

SPECTRAL ASYMPTOTICS OF A SINGULAR HARMONIC OSCILLATOR WITH AHARONOV–BOHM FLUX

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ABSTRACT. Motivated by a recent article of R. Vanlaere (2026), we investigate the spectral asymptotics of a singular harmonic oscillator arising from the radial reduction of the magnetic Schrödinger operator on the unit disk in the presence of a constant magnetic field and an Aharonov–Bohm flux. The corresponding eigenvalue problem is equivalent to the study of the zeros of the Kummer confluent hypergeometric function with respect to its first parameter. Combining now standard semiclassical methods with the uniform asymptotic theory of Whittaker and confluent hypergeometric functions developed by Dunster (1989), Gabutti-Gatteschi (2001) and improved quite recently by Dunster (2026), we obtain a comprehensive description of the different spectral regimes. When the boundary lies in the classically forbidden region, we derive the exponentially small asymptotic correction to the eigenvalue caused by the tunneling effect. For energy levels above the tunneling regime, we derive a shifted Bohr–Sommerfeld asymptotics for highly excited states at a fixed magnetic field. Finally, for the transition regime where the energy level touches the boundary, we derive a three-term asymptotic expansion of the eigenvalue. In an appendix, we translate these spectral results into explicit asymptotic formulas for the zeros of the Kummer function.

1. INTRODUCTION

1.1. Harmonic oscillator with inverse square potential. We study the semi-classical asymptotics of the high energy levels of the operator,

$$(1.1) \quad T_h = -h^2 \frac{d^2}{dx^2} + x^2 + h^2 \frac{\nu^2 - 1/4}{x^2}, \quad \nu \geq 0,$$

with Dirichlet boundary conditions at $x = 0$ and $x = 1$. Here $\nu \geq 0$ is a given parameter, which will be fixed throughout this paper.

Thanks to the Hardy inequality, T_h is semi-bounded on $C_c^\infty(0, 1)$. The Friedrichs extension yields a self-adjoint operator in $L^2(0, 1)$ which we denote by T_h hereafter.

If we would like to stress that the Dirichlet boundary condition is imposed at $x = 1$, we will add a superscript D and write T_h^D . Note that if we start with the domain $C_c^\infty((0, 1])$, the Friedrichs extension will be the Neumann realization and we denote it by T_h^N . For T_h^N , Neumann boundary condition is imposed at $x = 1$, while Dirichlet is maintained at $x = 0$.

By the Sturm–Liouville theory, the spectrum of T_h is purely discrete, consisting of simple eigenvalues

$$\lambda_1(h) < \lambda_2(h) < \cdots,$$

whose distribution we would like to understand in the semiclassical limit.

1.2. Motivation. One motivation behind the study of T_h is a problem in control theory [23, 2]. In this context, one encounters the operator T_h , with $h = \xi^{-1}$, which arises by taking the partial Fourier transform of the y -variable in the Baouendi–Grushin operator

$$-\partial_x^2 - x^2 \partial_y^2 + \frac{\nu^2 - 1/4}{x^2}.$$

Another motivation of our study comes from the magnetic Laplacian. It is known that the presence of a magnetic field leads to an interesting interaction between the magnetic field, the

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geometry of the domain, and its topology. A fundamental example is the magnetic Dirichlet Laplacian on the unit disk

$$(1.2) \quad D = \{x \in \mathbb{R}^2; |x| < 1\},$$

submitted to a constant magnetic field together with an Aharonov–Bohm solenoid located at the origin. Passing to polar coordinates and separating variables (see [12] and [22, Section 1.3]), the two-dimensional problem decomposes into a family of one-dimensional singular Sturm–Liouville operators indexed by the angular momentum. After a standard Liouville transformation, one obtains the singular harmonic oscillator acting in $L^2(0, 1)$,

$$-\frac{d^2}{dx^2} + \frac{B^2 x^2}{4} + \frac{(m + \alpha)^2 - 1/4}{x^2},$$

where B is the intensity of the magnetic field, $m \in \mathbb{Z}$ is the angular momentum, and α is the Aharonov–Bohm flux. Setting $h = 2/B$ and $\nu = |m + \alpha|$, and imposing Dirichlet boundary condition on ∂D , we recover the operator T_h . The semi-classical limit $h \rightarrow 0^+$ considered here is equivalent to the strong field limit, $B \rightarrow +\infty$ in the context of the magnetic Laplacian.

1.3. Universal comparison. We first observe that, for any integer $k \geq 1$, we have

$$(1.3a) \quad \lambda_k(h) \geq e_k(h),$$

where

$$(1.3b) \quad e_k(h) := (4k + 2\nu - 2)h$$

is the k 'th eigenvalue of the operator (see [8])

$$T_h^\infty = -h^2 \frac{d^2}{dx^2} + x^2 + h^2 \frac{\nu^2 - 1/4}{x^2} \quad \text{in } L^2(\mathbb{R}^+).$$

The proof is easy. Extending functions in the domain of T_h by zero, we get by the min-max principle that the eigenvalues of T_h are bounded below by the eigenvalues of T_h^∞ , the operator in $\mathbb{R}_+$.

1.4. Low lying energy levels. Fixing the eigenvalue label k , the following asymptotics hold as $h \rightarrow 0^+$,

$$(1.4) \quad \lambda_k(h) = (4k + 2\nu - 2)h + \frac{4h^{-2k-\nu+2}}{(k-1)!\Gamma(k+\nu)} e^{-1/h} (1 + \mathcal{O}(h)).$$

This is a special case of a theorem by O. Gannot [10], which holds for a generic class of scalar potentials, beyond the harmonic potential discussed in this paper¹. Therefore, the asymptotics (1.4) reflects the exponentially weak influence of the boundary as h tends to 0. This is a manifestation of tunneling, since the Dirichlet boundary condition models the hard wall potential².

1.5. Persistence of tunneling. Vanlaere recently observed that tunneling persists up to higher energy levels. More precisely, given $0 < \rho^* < 1$, it follows from [22, Theorem 1.4] that there exist constants $c, h_0 > 0$ such that for all pairs $(h, k) \in (0, h_0] \times \mathbb{N}^*$ satisfying

$$e_k(h) \leq \rho^*,$$

the following bounds hold

$$(1.5) \quad \lambda_k(h) \leq e_k(h) + e^{-c/h}.$$

Our objective is to prove an accurate asymptotics, specifically to obtain the optimal exponential decay in (1.5) and to exhibit the influence of the parameter ν (recall that ν encapsulates the Aharonov–Bohm flux). This quantifies the tunneling mechanism for excited states corresponding to these higher energy levels.

¹Recently, M. Baur and T. Weidl [1] derived (1.4) via special functions, and the second author together with G. Miranda [17] derived it and the analogous formula for the Neumann boundary condition via the Temple inequality (see also [12]).

²If we add to T_h the potential $V_a = a\mathbf{1}_{[1,+\infty)}$, then the Dirichlet boundary condition at $x = 1$ corresponds to taking $a = +\infty$.

Our first result is to prove that (1.4) continues to hold when $e_k(h) \ll 1$, up to $k \lesssim h^{-\gamma}$, for any fixed $0 < \gamma < 1/2$.

Theorem 1.1. *For any fixed $\gamma \in (0, 1/2)$ and for any integer k satisfying*

$$1 \leq k \leq h^{-\gamma},$$

we have

$$\lambda_k(h) = e_k(h) + \frac{4}{(k-1)!\Gamma(k+\nu)} h^{-2k-\nu+2} e^{-1/h} (1 + \varepsilon_k(h)),$$

where

$$\sup_{k \in [0, h^{-\gamma}]} |\varepsilon_k(h)| \longrightarrow 0 \quad \text{as } h \longrightarrow 0^+.$$

Following the terminology of [20], the eigenvalues of order $\mathcal{O}(h^\delta)$, with $0 < \delta < 1$, are called semi-excited states.

Our next task is to explore the regime beyond the semi-excited states, which are not covered by Theorem 1.1.

For a given $E \in (0, 1)$, we introduce (see for example [13, 15]) the Agmon distance (associated with the potential $V(s) = s^2$) to the “classical region” $\{s \in (0, 1), s^2 \leq E\}$:

$$(1.6) \quad S(x; E) = \int_0^x \sqrt{(s^2 - E)_+} ds,$$

whose maximum on $[0, 1]$ is

$$(1.7) \quad S_*(E) = \int_0^1 \sqrt{(s^2 - E)_+} ds,$$

which is the Agmon distance from the boundary $\{s = 1\}$ to the classical region. We can extend by continuity $S(x, E)$ and $S_*(E)$ at 0 and notice that $S(x, 0) = x^2/2$ and $S_*(0) = 1/2$. When $E = e_k(h)$, we write

$$S_{k,h} = S_*(e_k(h)).$$

Our second result is

Theorem 1.2 (Tunneling beyond semi-excited states). *Let $0 < \rho^* < 1$ be given. Let*

$$\ell : (0, 1] \longrightarrow \mathbb{R}_+$$

satisfy

$$\ell(h) \longrightarrow 0 \quad \text{and} \quad h^{-1}\ell(h) \longrightarrow +\infty \quad \text{as } h \longrightarrow 0^+.$$

For every $h \in (0, 1]$, define

$$(1.8) \quad \mathcal{J}(h) = \{k \in \mathbb{N} : \ell(h) \leq e_k(h) \leq \rho^*\}.$$

Then, for $k \in \mathcal{J}(h)$,

$$(1.9) \quad \lambda_k(h) = e_k(h) + \frac{2}{\pi} h \exp\left(-\frac{2S_{k,h}}{h}\right) (1 + r_k(h)),$$

where

$$\sup_{k \in \mathcal{J}(h)} |r_k(h)| \longrightarrow 0 \quad \text{as } h \longrightarrow 0^+.$$

Remark 1.3. A few remarks on (1.8) and (1.9) are in order.

- i) We may choose $\ell(h) = h^{1-\delta}$, with $0 < \delta < 1$. If, in addition, $\delta < 1/2$, then for any $\gamma \in (\delta, 1/2)$, the regimes covered by Theorems 1.1 and 1.2 overlap in the range $h^{-\delta} \lesssim k \lesssim h^{-\gamma}$.
- ii) When $k = k(h) \rightarrow +\infty$ and $kh \rightarrow 0^+$, we get that $S_{k,h}/h \sim 1/2h$, in accordance with (1.4). In Remark 2.12, we will verify that the asymptotics in Theorem 1.1 and Theorem 1.2 match in their overlapping region of validity.

- iii) The case $\rho^* \geq 1$ corresponds to energy levels above the ones allowed by (1.8), for which the tunneling effect disappears. One heuristic explanation is the following. Tunneling towards the boundary $x = 1$ occurs when the outer classical turning point lies in the interior of the domain. For

$$V_h(x) := x^2 + h^2 \frac{\nu^2 - 1/4}{x^2},$$

and for an energy $E \gg h$, the outer turning point is given by

$$x_+(E, h) = \left(\frac{E}{2} + \sqrt{\frac{E^2}{4} - h^2 \left(\nu^2 - \frac{1}{4} \right)} \right)^{1/2} = \sqrt{E} + \mathcal{O} \left(\frac{h^2}{E^{3/2}} \right).$$

Depending on the sign of $\nu^2 - 1/4$, there may also be an inner turning point near $x = 0$, which is irrelevant for the tunneling towards the boundary $x = 1$. Thus, (1.8) ensures that the outer turning point remains in the interior of $(0, 1)$ and stays away from the boundary $x = 1$.

- iv) A similar result holds for the Neumann realization, namely

$$\lambda_k^N(h) = e_k(h) - \frac{2}{\pi} h \exp \left(-\frac{2\mathcal{S}_{k,h}}{h} \right) (1 + o(1)),$$

which leads to

$$\lambda_k^D(h) - \lambda_k^N(h) = \frac{4}{\pi} h \exp \left(-\frac{2\mathcal{S}_{k,h}}{h} \right) (1 + o(1)).$$

Application. Like in [10, Corollary 1.3], for a given integer $l \geq 0$, we can consider the Coulomb Hamiltonian in $L^2(0, 1)$,

$$\mathcal{H}_h = -h^2 \frac{d^2}{dy^2} + h^2 \frac{l(l+1)}{y^2} - \frac{2}{y},$$

with Dirichlet boundary conditions at $y = 0$ and $y = 1$. Denoting the negative eigenvalues by $E_n(h)$, $n \geq l+1$ and applying a change of function, we map an eigenspace of $\mathcal{H}_h$ to an eigenspace of T_h with $\nu = 2l + 1$ (see the proof of Corollary 1.3 in [10]). More precisely, it holds

$$\frac{4}{\sqrt{-E_n(h)}} = L_n^2 \lambda_k(h L_n^{-2}) \quad \text{where } L_n^2 = 2\sqrt{-E_n(h)} \text{ and } n = k + l.$$

For $n = k + l$ and k as in Theorem 1.2, we have $h L_n^{-2} \rightarrow 0^+$ and $e_k(h) = 4nh L_n^{-2}$. Applying Theorem 1.2 with $\nu = 2l + 1$, we get for $n = k + l$,

- the approximation modulo $\mathcal{O}(h^\infty)$

$$L_n^{-2} = \frac{nh}{2} + \mathcal{O}(h^\infty),$$

- the asymptotics with exponentially small second term:

$$E_n(h) = -\frac{1}{n^2 h^2} + \frac{2}{2n^2 h \pi} \exp \left(-\frac{1}{h} \int_0^1 \sqrt{(s^2 - 2n^2 h^2)_+} ds \right) (1 + o(1)).$$

1.6. Bohr-Sommerfeld rule. We turn now to the study of eigenvalues corresponding to energies above $\sup_{x \in (0,1)} V(x)$, i.e. when the energy satisfies $E > 1$, since $V(x) = x^2$. This implies in particular that the condition in (1.8) is violated. In simple terms, we consider eigenvalues with label k satisfying

$$1 < \rho_0 \leq 4kh \leq \rho_1,$$

for some constants ρ_0, ρ_1 . This condition emerges naturally if we think of a Bohr-Sommerfeld quantization rule for energy levels that are > 1 (see Section 3).

The asymptotic distribution of the eigenvalues is described in the Theorem 1.4 below, in a regime where there is no tunneling phenomenon as described in Theorem 1.2.

Theorem 1.4 (Shifted Bohr-Sommerfeld rule). *Let $1 < \rho_0 < \rho_1 < +\infty$. Then, uniformly for integers k satisfying*

$$\rho_0 \leq 4kh \leq \rho_1,$$

we have, as $h \rightarrow 0^+$,

$$\lambda_k(h) = E_0 \left(4h \left(k + \frac{\nu}{2} - \frac{1}{4} \right) \right) + \mathcal{O}(h^2),$$

where, for $\rho > 1$, $E_0(\rho) > 1$ is uniquely determined by

$$I(E_0(\rho)) = \frac{\pi\rho}{4},$$

with

$$I(E) = \int_0^1 \sqrt{E - y^2} dy = \frac{1}{2} \sqrt{E - 1} + \frac{E}{2} \arcsin(E^{-1/2}), \quad E > 1.$$

Remark 1.5. Theorem 1.4 exhibits a Bohr-Sommerfeld quantization rule adapted to the singular potential, in which the eigenvalue label enters through the shifted quantity $k + \frac{\nu}{2} - \frac{1}{4}$, as predicted in (1.17). The correction $\nu/2 - 1/4$ reflects the contribution of the inverse square singularity at $x = 0$ and can be interpreted as a Bohr-Sommerfeld-Langer correction. Notice in particular that, when $\nu = 1/2$, the inverse square term vanishes and the shifted quantity $k + \frac{\nu}{2} - \frac{1}{4}$ reduces simply to k . Accordingly, the Bohr-Sommerfeld quantization condition takes the form $I(E) = \pi kh$. Since $E > 1$, there are no turning points in $(0, 1)$, and this is precisely the usual semiclassical quantization rule for a classically allowed interval with Dirichlet boundary conditions at both endpoints, (see, for instance, [19, Chapter 6, Section 9, Eq. (9.05)]). This interpretation will be discussed in more detail in Section 3 and in Appendix A.

1.7. Transition regime. In order to understand the transition between the tunneling asymptotics (Theorem 1.2) and the shifted Bohr-Sommerfeld rule (Theorem 1.4), we investigate the regime

$$|4kh - 1| = th^\delta, \quad t \in [t_0, t_1], \quad 0 < \delta < \frac{2}{3}.$$

which we refer to it as critical. This regime is interesting because the classical turning point can reach the boundary both from the interior of the domain (when $4kh - 1 < -t_0 h^\delta$) and also from the exterior (when $4kh - 1 > t_0 h^\delta$). Actually the nature of the transition from the left will be treated in Proposition 2.5 and Remark 2.6 and is quite different from the right which is treated for $\delta < 2/3$ in Proposition 4.1. Nevertheless, when $\delta = 2/3$ and $|4kh - 1| \leq Th^{2/3}$, we obtain a three-term expansion for the k 'th eigenvalue.

Theorem 1.6. *There exists a smooth decreasing function $\theta : \mathbb{R} \rightarrow \mathbb{R}^+$ such that, for every $T > 0$ and for integers k satisfying*

$$(1.10) \quad |4kh - 1| \leq Th^{2/3},$$

setting

$$(1.11) \quad \tau_k(h) := \frac{1 - 4kh}{2^{2/3} h^{2/3}},$$

then, uniformly for these k 's, the following holds

$$(1.12) \quad \begin{aligned} \lambda_k(h) = e_k(h) &+ \frac{4h}{\pi} \theta(\tau_k(h)) \\ &- \frac{4}{\pi 2^{2/3}} h^{4/3} \left(2\nu - 2 + \frac{4}{\pi} \theta(\tau_k(h)) \right) \theta'(\tau_k(h)) + \mathcal{O}(h^{5/3}), \end{aligned}$$

as $h \rightarrow 0^+$.

The function θ is explicitly introduced in (4.19), as a phase associated with the Airy functions. When we restrict to $|4kh - 1| = th^\delta$, with t fixed, we get by the smoothness of θ the leading order asymptotics of $\lambda_k(h) - e_k(h)$, which recovers the tunneling asymptotics of Theorem 1.2 in the subcritical side ($4kh - 1 = -th^\delta$ and $t \gg 1$), while on the supercritical side ($4kh - 1 = th^\delta$

and $t \gg 1$) it recovers the threshold expansion of the Bohr–Sommerfeld formula in Theorem 1.4 (more details are given in Proposition 4.1 and Subsection 4.4).

In the regime $\delta = 2/3$ and $t = t(h)$ large, we do not obtain an asymptotics, but we prove that $\lambda_k(h)$ is still exponentially close to $e_k(h)$, see Proposition 2.5 and Remark 2.6.

