A constructive procedure to recover asymptotics of short-range or long-range potentials.

François Nicoleau

Département de Mathématiques
Laboratoire Jean Leray
U.M.R 6629 - Université de Nantes
2, rue de la Houssinière BP 92208
F-44322 Nantes cedex 03

e-mail : nicoleau@math.univ-nantes.fr

Abstract

We study an inverse scattering problem for a pair of Hamiltonians (H, H_0) on $L^2(I\!\!R^n)$, where $H_0 = -\Delta$ and $H = H_0 + V$, V being a short-range or long-range potential. By an elementary constructive method, we show that the scattering operator S, which is localized near a fixed energy $\lambda > 0$, determines the asymptotics of the potential V at infinity, in dimension $n \geq 3$. This is done by studying the action of the scattering operator on suitable wave packets.

1 Introduction.

In this short note, we study an inverse scattering problem for the pair (H, H_0) on $L^2(I\!\!R^n)$, $n \geq 2$, where the free operator is $H_0 = -\Delta$ with domain $D(H_0) = H^2(I\!\!R^n)$, and

(1.1) $$H = H_0 + V,$$

with a potential $V \in C^\infty(I\!\!R^n)$ satisfying for each $\alpha \in I\!\!N^n$,

(H_1) $$\mid \partial_x^\alpha V(x) \mid \leq \ C_\alpha < x >^{-\rho-|\alpha|} \ , \ \rho > 1,$$

where $< x >= (1+ \mid x \mid^2)^{\frac{1}{2}}$. Under the hypothesis (H_1), the wave operators :

(1.2) $$W^\pm(H, H_0) \ = \ s - \lim_{t \to \pm\infty} \ e^{itH} e^{-itH_0},$$

exist and they are complete, i.e $Ran\ W^{\pm}(H,H_0) = \mathcal{H}_{ac}(H)$ = subspace of absolute continuity of H, (see [13] for example).

Let S be the scattering operator defined by :

$$S = W^{+*}(H,H_0)\ W^{-}(H,H_0). \tag{1.3}$$

In order to localize the scattering operator near a fixed energy $\lambda > 0$, we introduce a cut-off function $\chi \in C_0^\infty(]0,+\infty[)$, $\chi = 1$ in a neighborhood of $\lambda > 0$.

The goal of this note is to obtain information about the potential from $S\chi(H_0)$. We show that for $n \geq 3$, the operator $S\chi(H_0)$ determines the asymptotics of the potential at infinity, by studying :

$$F(h) = < S\chi(H_0)\Phi_{h,\omega}, \Psi_{h,\omega} > , \tag{1.4}$$

where $<,>$ is the usual scalar product in $L^2(I\!R^n)$, $\Phi_{h,\omega}$, $\Psi_{h,\omega}$ are suitable test functions, (see section 2), and $h \to 0$.

Our method is close to [12] where a scattering process with an obstacle and electromagnetic potentials at high energy is studied. As in [11]-[12], we use a stationary approach and we obtain the asymptotics of $F(h)$. We emphasize that the complete asymptotic expansion of $F(h)$ is necessary to recover the complete asymptotics of the potential V. Note that in [1], the authors study a scattering problem with potentials at high energies with a nice time-dependent method, but they only obtain the leading term of the high energies asymptotics.

In [4], it is shown using Fadeev's approach, under the hypothesis (H_1) with $\rho > n$, that the scattering amplitude $T(\mu,\theta,\omega)$, $\mu \in [\lambda, \lambda+\delta]$, $\delta > 0$, $\theta, \omega \in S^{n-1}$, determines the Fourier transform of V on $\{\ \xi \in I\!R^n,\ |\ \xi\ | \leq 2\sqrt{\lambda}\ \}$. In [5], this result is extended to the case $\rho > \frac{3}{2}$.

