, and K is compact, $(\mathcal{I}^* \mathcal{I} - 1) e^{-itH_0} \Phi \to 0$ when $t \to \pm\infty$. $\square$

B. Completeness of the wave operators.

First, let us recall the notion of completeness for the wave operators $W^{\pm} = W^{\pm}(H, H_0, \mathcal{I})$ in a two-Hilbert spaces setting[12,13,14]. The wave operators $W^{\pm}(H, H_0, \mathcal{I})$ are complete if $Im \ W^{\pm} \ = \ \mathcal{H}^{(ac)}(H)$, (subspace of absolute continuity of H).

Remarks
By the intertwining principle, $Im \ W^{\pm} \subset \mathcal{H}^{(ac)}(H)$. To show the inverse inclusion, we follow a method close to White's[13], which uses the stationary approach of Isozaki and Kitada[17]. This approach is also used in section III to study the inverse scattering problem. We only give the main steps.
First, we write the wave operator $W^{\pm}$ differently, using a Fourier Integral Operator, (F.I.O), in the definition (2.1). As in Ref. 16, we introduce the Fourier integral operator $J^{\pm}$ with phase $\varphi^{\pm}$ defined on $\Gamma^{\pm}$ by :

$$\varphi^{\pm}(x, \xi) \ = \ x.\xi + c_A^{\pm}(x, \xi) \ , \tag{2.2}$$

where

$$c_A^{\pm}(x, \xi) \ = \ - \int_0^{\pm\infty} \ A(x + t\xi).\xi \ dt \ , \tag{2.3}$$

and

$$\Gamma^{\pm} \ = \ \{ \ (x, \xi) \in \mathbb{R}^{2n} \ : | \ x \ |\geq R \ , | \ \xi \ |\geq \theta \ , \pm \ x.\xi \geq \pm \ \sigma \ | \ x \ | \ | \ \xi \ | \ \} \ , \sigma \in]-1, 1[\ .$$

Γ^- (resp. Γ^+) is called an incoming (resp. outgoing) zone. The amplitude of the F.I.O is given by $\chi_0(x)$ where $\chi_0(x) = 0$ in a neighborhood of K and $\chi_0(x) = 1$ in a neighborhood of infinity.
We have[16] :

$$W^\pm\ \chi(H_0)\ =\ s - \lim_{t \to \pm\infty}\ e^{itH}\ \mathcal{I}\ J^\pm\ e^{-itH_0}\ \chi(H_0)\ ,\ \forall \chi \in C_0^\infty([2\ \theta^2, \infty[)\ . \tag{2.4}$$

Now, we are going to prove that $\mathcal{I}J^\pm$ is a suitable approximation of $W^\pm$, when we localize in incoming (resp. outgoing) zones. We follow the Derezinski-Gérard's presentation[18]. We denote by $p^\pm$ a cut-off function with support in $\Gamma^\pm$ and $p^\pm(x,D)$ is the usual pseudo-differential operator with symbol $p^\pm$, (left-quantification).

Lemma 2

$$W^\pm\ \chi(H_0)\ p^\pm(x,D)\ =\ \mathcal{I}J^\pm \chi(H_0) p^\pm(x,D) + R^\pm\ , \tag{2.5}$$

where $R^\pm : L^2(\mathbb{R}^n) \to L^2(\Omega)$ is a compact operator.

Proof
We only sketch the proof. For more details, (see Ref. 18 Theorem 4.17.2, and Ref. 16). We treate the case (+). By (2.4),

$$W^+\ \chi(H_0)\ p^+(x,D)\ =\ \mathcal{I}J^+\chi(H_0)p^+(x,D) + \int_0^\infty\ R^+(t)\ dt\ ,$$

where

$$R^+(t)\ =\ ie^{itH}\ (H\mathcal{I}J^+ - \mathcal{I}J^+H_0)\ e^{-itH_0}\chi(H_0)p^+(x,D).$$

We easily see that

$$||\ R^+(t)\ ||_{\mathcal{L}(L^2(\mathbb{R}^n),L^2(\Omega))} \leq ||\ \mathcal{I}S^+(t)\ ||_{\mathcal{L}(L^2(\mathbb{R}^n),L^2(\Omega))}\ ,$$

with

$$|| < t >^{\nu-\epsilon} < D >^N < x >^\epsilon\ S^+(t)\ ||_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C_{N,\epsilon}\ ,$$

where $\epsilon > 0$ and $\nu = min\ (\delta, 2\rho, \rho+1) > 1$. We conlude by Rellich's theorem. □

Assuming the following technical lemma, (the proof is given in Appendix), we can formulate the main theorem of this section.

Lemma 3
Let $\Phi \in \mathcal{H}^{ac}(H)$ such that $\chi(H)\Phi = \Phi$. Then :

$$\lim_{t \to \pm\infty}\ (W^\pm\ \chi(H_0)\ p^\pm(x,D)J^{\pm *}\ \mathcal{I}^* - 1)\ e^{-itH}\Phi = 0\ . \tag{2.6}$$

Theorem 4
The wave operators $W^\pm$ are complete.

Proof
We have to prove that $\mathcal{H}^{ac}(H) \subset Im\ W^\pm$. It suffices to show that :

$$\forall \Phi \in \mathcal{H}^{ac}(H) \text{ such that } \chi(H)\Phi = \Phi, \text{with } \ \chi \in C_0^\infty([2\theta^2, \infty[)\ ,\ \text{ we have } \Phi \in Im\ W^\pm\ .$$

Using (2.6), we have, (for the case (+)),

$$W^+ \ \chi(H_0)\ p^+(x,D)J^{+*}\mathcal{I}^*\ e^{-itH}\Phi - e^{-itH}\Phi \ \to 0 \ \ , \ \ t\to+\infty \ ,$$

and since e^{itH} is unitary,

$$e^{itH}\ W^+\ \chi(H_0)\ p^+(x,D)J^{+*}\mathcal{I}^*\ e^{-itH}\Phi - \Phi \ \to 0 \ \ , \ \ t\to+\infty \ .$$

Using the intertwining principle,

$$\Phi \ = \ \lim_{t\to+\infty} W^+ \ [\ e^{itH_0}\ \chi(H_0)\ p^+(x,D)J^{+*}\mathcal{I}^*\ e^{-itH}\Phi].$$

Since $Im\ W^+$ is closed, Theorem 4 is proved. □

By Theorem 4, we can define the scattering operator, $S = W^{+*}W^- \ : \ L^2(I\!R^n) \to L^2(I\!R^n)$.