1.8. Spectral gap. For the applications in control theory, one needs a uniform estimate of the gap³

$$\lambda_{k+1}(h) - \lambda_k(h)$$

valid for all $k \geq 1$ and $h > 0$. The asymptotics in Theorems 1.1, 1.2 and 1.6 yield the asymptotic lower bound

$$\lambda_{k+1}(h) - \lambda_k(h) \geq 4h + o(h)$$

in the relevant spectral windows.

We present a non-asymptotic bound consistent with the eigenvalue asymptotics (also discovered independently by R. Vanlaere⁴).

Theorem 1.7. *For all $\nu \geq 0$, $h > 0$ and $k \geq 1$, it holds*

$$(1.13) \quad \lambda_{k+1}(h) - \lambda_k(h) > 4h.$$

The proof of this theorem relies on the Sturm-Liouville theory and the introduction of a creation-like operator.

1.9. Possible extensions and heuristics.

1.9.1. Origin of the study. Our operator T_h arises naturally by fixing the angular momentum and keeping the radial variable, leaving us with the inverse square potential plus the harmonic potential. More generally, we can study the operator on $(0, 1)$ with a general potential

$$(1.14) \quad T_{h,V} = -h^2 \frac{d^2}{dx^2} + h^2 \frac{\nu^2 - 1/4}{x^2} + V.$$

1.9.2. Tunneling. Denoting the Dirichlet and Neumann eigenvalues of $T_{h,V}$ by $\lambda_k^D(h, V)$ and $\lambda_k^N(h, V)$ respectively, and looking at the energy levels satisfying

$$\min_{[0,1]} V < \rho_* \leq \lambda_k^N(h, V) \leq \rho^* < \max_{[0,1]} V,$$

it would be desirable to study whether we have

$$(1.15) \quad \lambda_k^D(h, V) - \lambda_k^N(h, V) \sim C_k(h) e^{-2S_{k,h}(V)/h}.$$

The exponential decay is defined by the Agmon distance,

$$S_{k,h}(V) = \int_0^1 \sqrt{(V(s) - E)_+} ds, \quad E = \frac{1}{2}(\lambda_k^D(h, V) + \lambda_k^N(h, V)),$$

while the prefactor $C_k(h)$ is expected to be of the form αh^β , where both α ($\alpha \neq 0$) and β depend possibly on k , V and ν .

Note that it is possible to prove the weaker upper bound

$$\log(\lambda_k^D(h, V) - \lambda_k^N(h, V)) \leq -\frac{2S_{k,h}(V)}{h} (1 + o(1)),$$

via Agmon estimates (we illustrate this for the harmonic potential in Subsection 2.2). If $V \in C^\infty(\mathbb{R})$ is even, vanishes at $x = 0$, positive for $x \neq 0$, and satisfies

$$V''(0) > 0, \quad \lim_{|x| \rightarrow +\infty} V(x) > 0,$$

then for a fixed k , the following asymptotics is proven by Gannot [10, Theorem 2]

$$(1.16) \quad \lambda_k^D(h, V) = e_k(h, V) + \alpha_0(V) h^{-2k-\nu+2} e^{-2S_0(V)/h} (1 + o(1)),$$

³We thank R. Vanlaere for pointing this question to us.

⁴Personal communication.

where $e_k(h, V)$ is the eigenvalue of the operator

$$T_h^\infty = -h^2 \frac{d^2}{dx^2} + h^2 \frac{\nu^2 - 1/4}{x^2} + V \quad \text{in } \mathbb{R}_+.$$

The constants $S_0(V)$ and $\alpha_0(V)$ are explicit,

$$S_0(V) = \int_0^1 \sqrt{V(t)} dt, \quad \alpha_0(V) = \frac{4A\sqrt{V(1)}}{(k-1)!\Gamma(k+\nu)} \left(\frac{V''(0)}{2} \right)^{2k+\nu-1},$$

where A is dependent on V and computed as a limit.

We expect that

$$\lambda_k^N(h, V) = e_k(h, V) - \alpha_0(V)h^{-2k-\nu+2}e^{-2S_0(V)/h}(1 + o(1)),$$

which would answer (1.15) for fixed k . That said, it would be interesting to extend (1.16) for higher energy levels as we did in Theorem 1.2 for the harmonic potential $V(x) = x^2$.

The formula (1.15) is related to tunneling in double wells. In the particular case of a unique non-degenerate well at $x = 0$, we extend

$$V_h(x) = h^2 \frac{\nu^2 - 1/4}{x^2} + V(x),$$

by even symmetry about $x = 1$. Neglecting the inverse square potential by taking $\nu = 1/2$, we obtain a potential $\tilde{V}$ with double wells at $x = 0$ and $x = 2$, and the operator

$$\tilde{T}_h = -h^2 \frac{d^2}{dx^2} + \tilde{V},$$

in $L^2(0, 2)$, with Dirichlet boundary condition at $x = 0, 2$. By the min-max principle, $\lambda_1^N(h, V)$ is the lowest eigenvalue of $\tilde{T}_h$, while $\lambda_1^D(h, V)$ is the second eigenvalue of $\tilde{T}_h$. Thus, we have reduced to analyze the tunneling in double wells [11, 15]. Thanks to well known tunneling asymptotics ([13, p. 59, Eq. (4.5.21)])

$$\lambda_1^D(h, V) - \lambda_1^N(h, V) = \lambda_2(h, \tilde{V}) - \lambda_1(h, \tilde{V}) = c_0 \pi^{-1/2} h^{1/2} e^{-2S_0(V)/h} (1 + o(1)),$$

where $c_0 > 0$ is a constant dependent on V and computed as a limit. Notice that this sketch is only heuristic since the potentials are not regular.

1.9.3. Formal derivation of a Bohr-Sommerfeld rule with Langer correction. We turn to the study of the eigenvalues of the Dirichlet realization on $(0, 1)$ of the operator $T_{h,V}$ introduced in (1.14). For $\nu = \frac{1}{2}$, the inverse square potential disappears and the problem becomes regular. We follow the dependence on the parameter ν in the vicinity of the regular case $\nu = 1/2$. We denote the k 'th eigenvalue by

$$\mathcal{E}_k(\nu) = \lambda_k^D(h, V)$$

and we denote by $u_{k,\nu}$ an associated L^2 -normalized eigenfunction. We set

$$E = \mathcal{E}_k\left(\frac{1}{2}\right),$$

and we assume that

$$E > \max_{x \in [0,1]} V(x).$$

By the Feynman–Hellmann formula,

$$\partial_\nu \mathcal{E}_k(\nu) = 2h^2 \nu \int_0^1 \frac{|u_{k,\nu}(x)|^2}{x^2} dx.$$

We evaluate this expression heuristically at $\nu = \frac{1}{2}$. We introduce

$$p(x, E) = \sqrt{E - V(x)}, \quad I_V(E) = \int_0^1 p(x, E) dx.$$

For $\nu = 1/2$, a normalized WKB eigenfunction has the form

$$u_{k,1/2}(x) \sim \frac{A}{\sqrt{p(x, E)}} \sin \left(\frac{1}{h} \int_0^x p(s, E) ds \right).$$

Averaging the oscillations and using the L^2 -normalization gives

$$1 \sim \frac{A^2}{2} \int_0^1 \frac{dx}{p(x, E)} = A^2 I'_V(E),$$

and hence

$$A^2 \sim \frac{1}{I'_V(E)}.$$

Making the change of variable $x = h y$, we obtain

$$\int_0^1 \frac{|u_{k,1/2}(x)|^2}{x^2} dx = \frac{1}{h} \int_0^{1/h} \frac{|u_{k,1/2}(hy)|^2}{y^2} dy.$$

The singular weight suggests that the leading contribution comes from the boundary scale $x = \mathcal{O}(h)$. Writing

$$p_0 = \sqrt{E - V(0)},$$

we obtain, for $y = \mathcal{O}(1)$,

$$u_{k,1/2}(hy) \sim \frac{A}{\sqrt{p_0}} \sin(p_0 y).$$

Therefore, formally,

$$\int_0^1 \frac{|u_{k,1/2}(x)|^2}{x^2} dx \sim \frac{A^2}{h} \int_0^\infty \frac{\sin^2 z}{z^2} dz = \frac{\pi A^2}{2h} = \frac{\pi}{2h I'_V(E)},$$

where we used the change of variable $z = p_0 y$ and the classical Dirichlet integral. It follows that

$$\partial_\nu \mathcal{E}_k(\nu)|_{\nu=1/2} \sim \frac{\pi h}{2 I'_V(E)}.$$

Since

$$\partial_\nu I_V(\mathcal{E}_k(\nu)) = I'_V(\mathcal{E}_k(\nu)) \partial_\nu \mathcal{E}_k(\nu),$$

we obtain

$$\partial_\nu I_V(\mathcal{E}_k(\nu))|_{\nu=1/2} \sim \frac{\pi h}{2}.$$

A first-order Taylor expansion with respect to ν at $\nu = \frac{1}{2}$ then gives, formally,

$$I_V(\mathcal{E}_k(\nu)) \sim I_V(\mathcal{E}_k(\tfrac{1}{2})) + \frac{\pi h}{2} \left(\nu - \frac{1}{2} \right).$$

For $\nu = \frac{1}{2}$, the problem is regular and the usual Bohr–Sommerfeld quantization (see Olver [19]) gives, since we have the Dirichlet condition at the two ends of the interval,

$$I_V(\mathcal{E}_k(\tfrac{1}{2})) = \pi h k + \mathcal{O}(h^2).$$

Consequently, we are led heuristically to

$$(1.17) \quad I_V(\mathcal{E}_k(\nu)) \sim \pi h \left(k + \frac{\nu}{2} - \frac{1}{4} \right).$$

We have established (1.17) for all $\nu \geq 0$ in the harmonic potential case, $V(x) = x^2$, by exploiting the explicit solvability in terms of special functions; see Theorem 1.4 above.

1.9.4. *Role of Special functions.* In the case of the harmonic potential $V(x) = x^2$, the eigenvalues are characterized in terms of the zeros of the Kummer confluent hypergeometric function. That remarkable observation has been used in several papers [1, 22].

For a given eigenvalue λ of T_h , a corresponding eigenfunction regular at the origin is given by

$$(1.18) \quad u(x) = x^{\nu+\frac{1}{2}} e^{-x^2/2h} M\left(-\frac{\lambda - 2(1+\nu)h}{4h}, 1+\nu, x^2/h\right),$$

where $M(a, b, z)$, also denoted by ${}_1F_1(a; b; z)$, is the Kummer confluent hypergeometric function (see [18, pp. 262–264] or [5, §13.2, Eq. 13.2.4 and §13.3]). Imposing the Dirichlet boundary condition at $x = 1$ leads to the transcendental equation

$$(1.19) \quad M\left(-\frac{\lambda - 2(1+\nu)h}{4h}, 1+\nu, 1/h\right) = 0,$$

and reduces the eigenvalue problem to the study of the zeros of the Kummer function with respect to its first parameter (see [22, Theorem 1.2]).

Although the eigenvalue condition is naturally expressed in terms of the Kummer function, its asymptotic analysis is more conveniently performed in the Whittaker setting. Indeed, the classical identity [5, §13.14, Eq. 13.14.2]

$$(1.20) \quad M_{\kappa,\mu}(z) = e^{-z/2} z^{\mu+\frac{1}{2}} M\left(\mu - \kappa + \frac{1}{2}, 2\mu + 1, z\right)$$

shows that the two formulations are equivalent, where $M_{\kappa,\mu}$ denotes the Whittaker function. With

$$(1.21) \quad \kappa = \frac{\lambda}{4h}, \quad \mu = \frac{\nu}{2},$$

the equation (1.19) becomes

$$(1.22) \quad M_{\kappa,\mu}(1/h) = 0.$$

Thus, for λ to be an eigenvalue, $z_h = 1/h \gg 1$ must be a zero of the Whittaker function. For large κ , the asymptotic distribution of the positive zeros of $M_{\kappa,\mu}(z)$ with respect to the variable z was described by Gabutti and Gatteschi [9]. By inverting their asymptotic expansion in the spectral relation $M_{\kappa,\mu}(1/h) = 0$, we obtain the eigenvalue asymptotics in Theorem 1.4.

1.9.5. Summarizing picture.

The different spectral regimes considered in this paper, together with the corresponding asymptotic results, are summarized in Figure 1.

1.10. **Organization.** The paper is organized as follows. Section 2 is devoted to the tunneling regime and to the proofs of Theorems 1.1 and 1.2, including the low-energy regime and a brief discussion of the Neumann problem. Section 3 is devoted to the supercritical regime and the proof of Theorem 1.4, using asymptotic properties of Whittaker functions and their zeros. Section 4 treats the transition near the critical energy 1 as in Theorem 1.6, described by uniform Airy asymptotics, and discusses its matching with the subcritical and supercritical regimes. Section 5 proves Theorem 1.7. Appendix A discusses the Bohr–Sommerfeld–Langer quantization, while Appendix B reformulates our spectral results as asymptotic formulas for the zeros of the Kummer function with respect to its first parameter.

2. TUNNELING ASYMPTOTICS

This section is devoted to the proofs of Theorems 1.1 and 1.2. The proof of Theorem 1.2 relies on three ingredients:

- Agmon estimates (Subsection 2.2),
- Plancherel–Rotach asymptotics in the classically forbidden region (Subsection 2.3),
- Construction of an explicit trial state in the domain of the operator T_h (Subsection 2.4).

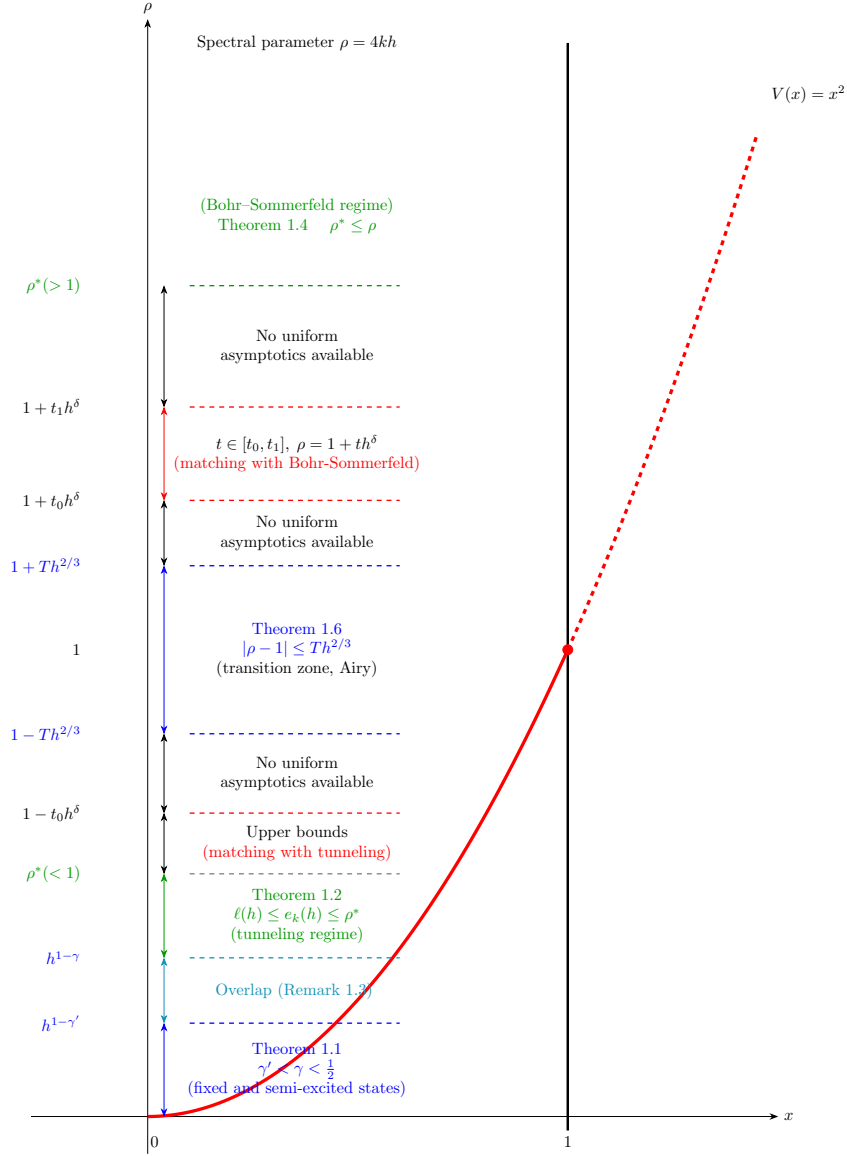


FIGURE 1. Summary of the different spectral regimes and the corresponding asymptotic results.

The proof of Theorem 1.1 follows the same scheme, with the Plancherel–Rotach asymptotics being replaced by elementary properties of the associated Laguerre polynomials (Subsection 2.5). The section is concluded with a brief presentation of the Neumann problem (Subsection 2.6).

2.1. Eigenfunction on $\mathbb{R}_+$.

Given a positive integer k , we introduce the function

$$(2.1) \quad \Phi_k(z) = C_k z^{\nu+\frac{1}{2}} L_{k-1}^{(\nu)}(z^2) e^{-z^2/2},$$

where $L_{k-1}^{(\nu)}$ is the associated Laguerre polynomial of degree $k-1$ and C_k is a positive constant chosen so that Φ_k is normalized in $L^2(\mathbb{R}_+)$. We calculate C_k from the orthogonality relations for Laguerre polynomials

$$\int_0^\infty y^\nu L_m^{(\nu)}(y) L_n^{(\nu)}(y) e^{-y} dy = \frac{\Gamma(n+\nu+1)}{n!} \delta_{mn},$$

which yield

$$\int_0^\infty y^\nu |L_n^{(\nu)}(y)|^2 e^{-y} dy = \frac{\Gamma(n+\nu+1)}{n!}, \quad n = 0, 1, 2, \dots$$

Thus, for $n = k - 1$, we obtain by performing the change of variable $y = z^2$ that

$$(2.2) \quad C_k = \sqrt{\frac{2(k-1)!}{\Gamma(k+\nu)}},$$

and by Stirling's approximation as $k \rightarrow +\infty$, we have

$$(2.3) \quad C_k = 2^{\frac{1}{2}} k^{-\frac{\nu}{2}} \left(1 - \frac{\nu(\nu-1)}{4k} + \mathcal{O}\left(\frac{1}{k^2}\right) \right).$$

Now we introduce an h -dependent function defined on $\mathbb{R}_+$ as

$$(2.4) \quad \Phi_{k,h}(x) = h^{-1/4} \Phi_k(x/\sqrt{h}),$$

where Φ_k is the function in (2.1). Note that $\Phi_{k,1} = \Phi_k$. The function $\Phi_{k,h}$ is normalized in $L^2(\mathbb{R}_+)$ and satisfies on $\mathbb{R}_+$,

$$T_h \Phi_{k,h} = e_k(h) \Phi_{k,h}.$$

It will be used to construct a trial state satisfying the Dirichlet boundary condition at $x = 1$. Towards that end, we need accurate information about the decay of $\Phi_{k,h}$ in the classically forbidden region $\{x \geq \sqrt{e_k(h)}\}$.