In ([8], Corollary 1.1), M. S. Joshi and A. Sa Barreto show that the scattering amplitude at a fixed energy $\lambda > 0$ determines the asymptotics of the potential. More precisely, they show that if $V_1,\ V_2 \in S_{cl}^{-2}(I\!R^n)$ for $n \geq 3$, and if the associated matrices at some non-zero fixed energy are equal up to smooth terms, then $V_1 - V_2 \in S^{-\infty}(I\!R^n)$. Their proof uses Hörmander's calculus of Fourier operators and the calculus of Legendrian distributions developed by Melrose and Zworski [10]. In [9], this approach is used to recover asymptotics of metrics from fixed energy scattering data. In [7], a more explicit approach for short-range potentials on $I\!R^n$ is proposed, based on the resolution of an eikonal equation.

We emphasize that the approach used in [7]-[8] for classical symbols is more general than our method : the authors only assume that the scattering matrices associated to the potentials $V_j \in S_{cl}^{-2}(I\!R^n)$ are equal at some fixed energy $\lambda > 0$ up to smooth terms.

Nevertheless it seems that our method is very elementary and constructive : the complete asymptotics of the potential appears simply and entirely in Theorem 2. Moreover, it allows us to study generic short-range potentials, (see Theorem 1 and Corollary 6).

Finally, working with modified wave operators, it is not difficult to extend our results to the case of long-range potentials ($\rho > 0$), (see section 4 for details).

2 Asymptotics for the localized scattering operator.

2.1 Definition of the test functions.

First, let us define $U(h)$, $h > 0$, as an unitary operator on $L^2(I\!R^n)$ by :

$$U(h)\Phi(x) = h^{\frac{n}{2}}\ \Phi(hx). \tag{2.1}$$

We also need a technical energy cut-off function $\chi_0 \in C_0^\infty(I\!R^n)$ such that $\chi_0(\xi) = 1$ if $|\ \xi\ | \leq 1$, $\chi_0(\xi) = 0$ if $|\ \xi\ | \geq 2$. As in [12], χ_0 is only introduced to control the spreading of the wave packets $\Phi_{h,\omega}$.

For $\omega \in S^{n-1}$, we write $x \in I\!R^n$ as $x = y + t\omega$, $y \in \Pi_\omega =$ orthogonal hyperplane to ω and we consider :

$$X_\omega = \{x = y + t\omega \in I\!R^n\ :\ |\ y\ | \geq 1\ \}. \tag{2.2}$$

Definition
For $\Phi \in C_0^\infty(X_\omega)$ and suitable $\epsilon > 0$,

$$\Phi_{h,\omega} = e^{i\sqrt{\lambda}x.\omega}\ U(h)\ \chi_0(h^\epsilon D)\ \Phi\ , \tag{2.3}$$

where $D = -i\nabla$, ($\Psi_{h,\omega}$ is defined in the same way with $\Psi \in C_0^\infty(X_\omega)$).

2.2 Asymptotic expansion of the scattering operator.

In this section, we prove the following theorems :

Theorem 1
Assume that $\epsilon < 1$. We have, when $h \to 0$,

$$< (S-1)\chi(H_0)\Phi_{h,\omega}, \Psi_{h,\omega} >= \frac{1}{2i\sqrt{\lambda}}\ < \int_{-\infty}^{+\infty}\ V(h^{-1}x + t\omega)\ dt\ \Phi, \Psi > +O(h^\mu), \tag{2.4}$$

where $\mu = min\ (2(\rho-1),\ \rho)$.

Remarks
a - Using (H_1), it is easy to see that the first term of the (R.H.S) of (2.4) is equal to $O(h^{\rho-1})$ since Φ, Ψ have their support in X_ω.

b - In section 3, using (2.4), we shall recover the leading term of the asymptotics of V at infinity.

If $V \in S_{cl}^{-2}(I\!R^n)$, a similar approach allows us to obtain a complete asymptotic expansion of $< (S-1)\chi(H_0)\Phi_{h,\omega}, \Psi_{h,\omega} >$.