III. THE INVERSE SCATTERING PROBLEM.

In the previous section, we have defined the scattering operator $S = S(A,V)$. The goal of this section is to answer the natural question : can one determine the electromagnetic field in Ω from the S-operator ?

In order to solve this problem, we have to make some technical assumptions. We suppose in this section that K is a convex set and moreover :

$$B \ \in \ C_0^\infty(I\!R^n) \qquad\qquad (H_2')$$

In this section, we prefer to work with the 1-form magnetic potential $A = (A_1,...,A_n)$ with the Coulomb gauge $(div\ A = 0)$, given by :

$$A_j(x) \ = \ \frac{1}{\sigma_{n-1}} \sum_{k=1}^{n} \int_{I\!R^n} \frac{x_k - y_k}{\mid x-y \mid^n}\ b_{jk}(y)\ dy \ , \ \ \forall x \in I\!R^n, \qquad (3.1)$$

where σ_{n-1} is the area of the unit sphere in $I\!R^n$.

In order to give some properties of these potentials, we define

$$E(x) = \int_1^{+\infty} sB(sx).x\ ds \ , \qquad (3.2)$$

and in the two dimensional case let β be the normalized magnetic flux given by :

$$\beta = \frac{1}{2\pi} \int_{I\!R^2} b(x)\ dx \ \ \text{where} \ \ b = b_{12} = -b_{21}. \qquad (3.3)$$

We have the following lemma[19] :

Lemma 5
The magnetic potential A is smooth and satisfies in Ω :

$$For\ n \geq 3,\ A(x) = E(x) + \nabla\varphi(x). \qquad (i)$$

$$For\ n = 2,\ A(x) = \frac{\beta}{\mid x \mid^2}(-x_2, x_1) + E(x) + \nabla\varphi(x). \qquad (ii)$$

where $\partial_x^\alpha \varphi(x) = O(<x>^{-1-|\alpha|})$.

Remark
In dimension $n \geq 3$, $A(x)$ is a classical short-range perturbation, and in the two dimensional case, $A(x)$ is a short range perturbation of the transversality gauge given by (1.2), so everything done in section II works and we can define the scattering operator $S = S(A,V)$.

A. Results.

Let V_1, V_2 be potentials satisfying (H_1). Let B_1, B_2 be magnetic fields satisfying (H_2') and let A_j, $j = 1, 2$ be the associated magnetic potentials with the Coulomb gauge.
For elementary topological reasons, we will see that we obtain different results in the n-dimensional case, $n \geq 3$, and in the two-dimensional case.

Theorem 6 : *(the n-dimensional case, $n \geq 3$).*
Under the hypotheses (H_1), (H_2'), we have :

$$S(A_1, V_1) = S(A_2, V_2) \iff B_1 = B_2 \text{ and } V_1 = V_2 \text{ in } \Omega \ .$$

In the two dimensional case, the result is more complicated; a quantification condition on the magnetic flux appears. Moreover, in order to use a support theorem for the Radon transform[20], we have to assume that :

$$V \text{ satisfies } (H_1) \ , \text{ and } \forall m \geq 0 \ , \ < x >^m V \text{ is bounded} \qquad (H_1')$$

Finally, let us denote by $\theta(x) = \tan^{-1}(x_2/x_1) \in [0, 2\pi[$ the azimuthal angle from the positive x_1 axis, let β_j be the flux over 2π of B_j, and $B_j = b_j \ dx_1 \wedge dx_2$.

Theorem 7 : *(the two-dimensional case).*
Under the hypotheses (H_1'), (H_2'), we assume that $S(A_1, V_1) = S(A_2, V_2)$.
Then : $b_1 = b_2 := b$ in Ω, $\beta_1 = \beta_2$ mod 2, $(\beta_1 - \beta_2)b + V_1 - V_2 = 0$ in Ω.

Proposition 8 : *(the Aharonov-Bohm effect).*
Under the hypotheses (H_1'), (H_2') with supp $B_j \subset K$,
(i) $S(A_1, V_1) = S(A_2, V_2) \Longrightarrow \beta_1 = \beta_2$ mod 2, $V_1 = V_2$ in Ω.
(ii) $\beta_1 = \beta_2$ mod 2, $V_1 = V_2$ in $\Omega \Longrightarrow S(A_1, V_1) = e^{-i(\beta_1 - \beta_2)\theta(D)} \ S(A_2, V_2) \ e^{i(\beta_1 - \beta_2)\theta(D)}$.

Comments
Ruijsenaars[3] obtained similar results in $\mathbb{R}^2$: when the electric potential $V \equiv 0$, $K = D(0, R)$, B is radial with support in K and $A(x_1, x_2) = \dfrac{\beta}{| x |^2}(-x_2, x_1)$ in Ω, we can calculate explicitly the scattering amplitude.
Actually, Ruijsenaars obtains two different formulas for the scattering amplitude. This difference comes from the interpretation of the operator $i\partial_\theta$, (different boundary conditions for the ray $\theta = \pi$), where θ is the azimuthal angle from the positive x_1 axis.
In the first interpretation, $i\partial_\theta$ acts on

$$D_1 = \{ \ \varphi \ s.t \ \lim_{\theta \to -\pi} \ \varphi(\theta) = \lim_{\theta \to \pi} \ \varphi(\theta) \ \}.$$