2.2. Agmon estimate and applications.

We start with an Agmon estimate⁵. The novelty in comparison with for example [15] is that there is a singularity due to ν .

Proposition 2.1 (rough decay of $\Phi_{k,h}$). *Let $0 < \rho^* < \bar{\rho} < 1$. There exist constants $C, h_0 > 0$ such that, for all $h \in (0, h_0]$ and for all integers k satisfying $e_k(h) \leq \rho^*$, it holds*

$$\int_0^{+\infty} |e^{\beta/h} \Phi_{k,h}|^2 dx \leq C, \quad \int_{\bar{\rho}^{1/2}}^{+\infty} |e^{\beta/h} \Phi'_{k,h}|^2 dx \leq C h^{-2},$$

where

$$\beta(x) = \int_0^x \sqrt{(s^2 - \bar{\rho})_+} ds.$$

Proof. For simplicity, we skip the reference to k and h and write $\Phi = \Phi_{k,h}$. Let $u = e^{\beta/h} \Phi$ and $\tilde{u} = e^{\tilde{\beta}/h} \Phi$, where $\tilde{\beta}$ is defined by

$$\tilde{\beta}(x) = \begin{cases} \beta(x) & x < R, \\ \beta(R) & x \geq R. \end{cases}$$

Here $R > 1$. Writing

$$\langle (T_h - e_k(h)) \Phi, e^{2\tilde{\beta}/h} \Phi \rangle = 0,$$

and integrating by parts (on $\mathbb{R}_+$), we obtain the identity

$$\int_0^{+\infty} \left(|h\tilde{u}'|^2 + h^2 \frac{\nu^2 - 1/4}{x^2} |\tilde{u}|^2 + (x^2 - e_k(h) - |\tilde{\beta}'|^2) |\tilde{u}|^2 \right) dx = 0,$$

which can be written in the form

$$\begin{aligned} \int_0^{+\infty} |h\tilde{u}'|^2 dx + h^2 \int_0^{+\infty} \frac{\nu^2 - 1/4}{x^2} |\tilde{u}|^2 dx + \int_{\bar{\rho}^{1/2}}^{+\infty} (x^2 - e_k(h) - |\tilde{\beta}'|^2) |\tilde{u}|^2 dx \\ = \int_0^{\bar{\rho}^{1/2}} (e_k(h) - x^2) |\tilde{u}|^2 dx. \end{aligned}$$

Using $|\tilde{\beta}'(x)|^2 \leq (x^2 - \bar{\rho})_+$, $e_k(h) \leq \rho^*$, and the Hardy inequality

$$\int_0^{\rho_1} \frac{|\tilde{u}(x)|^2}{x^2} dx \leq 4 \int_0^{\rho_1} |\tilde{u}'(x)|^2 dx,$$

⁵The Agmon estimate in Proposition 2.1 is valid in a neighborhood of $+\infty$ and can be easily extended to more general potentials.

with $\rho_1 = \bar{\rho}^{1/2}$, we deduce that,

$$\int_{\bar{\rho}^{1/2}}^{+\infty} |h\tilde{u}'|^2 dx + \int_{\bar{\rho}^{1/2}}^{+\infty} \left(\bar{\rho} - \rho^* - \frac{h^2}{4\bar{\rho}} \right) |\tilde{u}|^2 dx \leq \rho^* \int_0^{\bar{\rho}^{1/2}} |\tilde{u}|^2 \leq \rho^*.$$

Here we have also used that, for $x^2 \leq \bar{\rho}$, we have $\tilde{\beta}(x) = \beta(x) = 0$ and $\tilde{u} = \Phi$, where Φ is normalized in $\mathbb{R}_+$.

We put $A = \frac{\bar{\rho} - \rho^*}{2} > 0$ and we choose h_0 sufficiently small so that, for $h \in (0, h_0]$, we have $\bar{\rho} - \rho^* - \frac{h^2}{4\bar{\rho}} \geq A$. Thus,

$$\int_{\bar{\rho}^{1/2}}^{+\infty} |h\tilde{u}'|^2 dx + A \int_0^{+\infty} |\tilde{u}|^2 dx \leq 2A \int_0^{\bar{\rho}^{1/2}} |\Phi|^2 dx + \rho^* \leq A + 1.$$

Since $\tilde{u} = u$ on $(0, R)$, we get

$$\int_{\bar{\rho}^{1/2}}^R |hu'|^2 dx + A \int_0^R |u|^2 dx \leq 2A + 1.$$

Taking $R \rightarrow +\infty$, we get by monotone convergence that

$$\int_{\bar{\rho}^{1/2}}^{+\infty} |hu'|^2 dx + A \int_0^{+\infty} |u|^2 \leq C.$$

With $C = (2A + 1)$, we obtain the desired inequality. $\square$

We present a short proof of Vanlaere's bound (1.5) via the Agmon estimate in Proposition 2.1 with a nearly optimal decay rate.

Proposition 2.2. *Let $0 < \rho^* < 1$ and set*

$$\alpha = 2 \int_0^1 \sqrt{(s^2 - \rho^*)_+} ds.$$

For any $\eta \in (0, 1)$, there exists $h_0 > 0$ such that, for all $h \in (0, h_0]$ and for all integers k satisfying $e_k(h) \leq \rho^$, we have*

$$e_k(h) \leq \lambda_k(h) \leq e_k(h) + e^{-(1-\eta)\alpha/h}.$$

Proof. The lower bound was established in (1.3).

We turn to the proof of the upper bound. Let

$$0 < \varepsilon < \frac{1 - \rho^*}{4}, \quad \bar{\rho} = \rho^* + \frac{\varepsilon}{2}, \quad \alpha_\varepsilon = 2\beta(1 - \varepsilon),$$

where β is defined in Proposition 2.1. Choose a cut-off function $\chi \in C^\infty(\mathbb{R})$ such that

$$\chi = 1 \text{ on } (-\infty, 1 - \frac{\varepsilon}{2}], \quad \chi = 0 \text{ on } [1 - \frac{\varepsilon}{4}, +\infty).$$

Consider the subspace $M = \{\chi\Phi_{j,h} : 1 \leq j \leq k\}$. For sufficiently small h , it follows from Proposition 2.1 that $\dim M = k$ and that

$$\sup_{u \in M \setminus \{0\}} \frac{\langle T_h \phi, \phi \rangle}{\|\phi\|^2} \leq e_k(h) + e^{-\alpha_\varepsilon/h},$$

since the commutator terms in $[\chi_h, T_h]\Phi_{j',h}\Phi_{j,h}$ (with $j \leq k, j' \leq k$) are supported in the interval $(1 - \frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{4})$, and $2\beta(1 - \frac{\varepsilon}{2}) > \alpha_\varepsilon$.

Thus, by the min-max principle, we obtain that $\lambda_k(h) \leq e_k(h) + e^{-\alpha_\varepsilon/h}$. This proves the claimed upper bound, since for a given $\eta \in (0, 1)$, we can choose ε sufficiently small so that $|\alpha_\varepsilon - \alpha| < \eta$. $\square$

Remark 2.3. The following almost optimal upper bound results from Proposition 2.2,

$$0 \leq \lambda_k(h) - e_k(h) \leq \exp\left(-\frac{\alpha + o(1)}{h}\right).$$

To write down the asymptotics, we will use precise decay estimates given by the Plancherel-Rotach asymptotics for the associated Laguerre polynomials.

We will need the following direct consequence of Proposition 2.2.

Corollary 2.4. *Let $0 < \rho^* < 1$. There exists $h_0 > 0$ such that, if $h \in (0, h_0]$ and $e_k(h) \leq \rho^*$, then*

$$|\lambda_j(h) - e_k(h)| \geq 2h \quad \text{for all } j \neq k.$$

Proof. For $j \geq k+1$, the result in the corollary follows from the universal lower bound (1.3):

$$\lambda_j(h) \geq e_{k+1}(h) = e_k(h) + 4h.$$

For $j \leq k-1$, the result follows from the asymptotic upper bound in Proposition 2.2,

$$\lambda_j(h) \leq \lambda_{k-1}(h) \leq e_{k-1}(h) + e^{-\alpha/h} = e_k(h) - 4h + e^{-\alpha/h}.$$

□

We can push the upper bound in Proposition 2.2 up to $\rho^* \sim 1$.

Proposition 2.5. *Let $t > 0$ and $0 < \delta < 2/3$ be fixed. For every $\eta \in (0, 1)$, there exist $h_0 > 0$ such that, for all $h \in (0, h_0]$ and for all integers k satisfying $e_k(h) \leq 1 - th^\delta$, we have*

$$\lambda_k(h) \leq e_k(h) + \exp\left(-\frac{2(1-\eta)}{3} t^{3/2} h^{\frac{3\delta}{2}-1}\right).$$

Proof. The argument used in the proof of Proposition 2.1 applies with $\rho^* = 1 - th^\delta$ and $\bar{\rho} = 1 - t_1 h^\delta$, where $0 < t_1 < t$. It yields that $\Phi_{k,h}$ satisfies

$$\int_0^{+\infty} |e^{\beta/h} \Phi_{k,h}|^2 dx \leq C h^{-\delta}, \quad \int_{\bar{\rho}^{1/2}}^{+\infty} |e^{\beta/h} \Phi'_{k,h}|^2 dx \leq C h^{-2-\delta},$$

where

$$\beta(x) = \int_0^x \sqrt{(s^2 - \bar{\rho})} ds = \int_0^x \sqrt{(s^2 - 1 + t_1 h^\delta)_+} ds.$$

Note that, for $0 \leq t_2 < \frac{1}{2}t_1$, we have

$$\beta(1 - t_2 h^\delta) = \frac{1}{3}(t_1 - 2t_2)^{3/2} h^{3\delta/2} + \mathcal{O}(h^{5\delta/2}).$$

We choose $\varepsilon \in (0, \frac{1}{2})$ sufficiently small but fixed, $t_1 = (1 - \varepsilon)t$ and $t_2 = \varepsilon t$. Let χ be a cut-off function satisfying

$$\chi = 1 \text{ on } [0, 1 - t_2 h^\delta], \quad \chi = 0 \text{ on } [1, +\infty).$$

With $M = \text{Span}\{\chi \Phi_{j,h} : 1 \leq j \leq k\}$, and $\alpha_\varepsilon = 2\beta(1 - \varepsilon t h^\delta)$, we obtain

$$\sup_{u \in M \setminus \{0\}} \frac{\langle T_h \phi, \phi \rangle}{\|\phi\|^2} \leq e_k(h) + h^{-2\delta} e^{-\alpha_\varepsilon/h}.$$

Note that α_ε depends on h . For a given $\eta \in (0, 1)$, we can choose ε, h_0 sufficiently small such that

$$\left| \alpha_\varepsilon + 2\delta h \ln h - \frac{t^{3/2}}{3} h^{3\delta/2} \right| < \eta \frac{t^{3/2}}{2} h^{3\delta/2} \quad \text{for } h \in (0, h_0].$$

□

It follows from Proposition 2.5 that for a label $k = k(h)$ satisfying

$$4kh = 1 - th^\delta + o(h^\delta),$$

with $0 < \delta < 2/3$, the corresponding eigenvalue $\lambda_k(h)$ satisfies

$$(2.5) \quad e_k(h) \leq \lambda_k(h) \leq e_k(h) + \exp\left(-\frac{2t^{3/2}h^{\frac{3\delta}{2}-1}}{3}(1+o(1))\right).$$

Hence we are still in a regime where the correction is exponentially small.

Remark 2.6. A close inspection of the proof of Proposition 2.5 shows that we can obtain the following generalization for $\delta = 2/3$ and $t = t(h) \rightarrow +\infty$: For any $\eta \in (0, 1)$ and k satisfying $e_k(h) \leq 1 - t(h)h^{2/3}$, we have

$$e_k(h) \leq \lambda_k(h) \leq e_k(h) + \exp\left(-\frac{2(1-\eta)}{3}t(h)^{3/2}\right),$$

for h in a right neighborhood of 0.

2.3. Plancherel-Rotach asymptotics. We recall the formulas of Plancherel-Rotach type for Laguerre polynomials [21, Theorem 8.22.8 (b)]. After a change of variable, this gives accurate asymptotics for the function in (2.4) that are much stronger than the rough Agmon estimate in Proposition 2.1.

Let ϵ and ω be fixed positive numbers such that $\epsilon < \omega$. For n sufficiently large, $y = (4n + 2\nu + 2) \cosh^2 \phi$, and $\epsilon \leq \phi \leq \omega$, we have

$$(2.6) \quad e^{-y/2} L_n^{(\nu)}(y) = \frac{1}{2} (-1)^n (\pi \sinh \phi)^{-1/2} y^{-\frac{\nu}{2}-\frac{1}{4}} n^{\frac{\nu}{2}-\frac{1}{4}} \exp(E_n(y)) \{1 + \mathcal{O}(n^{-1})\}.$$

where

$$E_n(y) = \left(n + \frac{\nu+1}{2}\right) (2\phi - \sinh 2\phi) = - \int_{\sqrt{4n+2\nu+2}}^{\sqrt{y}} \sqrt{s^2 - (4n + 2\nu + 2)} ds,$$

and $\mathcal{O}(n^{-1})$ is uniform with respect to $\phi \in [\epsilon, \omega]$.

We show that the Plancherel-Rotach asymptotics continue to hold if we take $\omega = +\infty$.

Proposition 2.7. *Let $\epsilon > 0$ be fixed. There exist $C, N_0 > 0$ such that, if $n \geq n_0$, $\phi \geq \epsilon$, and $y = (4n + 2\nu + 2) \cosh^2 \phi$, then*

$$e^{-y/2} L_n^{(\nu)}(y) = \frac{1}{2} (-1)^n (\pi \sinh \phi)^{-1/2} y^{-\frac{\nu}{2}-\frac{1}{4}} n^{\frac{\nu}{2}-\frac{1}{4}} \exp(E_n(y)) \{1 + R_n(y)\},$$

with

$$|R_n(y)| \leq Cn^{-1}.$$

Proof. Fix $\nu \in \mathbb{R}$ and set

$$N = 4n + 2\nu + 2, \quad y = N \cosh^2 \phi, \quad X = \cosh^2 \phi.$$

For $\phi \geq \epsilon$, we have $X > 1$, and we define $\zeta > 0$ as in [5, Eq. (18.15.21)] by

$$\zeta^{3/2} = \frac{3}{4} \left(\sqrt{X^2 - X} - \operatorname{arccosh} \sqrt{X} \right) = \frac{3}{8} (\sinh(2\phi) - 2\phi).$$

Thus,

$$(2.7) \quad -\frac{2}{3} N \zeta^{3/2} = \frac{N}{4} (2\phi - \sinh(2\phi)) = E_n(y).$$

By [5, Eq. (18.15.22)] for $M = 1$, we have

$$\begin{aligned} L_n^{(\nu)}(NX) = & (-1)^n \frac{e^{\frac{1}{2}NX}}{2^{\nu-\frac{1}{2}} X^{\frac{1}{2}\nu+\frac{1}{4}}} \left(\frac{\zeta}{X-1} \right)^{\frac{1}{4}} \times \\ & \left(\frac{\operatorname{Ai}\left(N^{\frac{2}{3}}\zeta\right)}{N^{\frac{1}{3}}} + \frac{\operatorname{Ai}'\left(N^{\frac{2}{3}}\zeta\right)}{N^{\frac{5}{3}}} F_0(\zeta) + \operatorname{envAi}\left(N^{\frac{2}{3}}\zeta\right) R_n(X) \right), \end{aligned}$$

where envAi is the envelop of the Airy function (see [5, Eq. (2.8.20)])

$$\text{env Ai}(x) = \begin{cases} (\text{Ai}(x)^2 + \text{Bi}(x)^2)^{1/2}, & x \leq c, \\ \sqrt{2} \text{Ai}(x), & x \geq c, \end{cases}$$

where c is the real root of smallest absolute value of $\text{Ai}(c) = \text{Bi}(c)$, namely $c \sim -0.36 \dots$, F_0 is defined in [5, Eq. (18.15.23)] as

$$F_0(\zeta) = -\frac{5}{48\zeta^2} + \left(\frac{X-1}{X\zeta}\right)^{\frac{1}{2}} \left(\frac{1}{2}\nu^2 - \frac{1}{8} - \frac{1}{4}\frac{X}{X-1} + \frac{5}{24}\left(\frac{X}{X-1}\right)^2\right),$$

and

$$R_n(X) = \mathcal{O}\left(\frac{1}{N^{\frac{4}{3}}}\right)$$

uniformly for $\delta \leq X < +\infty$, for any $\delta > 0$, hence for $\epsilon \leq \phi < +\infty$.

Thus, we have an expansion of $L_n^{(\nu)}(y)$, since $y = NX$. Using the Airy asymptotic formulas as $z \rightarrow +\infty$ (see [5, Eq. (9.7.5) and (9.7.6)])

$$\text{Ai}(z) \sim \frac{e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}z^{1/4}}, \quad \text{Ai}'(z) \sim -\frac{z^{1/4}e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}},$$

we obtain by (2.7),

$$e^{-y/2}L_n^{(\nu)}(y) = \frac{(-1)^n}{2\sqrt{\pi}} 2^{-\nu+\frac{1}{2}} N^{-\frac{1}{2}} (\cosh \phi)^{-\nu-\frac{1}{2}} (\sinh \phi)^{-\frac{1}{2}} \exp(E_n(y)) (1 + \mathcal{O}(N^{-1})),$$

uniformly for $\epsilon \leq \phi < +\infty$.