Recall that the class $S^{-2}_{cl}(I\!R^n)$ of classical symbols of order -2 is the set of the smooth functions $V(x)$ such that there exist a sequence of smooth homogeneous functions f_j of order $-j$ such that

$$(2.5)\qquad \mid \partial_x^\alpha \left(V(x) - \sum_{j=2}^{N-1} f_j(x) \right) \mid \leq C_{\alpha,N} < x >^{-N-|\alpha|}$$

Now, we can formulate :

Theorem 2
Assume that $V \in S^{-2}_{cl}(I\!R^n)$*. We have the following asymptotic expansion when* $h \to 0$ *:*

$$(2.6)\qquad < (S-1)\chi(H_0)\Phi_{h,\omega}, \Psi_{h,\omega} > \sim \sum_{j\geq 1} h^j < \Phi, A_j(x,\omega,D)\ \Psi >,$$

where $A_j(x,\omega,D)$ *is a differential operator given by :*

$$(2.7)\qquad A_j(x,\omega,D) = \frac{i}{2\sqrt{\lambda}} \int_{-\infty}^{+\infty} f_{j+1}(x+t\omega)\ dt + B_j(x,\omega,D),$$

with $B_1 = 0$ *and for* $j \geq 2$, $B_j(x,\omega,D)$ *is a differential operator only depending on the functions* f_k, $2 \leq k \leq j$.

With this asymptotic expansion, we shall recover V modulo $S^{-\infty}$, (see section 3.2).

2.3 Proof of Theorem 1.

• Step 1 :

Let us begin by an elementary lemma.

Lemma 3
For $\epsilon < 1$ *and* $h \ll 1$, *we have :*

$$(2.8)\qquad \chi(H_0)\Phi_{h,\omega} = \Phi_{h,\omega}.$$

Proof
We easily obtain :

$$(2.9)\qquad \mathcal{F}\ [\chi(H_0)\Phi_{h,\omega}](\xi) = h^{-\frac{n}{2}}\chi(\xi^2)\chi_0(h^{\epsilon-1}(\xi-\sqrt{\lambda}\omega))\mathcal{F}\Phi(h^{-1}(\xi-\sqrt{\lambda}\omega)),$$

where $\mathcal{F}$ is the usual Fourier transform. Then, on *Supp* χ_0, we have $\mid \xi - \sqrt{\lambda}\omega \mid \leq 2h^{1-\epsilon}$. So, for $\epsilon < 1$ and h small enough, we have $\chi(\xi^2) = 1$. □

Then, by Lemma 3, we obtain,

$$(2.10)\qquad F(h) =< W^-(H,H_0)\Phi_{h,\omega}, W^+(H,H_0)\Psi_{h,\omega} > .$$

An easy calculation shows that :

$$F(h) = < \Omega^-(h,\omega)\chi_0(h^\epsilon D)\Phi, \Omega^+(h,\omega)\chi_0(h^\epsilon D)\Psi >, \tag{2.11}$$

where

$$\Omega^\pm(h,\omega) = s - \lim_{t\to\pm\infty} e^{itH(h,\omega)}e^{-itH_0(h,\omega)}, \tag{2.12}$$

with

$$H_0(h,\omega) = (D + \sqrt{\lambda}h^{-1}\omega)^2, \tag{2.13}$$

and

$$H(h,\omega) = H_0(h,\omega) + h^{-2}V(h^{-1}x). \tag{2.14}$$

So, by (2.11), we have to find the asymptotics of $\Omega^\pm(h,\omega)\chi_0(h^\epsilon D)\Phi$. We follow the same strategy as in ([11], [12]), and we only treate the case (+).