In this case, the scattering amplitude is given by :

$$T_\beta^1(\lambda, \theta, \theta') \ = (2i\pi k)^{-\frac{1}{2}} \sum_{m=-\infty}^{+\infty} (-e^{2i\delta_m(\beta)} \frac{H^{(2)}_{|m+\beta|}(kR)}{H^{(1)}_{|m+\beta|}(kR)} - 1) \ e^{im(\theta - \theta')}, \qquad (3.4)$$

where $k = \sqrt{\lambda}$, $\delta_m(\beta) = \frac{\pi}{2}(| \ m \ | - | \ m + \beta \ |)$ and $H_\nu^{(1)}, H_\nu^{(2)}$ are the Hankel functions.
In the second interpretation, $i\partial_\theta$ acts on

$$D_2 = \{ \ \varphi \ s.t \ \lim_{\theta \to -\pi} \ \varphi(\theta) = e^{i2\beta\pi} \ \lim_{\theta \to \pi} \ \varphi(\theta) \ \},$$

allowing a singular gauge transformation. Then, the scattering amplitude is :

$$T_\beta^2(\lambda, \theta, \theta') \ = e^{-i\beta(\theta - \theta')} \ e^{-i\beta\pi\epsilon(\theta')} \ T_0^1(\lambda, \theta, \theta') \ , \qquad (3.5)$$

where T_0^1 is given by (3.4) with $\beta = 0$, $\epsilon(\theta') = 1$ if $\theta' > 0$ and $\epsilon(\theta') = -1$ if $\theta' < 0$. In these two cases, one has for n integer, $\theta \neq \theta'$, $j = 1, 2$,

$$T_{\beta+n}^j(\lambda, \theta, \theta') = (-1)^n \ T_\beta^j(\lambda, \theta, \theta') e^{-in(\theta-\theta')},$$

and the cross sections, (scattering measurable quantities), given by

$$\left(\frac{d\sigma}{d\theta}\right)_{\beta,j} = | \ T_\beta^j(\lambda, \theta, \theta') \ |^2$$

are periodic in β with period 1.

In Theorem 7 and Proposition 8 (i), we consider an equality between the scattering operators : this is stronger than an equality between the cross sections. So, our problem is not well-posed from a physical point of view; we make the conjecture that $\beta_1 = \beta_2$, but Proposition 8 (ii), which is certainly known, shows that with our approach (one studies the high energy asymptotics of the scattering operator) we cannot prove uniqueness of the magnetic flux. Indeed, let B be a magnetic field with compact support in K and with $\beta \in 2\mathbb{Z}$, and let A be the magnetic potential satisfying the Coulomb gauge. By Lemma 5 (ii) and Proposition 8 (ii), we have off the 0-energy :

$$S(A,0) = e^{-i\beta\theta(D)} \ S(0,0) \ e^{i\beta\theta(D)}.$$

Using Theorem 9, (see below for the notation), one cannot obtain a more precise information about β. Indeed,

$$< e^{i\sqrt{\lambda}x.\omega}\Phi \ , \ (S(A,0)-1) \ e^{i\sqrt{\lambda}x.\omega}\Psi > = < e^{i\sqrt{\lambda}x.\omega} \ e^{i\beta\theta(\frac{D}{\sqrt{\lambda}}+\omega)}\Phi \ , \ (S(0,0)-1) \ e^{i\sqrt{\lambda}x.\omega} \ e^{i\beta\theta(\frac{D}{\sqrt{\lambda}}+\omega)}\Psi >$$

and $\forall N \geq 1$,

$$e^{i\beta\theta(\frac{D}{\sqrt{\lambda}}+\omega)}\Phi = e^{i\beta\theta(\omega)} \sum_{|\alpha| \leq N-1} a_\alpha \ \lambda^{-\frac{|\alpha|}{2}} D^\alpha \Phi + O(\lambda^{-\frac{N}{2}}),$$

for suitable constants a_α. Since Supp $D^\alpha\Phi \subset X_\omega$, we obtain :

$$< e^{i\sqrt{\lambda}x.\omega}\Phi \ , \ (S(A,0)-1) \ e^{i\sqrt{\lambda}x.\omega}\Psi > = \ O(\lambda^{-\infty}).$$

The proof of these theorems is based on the study of the high energy asymptotics of the scattering operator off the obstacle. This is the subject of the next subsection.

B. High energy asympotics of the S-operator.

The method is very close to the approach in Ref. 11. Before giving the main proposition of this section, we need some notation.

Notation

Let $\omega \in S^{n-1}$ fixed, S^{n-1} is the unit sphere of $\mathbb{R}^n$. Let X_ω be the following set :

$$X_\omega \ = \ \{ \ x \in \mathbb{R}^n \ : \ \forall t \in \mathbb{R}, \ x + t\omega \ \in \Omega \ \}.$$

We define

$$c_A(x,\xi) = c_A^-(x,\xi) - c_A^+(x,\xi) = \int_{-\infty}^{+\infty} \ A(x+t\xi).\xi \ dt \ .$$

$$R_\pm(x,\xi) \ = \ - \int_0^{\pm\infty} \ B(x+t\xi).\xi \ dt \ \ ,$$

and

$$f^\pm(x,\xi) \ = \frac{1}{2} \ (| \ R_\pm(x,\xi) \ |^2 - i \ \mathrm{div} \ R_\pm(x,\xi)) \ \ .$$

We have the following result, where $<,>$ is the usual scalar product in $L^2(\mathbb{R}^n)$.

Theorem 9
Under the hypotheses (H_1), (H_2), $\forall \Phi,\ \Psi \in C_0^\infty(X_\omega)$, *we have the asymptotic expansion for* $\lambda \to +\infty$ *:*

$$< e^{i\sqrt{\lambda}x.\omega}\Phi\ ,\ S\ e^{i\sqrt{\lambda}x.\omega}\Psi > \sim \sum_{j=0}^{+\infty} \lambda^{-\frac{j}{2}}\ < \Phi,\ a_{j,\omega}(x,D)\ \ \Psi > ,$$

where $a_{j,\omega}(x,D)$ *are differential operators. In particular,*

$$a_{0,\omega}(x,D)\ =\ e^{ic_A(x,\omega)}\ ,$$

$$a_{1,\omega}(x,D)\ =\ -i\ e^{ic_A(x,\omega)}\left(\int_{-\infty}^{+\infty} V(x+t\omega)\ dt\ + a_{A,\omega}(x,D)\right)\ ,$$

where $a_{A,\omega}(x,D)$ *is a differential operator depending only on* A *given by :*

$$a_{A,\omega}(x,\xi)\ =\ \int_0^{+\infty} \overline{f}^+(x+t\omega,\omega)\ dt + \int_{-\infty}^0 f^-(x+t\omega,\omega)\ dt$$

$$-i\sum_{k=1}^n \partial^2_{x_k\xi_k} c_A^+(x,\omega) + \partial_\xi c_A^+(x,\omega).\partial_x c_A(x,\omega) - \partial_\xi c_A(x,\omega).\xi\ .$$

Proof
We only sketch the proof, (see Ref. 11 for details).