It remains to observe that

$$y^{-\nu/2-1/4} n^{\nu/2-1/4} = N^{-\nu/2-1/4} n^{\nu/2-1/4} (\cosh \phi)^{-\nu-1/2},$$

and

$$2^{-\nu+1/2} \left(\frac{N}{n}\right)^{\nu/2-1/4} = 1 + \mathcal{O}(n^{-1}).$$

Therefore

$$e^{-y/2}L_n^{(\nu)}(y) = \frac{(-1)^n}{2\sqrt{\pi} \sinh \phi} y^{-\nu/2-1/4} n^{\nu/2-1/4} \exp(E_n(y)) (1 + \mathcal{O}(n^{-1})),$$

uniformly for $\epsilon \leq \phi < +\infty$. □

Let $1 < A < +\infty$. In terms of the function Φ_k in (2.1), the asymptotics in (2.6) reads as

$$\Phi_k(z) = C_k \frac{(-1)^{k-1} k^{\frac{\nu}{2}-\frac{1}{4}}}{2\sqrt{\pi}} \left(\frac{z^2}{e_k(1)} - 1\right)^{-1/4} \exp(E_{k-1}(z^2)) \{1 + \mathcal{O}(k^{-1})\},$$

for $z \geq A\sqrt{e_k(1)}$.

Furthermore, by Proposition 2.7, $\mathcal{O}(k^{-1})$ is uniform with respect to $z \geq A\sqrt{e_k(1)}$.

Doing the change of variable $z = x/\sqrt{h}$ and noting that

$$e_k(h) = h e_k(1), \quad E_{k-1}(z^2) = -S(x; e_k(h))/h,$$

we obtain, for $x \geq A\sqrt{e_k(h)}$,

$$\Phi_{k,h}(x) = \hat{C}_{k,h} \left(\frac{x^2}{e_k(h)} - 1\right)^{-1/4} \exp(-S(x; e_k(h))/h) \{1 + \mathcal{O}(k^{-1})\},$$

where $\Phi_{k,h}$ is introduced in (2.4) and

$$\hat{C}_{k,h} = C_k \frac{(-1)^{k-1} k^{\frac{\nu}{2}-\frac{1}{4}} h^{-1/4}}{2\sqrt{\pi}}.$$

Later, we need an accurate asymptotics of $\Phi_{k,h}$ in a neighborhood of $x = 1$. We manage to do this by imposing the condition

$$(2.8) \quad \ell(h) \leq e_k(h) \leq \rho^* < 1,$$

where $\rho^* > 0$ is given, $\ell(h) \rightarrow 0$ and $h^{-1}\ell(h) \rightarrow +\infty$. This in particular implies that $k \rightarrow +\infty$ as $h \rightarrow 0^+$. Fixing $a \in (\sqrt{\rho^*}, 1)$, we can select $A > 1$ in such a way that

$$A\sqrt{e_k(h)} < a.$$

Consequently, we obtain the following asymptotic formula

$$(2.9) \quad \Phi_{k,h}(x) = \hat{C}_{k,h} \left(\frac{x^2}{e_k(h)} - 1 \right)^{-1/4} e^{-S_{k,h}(x)/h} (1 + \mathcal{O}(k^{-1})),$$

uniformly with respect to $x \in [a, +\infty)$.

Here

$$(2.10) \quad S_{k,h}(x) = S(x; e_k(h)),$$

and $S(x; E)$ is introduced in (1.6). Furthermore, thanks to (2.3), we have

$$(2.11) \quad \hat{C}_{k,h} = \frac{(-1)^{k-1}}{\sqrt{2\pi}(kh)^{1/4}} (1 + \mathcal{O}(k^{-1})) \sim (-1)^{k-1} \pi^{-\frac{1}{2}} e_k(h)^{-\frac{1}{4}}.$$

We prove next that we can differentiate (2.9).

Proposition 2.8. *Let $a \in (\sqrt{\rho^*}, 1)$ be fixed. There exists $C, h_0 > 0$ such that, if (k, h) satisfies (2.8), $h < h_0$, and $x \in [a, 1]$, then*

$$\Phi'_{k,h}(x) = h^{-1} S'_{k,h}(x) \Phi_{k,h}(x) (1 + R_{k,h}(x)),$$

with

$$|R_{k,h}(x)| \leq Ck^{-1}.$$

Proof. For simplicity, we will omit the reference to (k, h) and write $S(x)$ instead of $S_{k,h}(x)$, and $\Phi(x)$ instead of $\Phi_{k,h}(x)$. Let $E = e_k(h)$ and $\Phi_*(x) = \hat{C}_{k,h} \left(\frac{x^2}{E} - 1 \right)^{-1/4} e^{-S(x)/h}$. Fixing $b > 1$, we know that $\Phi(x) = \Phi_*(x) (1 + \mathcal{O}(k^{-1}))$ for $x \in [a, b]$.

Note that $\Phi'_*(x) \sim -h^{-1} S'(x) \Phi_*(x)$ and that $S'(x) = \sqrt{x^2 - E}$ is bounded in $[a, b]$. Moreover, $\Phi_*(b)/\Phi_*(x) \rightarrow 0$ uniformly for $x \in [a, 1]$.

The function Φ_* satisfies

$$-h^2 \Phi''_* + h^2 \frac{\nu^2 - 1/4}{x^2} \Phi_* + (x^2 - E) \Phi_* = R_h \Phi_*,$$

with $R_h = \mathcal{O}(h^2)$, on $[a, b]$. Indeed, writing

$$q(x) = \left(\frac{x^2}{E} - 1 \right)^{-1/4},$$

we have

$$2S'(x)q'(x) + S''(x)q(x) = 0,$$

and

$$R_h = h^2 \left(-\frac{q''(x)}{q(x)} + \frac{\nu^2 - 1/4}{x^2} \right) = \mathcal{O}(h^2),$$

uniformly on $[a, b]$.

Thus

$$(\Phi' \Phi_* - \Phi \Phi'_*)' = -h^{-2} R_h \Phi \Phi_*.$$

Consequently

$$\Phi'(x) \Phi_*(x) - \Phi(x) \Phi'_*(x) = h^{-2} \int_x^1 R_h(t) \Phi(t) \Phi_*(t) dt.$$

Dividing both sides by $\Phi'_*(x)\Phi(x)$ and using that $\Phi_*(t) \sim -h\Phi'_*(t)/S'(t)$, we infer from the aforementioned formula that

$$\frac{\Phi'(x)}{\Phi'_*(x)} - 1 = \left(\frac{1}{\Phi(x)\Phi_*(x)} \int_x^b \Phi(t)\Phi'_*(t)dt \right) \mathcal{O}(h^2).$$

Using that $\Phi(t) \sim \Phi_*(t)$, we obtain

$$\frac{1}{\Phi(x)\Phi_*(x)} \int_x^b \Phi(t)\Phi'_*(t)dt \sim \frac{1}{\Phi_*(x)^2} \int_x^b \Phi_*(t)\Phi'_*(t)dt = \frac{1}{2} - \frac{\Phi_*(b)^2}{\Phi_*(x)^2} = \frac{1}{2} + o(1).$$

Thus, $\Phi'(x) = \Phi'_*(x)(1 + \mathcal{O}(h^2))$, uniformly for $x \in [a, 1]$. $\square$

2.4. The trial state. We will adjust the function $\Phi_{k,h}$ by adding to it another function, defined by a growing solution of (2.1), resulting in a trial state $v_{k,h}$ satisfying the Dirichlet boundary condition at $x = 1$ (see (2.16)). This line of thought goes back to [3, 4].

Choose $a \in (\sqrt{\rho^*}, 1)$. Thanks to (2.9), $\Phi_{k,h}$ has a constant sign on $[a, 1]$, for sufficiently small h . We define a second solution of

$$(T_h - \mu)u = 0$$

on $[a, 1]$ by reduction of order:

$$(2.12) \quad \Psi_{k,h}(x) = \Phi_{k,h}(x) \int_a^x \frac{dt}{\Phi_{k,h}(t)^2}.$$

Then

$$(2.13) \quad W(\Phi_{k,h}, \Psi_{k,h}) := \Phi_{k,h} \Psi'_{k,h} - \Phi'_{k,h} \Psi_{k,h} = 1.$$

Choose (fixed) numbers ϵ, a_0 and a_1 such that

$$0 < \epsilon < a < a_0 < a_1 < 1$$

and

$$(2.14) \quad 2S_{k,h}(a_1) - S_{k,h}(a_0) - S_{k,h}(1) < -\epsilon.$$

Then, choose a function $\chi \in C^\infty([0, 1])$ such that

$$\chi = 0 \text{ on } [0, a_0], \quad \chi = 1 \text{ on } [a_1, 1].$$

Set

$$(2.15) \quad J_{k,h} := \int_a^1 \frac{1}{\Phi_{k,h}(t)^2} dt,$$

and

$$(2.16) \quad c_{k,h} := -\frac{1}{J_{k,h}}, \quad v_{k,h} := \Phi_{k,h} + c_h \chi \Psi_{k,h}.$$

Since

$$\Psi_{k,h}(1) = \Phi_{k,h}(1)J_{k,h},$$

we have

$$v_{k,h}(1) = \Phi_{k,h}(1) + c_{k,h}\Psi_{k,h}(1) = 0.$$

Moreover, $v_{k,h}$ has the regular behavior at the singular endpoint $x = 0$, since $\chi\Psi_{k,h}$ vanishes in a neighborhood of 0. Consequently, $v_{k,h}$ belongs to the domain of T_h .

Recall that we fixed $\rho^* \in (0, 1)$ and we introduced the set $\mathcal{J}(h)$ in (1.8).

Proposition 2.9. *As $h \rightarrow 0^+$, and for $k \in \mathcal{J}(h)$, we have*

$$(2.17) \quad c_{k,h} = -\frac{2}{\pi h} e^{-2S_{k,h}(1)/h} (1 + o(1)),$$

and

$$(2.18) \quad \lambda_k(h) - e_k(h) = -h^2 c_{k,h} (1 + o(1)).$$

With Proposition 2.9 in hand, we have:

Proof of Theorem 1.2. Combine (2.17) and (2.18), and use that $S_{k,h} = S_{k,h}(1)$. $\square$

Proof of Proposition 2.9. We divide the proof into several steps. For simplicity, we write $E = e_k(h)$, $S(x)$ instead of $S_{k,h}(x) = S(x; E)$, and $\Phi(x)$, $\Psi(x)$, $v(x)$ instead of $\Phi_{k,h}(x)$, $\Psi_{k,h}(x)$, $v_{k,h}(x)$. Also, $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ denote the standard norm and inner product in $L^2(0, 1)$.

Step 1. We derive (2.17). Using (2.9), we have

$$\frac{1}{\Phi(t)^2} = \hat{C}_{k,h}^{-2} \left(\frac{t^2}{E} - 1 \right)^{1/2} e^{2S(t)/h} (1 + o(1)).$$

Hence, by the Laplace method at the endpoint $t = 1$,

$$J_{k,h} = \frac{\hat{C}_{k,h}^{-2} h}{2S'(1)} \left(\frac{1}{E} - 1 \right)^{1/2} e^{2S(1)/h} (1 + o(1)).$$

Since

$$S'(1) = \sqrt{1 - E}, \quad \left(\frac{1}{E} - 1 \right)^{1/2} = \frac{\sqrt{1 - E}}{\sqrt{E}},$$

we obtain

$$J_{k,h} = \frac{\hat{C}_{k,h}^{-2} h}{2\sqrt{E}} e^{2S(1)/h} (1 + o(1)).$$

Finally, using (2.11),

$$\hat{C}_{k,h}^{-2} = \pi E^{1/2} (1 + o(1)),$$

and therefore

$$J_{k,h} = \frac{\pi h}{2} e^{2S(1)/h} (1 + o(1)).$$

In light of (2.11), the coefficient $c_{k,h} = -J_{k,h}^{-1}$ satisfies

$$(2.19) \quad c_{k,h} = -\frac{2}{\pi h} e^{-2S(1)/h} (1 + o(1)) \text{ as } h \rightarrow 0^+.$$

Since $S(1) = S_{k,h}(1)$, (2.17) follows from (2.19).

This allows us to show that v is almost normalized in $L^2(0, 1)$. Since Φ is normalized in $L^2(\mathbb{R}_+)$, it follows from (2.19) and from Proposition 2.1 that

$$(2.20) \quad \|v\| = 1 + o(1) \quad (h \rightarrow 0^+).$$

Step 2. We calculate the inner product $\langle L_h v, v \rangle$, where

$$L_h := T_h - E.$$

Since $L_h \Phi = L_h \Psi = 0$, we have pointwise

$$(2.21) \quad L_h v = c_{k,h} L_h (\chi \Psi) = c_{k,h} [L_h, \chi] \Psi,$$

where

$$(2.22) \quad [L_h, \chi] \Psi = -h^2 (2\chi' \Psi' + \chi'' \Psi).$$

We first compute $\langle L_h (\chi \Psi), \Phi \rangle$. Since $\chi \Psi$ vanishes in a neighborhood of 0, whereas $\chi = 1$ in a neighborhood of 1, two integrations by parts and the identity $L_h \Phi = 0$ yield

$$\begin{aligned} \langle L_h (\chi \Psi), \Phi \rangle &= -h^2 [(\chi \Psi)' \Phi - (\chi \Psi) \Phi']_0^1 \\ &= -h^2 (\Psi'(1) \Phi(1) - \Psi(1) \Phi'(1)) \\ &= -h^2 W(\Phi, \Psi) = -h^2. \end{aligned}$$

Therefore, we infer from (2.21),

$$(2.23) \quad \langle L_h v, v \rangle = -h^2 c_{k,h} + c_{k,h}^2 \langle [L_h, \chi] \Psi, \chi \Psi \rangle.$$

It remains to prove that the second term on the right hand side is negligible compared to the first term. Since

$$\text{supp } \chi' \cup \text{supp } \chi'' \subset [a_0, a_1],$$

it follows from (2.9) and Proposition 2.8,

$$|\langle [L_h, \chi] \Psi, \chi \Psi \rangle| = \mathcal{O}(h^{-1/2} e^{\frac{3S(a_1) - S(a_0)}{h}}).$$

Thanks to (2.19) and (2.14), it holds

$$(2.24) \quad c_{k,h}^2 \langle [L_h, \chi] \Psi, \chi \Psi \rangle = o(h^2 c_{k,h}) \quad \text{as } h \rightarrow 0^+.$$

In a similar fashion, we prove that

$$(2.25) \quad \|L_h v\| = \mathcal{O}(h^{-1} |c_{k,h}| e^{\frac{2S(a_1) - S(a_0)}{h}}) = o(h^2 \sqrt{|c_{k,h}|}).$$

Step 3. It follows from (2.23), (2.24) and (2.20) that the Rayleigh quotient satisfies

$$(2.26) \quad \frac{\langle L_h v, v \rangle}{\|v\|^2} = -h^2 c_{k,h} (1 + o(1)).$$

An argument via the spectral theorem yields that

$$\lambda_k(h) = E - h^2 c_{k,h} + o(h^2 |c_{k,h}|).$$

We provide the details for the convenience of the reader. We write the spectral resolution of T_h in $L^2(0, 1)$ as

$$T_h = \sum_{j=1}^{+\infty} \lambda_j(h) \Pi_j, \quad I = \sum_{j=1}^{+\infty} \Pi_j,$$

where Π_j is the orthogonal projection on the normalized eigenfunction associated to $\lambda_j(h)$. It follows then that

$$\begin{aligned} \langle L_h(v - \Pi_k v), v - \Pi_k v \rangle &= \langle L_h v, v - \Pi_k v \rangle, \\ \|L_h v\|^2 &= \sum_{j=1}^{+\infty} |\lambda_j(h) - E|^2 \|\Pi_j v\|^2 \geq 4h^2 \sum_{j \neq k} \|\Pi_j v\|^2, \end{aligned}$$

where we used the result of Corollary 2.4 in the last step. Thus, we obtain

$$(2.27) \quad |\langle L_h(v - \Pi_k v), v - \Pi_k v \rangle| \leq \|L_h v\| \|v - \Pi_k v\|, \quad \|v - \Pi_k v\| \leq \frac{1}{2h} \|L_h v\|.$$

We use next the orthogonal decomposition

$$v = \Pi_k v + (v - \Pi_k v)$$

to write

$$\langle L_h v, v \rangle = (\lambda_k(h) - E) \|\Pi_k v\|^2 + \langle L(v - \Pi_k v), v - \Pi_k v \rangle - E \|v - \Pi_k v\|^2.$$

Using (2.25) and (2.27), we get

$$\langle L_h v, v \rangle = (\lambda_k(h) - E) \|\Pi_k v\|^2 + o(h^2 c_{k,h}), \quad \|\Pi_k v\|^2 = \|v\|^2 + o(1).$$

Inserting these estimates into (2.26), we obtain the formula in (2.18). $\square$

Remark 2.10 (Explicit dependence on the flux term). With $\rho = 4kh$ and

$$E = e_k(h) = \rho + 2(\nu - 1)h,$$

we have, by a straightforward calculation,

$$S(1; E) = S_0(\rho) - (\nu - 1)h \log\left(\frac{1 + \sqrt{1 - \rho}}{\sqrt{\rho}}\right) + \mathcal{O}(h^2),$$

where

$$S_0(\rho) = \frac{\sqrt{1 - \rho}}{2} - \frac{\rho}{2} \log\left(\frac{1 + \sqrt{1 - \rho}}{\sqrt{\rho}}\right).$$

Thus,

$$e^{-2S(1; E)/h} = \left(\frac{1 + \sqrt{1 - \rho}}{\sqrt{\rho}}\right)^{2\nu - 2} e^{-2S_0(\rho)/h} (1 + \mathcal{O}(h)).$$

Consequently, the splitting asymptotics in (2.18) becomes

$$(2.28a) \quad \lambda_k(h) = (4k + 2\nu - 2)h + C(\rho, \nu)he^{-2S_0(\rho)/h}(1 + o(1)),$$

where

$$(2.28b) \quad C(\rho, \nu) = \frac{2}{\pi} \left(\frac{1 + \sqrt{1 - \rho}}{\sqrt{\rho}} \right)^{2\nu - 2}, \quad h \ll \rho = 4kh \leq \rho_1 < 1.$$

In particular, the leading term $S_0(\rho)$ is independent of the flux term ν . This is expected since the Aharonov–Bohm flux appears in the potential as a term of order h^2 , hence it contributes as an $\mathcal{O}(h)$ correction, yielding the explicit algebraic prefactor in the exponentially small asymptotics.