- **Step 2 :**

As in ([11], [12]), we construct a modifier $J^+(h,\omega)$ in the form :

$$J^+(h,\omega) = \ 1 + d^+(h^{-1}x,\omega), \tag{2.15}$$

We denote :

$$T^+(h,\omega) = H(h,\omega)J^+(h,\omega) - J^+(h,\omega)H_0(h,\omega). \tag{2.16}$$

A direct calculation shows that :

$$\begin{aligned} T^+(h,\omega) = {} & h^{-2}V(h^{-1}x) - 2i\sqrt{\lambda}h^{-2}\omega.\nabla d^+(h^{-1}x,\omega) \\ & +h^{-2}V(h^{-1}x)d^+(h^{-1}x,\omega) - h^{-2}\Delta d^+(h^{-1}x,\omega) \\ & -2h^{-1}\nabla d^+(h^{-1}x,\omega).\nabla \ . \end{aligned} \tag{2.17}$$

We solve the transport equation :

$$\omega.\nabla d^+(x,\omega) = \frac{1}{2i\sqrt{\lambda}}V(x). \tag{2.18}$$

The solution of (2.18) is given by :

$$d^+(x,\omega) = \frac{i}{2\sqrt{\lambda}} \int_0^{+\infty} V(x+t\omega)\ dt. \tag{2.19}$$

We easily obtain :

$$\mid \partial_x^\alpha \ d^+(x,\omega) \mid \leq C_\alpha \quad < x >^{1-\rho-|\alpha|}, \ \forall\ x \in \Gamma^+(\omega), \tag{2.20}$$

where

$$\Gamma^+(\omega) = \{ \ x \in I\!R^n : \mid x \mid \geq R, \ x.\omega \geq -\sigma \mid x \mid \}, \ \sigma \in]-1,1[.$$

From this, we deduce the following lemma, see ([11], Proposition 9) for details.

Lemma 4
For $\epsilon < 1$ and $h \ll 1$,

$$\Omega^+(h,\omega)\chi_0(h^\epsilon D)\Phi = \lim_{t\to+\infty} e^{itH(h,\omega)}\ J^+(h,\omega)\ e^{-itH_0(h,\omega)}\chi_0(h^\epsilon D)\Phi. \tag{2.21}$$

• **Step 3 :**

Recall that $\mu = min\ (2(\rho-1),\ \rho)$. Now, we can formulate :

Lemma 5
Assume $\epsilon < 1$. We have :

$$\|\ (\Omega^+(h,\omega) - J^+(h,\omega))\ \chi_0(h^\epsilon D)\Phi\ \| =\ O(h^\mu). \tag{2.22}$$

Proof
First, we write :

$$(\Omega^+(h,\omega) - J^+(h,\omega))\ \chi_0(h^\epsilon D)\Phi\ =\ i\int_0^{+\infty} e^{itH(h,\omega)}\ T^+(h,\omega)\ e^{-itH_0(h,\omega)}\chi_0(h^\epsilon D)\Phi\ dt. \tag{2.23}$$

We want to construct a cut-off χ^+ in space variables which localizes far from the origin. Let V_ω be a neighborhood of ω in S^{n-1}. We define :

$$O^+ = \{\ x + t\omega'; x \in Supp\ \Phi, t \geq 0,\ \omega' \in V_\omega\ \}. \tag{2.24}$$

If V_ω is rather small, it is clear that $O^+ \subset I\!R^n \backslash B$ where B is the unit ball.

Let $\chi^+ \in C^\infty(I\!R^n\backslash B)$ be a cut-off function such that $\chi^+ = 1$ in a conical neighborhood of O^+.

Now, we have the following estimation which is obtained by using a standard non stationary phase argument :

$$\forall \epsilon < 1,\ \forall N \geq 1,\ (\chi^+ - 1)\ e^{-itH_0(h,\omega)}\chi_0(h^\epsilon D)\Phi\ = O(< t >^{-N}\ h^N), \tag{2.25}$$

in the sense of the L^2-norm. For more details, see ([12], lemma 10).