• **Step 1 :**
In Ref. 11, we work with functions Φ, Ψ s.t $\widehat{\Phi}, \widehat{\Psi}$ have compact support. So, at high energies, translation of wave packets dominates over spreading.
In this paper, since we localize Φ, Ψ in X_ω, we have to introduce an energy cut-off function (depending on λ). We consider $\chi_0 \in C_0^\infty(I\!R^n)$ such that $\chi_0(\xi) = 1$ if $|\ \xi\ | \leq 1$, $\chi_0(\xi) = 0$ if $|\ \xi\ | \geq 2$. It is clear that : $\forall \Phi \in C_0^\infty(I\!R^n)$, $\forall \epsilon > 0, \forall N \geq 1$,

$$||\ [\chi_0(\frac{D}{\lambda^\epsilon}) - 1]\ \Phi\ ||_{L^2(I\!R^n)} = O(\lambda^{-N}) \tag{3.6}$$

• **Step 2** :
Using (3.6), it suffices to calculate the asymptotic expansion of :

$$\Omega^\pm(\lambda,\omega)\ \chi_0(\frac{D}{\lambda^\epsilon})\Phi\ =\ e^{-i\sqrt{\lambda}x.\omega} W^\pm e^{i\sqrt{\lambda}x.\omega}\ \chi_0(\frac{D}{\lambda^\epsilon})\Phi\ ,$$

$$=\ \lim_{t\to\pm\infty}\ e^{itH(\lambda,\omega)}\ \mathcal{I}\ e^{-itH_0(\lambda,\omega)}\ \chi_0(\frac{D}{\lambda^\epsilon})\Phi\ ,$$

since

$$< e^{i\sqrt{\lambda}x.\omega}\Phi\ ,\ S\ e^{i\sqrt{\lambda}x.\omega}\Psi > = < \Omega^+(\lambda,\omega)\ \chi_0(\frac{D}{\lambda^\epsilon})\Phi,\ \Omega^-(\lambda,\omega)\ \chi_0(\frac{D}{\lambda^\epsilon})\Psi > + O(\lambda^{-N})\ ,$$

where

$$H(\lambda,\omega)\ =\ e^{-i\sqrt{\lambda}x.\omega}\ H\ e^{i\sqrt{\lambda}x.\omega}\ \text{ on } L^2(\Omega)\ ,$$

$$H_0(\lambda,\omega)\ =\ e^{-i\sqrt{\lambda}x.\omega}\ H_0\ e^{i\sqrt{\lambda}x.\omega}\ \text{ on } L^2(I\!R^n)\ .$$

• **Step 3** :

In Ref. 11, in order to calculate the asymptotic expansion of $\Omega^{\pm}(\lambda,\omega)$, we have constructed an energy modifier (F.I.O) $J_N^{\pm}(\lambda,\omega)$ for $N \geq 1$. We follow the same strategy, but since we work with an obstacle, we have to introduce a cut-off function $\chi^{\pm}$ which localizes in the classical propagation zone.

First, let us begin with some notation : for V_ω neighborhood of ω in S^{n-1}, let

$$O^{\pm} \ = \ \{ \ x + t\omega' \ ; \ x \in Supp \ \Phi, \ t \in \mathbb{R}^{\pm}, \ \omega' \in V_\omega \ \} \ .$$

If V_ω is rather small, $O^{\pm} \subset \Omega$. We define a cut-off function $\chi^{\pm} \in C^\infty(\Omega)$, $\chi^{\pm} \equiv 1$ in a conical neighborhood of $O^{\pm}$. Now, we can define the energy modifier $J_N^{\pm}(\lambda,\omega)$, $N \geq 1$.

The amplitude of the F.I.O $J_N^{\pm}(\lambda,\omega)$ is given by :

$$e_N^{\pm}(x,\xi,\lambda,\omega) \ = \ \chi^{\pm}(x) \sum_{m=0}^{N-1} \lambda^{-\frac{m}{2}} \ d_m^{\pm}(x,\xi,\omega) \ ,$$

where $d_0^{\pm} = 1$ and the functions $d_m^{\pm}$ satisfy for $m \geq 0$ the transport equations :

$$\omega.\nabla d_{m+1}^{\pm} \ = \ -i \ V(x) \ d_m^{\pm} - i \sum_{|\alpha|+j=m} \frac{1}{\alpha!} \ \xi^\alpha \ \partial_\xi^\alpha f^{\pm}(x,\omega) \ d_j^{\pm}$$

$$- \sum_{|\alpha|+j=m} \frac{1}{\alpha!} \ \xi^\alpha \ \partial_\xi^\alpha R^{\pm}(x,\omega) \ . \ \nabla d_j^{\pm} \ + \frac{i}{2} \ \Delta d_m^{\pm} \ - \ \xi \ . \ \nabla d_m^{\pm} \ ,$$

where $d_m^{\pm}$ is written for $d_m^{\pm}(x,\xi,\omega)$. We can easily solve these equations for $x \in Supp \ \chi^{\pm}$ remarking that $x \in Supp \ \chi^{\pm}$ implies $x + t\omega \in Supp \ \chi^{\pm}, \ t \in \mathbb{R}^{\pm}$.