Remark 2.11 (Low energy regime). Let us consider more precisely the regime

$$h \ll \rho = 4kh \ll 1.$$

As $\rho \rightarrow 0$, we have

$$S_0(\rho) = \frac{1}{2} + \frac{\rho}{4} \log \rho - \frac{\rho}{2} \log 2 - \frac{\rho}{4} + \frac{\rho^2}{16} + \mathcal{O}(\rho^3).$$

Consequently, since $\rho = 4kh$,

$$-\frac{2S_0(\rho)}{h} = -\frac{1}{h} - 2k \log(kh) + 2k - 2k^2h + \mathcal{O}(k^3h^2).$$

In particular, in the regime

$$k \rightarrow +\infty, \quad k^2h \rightarrow 0,$$

we obtain

$$e^{-2S_0(\rho)/h} = e^{-1/h} e^{2k} (kh)^{-2k} (1 + o(1)).$$

Moreover,

$$\left(\frac{1 + \sqrt{1 - \rho}}{\sqrt{\rho}} \right)^{2\nu - 2} = (kh)^{1 - \nu} (1 + o(1)).$$

Hence, the asymptotics in (2.28) reads as

$$(2.29) \quad \lambda_k(h) - e_k(h) = \frac{2}{\pi} e^{2k} k^{1 - \nu - 2k} h^{2 - \nu - 2k} e^{-1/h} (1 + o(1)).$$

Remark 2.12 (Comparison with Gannot’s formula). It is interesting to compare (2.29) with the fixed- k asymptotics in (1.4). Indeed, as $k \rightarrow +\infty$, Stirling’s formula gives

$$(k - 1)! \Gamma(k + \nu) = 2\pi k^{2k + \nu - 1} e^{-2k} (1 + o(1)),$$

and therefore the explicit coefficient in (1.4) satisfies

$$\frac{4h^{-2k - \nu + 2}}{(k - 1)! \Gamma(k + \nu)} e^{-1/h} = \frac{2}{\pi} e^{2k} k^{1 - \nu - 2k} h^{2 - \nu - 2k} e^{-1/h} (1 + o(1)).$$

Thus, formally, the asymptotics in (1.4) and (2.29) match in the overlap regime

$$k \rightarrow +\infty, \quad k^2h \rightarrow 0.$$

We emphasize, however, that (1.4) is proved for fixed k , so this comparison does not by itself provide a uniform version of the fixed- k asymptotics. We address this rigorously in the next subsection.

2.5. Semi-excited states. We now consider the energy levels satisfying

$$(2.30) \quad 0 < e_k(h) < \ell(h),$$

where

$$(2.31) \quad \ell(h) \longrightarrow 0, \quad \frac{\ell(h)}{h} \longrightarrow +\infty, \quad \frac{\ell(h)^2}{h} \longrightarrow 0 \quad (h \rightarrow 0^+).$$

Proposition 2.13. *If (2.30) and (2.31) hold, then*

$$(2.32) \quad \lambda_k(h) - e_k(h) = \frac{4h^{-2k-\nu+2}}{(k-1)!\Gamma(k+\nu)} e^{-1/h} (1 + o(1)),$$

where $o(1)$ is uniform with respect to k .

Remark 2.14.

- For fixed k , (2.32) is precisely Gannot's asymptotic formula (1.4). The argument above shows, moreover, that this formula remains uniform in the regime $k^2 h \rightarrow 0$.
- We may eventually choose

$$\ell(h) = h^q, \quad \frac{1}{2} < q < 1.$$

The proof of Theorem 1.1 is now straightforward.

Proof of Theorem 1.1. Apply Proposition 2.13 with $\ell(h) = 4h^{1-\gamma} + (2\nu - 2)h$. $\square$

The proof of Proposition 2.13 does not use the Plancherel–Rotach asymptotics. Instead, we exploit directly the exact Laguerre representation of the normalized eigenfunction on the half-line.

We start with an elementary uniform estimate for Laguerre polynomials. Later on we shall apply this result with $m = k - 1$. The C^1 estimate, and in particular the logarithmic derivative estimate below, provide the estimates on the second solution obtained by reduction of order.

Lemma 2.15. *Let $a \in (0, 1)$ be fixed and let $m = m(h) \geq 0$ satisfy*

$$m^2 h \longrightarrow 0 \quad \text{as } h \rightarrow 0^+.$$

Then, for $x \in [a, 1]$,

$$(2.33) \quad L_m^{(\nu)}\left(\frac{x^2}{h}\right) = \frac{(-1)^m}{m!} \left(\frac{x^2}{h}\right)^m (1 + R_{m,h}(x)),$$

where $R_{m,h}(x)$ belongs to $C^1([a, 1])$ and satisfies

$$(2.34) \quad \|R_{m,h}\|_{C^1([a,1])} = \mathcal{O}(m^2 h).$$

In particular,

$$(2.35) \quad \frac{d}{dx} \log \left| L_m^{(\nu)}\left(\frac{x^2}{h}\right) \right| = \frac{2m}{x} + \mathcal{O}(m^2 h),$$

uniformly on $[a, 1]$.

Proof. We use the explicit finite-sum representation of the generalized Laguerre polynomials (see [5], Eq. (18.5.12)):

$$L_m^{(\nu)}(X) = \sum_{j=0}^m (-1)^j \binom{m+\nu}{m-j} \frac{X^j}{j!}.$$

Here

$$\binom{m+\nu}{m-j} = \frac{(m+\nu)(m+\nu-1)\cdots(\nu+j+1)}{(m-j)!}.$$

Setting $r = m - j$, we obtain

$$L_m^{(\nu)}(X) = \frac{(-1)^m X^m}{m!} \left(1 + \sum_{r=1}^m (-1)^r \frac{m!}{(m-r)!} \binom{m+\nu}{r} X^{-r} \right).$$

Thus, for $x \in [a, 1]$, we define

$$R_{m,h}(x) := \sum_{r=1}^m (-1)^r \frac{m!}{(m-r)!} \binom{m+\nu}{r} \left(\frac{h}{x^2}\right)^r.$$

Since ν is fixed and non negative, for $1 \leq r \leq m$, we have

$$\frac{m!}{(m-r)!} \leq m^r, \quad \binom{m+\nu}{r} \leq \frac{(m+\nu)^r}{r!}.$$

Consequently, since $x \in [a, 1]$,

$$|R_{m,h}(x)| \leq \sum_{r=1}^m \frac{1}{r!} \left(\frac{m(m+\nu)h}{a^2}\right)^r \leq \exp\left(\frac{m(m+\nu)h}{a^2}\right) - 1.$$

Since ν and $a > 0$ are fixed and $m^2h \rightarrow 0$, we obtain

$$|R_{m,h}(x)| = \mathcal{O}(m^2h),$$

uniformly for $x \in [a, 1]$. Differentiating the finite sum defining $R_{m,h}$ gives

$$R'_{m,h}(x) = -\frac{2}{x} \sum_{r=1}^m r(-1)^r \frac{m!}{(m-r)!} \binom{m+\nu}{r} \left(\frac{h}{x^2}\right)^r.$$

Hence, since $x \in [a, 1]$,

$$|R'_{m,h}(x)| \leq \frac{2}{a} \sum_{r=1}^m \frac{1}{(r-1)!} \left(\frac{m(m+\nu)h}{a^2}\right)^r.$$

Since

$$\sum_{r=1}^{\infty} \frac{t^r}{(r-1)!} = te^t,$$

we obtain

$$|R'_{m,h}(x)| \leq \frac{2}{a} \frac{m(m+\nu)h}{a^2} \exp\left(\frac{m(m+\nu)h}{a^2}\right) = \mathcal{O}(m^2h),$$

uniformly for $x \in [a, 1]$. This proves (2.34). Since $R_{m,h} = o(1)$ uniformly, we also have

$$\frac{R'_{m,h}(x)}{1 + R_{m,h}(x)} = \mathcal{O}(m^2h),$$

and (2.35) follows by taking the logarithmic derivative in (2.33). $\square$

Proof of Proposition 2.13. We split the proof into two steps, working throughout under the assumption $k^2h \rightarrow 0$. We first derive accurate asymptotics for $\Phi_{k,h}$ and its derivative near $x = 1$. We then use the tunneling argument of the previous subsection to obtain the eigenvalue asymptotics.

Step 1. Asymptotics of $\Phi_{k,h}$ and $\Phi'_{k,h}$ near $x = 1$.

We apply Lemma 2.15 with $m = k - 1$. Recall that

$$\Phi_{k,h}(x) = h^{-1/4} C_k \left(\frac{x}{\sqrt{h}}\right)^{\nu+1/2} L_{k-1}^{(\nu)}\left(\frac{x^2}{h}\right) e^{-x^2/(2h)},$$

where C_k is introduced in (2.2). Setting

$$p_k = 2k + \nu - \frac{3}{2},$$

and

$$D_{k,h} = \frac{(-1)^{k-1} C_k}{(k-1)!} h^{-k-\frac{\nu}{2}+\frac{1}{2}},$$

we get by Lemma 2.15,

$$(2.36) \quad \Phi_{k,h}(x) = D_{k,h} x^{p_k} e^{-x^2/(2h)} (1 + R_{k-1,h}(x)),$$

uniformly for $x \in [a, 1]$, with

$$\|R_{k-1,h}\|_{C^1([a,1])} = \mathcal{O}(k^2h) = o(1).$$

In particular, for h sufficiently small, $\Phi_{k,h}$ has a constant sign on $[a, 1]$. Taking the logarithmic derivative in (2.36), we obtain

$$(2.37) \quad \frac{\Phi'_{k,h}(x)}{\Phi_{k,h}(x)} = \frac{p_k}{x} - \frac{x}{h} + \frac{R'_{k-1,h}(x)}{1 + R_{k-1,h}(x)}.$$

Since

$$\|R_{k-1,h}\|_{C^1([a,1])} = \mathcal{O}(k^2h),$$

it follows that

$$(2.38) \quad \frac{\Phi'_{k,h}(x)}{\Phi_{k,h}(x)} = \frac{p_k}{x} - \frac{x}{h} + \mathcal{O}(k^2h) = -\frac{x}{h} \left(1 + \mathcal{O}(k^2h)\right),$$

uniformly for $x \in [a, 1]$.

Step 2. Tunneling argument.

We now follow the argument of Subsection 2.4, replacing the Plancherel-Rotach estimates by (2.36) and (2.38). Define, as in (2.12),

$$\Psi_{k,h}(x) = \Phi_{k,h}(x) \int_a^x \frac{dt}{\Phi_{k,h}(t)^2}.$$

Using (2.36), we have

$$\frac{1}{\Phi_{k,h}(t)^2} = D_{k,h}^{-2} t^{-2p_k} e^{t^2/h} (1 + \mathcal{O}(k^2h)),$$

uniformly for $t \in [a, 1]$. An endpoint Laplace estimate then gives, uniformly on fixed compact subintervals of $(a, 1]$,

$$(2.39) \quad \Psi_{k,h}(x) = \frac{h}{2x - 2p_k h/x} \frac{1}{\Phi_{k,h}(x)} (1 + o(1)) = \frac{h}{2x} \Phi_{k,h}(x) (1 + o(1)).$$

Moreover, using the Wronskian identity

$$W(\Phi_{k,h}, \Psi_{k,h}) = 1,$$

together with (2.38) and (2.39), we obtain

$$(2.40) \quad \Psi'_{k,h}(x) = \frac{1}{2\Phi_{k,h}(x)} (1 + o(1)).$$

Thus, the estimates required in the reduction-of-order and quasimode argument of Subsection 2.4 remain valid in the present regime. Define

$$v_{k,h} = \Phi_{k,h} + c_{k,h} \chi \Psi_{k,h}$$

with $\chi = 0$ on $[0, a]$ and $\chi = 1$ on $[1, +\infty)$. Then $v_{k,h}$ satisfies the Dirichlet boundary condition at $x = 1$ when

$$c_{k,h} = -\frac{\Phi_{k,h}(1)}{\Psi_{k,h}(1)}.$$

Using (2.39), we obtain

$$c_{k,h} = -\frac{2}{h} \Phi_{k,h}(1)^2 (1 + o(1)).$$

The proof of Proposition 2.9 can then be repeated with the same argument to obtain

$$\lambda_k(h) - e_k(h) = 2h \Phi_{k,h}(1)^2 (1 + o(1)),$$

uniformly in the regime $k^2h \rightarrow 0$.

Using (2.36) at $x = 1$ and the definition of $D_{k,h}$, we obtain

$$\Phi_{k,h}(1)^2 = \frac{2h^{-2k-\nu+1}}{(k-1)!\Gamma(k+\nu)} e^{-1/h} (1 + o(1)),$$

which finishes the proof of Proposition 2.13. $\square$

2.6. Neumann realization. If we impose Neumann boundary condition at $x = 1$, we obtain the Neumann realization T_h^N with eigenvalues

$$\lambda_1^N(h) < \lambda_2^N(h) < \dots$$

We establish the analogue of Theorem 1.2 for the Neumann realization.

Proposition 2.16. *Let $0 < \rho^* < 1$ be given. If (k, h) satisfy*

$$\ell(h) \leq e_k(h) \leq \rho^*, \quad \ell(h) = |\ln h|^{-1},$$

then

$$\lambda_k^N(h) = e_k(h) - \frac{2h}{\pi} \exp\left(-\frac{2S_{k,h}(1)}{h}\right) (1 + \delta_k(h)),$$

where

$$\sup_{\ell(h) \leq e_k(h) \leq \rho^*} |\delta_k(h)| \longrightarrow 0 \quad \text{as } h \longrightarrow 0^+.$$

Proof. Let $\bar{\rho} \in (\rho^*, 1)$. We start with rough bounds for energy levels with label j satisfying $e_j(h) \leq \bar{\rho}$. Firstly, by the min-max principle and Proposition 2.2, we have

$$\lambda_j^N(h) \leq \lambda_j^D(h) \leq e_j(h) + \mathcal{O}(h^\infty).$$

Secondly, by Agmon estimates, the normalized eigenfunction corresponding to $\lambda_j^N(h)$ obeys an exponential decay law in $[\bar{\rho}, 1]$. Thus, by the min-max principle,

$$\lambda_j^N(h) \geq \lambda_k^D(h) + \mathcal{O}(h^\infty) \geq e_j(h) + \mathcal{O}(h^\infty).$$

Therefore, there exists $h_0 > 0$ such that, for $h \in (0, h_0]$ and for every integer $j \neq k$, we have

$$|\lambda_j^N(h) - E| \geq 2h.$$

Now we consider a trial state

$$u = \Phi + d_{k,h} \chi \Psi$$

with $\Phi = \Phi_{k,h}$ and $\Psi = \Psi_{k,h}$ as in (2.4) and (2.12) respectively. We require that u satisfies Neumann boundary condition, $u'(1) = 0$, which yields

$$d_{k,h} = -\frac{\Phi'(1)}{\Psi'(1)}.$$

By the Plancherel-Rotach asymptotics for Φ and Φ' in (2.9) and Proposition 2.8, we obtain that

$$d_{k,h} = \frac{1}{J_{k,h}} (1 + o(1)) = -c_{k,h} (1 + o(1)).$$

A slight adjustment of the proof of Proposition 2.9 yields

$$\lambda_k^N(h) - e_k(h) = h^2 c_{k,h} (1 + o(1)).$$

To finish the proof, we use the asymptotics for $c_{k,h}$ in Proposition 2.9. $\square$

3. ABOVE THE TUNNELING REGIME

3.1. Preliminaries. In this section, we consider eigenvalues

$$E > 1,$$

which are not covered by Theorem 1.2. Neglecting the term of order h^2 in the potential, and identifying the Dirichlet boundary condition with a hard wall, our operator T_h corresponds to

$$T_h^{\text{eff}} = -h^2 \frac{d^2}{dx^2} + V,$$

with

$$V(x) = \begin{cases} x^2 & 0 \leq x \leq 1, \\ +\infty & x > 1. \end{cases}$$

Since $\{V(x) \leq E\} = [0, 1]$, the Bohr-Sommerfeld quantization condition reads [19, 14]

$$(3.1) \quad \int_0^1 \sqrt{E - x^2} dx \sim k\pi h,$$

that is, the energy level $E > 1$ satisfies

$$(3.2) \quad \frac{E}{2} \arcsin\left(\frac{1}{\sqrt{E}}\right) + \frac{1}{2}\sqrt{E-1} \sim k\pi h,$$

where the term on the left hand side is bounded below by $\pi/4$, thereby enforcing the following condition on the label k

$$(3.3) \quad \rho := 4kh \geq 1 + \mathcal{O}(h).$$

We will analyze this regime directly by means of properties of the Whittaker function, without passing through the hard wall potential. Unlike the tunneling regime covered in Theorem 1.2, we will not use a semiclassical argument. Instead, our analysis relies entirely on the uniform asymptotic expansions of Whittaker functions established by Dunster [6] together with the corresponding asymptotic description of their positive zeros obtained by Gabutti and Gatteschi [9]. The final inversion step is combined with the classical McMahon expansion for the positive zeros of Bessel functions (see, e.g., [5, §10.21]), which we simply invert in the present spectral setting.

Throughout this section, we assume that

$$(3.4) \quad \rho_0 \leq \rho = 4kh \leq \rho_1, \quad 1 < \rho_0 < \rho_1 < +\infty.$$

3.2. Supercritical regime.

3.2.1. Preliminaries and main statements. We assume that (3.4) holds and derive an accurate asymptotic expansion of the k -th eigenvalue of T_h .

Theorem 3.1. *Assume that (3.4) holds, and let*

$$\mu = \frac{\nu}{2}, \quad \rho = 4kh.$$

Then, as $h \rightarrow 0^+$,

$$\lambda_k(h) = E_0(\rho) + hE_1(\rho) + \mathcal{O}(h^2),$$

where

- $E_0(\rho) > 1$ *is the solution of*

$$I(E_0(\rho)) = \frac{\pi\rho}{4},$$

- *For $E > 1$,*

$$I(E) = \frac{1}{2}\sqrt{E-1} + \frac{E}{2} \arcsin(E^{-1/2}),$$

- $E_1(\rho)$ *is defined by*

$$E_1(\rho) = \frac{2\pi(\mu - \frac{1}{4})}{\arcsin(E_0(\rho)^{-1/2})}.$$

Remark 3.2. The function I admits the action representation

$$(3.5) \quad I(E) = \int_0^1 \sqrt{E - y^2} dy, \quad E > 1,$$

and is the classical action of the harmonic oscillator on the interval $(0, 1)$. Thus, the leading-order equation

$$(3.6) \quad I(E_0(\rho)) = \frac{\pi\rho}{4}$$

can be viewed as the principal Bohr–Sommerfeld quantization condition. The subprincipal term in Theorem 3.1 can moreover be absorbed into a shift of the quantization parameter. Indeed,

$$I'(E) = \frac{1}{2} \arcsin(E^{-1/2}),$$

and differentiating (3.6) gives

$$E'_0(\rho) = \frac{\pi}{2 \arcsin(E_0(\rho)^{-1/2})}.$$

Since $\mu = \nu/2$, we therefore have

$$(3.7) \quad E_1(\rho) = 4 \left(\mu - \frac{1}{4} \right) E'_0(\rho).$$

Consequently, Taylor's formula yields

$$E_0(\rho) + hE_1(\rho) = E_0 \left(\rho + 4h \left(\mu - \frac{1}{4} \right) \right) + \mathcal{O}(h^2),$$

uniformly for $\rho \in [\rho_0, \rho_1]$. Since $\rho = 4kh$, the asymptotic formula of Theorem 3.1 can equivalently be written as

$$(3.8) \quad \lambda_k(h) = E_0 \left(4h \left(k + \frac{\nu}{2} - \frac{1}{4} \right) \right) + \mathcal{O}(h^2),$$

which is the form stated in Theorem 1.4. The shift $\nu/2 - 1/4$ is the Bohr–Sommerfeld–Langer correction associated with the singular inverse square term, and the dependence on the magnetic field appears (modulo $\mathcal{O}(h^2)$) through the shift in the quantization rule.