So, we deduce using (2.17) and (2.25) :

$$\|\ (\Omega^+(h,\omega) - J^+(h,\omega))\ \chi_0(h^\epsilon D)\Phi\ \| \leq \int_0^{+\infty} \|\ \chi^+ T^+(h,\omega)\ e^{-itH_0(h,\omega)}\chi_0(h^\epsilon D)\Phi\ \|\ \ dt, \tag{2.26}$$

modulo $O(h^\infty)$. Since $Supp\ \chi^+ \subset I\!R^n\backslash B$, using (2.17), (2.20) and (2.26), as in ([12], Lemma 11, step 4), we obtain :

$$\|\ (\Omega^+(h,\omega) - J^+(h,\omega))\ \chi_0(h^\epsilon D)\Phi\ \| = O(h^{2(\rho-1)}) + O(h^\rho) = O(h^\mu). \tag{2.27}$$

• **Step 4 :**

Using the following estimation on $L^2(I\!R^n)$: $\forall N \geq 1$,

$$|| \, (\chi_0(h^\epsilon D) - 1)\Phi \;\; ||= \; O(h^N), \tag{2.28}$$

and Lemma 5, we obtain :

$$\Omega^+(h,\omega)\chi_0(h^\epsilon D)\Phi = \Phi + \frac{i}{2\sqrt{\lambda}} \; d^+(h^{-1}x,\omega) \; \Phi + O(h^\mu). \tag{2.29}$$

This concludes the proof of Theorem 1. □

2.4 Proof of Theorem 2.

We only sketch the proof. In this case, since V is a classical symbol of order -2, we construct the modifier $J^+(h,\omega)$, (see section 2.3) as a pseudodifferential operator, (actually a differential operator), having the asymptotic expansion :

$$J^+(h,\omega) = op \; (\sum_{k\geq 0} \; h^k \; d_k^+(x,\xi,\omega)) \;\; , \;\; d_0^+ = 1. \tag{2.30}$$

Roughly speaking, we construct d_k^+ in order to obtain $T^+(h,\omega) = O(h^\infty)$. An easy calculation shows us we need to solve the transport equations :

$$-2i\sqrt{\lambda}\omega.\nabla d_1^+ + f_2 = 0, \tag{2.31}$$

and for $m \geq 1$,

$$-2i\sqrt{\lambda}\omega.\nabla d_{m+1}^+ - 2i\xi.\nabla d_m^+ - \Delta d_m^+ + \sum_{k=0}^{m} \; f_{m-k+2} \; d_k^+ = 0. \tag{2.32}$$

Then Theorem 2 follows as in section 2.2, (see also [11], [12] for details). □

3 Recovering asymptotics of the potential.

3.1 A first application.

In this section, we use (2.4) in the particular case where V is an asymptotic homogeneous function.
Let $V_j, \; j = 1,2$ be two potentials satisfying (H_1) and when $| \, x \, | \to +\infty$,

$$V_j(x) = \; f_j(x) \; + \; o(| \, x \, |^{-\rho}), \; \rho > 1, \tag{3.1}$$

where f_j is a homogeneous function of order $-\rho$.

We denote S_j the scattering operator associated with the pair $(H_0 + V_j, H_0)$. We have the following result :

Corollary 6
Assume that the dimension $n \geq 3$ *and* $S_1\chi(H_0) = S_2\chi(H_0)$.
Then : $f_1 = f_2$.

Proof
By Theorem 1, we have for $\epsilon < 1$ and $h \to 0$,

$$(3.2) \qquad < (S_j - 1)\ \chi(H_0)\Phi_{h,\omega}, \Psi_{h,\omega} >= \frac{h^{\rho-1}}{2i\sqrt{\lambda}} < \int_{-\infty}^{+\infty} f_j(x+t\omega)\ dt\ \Phi, \Psi > +o(h^{\rho-1}).$$

So, if $S_1\chi(H_0) = S_2\chi(H_0)$, we deduce :

$$(3.3) \qquad \forall \omega \in S^{n-1},\ \forall x \in X_\omega, \int_{-\infty}^{+\infty} V_0(x+t\omega)\ dt\ = 0,$$

where $V_0 = f_1 - f_2$. So, using the uniqueness theorem for the Radon transform [3] in dimension $n \geq 3$, we obtain $V_0(x) = 0,\ \mid x \mid \geq 1$. □ .