The phase of the F.I.O $J_N^{\pm}(\lambda,\omega)$ is given for $(x, \xi + \sqrt{\lambda}\omega) \in \Gamma^{\pm}$ by :

$$\varphi_N^{\pm}(x,\xi,\lambda,\omega) \ = \ x.\xi + c_A^{\pm}(x, \xi + \sqrt{\lambda}\omega) \ .$$

We define

$$J(\lambda) \ = (\chi^{\pm} - 1) \ e^{-itH_0(\lambda,\omega)} \ \chi_0(\frac{D}{\lambda^\epsilon}) \ \Phi \ ,$$

and we have the following lemma :

Lemma 10
For $\lambda >> 1, \ \epsilon < \frac{1}{2}, \ t \in \mathbb{R}^{\pm}$, we have : $\forall N \geq 1$,

$$|| \ J(\lambda) \ ||_{L^2(\mathbb{R}^n)} = O(< t >^{-N} \ \lambda^{-N}).$$

Proof
We have :

$$J(\lambda) \ = (\chi^{\pm} - 1) \ \lambda^{n\epsilon} \int (\int e^{i\varphi(\xi)} \ \chi_0(\xi) \ d\xi) \Phi(y) \ dy,$$

where $\varphi(\xi) = \lambda^\epsilon (x-y).\xi - \frac{t}{2}(\lambda^\epsilon \xi + \sqrt{\lambda}\omega)^2$.
So, $\partial_\xi \varphi(\xi) \ = \ \lambda^\epsilon [x - (y + \lambda^{\frac{1}{2}} t(\omega + \lambda^{-\frac{1}{2}+\epsilon}\xi))]$. Since $| \ \xi \ | = O(1)$ and $\epsilon < \frac{1}{2}, \ y \in Supp \ \Phi$, we easily obtain for $x \in Supp \ (\chi^{\pm} - 1), \ \lambda >> 1$,

$$| \ \partial_\xi \varphi(\xi) \ | \geq c \ \lambda^\epsilon (1 + | \ t \ | \ \lambda^{\frac{1}{2}}) \geq c \ \lambda^\epsilon < t >,$$

for a suitable constant $c > 0$. We conclude by a standard non stationary phase argument. □

Lemma 11
For $\lambda >> 1$,

$$\Omega^{\pm}(\lambda,\omega)\ \chi_0(\frac{D}{\lambda^{\epsilon}})\Phi = \lim_{t\to\pm\infty}\ e^{itH(\lambda,\omega)}\ \mathcal{I}J_N^{\pm}(\lambda,\omega)\ e^{-itH_0(\lambda,\omega)}\ \chi_0(\frac{D}{\lambda^{\epsilon}})\Phi\ .$$

Proof
Let $K_N^{\pm}(\lambda,\omega)$=F.I.O with phase $\varphi_N^{\pm}(x,\xi,\lambda,\omega)$ and with amplitude 1. We have[11] :

$$\lim_{t\to\pm\infty}\ (K_N^{\pm}(\lambda,\omega)-1)\ e^{-itH_0(\lambda,\omega)}\ \chi_0(\frac{D}{\lambda^{\epsilon}})\Phi\ = 0\ . \tag{3.7}$$

Since $e_N^{\pm}(x,\xi,\lambda,\omega) = \chi^{\pm} + O(<x>^{-\nu+1})$, $\nu > 1$, it suffices to prove :

$$\lim_{t\to\pm\infty}\ (\chi^{\pm}\ K_N^{\pm}(\lambda,\omega)-1)\ e^{-itH_0(\lambda,\omega)}\ \chi_0(\frac{D}{\lambda^{\epsilon}})\Phi\ = 0\ .$$

Remarking that

$$\chi^{\pm}K_N^{\pm}(\lambda,\omega)-1 = \chi^{\pm}\ (K_N^{\pm}(\lambda,\omega)-1)+(\chi^{\pm}-1)\ .$$

we conclude by using Lemma 10. □

• **Step 4** :
We have :

$$||\ (\Omega^{\pm}(\lambda,\omega)-\mathcal{I}J_N^{\pm}(\lambda,\omega))\ \chi_0(\frac{D}{\lambda^{\epsilon}})\Phi\ || = O(\lambda^{-\frac{N}{2}})\ . \tag{3.8}$$

Indeed, everything done in (Ref. 11, Proposition 10) works for $\Omega^{\pm}(\lambda,\omega)\ \chi_0(\frac{D}{\lambda^{\epsilon}})\Phi$. It suffices to remark that all the contributions coming from the cut-off function $\chi^{\pm}$ are negligible using the same arguments as in Lemma 10.

• **Step 5** :
Using Steps 1-4,

$$< e^{i\sqrt{\lambda}x.\omega}\Phi\ ,\ S\ e^{i\sqrt{\lambda}x.\omega}\Psi > = <\ \mathcal{I}J_N^{+}(\lambda,\omega)\chi_0(\frac{D}{\lambda^{\epsilon}})\Phi,\ \mathcal{I}J_N^{-}(\lambda,\omega)\chi_0(\frac{D}{\lambda^{\epsilon}})\Psi\ >_{L^2(\Omega)}\ + O(\lambda^{-\frac{N}{2}})\ ,$$

$$= <\ J_N^{+}(\lambda,\omega)\chi_0(\frac{D}{\lambda^{\epsilon}})\Phi,\ J_N^{-}(\lambda,\omega)\chi_0(\frac{D}{\lambda^{\epsilon}})\Psi\ >_{L^2(\mathbb{R}^n)} + O(\lambda^{-\frac{N}{2}})\ ,$$

$$= <\ \chi_0(\frac{D}{\lambda^{\epsilon}})\Phi,\ J_N^{+*}(\lambda,\omega)\ J_N^{-}(\lambda,\omega)\chi_0(\frac{D}{\lambda^{\epsilon}})\Psi\ >_{L^2(\mathbb{R}^n)} + O(\lambda^{-\frac{N}{2}})\ ,$$

$$= <\ \Phi,\ J_N^{+*}(\lambda,\omega)\ J_N^{-}(\lambda,\omega)\ \Psi\ >_{L^2(\mathbb{R}^n)} + O(\lambda^{-\frac{N}{2}})\ ,$$

and we conclude as in Ref. 11. □

C. Proof of the n-dimensional case, $n \geq 3$.

We suppose that $S(A_1,V_1) = S(A_2,V_2)$. We denote by $A = A_1 - A_2$ and $V = V_1 - V_2$. The goal of this section is to prove that $B = dA = 0,\ V = 0$ in Ω.