3.2.2. Heuristics. We fix the notation as

$$(3.9) \quad \mu = \frac{\nu}{2}, \quad \rho = 4kh, \quad E(\rho) := \lambda_k(h).$$

The inverse-square potential in T_h is a lower-order term of order h^2 , so it does not contribute to the principal symbol of the operator. Consequently, the underlying model operator is simply the truncated harmonic oscillator

$$(3.10) \quad -h^2 \frac{d^2}{dx^2} + x^2$$

on the interval $(0, 1)$ with a Dirichlet condition at $x = 1$. Its associated classical action is

$$(3.11) \quad I(E) = \int_0^1 \sqrt{(E - x^2)_+} dx.$$

Forgetting a possible difficulty related to the singularity at the origin, the singular inverse-square potential only contributes to the subprincipal correction and is therefore responsible for the second term in the asymptotic expansion of Theorem 3.1, that is

$$E(\rho) = E_0(\rho) + \text{subleading terms}.$$

The condition in (3.4) ensures that the whole interval $(0, 1)$ is classically allowed for this principal Hamiltonian. Formally, the leading term $E_0(\rho)$ is characterized by the formal Bohr–Sommerfeld quantization rule (3.1), which reads as

$$I(E_0(\rho)) = \frac{\pi\rho}{4}.$$

Recall that the eigenvalues of T_h are given by

$$(3.12) \quad \lambda_k(h) = 4h\kappa_k,$$

where $\kappa_k = \kappa_k(h)$ is determined by the spectral equation

$$(3.13) \quad M_{\kappa_k, \mu}(\xi) = 0, \quad \xi = 1/h \gg 1.$$

3.2.3. Gabutti–Gatteschi’s asymptotic description of the positive Whittaker zeros. The starting point of our analysis is the asymptotic description of the positive zeros of the Whittaker function obtained by Gabutti–Gatteschi [9, Theorems 2.1 and 2.3], which is itself based on the uniform asymptotic expansions of Dunster [6].

Throughout this subsection, κ denotes the large asymptotic parameter, whereas μ is fixed. We introduce the scaled variable

$$(3.14) \quad X = \frac{\xi}{\kappa},$$

together with

$$(3.15) \quad \gamma = \frac{2\mu}{\kappa},$$

and define

$$(3.16) \quad \xi_1 = 2 - \sqrt{4 - \gamma^2}, \quad \xi_2 = 2 + \sqrt{4 - \gamma^2},$$

as well as

$$(3.17) \quad R = \sqrt{(X - \xi_1)(\xi_2 - X)}.$$

Furthermore, let $\zeta = \zeta(X)$ be defined implicitly by

$$(3.18) \quad \begin{aligned} \sqrt{\zeta - \gamma^2} - \gamma \arctan\left(\frac{\sqrt{\zeta - \gamma^2}}{\gamma}\right) &= \frac{1}{2}R - \frac{\gamma}{2} \arctan\left(\frac{2X - \gamma^2}{\gamma R}\right) \\ &\quad - \arctan\left(\frac{2 - X}{R}\right) + \frac{\pi}{2}\left(1 - \frac{\gamma}{2}\right). \end{aligned}$$

Finally, for every positive integer k , define

$$(3.19) \quad \zeta_k = \left(\frac{j_{2\mu,k}}{\kappa}\right)^2,$$

where $j_{\nu,k}$ denotes the k -th positive zero of the Bessel function J_ν .

Theorem 2.1 of [9] shows that, if X_k is obtained by inverting (3.18) with $\zeta = \zeta_k$, then the k -th positive zero m_k of $M_{\kappa,\mu}$ satisfies

$$(3.20) \quad m_k = \kappa X_k + \mathcal{O}(\kappa^{-1}).$$

More precisely, for every fixed $q \in (\xi_1, \xi_2)$, the estimate (3.20) holds uniformly for all zeros satisfying $m_k \leq q\kappa$.

Define

$$(3.21) \quad C_0(\zeta, \gamma, X) = \frac{1}{4} \frac{1}{\sqrt{\zeta - \gamma^2}} + \frac{5}{12} \frac{\gamma^2}{(\zeta - \gamma^2)^{3/2}} + \frac{1}{6} \frac{2X^3 + 3(\gamma^2 - 8)X^2 + 24X - 2\gamma^2(\gamma^2 - 2)}{R^3(\gamma^2 - 4)},$$

with R given by (3.17).

Then Theorem 2.3 of [9] yields the refined asymptotic expansion

$$(3.22) \quad m_k = \kappa X_k - \frac{X_k}{\kappa R_k} C_0(\zeta_k, \gamma, X_k) + \mathcal{O}(\kappa^{-3}),$$

where R_k is defined by (3.17) with X replaced by X_k . More precisely, for every fixed $q \in (\xi_1, \xi_2)$, the expansion (3.22) holds uniformly for all positive zeros satisfying $m_k \leq q\kappa$.

Remark 3.3. The implicit relation (3.18) admits a natural semiclassical interpretation. It can be written as the equality

$$(3.23) \quad S_W(X, \gamma) = S_B(\zeta, \gamma),$$

where S_W and S_B denote respectively the classical actions associated with the Whittaker equation and with its Bessel model. This identity is the key ingredient in Dunster's Liouville transformation and may be viewed as a Bohr–Sommerfeld–Langer quantization condition. For the convenience of the reader, its derivation and geometric interpretation are recalled in Appendix A.

3.2.4. Inversion strategy. The Gabutti–Gatteschi asymptotics describe, for large κ , the k -th positive zero m_k ($k = 1, 2, \dots$) of the Whittaker function $M_{\kappa, \mu}$ with respect to its argument.

In light of (3.12) and (3.13), we would like to analyze, as $\xi \rightarrow +\infty$, the spectral condition

$$M_{\kappa_k, \mu}(\xi) = 0.$$

Thus, for $\kappa = \kappa_k$, the parameter ξ is a positive zero of the Whittaker function $M_{\kappa, \mu}$. We prove below that this zero coincides with the k -th positive zero in the Gabutti–Gatteschi ordering.

Lemma 3.4. *For every $k \geq 1$, the parameter ξ is the k -th positive zero m_k of the Whittaker function $M_{\kappa_k, \mu}$.*

Proof. The regular solution of the singular harmonic oscillator is, up to a non-zero multiplicative constant,

$$(3.24) \quad u_k(x) = x^{\nu + \frac{1}{2}} e^{-\xi x^2/2} M_{\kappa_k, \mu}(\xi x^2).$$

By (1.18), the change of variable $z = \xi x^2$ is strictly increasing from $(0, 1)$ onto $(0, \xi)$, the zeros of u_k in $(0, 1)$ are in one-to-one correspondence with the positive zeros of $M_{\kappa_k, \mu}$ in $(0, \xi)$. By the Sturm oscillation theorem, the k -th eigenfunction has exactly $k - 1$ zeros in $(0, 1)$. Since $u_k(1) = 0$, it follows that ξ is the k -th positive zero of $M_{\kappa_k, \mu}$. $\square$

Consequently, the asymptotic expansion of the eigenvalues reduces to determining the corresponding value of κ_k . Since the spectral condition is equivalent to

$$(3.25) \quad m_k(\kappa_k, \mu) = \xi,$$

we invert the two-term asymptotic expansion of the Whittaker zero $m_k(\kappa, \mu)$ obtained by Gabutti and Gatteschi. In this expansion, the leading term is determined by the implicit action relation (3.18), with

$$(3.26) \quad \zeta_k = \left(\frac{j_{2\mu, k}}{\kappa} \right)^2.$$

Substituting the McMahon expansion of the Bessel zero $j_{2\mu, k}$ and then inverting the resulting relation yields the desired expansion of κ_k , and hence of $\lambda_k(h)$, in light of (3.12) and (3.13). This formulation keeps the classical action $I(E)$ visible while making clear where the Gabutti–Gatteschi correction enters.

3.2.5. Asymptotic inversion of the Whittaker zero expansion. We now derive the asymptotic expansion of the eigenvalues by inverting the two-term asymptotic expansion of the Whittaker zeros established by Gabutti and Gatteschi. Recall that

$$(3.27) \quad \kappa_k(h) = \frac{\lambda_k(h)}{4h}.$$

By Lemma 3.4, and writing $\kappa_k = \kappa_k(h)$ for short, we have

$$(3.28) \quad m_k(\kappa_k, \mu) = \xi,$$

where $m_k(\kappa, \mu)$ denotes the k -th positive zero of the Whittaker function with respect to its argument. Since

$$\lambda_k(h) \geq 4kh + 2(\nu - 1)h \geq \rho_0 + \mathcal{O}(h) > 1.$$

We can then choose a fixed $q \in (0, 4)$ such that, for h sufficiently small,

$$\xi_1 < \frac{\xi}{\kappa_k} < q < \xi_2.$$

We can then select a fixed $q \in (0, 4)$ such that $\xi = 1/h < q\kappa$, and therefore apply the refined asymptotic expansion (3.22) with $\kappa = \kappa_k$; we obtain

$$(3.29) \quad \xi = \kappa_k X_k - \frac{X_k}{\kappa_k R_k} C_0(\zeta_k, \gamma_k, X_k) + \mathcal{O}(\kappa_k^{-3}),$$

where ζ_k is given by (3.26), $\gamma_k = \frac{2\mu}{\kappa_k}$ and X_k is determined by the implicit relation (3.18).

We now introduce the scaled energy

$$(3.30) \quad E = \frac{4}{X_k}.$$

Since

$$(3.31) \quad X_k = \frac{\xi}{\kappa_k}, \quad \kappa_k = \frac{\xi E}{4},$$

it follows from (3.27) that

$$(3.32) \quad \lambda_k(h) = E.$$

Consequently, the asymptotic expansion of the eigenvalues reduces to determining the corresponding asymptotic expansion of the scaled energy E .

The next step consists in expanding the implicit relation (3.18) in the regime $\xi \rightarrow +\infty$, with

$$\rho = \frac{4k}{\xi} \in [\rho_0, \rho_1].$$

To this end, we first derive an asymptotic expansion of the quantity ζ_k appearing in (3.26) by using the classical McMahon expansion of the Bessel zeros, see [5, §10.21, Eq. (10.21.19)]:

$$(3.33) \quad j_{2\mu,k} = \beta_k - \frac{16\mu^2 - 1}{8\beta_k} + \mathcal{O}(\beta_k^{-3}), \quad \beta_k = \pi \left(k + \mu - \frac{1}{4} \right).$$

Using (3.26), we obtain

$$(3.34) \quad \zeta_k = \left(\frac{\beta_k}{\kappa_k} \right)^2 - \frac{16\mu^2 - 1}{4\kappa_k^2} + \mathcal{O}(\beta_k^{-2} \kappa_k^{-2}).$$

Moreover,

$$(3.35) \quad \frac{\beta_k}{\kappa_k} = \frac{\pi\rho}{E} + \frac{4\pi}{\xi E} \left(\mu - \frac{1}{4} \right).$$

Substituting (3.35) into (3.34), we get

$$(3.36) \quad \zeta_k = \left(\frac{\pi\rho}{E} \right)^2 + \frac{8\pi^2\rho}{\xi E^2} \left(\mu - \frac{1}{4} \right) + \mathcal{O}(\xi^{-2}).$$

Using the identity $X = \frac{4}{E}$, the implicit relation (3.18) is now expressed in terms of the scaled energy E . Moreover, since

$$\kappa_k = \frac{\xi E}{4},$$

we have

$$(3.37) \quad \gamma_k = \frac{2\mu}{\kappa_k} = \frac{8\mu}{\xi E}.$$

We shall now expand both sides of (3.18) as $\xi \rightarrow +\infty$, uniformly for

$$\rho = \frac{4k}{\xi} \in [\rho_0, \rho_1].$$

In what follows, the expansions are first performed for E in an arbitrary compact subset of $(1, +\infty)$. The leading equation obtained below will show that the relevant solutions E indeed remain in such a compact subset.

The right-hand side of (3.18) becomes

$$(3.38) \quad \begin{aligned} S_W\left(\frac{4}{E}, \gamma_k\right) &= \frac{1}{2} \sqrt{\frac{4}{E} \left(4 - \frac{4}{E}\right)} + 2 \arcsin\left(\frac{1}{\sqrt{E}}\right) - \frac{4\pi\mu}{\xi E} + \mathcal{O}(\xi^{-2}) \\ &= \frac{2}{E} \sqrt{E-1} + 2 \arcsin(E^{-1/2}) - \frac{4\pi\mu}{\xi E} + \mathcal{O}(\xi^{-2}). \end{aligned}$$

We next expand the Bessel action. Recall that

$$S_B(\zeta, \gamma) = \sqrt{\zeta - \gamma^2} - \gamma \arctan\left(\frac{\sqrt{\zeta - \gamma^2}}{\gamma}\right).$$

Using (3.36), we first get

$$(3.39) \quad \sqrt{\zeta_k} = \frac{\pi\rho}{E} + \frac{4\pi}{\xi E} \left(\mu - \frac{1}{4}\right) + \mathcal{O}(\xi^{-2}).$$

Moreover, we have

$$\sqrt{\zeta_k - \gamma_k^2} = \sqrt{\zeta_k} + \mathcal{O}(\xi^{-2}),$$

and

$$\arctan\left(\frac{\sqrt{\zeta_k - \gamma_k^2}}{\gamma_k}\right) = \frac{\pi}{2} + \mathcal{O}(\xi^{-1}).$$

Therefore,

$$(3.40) \quad \begin{aligned} S_B(\zeta_k, \gamma_k) &= \sqrt{\zeta_k} - \frac{\pi}{2} \gamma_k + \mathcal{O}(\xi^{-2}) \\ &= \frac{\pi\rho}{E} + \frac{4\pi}{\xi E} \left(\mu - \frac{1}{4}\right) - \frac{4\pi\mu}{\xi E} + \mathcal{O}(\xi^{-2}) \\ &= \frac{\pi\rho}{E} - \frac{\pi}{\xi E} + \mathcal{O}(\xi^{-2}). \end{aligned}$$

Identifying (3.38) and (3.40), we obtain

$$\frac{2}{E} \sqrt{E-1} + 2 \arcsin(E^{-1/2}) - \frac{4\pi\mu}{\xi E} = \frac{\pi\rho}{E} - \frac{\pi}{\xi E} + \mathcal{O}(\xi^{-2}).$$

Multiplying by $E/2$, we get

$$(3.41) \quad \sqrt{E-1} + E \arcsin(E^{-1/2}) = \frac{\pi\rho}{2} + \frac{\pi}{\xi} \left(2\mu - \frac{1}{2}\right) + \mathcal{O}(\xi^{-2}).$$

It is therefore natural to introduce the function

$$(3.42) \quad I(E) = \frac{1}{2} \sqrt{E-1} + \frac{E}{2} \arcsin(E^{-1/2}).$$

Equation (3.41) then becomes

$$(3.43) \quad I(E) = \frac{\pi\rho}{4} + \frac{\pi}{\xi} \left(\mu - \frac{1}{4}\right) + \mathcal{O}(\xi^{-2}).$$

We now invert the relation (3.43). One has

$$(3.44) \quad I'(E) = \frac{1}{2} \arcsin(E^{-1/2}) > 0, \quad E > 1.$$

Thus, I is a smooth increasing diffeomorphism from $(1, +\infty)$ onto $(\frac{\pi}{4}, +\infty)$. Hence, for every

$$\rho \in [\rho_0, \rho_1], \quad 1 < \rho_0 \leq \rho_1 < +\infty,$$

there exists a unique $E_0(\rho) > 1$ such that

$$(3.45) \quad I(E_0(\rho)) = \frac{\pi\rho}{4}.$$

Since $E_0(\rho)$ ranges over a compact subset of $(1, +\infty)$ as ρ varies in $[\rho_0, \rho_1]$, and since

$$I'(E_0(\rho)) \neq 0,$$

the implicit function theorem applies uniformly with respect to $\rho \in [\rho_0, \rho_1]$. It follows from (3.43) that

$$(3.46) \quad E = E_0(\rho) + \frac{E_1^\sharp(\rho)}{\xi} + \mathcal{O}(\xi^{-2}),$$

where

$$(3.47) \quad E_1^\sharp(\rho) = \frac{\pi(\mu - \frac{1}{4})}{I'(E_0(\rho))} = \frac{2\pi(\mu - \frac{1}{4})}{\arcsin(E_0(\rho)^{-1/2})}.$$

We now return to the correction term in the Gabutti–Gatteschi expansion (3.29). From (3.46), we know that E remains in a fixed compact subset of $(1, +\infty)$, uniformly for $\rho \in [\rho_0, \rho_1]$. Hence

$$X_k = \frac{4}{E}$$

remains in a fixed compact subset of $(0, 4)$. Hence Theorems 2.1 and 2.3 of [9] apply uniformly with respect to $\rho \in [\rho_0, \rho_1]$, and

$$\kappa_k = \frac{\xi E}{4} = \mathcal{O}(\xi), \quad \kappa_k^{-1} = \mathcal{O}(\xi^{-1}).$$

Since E remains in a compact subset of $(1, +\infty)$, the quantities X_k and ζ_k remain uniformly bounded away from the relevant singular endpoints, while $\gamma_k = \mathcal{O}(\xi^{-1})$. In particular,

$$R_k = R(X_k) \geq c > 0,$$

for some constant $c > 0$, and

$$C_0(\zeta_k, \gamma_k, X_k) = \mathcal{O}(1).$$

Consequently,

$$(3.48) \quad \frac{X_k}{\kappa_k R_k} C_0(\zeta_k, \gamma_k, X_k) = \mathcal{O}(\xi^{-1}).$$

We now explain why the correction term in (3.29) does not modify the coefficients already obtained in (3.46). By (3.48), equation (3.29) becomes

$$(3.49) \quad \kappa_k X_k = \xi + \mathcal{O}(\xi^{-1}).$$

Since $\kappa_k \asymp \xi$, a division by κ_k yields

$$(3.50) \quad X_k = \frac{\xi}{\kappa_k} + \mathcal{O}(\xi^{-2}).$$

Since X_k remains in a compact subset of $(0, 4)$, the map

$$X \mapsto \frac{4}{X}$$

is smooth in a neighborhood of X_k . Hence the estimate (3.50) implies that the corresponding correction to the scaled energy is also of order $\mathcal{O}(\xi^{-2})$. Consequently, the correction term in (3.29) does not affect the coefficients $E_0(\rho)$ and $E_1^\sharp(\rho)$ in (3.46). The previous analysis immediately yields the result of Theorem 3.1.