Comments
For $n = 2$, since f_j is a homogeneous function, it is not difficult to show using [5] that Corollary 6 is still true if $\rho > \frac{3}{2}$.
In [14], P. Sabatier constructed examples of smooth potentials decreasing at infinity as $\mid x \mid^{-\frac{3}{2}}$ with zero amplitude at a fixed energy. These examples do not contradict our results for two reasons. First, in [14], the potentials do not satisfy (3.1), (the leading term of the potential is oscillating). Secondly, in Corollary 6, the energy is not really fixed but belongs to a small neighborhood of $\lambda > 0$.

3.2 A constructive procedure to recover complete asymptotics.

As in section 3.1, if V is a classical symbol of order -2 and if $n \geq 3$, using the uniqueness theorem for the Radon transform [3] and Theorem 2, we reconstruct all the f_j's for $j \geq 2$.

Thus, if we denote $S^{-\infty}(I\!R^n)$ the set of functions rapidly decreasing at infinity, we recover a result close to ([9], Corollary 1.1 - Corollary 1.2), (actually, our result is weaker).

Corollary 7
Assume that $V_1,\ V_2 \in S_{cl}^{-2}(I\!R^n)$, $n \geq 3$ *and* $S_1\chi(H_0) = S_2\chi(H_0)$.
Then : $V_1 - V_2 \in S^{-\infty}(I\!R^n)$.
In particular, if $V_1,\ V_2$ *are rational potentials of order less than or equal to* -2*, then* $V_1 = V_2$.

Comments
Let us conclude this note by some remarks in the two dimensional case. As for Corollary 6, using the results of [5], Corollary 7 is true for $n = 2$.
Nevertheless, as it was pointed by P. Grinevich [2], for $n = 2$, we can construct examples of smooth rational potentials decaying at infinity as $\mid x \mid^{-2}$, with zero amplitude at a fixed energy.

4 The long-range case.

In this section, we generalize to the long-range case the results obtained in the preceding section for short-range interactions.
We consider generic long-range potentials V, (i.e V satisfies (H_1) with $\rho > 0$). Obviously, under such hypotheses, in general the usual wave operators do not exist. So, we have to define modified wave operators. For $\rho > \frac{1}{2}$, we can use the well-known Dollard wave operators. Since we consider general long-range interactions, we have to define other modified wave operators.
These modified wave operators, close to Isozaki-Kitada's wave operators [6], have been constructed in [10]. Roughly speaking, we define :

$$W^{\pm} = s - \lim_{t \to \pm\infty} \ e^{itH} \ J^{\pm} e^{-itH_0} \ , \tag{4.1}$$

where $J^{\pm}$ is a Fourier integral operator with phase $\varphi^{\pm}(x,\xi)$ satisfying an approximate eikonal equation :

$$\mid \partial_x \varphi^{\pm}(x,\xi) \mid^2 + V(x) \approx \xi^2 \ , \tag{4.2}$$

and with amplitude 1, (see [11] for details). When $\rho > \frac{1}{2}$, our modified wave operators are equal to the Dollard's one up to an energy phase ([11], Corollary 13), and in suitable incoming or outgoing zones,

$$\varphi^{\pm}(x,\xi) = x \cdot \xi + \int_0^{\pm\infty} \ (V(x+t\xi) - V(t\xi)) \ dt \ . \tag{4.3}$$

The modified scattering operator is defined as in (1.3) by : $S = W^{+*} W^{-}$.