Using the first term in the asymptotic expansion in Theorem 9, we obtain :

$$\forall\omega\in S^{n-1}\ ,\ \forall x\in X_{\omega}\ ,\quad \int_{-\infty}^{+\infty} A(x+t\omega).\omega\ dt\ \ =\ 2k(x,\omega)\pi\ , \tag{3.9}$$

where $k(x,\omega)\in\mathbb{Z}$. We have the following lemma :

Lemma 12
Let $B \in C_0^\infty(\mathbb{R}^n)$. We suppose that (3.9) is satisfied. Then : $B \equiv 0$ in Ω .

Proof
Step 1 :
Without loss of generality, working with $\partial_x^\alpha A(x)$ instead of $A(x)$, we can always assume that $A \in L^1(\mathbb{R}^n)$ and $k(x,\omega) = 0$. Indeed, differentiating (3.9), since $k(x,\omega)$ is locally constant, we have :

$$\forall \alpha \in \mathbb{N}^n \quad , \ \forall x \in X_\omega \ , \ \int_{-\infty}^{+\infty} \partial_x^\alpha A(x+t\omega)\ .\ \omega\ dt \ = \ 0 \quad , \tag{3.10}$$

and by Lemma 5, for $\mid \alpha \mid \geq n$, it is clear that $\partial_x^\alpha A(x) \ \in L^1(\mathbb{R}^n)$.

Step 2 :
Since K is a convex set, it suffices to show that $B \equiv 0$ on every

$$P_{a,\omega_0} \ = \ \{x \in \mathbb{R}^n \ : \ x\ .\ \omega_0 > a\ \} \ \subset \ \Omega \quad , \quad \text{where } a > 0,\ \omega_0 \in S^{n-1}\ .$$

Without loss of generality, we can suppose that $\omega_0 = e_1$ where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$. We denote by Π_0 the hyperplane generated by $(e_2, \ldots, e_n)$. We write $x = (x_1, x') \in \mathbb{R}^n$, $x' \in \Pi_0$. Finally Π_ω is the orthogonal hyperplane to ω.

For $x_1 > a,\ \omega \in S^{n-1} \cap \Pi_0,\ \xi' \in \Pi_\omega \cap \Pi_0$, we have :

$$\widehat{A}(x_1,\xi').\omega \ = \ \int_{\Pi_0} e^{-ix'.\xi'}\ A(x_1,x').\omega\ dx'\ , \tag{3.11}$$

where $\widehat{A}$ is the Fourier transform in the $x'-$variable. Writting $x' = y + t\omega$ with $y \in \Pi_\omega \cap \Pi_0$, we obtain :

$$\widehat{A}(x_1,\xi').\omega \ = \ \int_{\Pi_\omega \cap \Pi_0} e^{-iy.\xi'} \left(\int_{-\infty}^{+\infty} A(x_1, y+t\omega).\omega\ dt \right)\ dy \ = \ 0\ , \tag{3.12}$$

by hypothesis. So, we have showed :

$$\forall \omega \in S^{n-1} \cap \Pi_0\ ,\ \forall \xi' \in \Pi_\omega \cap \Pi_0\ ,\ \widehat{A}(x_1,\xi').\omega = 0\ , \tag{3.13}$$

or equivalently,

$$\forall \xi \in \Pi_0 \backslash 0,\ \forall \omega \in \Pi_\xi \cap \Pi_0\ ,\ \widehat{A}(x_1,\xi).\omega = 0 \quad . \tag{3.14}$$

Thus, there exist $c_1(\xi),\ c_1(\xi) \in \mathbb{C}$ such that :

$$\widehat{A}(x_1,\xi) \ = \ c_1(\xi)e_1 + c_2(\xi)\xi\ . \tag{3.15}$$

Since $b_{jk}(x) = \partial_j A_{(k)}(x) - \partial_k A_{(j)}(x)$, where $A = (A_{(1)}, ..., A_{(n)})$, we have by (3.15) for $j,k \geq 2,\ \forall \xi \in \Pi_0 \backslash 0$,

$$\widehat{b_{jk}}(x_1,\xi) \ = \ i(\xi_j \widehat{A_{(k)}}(x_1,\xi) - \xi_k \widehat{A_{(j)}}(x_1,\xi)) \ = \ 0\ .$$

By the injectivity of the Fourier transform on $L^1(\mathbb{R}^n)$, one has :

$$\forall j,k \geq 2,\ b_{jk} = 0 \text{ in } P_{a,e_1}\ . \tag{3.16}$$

Step 3 :
Now, we prove that $b_{j1}(x) = 0$ in P_{a,e_1}. To do this, we prove that $b_{j1}(x) = 0$ in $P_{a+\epsilon,e_1}$, $\forall \epsilon > 0$, using a support theorem for the Radon Transform.
Let V_{e_1} be a neighborood in S^{n-1} of e_1 such that : $\forall e \in V_{e_1}$,

$$P_{a+\epsilon,e} \cap Supp\ B \subset P_{a,e_1}\ .$$

We claim that for $x \in P_{a+\epsilon,e}$ and $\omega \in S^{n-1} \cap \Pi_e$,

$$\int_{-\infty}^{+\infty} b_{j1}(x+t\omega)\ dt \ = \ 0 \ . \tag{3.17}$$

Using the support theorem for the Radon transform, (Ref. 20, Lemma 2.11), we obtain $b_{j1}(x)=0$ in $P_{a+\epsilon,e_1}$. Now, we prove (3.17). By a standard continuity argument, it suffices to prove (3.17) for $e \in V_{e_1}$ and $\omega = (\omega_1, \dots \omega_n) \in S^{n-1} \cap \Pi_e$, $\omega_1 \neq 0$. First, let us remark that $\forall x \in P_{a+\epsilon,e}$, $\omega \in S^{n-1} \cap \Pi_e$:

$$\partial_x \ \left(\int_{-\infty}^{+\infty} A(x+t\omega)\ .\ \omega\ dt \right) \ = \ \int_{-\infty}^{+\infty} B(x+t\omega)\ .\ \omega\ dt \ = \ 0 \ . \tag{3.18}$$

The j^{th} component, $j \geq 2$, gives, using (3.16) :

$$\sum_{k=1}^{n} \int_{-\infty}^{+\infty} b_{jk}(x+t\omega)\ \omega_k\ dt \ = \ \int_{-\infty}^{+\infty} b_{j1}(x+t\omega)\ \omega_1\ dt = 0 \ . \ \Box$$

Proof of Theorem 6.
($\Longrightarrow$) Since $B_1 = B_2$ in Ω, we have by Lemma 5, $A_1 = A_2 + \nabla\varphi$. So, $S(A_1,V_1) = S(A_2+\nabla\varphi, V_1) = S(A_2,V_1) = S(A_2,V_2)$, by gauge invariance.
Studying the second term in the asymptotic expansion in Theorem 9 for $S(A_2,V_1) - S(A_2,V_2)$, we obtain :

$$\forall\omega \in S^{n-1} \ \ , \ \forall x \in X_\omega \ , \ \int_{-\infty}^{+\infty} V(x+t\omega)\ dt \ = \ 0 \ . \tag{3.19}$$

Working on hyperplanes P which do not intersect K and since $dim\ P \geq 2$, we obtain as in (Ref.11, Lemma 5), that $V=0$ in Ω.