4. THE TRANSITION REGIME

The Bohr–Sommerfeld–Langer quantization derived in the previous section is valid uniformly in the regime

$$1 < \rho_0 \leq \rho := 4kh \leq \rho_1.$$

As $\rho \downarrow 1$, the classical turning point reaches the boundary of the interval. Consequently, the standard semiclassical expansion ceases to be uniform and must be replaced by a uniform Airy approximation. We first recall the relevant Airy asymptotics and derive a localization result for the eigenvalues in an h^δ -neighborhood of the critical threshold, with $0 < \delta < 2/3$. We then specialize to the critical scale $\delta = 2/3$ and obtain an explicit transition asymptotic expansion in the window $|4kh - 1| = \mathcal{O}(h^{2/3})$.

4.1. Uniform Airy asymptotics. Throughout this section we use the notation

$$(4.1) \quad \kappa = \frac{b}{2} - a, \quad \mu = \frac{b-1}{2}, \quad \xi = 1/h,$$

so that

$$(4.2) \quad M(a, b, \xi) = e^{\xi/2} \xi^{-b/2} M_{\kappa, \mu}(\xi),$$

see [5, Eq. 13.14.4]. We assume throughout that

$$\kappa \rightarrow +\infty,$$

while μ remains fixed. Under these assumptions, we are precisely in the framework of the uniform Airy asymptotic theory of Dunster, reproduced in [5, §13.21(iii)].

In the transition regime, the Airy variable changes sign at the outer turning point

$$\xi = 2\kappa + 2\sqrt{\kappa^2 - \mu^2}.$$

For definiteness, we write below the expression corresponding to

$$\xi > 2\kappa + 2\sqrt{\kappa^2 - \mu^2},$$

for which $\hat{\zeta}$, introduced in (4.3) below, satisfies $\hat{\zeta} > 0$. The case on the other side of the turning point is treated similarly, using the corresponding expression for $\hat{\zeta}$ in [5, Eq. (13.21.20)].

We therefore introduce the Airy variable

$$(4.3) \quad \hat{\zeta} = \left(\frac{3}{2\kappa} \left(\frac{X}{2} + \mu \log \left(\frac{\xi \sqrt{\kappa^2 - \mu^2}}{\kappa \xi - 2\mu^2 - \mu X} \right) + \kappa \log \left(\frac{2\sqrt{\kappa^2 - \mu^2}}{\xi - 2\kappa + X} \right) \right) \right)^{2/3},$$

where

$$X = \sqrt{\xi^2 - 4\kappa\xi + 4\mu^2}.$$

Furthermore, define

$$\begin{aligned} \hat{c}(\kappa, \mu) &= \sqrt{2\pi} \kappa^{1/6} \left(\frac{\kappa - \mu}{\kappa + \mu} \right)^{\mu/2} \left(\frac{e}{\sqrt{\kappa^2 - \mu^2}} \right)^\kappa, \\ \hat{\Psi}(\kappa, \mu, \xi) &= \left(\frac{\hat{\zeta}}{\xi^2 - 4\kappa\xi + 4\mu^2} \right)^{1/4} (2\xi)^{1/2}. \end{aligned}$$

Then the uniform Airy expansion of Dunster takes the form [5, Eq. 13.21.22]

$$(4.4) \quad \begin{aligned} M_{\kappa, \mu}(\xi) &= \frac{1}{2\pi} \Gamma(2\mu + 1) \Gamma\left(\kappa - \mu + \frac{1}{2}\right) \hat{c}(\kappa, \mu) \hat{\Psi}(\kappa, \mu, \xi) \\ &\times \left[\sin(\pi(\kappa - \mu)) \operatorname{Ai}\left(\kappa^{2/3} \hat{\zeta}\right) + \cos(\pi(\kappa - \mu)) \operatorname{Bi}\left(\kappa^{2/3} \hat{\zeta}\right) \right. \\ &\quad \left. + \operatorname{envBi}\left(\kappa^{2/3} \hat{\zeta}\right) \mathcal{O}(\kappa^{-1}) \right]. \end{aligned}$$

Here and in the following, env Bi denotes the Airy envelope function of Olver (see [5, §2.8(iii)]). We recall its definition for convenience. If c denotes the real root of smallest absolute value of $\text{Ai}(c) = \text{Bi}(c)$, then

$$\text{env Bi}(x) = (\text{Ai}(x)^2 + \text{Bi}(x)^2)^{1/2}, \quad x \leq c,$$

whereas

$$\text{env Bi}(x) = \sqrt{2} \text{Bi}(x), \quad x \geq c.$$

4.2. Uniform quantization in the transition regime. We now investigate the transition regime for the zeros of the Kummer function

$$a \mapsto M(a, b, \xi).$$

In this regime, the Airy argument remains bounded and the Airy functions appearing in the uniform expansion (4.4) must therefore be kept in their exact form.

Using the relation (4.2), the uniform Airy expansion (4.4) becomes

$$\begin{aligned} M(a, b, \xi) &= \frac{e^{\xi/2} \xi^{-b/2}}{2\pi} \Gamma(b) \Gamma(1-a) \widehat{c}(\kappa, \mu) \widehat{\Psi}(\kappa, \mu, \xi) \\ &\quad \times \left[\cos(\pi a) \text{Ai}(\kappa^{2/3} \widehat{\zeta}) + \sin(\pi a) \text{Bi}(\kappa^{2/3} \widehat{\zeta}) \right. \\ &\quad \left. + R(\kappa, \mu, \xi) \right]. \end{aligned}$$

where

$$(4.5) \quad |R(\kappa, \mu, \xi)| \leq C \kappa^{-1} \text{env Bi}(\kappa^{2/3} \widehat{\zeta}),$$

uniformly in the regime under consideration. Hence, the zeros of $a \mapsto M(a, b, \xi)$ satisfy the uniform implicit quantization condition

$$(4.6) \quad \cos(\pi a) \text{Ai}(\kappa^{2/3} \widehat{\zeta}) + \sin(\pi a) \text{Bi}(\kappa^{2/3} \widehat{\zeta}) + R(\kappa, \mu, \xi) = 0,$$

where $R(\kappa, \mu, \xi)$ satisfies (4.5). Equation (4.6) provides a uniform quantization condition for the zeros of the Kummer function throughout the Airy transition regime. It forms the starting point for deriving various transition asymptotics by considering appropriate scalings of the Airy argument $\kappa^{2/3} \widehat{\zeta}$.

We first establish rough eigenvalue bounds near the critical energy. Besides providing the localization needed below, the argument explains the distinguished role of the scale $h^{2/3}$.

Proposition 4.1. *Let $0 < \delta < 2/3$ and $0 < t_0 < t_1 < +\infty$. Assume that*

$$\rho = 4kh = 1 + th^\delta, \quad t \in [t_0, t_1].$$

Then, uniformly for $t \in [t_0, t_1]$,

$$0 \leq \lambda_k(h) - e_k(h) = \mathcal{O}(h^{3\delta/2}).$$

Proof. By (1.3), we have

$$(4.7) \quad e_k(h) \leq \lambda_k(h).$$

We introduce a Dirichlet decoupling at $x = 1$ for the half-line operator T_h^∞ . The resulting operator is

$$T_h^{\text{dec}} = T_{h,(0,1)}^D \oplus T_{h,(1,\infty)}^D.$$

The first component is precisely the operator on $(0, 1)$ whose eigenvalues are $\lambda_j(h)$. We denote by

$$\lambda_1^{\text{ext}}(h) \leq \lambda_2^{\text{ext}}(h) \leq \dots$$

the eigenvalues of the second component, which we shall refer to as the exterior component. Thus

$$\sigma(T_h^{\text{dec}}) = \{\lambda_j(h)\}_{j \geq 1} \cup \{\lambda_j^{\text{ext}}(h)\}_{j \geq 1},$$

where the union is understood with multiplicities. Let

$$\mu_1(h) \leq \mu_2(h) \leq \dots$$

denote this union rearranged in increasing order. Since the decoupling amounts to imposing one additional Dirichlet condition on the form domain, the min-max principle gives

$$(4.8) \quad e_j(h) \leq \mu_j(h) \leq e_{j+1}(h).$$

We next estimate the spectrum of the exterior component. On $(1, \infty)$, we have

$$x^2 + h^2 \frac{\nu^2 - \frac{1}{4}}{x^2} \geq 1 + 2(x-1) - C_\nu h^2.$$

Hence, after the scaling $x = 1 + h^{2/3}y$, the exterior operator is bounded from below by

$$1 + h^{2/3}A - C_\nu h^2, \quad A = -\frac{d^2}{dy^2} + 2y,$$

on $\mathbb{R}_+$, with Dirichlet boundary condition at $y = 0$. Let

$$0 < \alpha_1 < \alpha_2 < \dots$$

denote the eigenvalues of A . By the min-max principle,

$$(4.9) \quad \lambda_j^{\text{ext}}(h) \geq 1 + h^{2/3}\alpha_j - C_\nu h^2.$$

The Airy asymptotics give

$$(4.10) \quad \alpha_j \sim (3\pi j)^{2/3}, \quad j \rightarrow +\infty.$$

By assumption,

$$(4.11) \quad \rho = 4kh = 1 + th^\delta.$$

Set

$$E_h := 1 + (t_1 + 1)h^\delta,$$

and let

$$L(h) := \#\{j \geq 1 : \lambda_j^{\text{ext}}(h) \leq E_h\}.$$

By (4.9), if $\lambda_j^{\text{ext}}(h) \leq E_h$, then

$$1 + h^{2/3}\alpha_j - C_\nu h^2 \leq 1 + (t_1 + 1)h^\delta.$$

Hence

$$\alpha_j \leq (t_1 + 1)h^{\delta-2/3} + C_\nu h^{4/3}.$$

Using (4.10), we infer that

$$(4.12) \quad L(h) = \mathcal{O}(h^{3\delta/2-1}).$$

We next show that the first $k + L(h)$ eigenvalues of the decoupled operator lie below E_h . Indeed,

$$e_{k+L(h)+1}(h) = 4kh + (4(L(h) + 1) + 2\nu - 2)h.$$

Using (4.11) and (4.12), we obtain

$$e_{k+L(h)+1}(h) = 1 + th^\delta + \mathcal{O}(h^{3\delta/2}) + \mathcal{O}(h).$$

Since $0 < \delta < 2/3$, both remainder terms are $o(h^\delta)$. Therefore, uniformly for $t \in [t_0, t_1]$,

$$e_{k+L(h)+1}(h) \leq 1 + (t_1 + 1)h^\delta = E_h,$$

for h sufficiently small. By (4.8),

$$\mu_{k+L(h)}(h) \leq e_{k+L(h)+1}(h) \leq E_h.$$

Thus the first $k + L(h)$ eigenvalues of the decoupled operator lie below E_h . By the definition of $L(h)$, at most $L(h)$ of them belong to the exterior spectrum. Hence at least k of them belong to the interior spectrum, and therefore

$$\lambda_k(h) \leq \mu_{k+L(h)}(h) \leq e_{k+L(h)+1}(h).$$

Together with (4.7), this gives

$$0 \leq \lambda_k(h) - e_k(h) \leq e_{k+L(h)+1}(h) - e_k(h) = 4(L(h) + 1)h.$$

Finally, using (4.12), we obtain

$$\lambda_k(h) - e_k(h) = \mathcal{O}(h^{3\delta/2}),$$

uniformly for $t \in [t_0, t_1]$. □

The preceding proof also makes the critical exponent transparent. Indeed, the exterior eigenvalues which may enter a window of size h^δ satisfy

$$h^{2/3}\alpha_j \lesssim h^\delta.$$

Since $\alpha_j \sim (3\pi j)^{2/3}$, this implies $j \lesssim h^{3\delta/2-1}$, and hence the number of such exterior eigenvalues is of order at most $h^{3\delta/2-1}$. This number may grow when $\delta < 2/3$, whereas it remains bounded when $\delta = 2/3$. At the latter scale, the same argument gives the following localization.

Corollary 4.2. *Let $C > 0$ and set*

$$(4.13) \quad \kappa_k(h) := \frac{\lambda_k(h)}{4h}.$$

Then, uniformly for integers k satisfying

$$(4.14) \quad |4kh - 1| \leq C h^{2/3},$$

we have

$$(4.15) \quad \kappa_k(h) = k + \mathcal{O}(1), \text{ as } h \rightarrow 0^+.$$

Proof. At the scale (4.14), the preceding counting argument shows that only a bounded number of exterior eigenvalues can occur in the relevant energy window. Hence, for some L independent of h ,

$$e_k(h) \leq \lambda_k(h) \leq e_{k+L+1}(h) = e_k(h) + 4(L+1)h.$$

Since

$$e_k(h) = (4k + 2\nu - 2)h,$$

division by $4h$ gives (4.15). □

4.3. A particular transition scaling. We now consider the classical Airy scaling near the turning point. Recall that $\xi = h^{-1}$. In terms of the Whittaker parameter κ , this scaling is

$$\xi = 4\kappa \left(1 + \frac{t}{(2\kappa)^{2/3}} \right), \quad t = \mathcal{O}(1).$$

This is also the scaling used in the transition formula of Magnus–Oberhettinger–Soni [18, p. 292, Case 3]. For k satisfying $|4kh - 1| \leq T h^{2/3}$, set

$$\kappa_k(h) := \frac{\lambda_k(h)}{4h},$$

and define t_k by

$$(4.16) \quad \xi = 4\kappa_k(h) \left(1 + \frac{t_k}{(2\kappa_k(h))^{2/3}} \right).$$

By Corollary 4.2,

$$\kappa_k(h) = k + \mathcal{O}(1).$$

Since $4kh = 1 + \mathcal{O}(h^{2/3})$, we have $4h\kappa_k = 1 + \mathcal{O}(h^{2/3})$, and therefore

$$t_k = \mathcal{O}(1).$$

Thus the transition scaling applies uniformly in the regime under consideration. Expanding the corresponding Airy variable on either side of the turning point under (4.16) gives, uniformly for bounded t_k ,

$$(4.17) \quad \kappa_k^{2/3} \hat{\zeta} = t_k + \mathcal{O}(\kappa_k^{-2/3}).$$

Hence

$$\text{Ai}(\kappa_k^{2/3} \widehat{\zeta}) = \text{Ai}(t_k) + \mathcal{O}(\kappa_k^{-2/3}),$$

and

$$\text{Bi}(\kappa_k^{2/3} \widehat{\zeta}) = \text{Bi}(t_k) + \mathcal{O}(\kappa_k^{-2/3}).$$

Substituting these estimates into (4.6), and using (4.5), we obtain

$$(4.18) \quad \cos(\pi a) \text{Ai}(t_k) + \sin(\pi a) \text{Bi}(t_k) = \mathcal{O}(\kappa_k^{-2/3}).$$

To write this condition in phase form, set

$$A(s) := (\text{Ai}(s)^2 + \text{Bi}(s)^2)^{1/2}.$$

Since the Airy functions Ai and Bi have no common zeros, $A(s) > 0$ for every $s \in \mathbb{R}$. We may therefore introduce a smooth phase $\theta(s)$ by

$$(4.19) \quad \text{Ai}(s) = A(s) \sin \theta(s), \quad \text{Bi}(s) = A(s) \cos \theta(s),$$

with the continuous determination such that $\theta(s) \rightarrow 0$ as $s \rightarrow +\infty$. Moreover, differentiating

$$\tan \theta(s) = \frac{\text{Ai}(s)}{\text{Bi}(s)}$$

and using the Wronskian identity $W(\text{Ai}, \text{Bi}) = 1/\pi$ (see [5, Eq. (9.2.7)]), we obtain

$$(4.20) \quad \theta'(s) = \frac{\text{Ai}'(s) \text{Bi}(s) - \text{Ai}(s) \text{Bi}'(s)}{\text{Ai}(s)^2 + \text{Bi}(s)^2} = -\frac{1}{\pi(\text{Ai}(s)^2 + \text{Bi}(s)^2)} < 0.$$

Thus θ is strictly decreasing. Using the standard Airy asymptotics, we also have

$$\theta(s) = \frac{2}{3}(-s)^{3/2} + \frac{\pi}{4} + o(1), \quad s \rightarrow -\infty,$$

whereas

$$\theta(s) = \frac{1}{2} \exp\left(-\frac{4}{3}s^{3/2}\right) \left(1 + \mathcal{O}(s^{-3/2})\right), \quad s \rightarrow +\infty.$$

Then (4.18) becomes

$$\sin(\pi a + \theta(t_k)) = \mathcal{O}(\kappa_k^{-2/3}).$$

Thus,

$$a = m - \frac{\theta(t_k)}{\pi} + \mathcal{O}(\kappa_k^{-2/3}), \quad \text{for } m \in \mathbb{Z}.$$

On the subcritical side, the k -th zero approaches $1 - k$. Hence $m = 1 - k$, and consequently

$$(4.21) \quad a_k(\xi) = 1 - k - \frac{\theta(t_k)}{\pi} + \mathcal{O}(\kappa_k^{-2/3}).$$

Since $\kappa_k^{-2/3} = \mathcal{O}(h^{2/3})$, we finally obtain

$$(4.22) \quad a_k(\xi) = 1 - k - \frac{\theta(t_k)}{\pi} + \mathcal{O}(h^{2/3}).$$

By Sturm's oscillation theorem, the zero $a_k(\xi)$ in (4.22) corresponds to the k -th eigenvalue of the singular harmonic oscillator. We can now state the resulting transition asymptotics directly in terms of k and h .

Proposition 4.3. *For every $T > 0$ and for integers k satisfying*

$$(4.23) \quad |4kh - 1| \leq Th^{2/3},$$

set

$$(4.24) \quad \tau_k(h) := \frac{1 - 4kh}{2^{2/3}h^{2/3}}.$$

Then, uniformly for these k 's, we have

$$(4.25) \quad \begin{aligned} \lambda_k(h) &= e_k(h) + \frac{4h}{\pi} \theta(\tau_k(h)) \\ &\quad - \frac{4}{\pi 2^{2/3}} h^{4/3} \left(2\nu - 2 + \frac{4}{\pi} \theta(\tau_k(h)) \right) \theta'(\tau_k(h)) + \mathcal{O}(h^{5/3}), \end{aligned}$$

as $h \rightarrow 0^+$.