Using the same approach as in Theorem 1, replacing $J^{+}(h,\omega)$ in (2.30) with a Fourier integral operator as in [11]-[12], we easily obtain :

Theorem 8
We have, when $h \to 0$,

$$< [S,D] \chi(H_0) \Phi_{h,\omega}, \Psi_{h,\omega} >= \frac{1}{2\sqrt{\lambda}} \ < \int_{-\infty}^{+\infty} \ \nabla V(h^{-1}x + t\omega) \ dt \ \Phi, \Psi > + o(h^{\rho}), \tag{4.4}$$

As in Corollary 6, in the particular case where V is an asymptotic homogeneous function, we can recover the leading term of the potential at infinity :

Let $V_j, \ j = 1,2$ be two potentials satisfying (H_1) and (3.1) with $\rho > 0$. We denote S_j the modified scattering operator associated with the pair $(H_0 + V_j, H_0)$. We have the following result :

Corollary 9
Assume that the dimension $n \geq 3$ and $S_1 \chi(H_0) = S_2 \chi(H_0)$.
Then : $f_1 = f_2$.

Moreover, using the same approach as in Theorem 2 - Corollary 7, we have :

Corollary 10
Assume that $V_1,\ V_2 \in S_{cl}^{-1}(I\!R^n)$ *,* $n \geq 3$ *and* $S_1\chi(H_0) = S_2\chi(H_0)$.
Then : $V_1 - V_2 \in S^{-\infty}(I\!R^n)$.
In particular, if V_1*,* V_2 *are rational potentials of order less than or equal to* -1*, then* $V_1 = V_2$.

References

[1] V. Enss - R. Weder, "The geometrical approach to multidimensional inverse scattering", J. Math. Phys. 36 (8), 3902-3921 (1995).

[2] P.G. Grinevich, "Rational solutions of the Veselov-Novikov equations are reflectionless two- dimensional potentials at fixed energy", Theor. and Math. Phys. 69 (2), 307-310 (1986).

[3] S. Helgason, "The Radon Transform", Progress in Mathematics 5, Birkhäuser, (1980).

[4] G. M. Henkin - R. G. Novikov, "The $\overline{\partial}$-equation in the multidimensional inverse scattering problem", Russ. Math. Surv. 42 (4), 109-180, (1987).

[5] H. Isozaki, "Multi-dimensional inverse scattering theory for Schrödinger operators", Reviews in Mathematical Physics, vol. 8, (4), 591-622, (1996).

[6] H. Isozaki - H. Kitada, "Modified wave operators with time-independent modofiers", Papers of the College of Arts and Sciences, Tokyo Univ, Vol.32, 81-107, (1985).

[7] M. S. Joshi, "Explicitely recovering asymptotics of short-range potentials", Commun. in Partial Differential equations, 25 (9-10), 1907-1923 (2000).

[8] M. S. Joshi - A.Sa Barreto, "Recovering asymptotics of short range potentials", Communications in Mathematical Physics, 193, 197-208 (1998).

[9] M. S. Joshi - A.Sa Barreto, "Recovering asymptotics of metrics from fixed energy scattering data", Invent. 137, 127-143 (1999).

[10] R. B. Melrose - M. Zworski, "Scattering metrics and geodesic flow at infinity", Invent. Math. 124, 389-436 (1996).

[11] F. Nicoleau, "A stationary approach to inverse scattering for Schrödinger operators with first order perturbation", Communication in P.D.E, Vol 22 (3-4), 527-553, (1997).

[12] F. Nicoleau, "An inverse scattering problem with the Aharonov-Bohm effect", Journal of Mathematical Physics, Issue 8, 5223-5237, (2000).

[13] M. Reed - R. Simon, "Methods of mathematical physics", Vol. 3, Academic Press, (1978).

[14] P. Sabatier, "Asymptotic properties of the potentials in the inverse scattering problem at fixed energy", J. Math. Phys., 1515-1531 (1966).