($\Longleftarrow$) Obvious by gauge invariance. $\Box$

D. Proof of Theorem 7.

First, we prove that $B = d(A_1 - A_2) \equiv 0$ in Ω. We could use Lemma 12 but we prefer to give an elementary proof. As in (3.18), we have :

$$\forall\omega \in S^1 \ \ , \ \ \forall x \in X_\omega \ , \ \int_{-\infty}^{+\infty} B(x+t\omega)\ .\ \omega\ dt \ = \ 0 \ . \tag{3.20}$$

Thus, since $b_{12} = -b_{21}$, we obtain :

$$\forall j = 1,2, \ \forall\omega \in S^1 \ \ , \ \ \forall x \in X_\omega \ , \ \int_{-\infty}^{+\infty} b_{12}(x+t\omega)\ .\ \omega_j\ dt \ = \ 0 \ .$$

So, since $\mid \omega \mid = 1$, we have :

$$\forall\omega \in S^1 \ \ , \ \ \forall x \in X_\omega \ , \ \int_{-\infty}^{+\infty} b_{12}(x+t\omega)\ dt \ = \ 0 \ . \tag{3.21}$$

By the support theorem for the Radon transform[20], we obtain $B=0$ in Ω. Thus, in Ω, by Lemma 5, $A_1 = A_2 + A_\beta + \nabla\varphi$ where :

$$A_\beta(x) = \frac{\beta}{\mid x \mid^2}(-x_2, x_1), \quad \beta = \beta_1 - \beta_2. \tag{3.22}$$

So, $S(A_1,V_1) = S(A_2 + A_\beta + \nabla\varphi, V_1) = S(A_2 + A_\beta, V_1) = S(A_2, V_2)$. Using Theorem 9 again, we obtain $c_{A_\beta}(x,\omega) \in 2\pi\mathbb{Z}$ and an easy calculation shows us that $c_{A_\beta}(x,\omega) = \pm\beta\pi$ if $\pm x.\omega' > 0$ where (ω,ω') is a direct orthogonal basis of $I\!R^2$. So, $\beta = 0 \ mod \ 2$. Finally, the last equality follows from studying the second term in the asymptotic expansion in Theorem 9; using the relations for $j = 1, 2$,

$$\partial_x c_{A_j}(x, \omega) = \int_{-\infty}^{+\infty} b_j(x + t\omega) \ dt \ \ (-\omega_2, \ \omega_1) \ ,$$

and $\partial_\xi c_{A_j}^{\pm}(x, \omega) = \beta_j(-\omega_2, \omega_1)$, we have :

$$\forall \omega \in S^1 \ \ , \ \ \forall x \in X_\omega \ , \ \int_{-\infty}^{+\infty} (\beta b + V_1 - V_2)(x + t\omega) \ dt \ = \ 0 \ .$$

where $b = b_1 = b_2$. We conclude by using the support theorem for the Radon transform[20].

Proof of Proposition 8.
(i) Obvious using Theorem 7.
(ii) Conversely, this result is certainly well-known. If we suppose that $\beta_1 = \beta_2 \ mod \ 2$, and $V_1 = V_2$ in Ω, then we easily see[19], that

$$H(A_1, V_1) \ = \ e^{-i\beta\theta(x)} \ H(A_2 + \nabla\varphi, V_2) \ e^{i\beta\theta(x)} \quad \text{on } L^2(\Omega) \ , \tag{3.23}$$

where $\theta(x) = \ \tan^{-1}(x_2/x_1) \in [0, 2\pi[$ is the azimuthal angle from the positive x_1 axis. Let us remark that since β is an integer,

$$e^{-i\beta\theta(x)} \in C^\infty(I\!R^2 \backslash 0) \ . \tag{3.24}$$

Moreover, $\theta(x)$ is 0-homogeneous. So, it follows that far from the 0-energy,

$$S(A_1, V) \ = \ e^{-i\beta\theta(D)} \ S(A_2, V) \ e^{i\beta\theta(-D)} \ . \tag{3.25}$$

Since $\theta(-D) = \theta(D) + \pi$ and $\beta \in 2Z\!\!Z$, we obtain the result. □

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APPENDIX

In this section, we give the proof of Lemma 3. We denote by $\gamma_\infty(\mathcal{H})$ (resp. $\gamma_1(\mathcal{H})$) the set of compact (resp. class-trace) operators on $\mathcal{H}$.

1. A preliminary lemma.

Lemma 13
Let $\chi \in C_0^\infty(I\!R_+)$. Then :

$$\chi(H_0) - \mathcal{I}^* \chi(H) \mathcal{I} \ \in \gamma_\infty(L^2(I\!R^n)). \tag{i}$$

$$\chi(H_0)\mathcal{I}^* - \mathcal{I}^* \chi(H) \ , \ \ \mathcal{I}\chi(H_0)\mathcal{I}^* - \chi(H) \in \gamma_\infty(L^2(\Omega)). \tag{ii}$$

Proof
Since $\mathcal{I}\mathcal{I}^* = id_{L^2(\Omega)}$, it suffices to show (i). Let us denote by H_0^D the self-adjoint Dirichlet realization of $-\frac{1}{2}\Delta$ on $L^2(\Omega)$. Since H is a H_0^D-compact perturbation of H_0^D, it is well known that $\chi(H) - \chi(H_0^D) \in \gamma_\infty(L^2(\Omega))$. So, we have only to prove the following assertion :

$$\chi(H_0) - \mathcal{I}^*\chi(H_0^D)\mathcal{I} \in \gamma_\infty(L^2(I\!\!R^n)) \ .$$

On the other hand, we have[21],

$$\forall t > 0 \ , \ e^{-tH_0} - \mathcal{I}^* e^{-tH_0^D}\mathcal{I} \in \gamma_1(L^2(I\!\!R^n)) \ .$$

Let $z \in \mathbb{C}$ such that $Re\ z < 0$. Since H_0 et H_0^D are positive, we can define $R_0(z) = (H_0 - z)^{-1}$, (resp. $R_0^D(z) = (H_0^D - z)^{-1}$). We deduce that :

$$R_0(z) - \mathcal{I}^* R_0^D(z)\mathcal{I} \ = \ \int_0^\infty \ (e^{-tH_0} - \mathcal{I}^* e^{-tH_0^D}\mathcal{I}) \ e^{tz} \ dt \in \gamma_1(L^2(I\!\!R^n)) \ .$$

By the Stone-Weierstrass theorem, we obtain the result. □

2. Proof of Lemma 3.

We consider the case $(+)$. By Lemma 2,

$$[W^+\chi(H_0)p^+(x,D)J^{+*}\mathcal{I}^* - 1]e^{-itH}\Phi \ = \ [(\mathcal{I}J^+\chi(H_0)p^+(x,D) + R^+)J^{+*}\mathcal{I}^* - 1]e^{-itH}\Phi \ ,$$

$$= \ [\mathcal{I}J^+\chi(H_0)p^+(x,D)J^{+*}\mathcal{I}^* - 1]e^{-itH}\Phi + o(1) \ ,$$

since $e^{-itH}\Phi \to 0$ weakly and $R^+J^{+*}\mathcal{I}^*$ is a compact operator. Thus,

$$(L.H.S) = \ [\mathcal{I}J^+\chi(H_0)J^{+*}\mathcal{I}^* - 1]e^{-itH}\Phi + \mathcal{I}J^+\chi(H_0)(p^+(x,D) - 1)J^{+*}\mathcal{I}^*e^{-itH}\Phi + o(1) \ ,$$

denoted by $(1) + (2) + o(1)$.

• **Step 1** :
We easily verify that $J^+\chi(H_0)J^{+*} - \chi(H_0)$ is a compact operator on $L^2(I\!\!R^n)$. Thus,

$$(1) \ = \ [\mathcal{I}\chi(H_0)\mathcal{I}^* - 1]e^{-itH}\Phi + o(1) \ ,$$

$$= \ [\mathcal{I}\chi(H_0)\mathcal{I}^* - \chi(H)]e^{-itH}\Phi + o(1) \ .$$

By Lemma 13, we have $(1) = o(1)$.

• **Step 2** :

$$(2) = -\mathcal{I}J^+\chi(H_0)(p^+(x,D) - 1)J^{+*}(J^-J^{-*} - 1)\mathcal{I}^*e^{-itH}\Phi$$

$$+\mathcal{I}J^+\chi(H_0)(p^+(x,D) - 1)J^{+*}J^-J^{-*}\mathcal{I}^*e^{-itH}\Phi \ .$$

By Lemma 13,

$$(J^-J^{-*} - 1)\mathcal{I}^*e^{-itH}\Phi = (J^-J^{-*} - 1)\mathcal{I}^*\chi(H)e^{-itH}\Phi \ ,$$

$$= (J^-J^{-*} - 1)\chi(H_0)\mathcal{I}^*e^{-itH}\Phi + o(1) = o(1) \ ,$$

since $(J^- J^{-*} - 1)\chi(H_0)$ is a compact operator. Thus,

$$(2) \ = \ \mathcal{I} J^+ \chi(H_0)(p^+(x,D)-1) J^{+*} J^- J^{-*} \mathcal{I}^* e^{-itH} \Phi + o(1) \ ,$$

$$= -\mathcal{I} J^+ \chi(H_0)(p^+(x,D)-1) J^{+*} J^- (p^-(x,D)-1) J^{-*} \mathcal{I}^* e^{-itH} \Phi$$

$$+\mathcal{I} J^+ \chi(H_0)(p^+(x,D)-1) J^{+*} J^- p^-(x,D) J^{-*} \mathcal{I}^* e^{-itH} \Phi + o(1) \ .$$

Let us verify that the last term of this expression is $= o(1)$. We have :

$$p^-(x,D) J^{-*} \mathcal{I}^* e^{-itH} \Phi = p^-(x,D) J^{-*} \mathcal{I}^* \chi(H) e^{-itH} \Phi \ ,$$

$$= p^-(x,D) J^{-*} \chi(H_0) \mathcal{I}^* e^{-itH} \Phi + o(1) \ ,$$

$$= p^{-*}(x,D) \chi(H_0) J^{-*} \mathcal{I}^* e^{-itH} \Phi + o(1) \ ,$$

since $[J^{-*}, \chi(H_0)]$ and $(p^-(x,D) - p^{-*}(x,D))\chi(H_0)$ are compact. Then, by Lemma 2,

$$p^-(x,D) J^{-*} \mathcal{I}^* e^{-itH} \Phi \ = \ p^{-*}(x,D) \chi(H_0) W^{-*} e^{-itH} \Phi + o(1) \ ,$$

$$= p^{-*}(x,D) \chi(H_0) e^{-itH_0} (W^{-*} \Phi) + o(1) = o(1) \ ,$$

(well-known propagation estimates for the free group). Thus,

$$(2) \ = \ -\mathcal{I} J^+ \chi(H_0)(p^+(x,D)-1) J^{+*} J^- (p^-(x,D)-1) J^{-*} \mathcal{I}^* e^{-itH} \Phi + o(1) \ .$$

We easily conclude that $(2) = o(1)$ since $\chi(H_0)(p^+(x,D)-1)J^{+*}J^-(p^-(x,D)-1)$ is a compact operator, (this is a pseudo-differential operator with symbol $= O(<x>^{-1}<\xi>^{-N})$). □

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