Remark 4.4 (Asymptotic scale). Since $|\tau_k(h)| \leq 2^{-2/3}T$, the functions $\theta(\tau_k(h))$ and $\theta'(\tau_k(h))$ remain uniformly bounded. Moreover, $\theta > 0$ on $\mathbb{R}$. Therefore, in (4.25), the first correction is of order h , whereas the second one is $\mathcal{O}(h^{4/3})$. Thus these terms form an asymptotic scale uniformly in the transition window (4.23).

Proof of Proposition 4.3. Let

$$\tau_k = \tau_k(h) := \frac{1 - 4kh}{2^{2/3}h^{2/3}}.$$

Under the assumption of the proposition, we have

$$|\tau_k| \leq 2^{-2/3}T.$$

With t_k defined by (4.16), formula (4.22) gives

$$(4.26) \quad a_k(h^{-1}) = 1 - k - \frac{\theta(t_k)}{\pi} + \mathcal{O}(h^{2/3}).$$

Since the k -th eigenvalue and the corresponding a -zero are related by

$$\lambda_k(h) = h(2b - 4a_k(h^{-1})),$$

we obtain

$$(4.27) \quad \lambda_k(h) = e_k(h) + \frac{4h}{\pi} \theta(t_k) + \mathcal{O}(h^{5/3}),$$

where

$$e_k(h) = (4k + 2\nu - 2)h.$$

We now eliminate the auxiliary parameter t_k . For convenience, set

$$(4.28) \quad q(s) := 2\nu - 2 + \frac{4}{\pi} \theta(s).$$

Since

$$4kh = 1 - 2^{2/3} \tau_k h^{2/3},$$

equation (4.27) yields

$$(4.29) \quad \lambda_k(h) = 1 - 2^{2/3} \tau_k h^{2/3} + hq(t_k) + \mathcal{O}(h^{5/3}).$$

On the other hand, since

$$4\kappa_k = \frac{\lambda_k(h)}{h},$$

the defining relation (4.16) is equivalently

$$(4.30) \quad 1 = \lambda_k(h) + 2^{2/3} t_k h^{2/3} \lambda_k(h)^{1/3}.$$

From (4.29) and the boundedness of τ_k and t_k , we have

$$\lambda_k(h) = 1 - 2^{2/3} \tau_k h^{2/3} + \mathcal{O}(h),$$

and therefore

$$(4.31) \quad \lambda_k(h)^{1/3} = 1 - \frac{2^{2/3}}{3} \tau_k h^{2/3} + \mathcal{O}(h).$$

Substituting (4.29) and (4.31) into (4.30), we obtain

$$0 = 2^{2/3} (t_k - \tau_k) h^{2/3} + hq(t_k) - \frac{2^{4/3}}{3} t_k \tau_k h^{4/3} + \mathcal{O}(h^{5/3}).$$

Since t_k and τ_k remain bounded, this first implies

$$(4.32) \quad t_k - \tau_k = \mathcal{O}(h^{1/3}).$$

The function θ , and hence q , is smooth. Thus

$$q(t_k) = q(\tau_k) + \mathcal{O}(h^{1/3}),$$

and a second use of the preceding identity gives

$$(4.33) \quad t_k = \tau_k - 2^{-2/3} q(\tau_k) h^{1/3} + \mathcal{O}(h^{2/3}).$$

The Taylor expansion of θ at τ_k now gives

$$(4.34) \quad \theta(t_k) = \theta(\tau_k) - 2^{-2/3} q(\tau_k) \theta'(\tau_k) h^{1/3} + \mathcal{O}(h^{2/3}),$$

uniformly for $|\tau_k| \leq T$.

Inserting (4.34) into (4.27), we finally obtain

$$\lambda_k(h) = e_k(h) + \frac{4h}{\pi} \theta(\tau_k) - \frac{4}{\pi 2^{2/3}} q(\tau_k) \theta'(\tau_k) h^{4/3} + \mathcal{O}(h^{5/3}).$$

Recalling (4.28), this is precisely (4.25). $\square$

Remark 4.5 (Eigenvalue spacing). Uniformly in the transition window,

$$\lambda_{k+1}(h) - \lambda_k(h) = 4h - \frac{2^{10/3}}{\pi} \theta'(\tau_k(h)) h^{4/3} + \mathcal{O}(h^{5/3}).$$

Since $\theta' < 0$, the first correction to the leading spacing $4h$ is positive.

4.4. Matching with the subcritical and supercritical regimes.

4.4.1. Accurate Dunster expansions. The classical uniform Airy expansions from the NIST Handbook [5], used in Section 4, are perfectly suited to the transition regime, where the Airy variable $s = \kappa_k(h)^{2/3} \widehat{\zeta}$ remains bounded. They could also be used to carry out the supercritical matching. On the subcritical side, however, the matching involves an exponentially small contribution, and the classical remainder estimates are too large to detect it. Indeed, in the matching regimes introduced below, Propositions 2.5 and 4.1 give, respectively on the subcritical and supercritical sides,

$$(4.35) \quad \kappa_k(h) = \frac{\lambda_k(h)}{4h} = \frac{1 + \mathcal{O}(h^\delta)}{4h}.$$

The proof of Proposition 4.3 starts from the uniform quantization condition (4.6), which in the present spectral setting reads

$$(4.36) \quad \cos(\pi a) \operatorname{Ai}(\kappa_k(h)^{2/3} \widehat{\zeta}) + \sin(\pi a) \operatorname{Bi}(\kappa_k(h)^{2/3} \widehat{\zeta}) + R(\kappa_k(h), \mu, \xi) = 0,$$

where $\widehat{\zeta}$ is the Liouville–Green variable. Moreover, by (4.5),

$$|R(\kappa_k(h), \mu, \xi)| \leq C \kappa_k(h)^{-1} \operatorname{env} \operatorname{Bi}(\kappa_k(h)^{2/3} \widehat{\zeta}).$$

Thus, dividing (4.36) by $\operatorname{Bi}(\kappa_k(h)^{2/3} \widehat{\zeta})$, the contribution of the remainder is only controlled by $\mathcal{O}(h)$, where we used (4.35). This estimate is, however, far too weak to detect the exponentially small $\operatorname{Ai}(s)/\operatorname{Bi}(s)$ ratio on the subcritical side, when $s \rightarrow +\infty$. Indeed, by the standard Airy asymptotics [5, (9.7.5), (9.7.7)],

$$(4.37) \quad \frac{\operatorname{Ai}(s)}{\operatorname{Bi}(s)} = \frac{1}{2} \exp\left(-\frac{4}{3} s^{3/2}\right) \left(1 + \mathcal{O}(s^{-3/2})\right), \text{ as } s \rightarrow +\infty.$$

We therefore use here the very recent uniform Airy expansions for Whittaker functions obtained by Dunster [7, Theorem 3.1, Eqs. (3.40)–(3.41)]. Our spectral condition can be written in the form

$$(4.38) \quad \mathcal{A}(s, -\pi a) \check{A} + \mathcal{A}'(s, -\pi a) \check{B} = 0,$$

where

$$s = \kappa_k(h)^{2/3} \widehat{\zeta}, \quad \mathcal{A}(s, -\pi a) := \cos(\pi a) \operatorname{Ai}(s) + \sin(\pi a) \operatorname{Bi}(s),$$

and $\mathcal{A}'$ denotes the derivative with respect to the first variable s . The coefficient functions $\check{A}$ and $\check{B}$ are those introduced by Dunster. Moreover, his expansions [7, Eqs. (3.37)–(3.38)], together with (4.35), imply

$$(4.39) \quad \frac{\check{B}}{\check{A}} = \mathcal{O}(\kappa_k(h)^{-4/3}) = \mathcal{O}(h^{4/3}).$$

Dividing (4.38) by $\check{A}$, we recover the quantization condition (4.6), with the new structured remainder

$$(4.40) \quad R_D = \mathcal{A}'(s, -\pi a) \frac{\check{B}}{\check{A}},$$

where R_D denotes the Dunster remainder.

The crucial point for the matching argument is the particular structure of this remainder. Indeed, setting $r = \check{B}/\check{A}$, we obtain

$$(4.41) \quad \tan(\pi a) = -\frac{\text{Ai}(s) + r \text{Ai}'(s)}{\text{Bi}(s) + r \text{Bi}'(s)}, \quad r = \mathcal{O}(h^{4/3}).$$

In the transition regime of Proposition 4.3, we have

$$s = \tau_k(h) + \mathcal{O}(h^{1/3}),$$

so that s remains bounded. In this case, the new Dunster form does not make a substantial difference. In the matching regions considered below, however, the Airy variable s is no longer bounded. More precisely, for fixed $t > 0$, we shall show that

$$(4.42) \quad s = \pm \frac{t}{2^{2/3}} h^{\delta-2/3} (1 + o(1)) \longrightarrow \pm \infty,$$

with the upper sign on the subcritical side and the lower sign on the supercritical side. On the subcritical side, the crucial point is that

$$(4.43) \quad \frac{\text{Ai}(s) + r \text{Ai}'(s)}{\text{Bi}(s) + r \text{Bi}'(s)} = \frac{\text{Ai}(s)}{\text{Bi}(s)} \left(\frac{1 + r \text{Ai}'(s)/\text{Ai}(s)}{1 + r \text{Bi}'(s)/\text{Bi}(s)} \right).$$

For $s \rightarrow +\infty$, the standard Airy asymptotics [5, Sec. 9.7] give

$$(4.44) \quad \frac{\text{Ai}'(s)}{\text{Ai}(s)} = -s^{1/2} \left(1 + \mathcal{O}(s^{-3/2}) \right), \quad \frac{\text{Bi}'(s)}{\text{Bi}(s)} = s^{1/2} \left(1 + \mathcal{O}(s^{-3/2}) \right).$$

Since

$$r = \mathcal{O}(h^{4/3}),$$

and using (4.42), the second factor in (4.43) is

$$1 + \mathcal{O}(h^{4/3} s^{1/2}) = 1 + \mathcal{O}(h^{1+\delta/2})$$

in the matching regime. Combining this with (4.37), we obtain

$$(4.45) \quad \frac{\text{Ai}(s) + r \text{Ai}'(s)}{\text{Bi}(s) + r \text{Bi}'(s)} = \frac{1}{2} \exp \left(-\frac{4}{3} s^{3/2} \right) \left(1 + \mathcal{O}(h^{1-3\delta/2}) \right).$$

Thus the exponentially small term is preserved. This is precisely what was lost in the additive remainder of the classical Airy expansion. Then, as in Proposition 4.3, to obtain the eigenvalue asymptotics, we write

$$(4.46) \quad a = 1 - k - \eta_k, \quad \eta_k = \frac{\lambda_k(h) - e_k(h)}{4h}.$$

The quantization condition (4.41), together with (4.46), gives

$$\tan(\pi \eta_k) = \frac{\text{Ai}(s) + r \text{Ai}'(s)}{\text{Bi}(s) + r \text{Bi}'(s)}.$$

Using (4.45), the right-hand side is exponentially small. Since we are on the branch for which $\eta_k \rightarrow 0$, we have $\tan(\pi\eta_k) = \pi\eta_k(1 + \mathcal{O}(\eta_k^2))$. Moreover, η_k is exponentially small, so that this additional error is negligible with respect to $\mathcal{O}(h^{1-3\delta/2})$. Therefore,

$$(4.47) \quad \lambda_k(h) - e_k(h) = \frac{2h}{\pi} \exp\left(-\frac{4}{3}s^{3/2}\right) \left(1 + \mathcal{O}(h^{1-3\delta/2})\right).$$

On the supercritical side, $s \rightarrow -\infty$ and both Airy functions are oscillatory with the same algebraic size. More precisely, setting

$$(4.48) \quad \Phi(s) = \frac{2}{3}(-s)^{3/2} + \frac{\pi}{4},$$

the standard Airy asymptotics [5, Sec. 9.7] give

$$(4.49) \quad \begin{aligned} \text{Ai}(s) &= \frac{(-s)^{-1/4}}{\sqrt{\pi}} \left(\sin \Phi(s) + \mathcal{O}((-s)^{-3/2}) \right), \\ \text{Bi}(s) &= \frac{(-s)^{-1/4}}{\sqrt{\pi}} \left(\cos \Phi(s) + \mathcal{O}((-s)^{-3/2}) \right), \end{aligned}$$

and

$$(4.50) \quad \begin{aligned} \text{Ai}'(s) &= -\frac{(-s)^{1/4}}{\sqrt{\pi}} \left(\cos \Phi(s) + \mathcal{O}((-s)^{-3/2}) \right), \\ \text{Bi}'(s) &= \frac{(-s)^{1/4}}{\sqrt{\pi}} \left(\sin \Phi(s) + \mathcal{O}((-s)^{-3/2}) \right). \end{aligned}$$

Thus, on the supercritical side, the matching is governed by the oscillatory phase $\Phi(s)$. In contrast with the subcritical case, we do not form ratios of Airy functions, which would be singular at their zeros. Instead, we insert these oscillatory expansions directly into the quantization condition.

4.4.2. Subcritical matching. We consider the subcritical matching regime

$$(4.51) \quad 4kh = 1 - th^\delta, \quad t \in [t_0, t_1], \quad 0 < \delta < \frac{2}{3}.$$

By Proposition 2.5, for any fixed $\eta \in (0, 1)$,

$$0 \leq \lambda_k(h) - e_k(h) \leq \exp\left(-\frac{2(1-\eta)}{3}t^{3/2}h^{3\delta/2-1}\right).$$

Since

$$e_k(h) = 1 - th^\delta + (2\nu - 2)h,$$

it follows that, uniformly for $t \in [t_0, t_1]$,

$$(4.52) \quad 1 - \lambda_k(h) = th^\delta(1 + \mathcal{O}(h^{1-\delta})).$$

We now recall the definition of the Liouville–Green variable entering the recent uniform Airy expansion of Dunster [7]. For

$$\kappa = \frac{\lambda}{4h}, \quad \mu = \frac{\nu}{2}, \quad z = \frac{4}{\lambda}, \quad \alpha = \frac{2\mu}{\kappa} = \frac{4\nu h}{\lambda},$$

the Liouville–Green variable $\widehat{\zeta} = \widehat{\zeta}(z, \alpha)$ is defined, in our notation, by

$$(4.53) \quad \frac{2}{3}\widehat{\zeta}(z, \alpha)^{3/2} = \int_{z_+(\alpha)}^z \sqrt{\frac{v^2 - 4v + \alpha^2}{4v^2}} dv, \quad z_+(\alpha) = 2 + \sqrt{4 - \alpha^2}.$$

Since $\alpha = \mathcal{O}(h)$, we have $z_+(\alpha) = 4 + \mathcal{O}(h^2)$. Therefore, in the regime considered here,

$$\widehat{\zeta}\left(\frac{4}{\lambda}, \frac{4\nu h}{\lambda}\right) = 2^{2/3}(1 - \lambda)(1 + \mathcal{O}(h^\delta)).$$

Evaluating at $\lambda = \lambda_k(h)$, we set

$$\kappa_k(h) = \frac{\lambda_k(h)}{4h}, \quad \widehat{\zeta}_k(h) = \widehat{\zeta}\left(\frac{4}{\lambda_k(h)}, \frac{4\nu h}{\lambda_k(h)}\right), \quad s_k(h) = \kappa_k(h)^{2/3} \widehat{\zeta}_k(h).$$

Then (4.52) gives

$$(4.54) \quad s_k(h) = \frac{t}{2^{2/3}} h^{\delta-2/3} \left(1 + \mathcal{O}\left(h^{\min\{\delta, 1-\delta\}}\right)\right) \longrightarrow +\infty.$$

Therefore, applying (4.47) with $s = s_k(h)$, we obtain

$$(4.55) \quad \lambda_k(h) - e_k(h) = \frac{2h}{\pi} \exp\left(-\frac{4}{3}s_k(h)^{3/2}\right) \left(1 + \mathcal{O}(h^{1-3\delta/2})\right).$$

It remains to compare the exponent in this formula with the tunneling action $S^*(e_k(h))$. We shall prove that

$$(4.56) \quad \frac{4}{3}s_k(h)^{3/2} = \frac{2S^*(e_k(h))}{h} + \mathcal{O}(h^{\delta/2}).$$

To prove this, we first consider a general λ . Setting $s = \kappa^{2/3} \widehat{\zeta}$, and using (4.53), we obtain

$$(4.57) \quad \frac{4}{3}s^{3/2} = 2\kappa \int_{z_+(\alpha)}^{4/\lambda} \sqrt{\frac{v^2 - 4v + \alpha^2}{4v^2}} dv.$$

The change of variables

$$v = \frac{4x^2}{\lambda}$$

gives

$$(4.58) \quad \frac{4}{3}s^{3/2} = \frac{2}{h} \int_{x_+(\lambda, h)}^1 \sqrt{x^2 - \lambda + \frac{\nu^2 h^2}{x^2}} dx,$$

where

$$(4.59) \quad x_+(\lambda, h)^2 = \frac{\lambda + \sqrt{\lambda^2 - 4\nu^2 h^2}}{2}.$$

Set

$$I_h(\lambda) := \int_{x_+(\lambda, h)}^1 \sqrt{x^2 - \lambda + \frac{\nu^2 h^2}{x^2}} dx.$$

Since

$$S^*(\lambda) = \int_{\sqrt{\lambda}}^1 \sqrt{x^2 - \lambda} dx,$$

a direct comparison of the two integrals yields, uniformly in the regime considered here,

$$(4.60) \quad I_h(\lambda) = S^*(\lambda) + \mathcal{O}(h^2 \sqrt{1 - \lambda}) + \mathcal{O}(h^3).$$

Indeed, on $[\sqrt{\lambda}, 1]$, we have

$$0 \leq \sqrt{x^2 - \lambda + \frac{\nu^2 h^2}{x^2}} - \sqrt{x^2 - \lambda} \leq \frac{\nu^2 h^2}{x^2 \sqrt{x^2 - \lambda}},$$

and

$$\int_{\sqrt{\lambda}}^1 \frac{dx}{x^2 \sqrt{x^2 - \lambda}} = \frac{\sqrt{1 - \lambda}}{\lambda}.$$

Finally, $\sqrt{\lambda} - x_+(\lambda, h) = \mathcal{O}(h^2)$ and the integrand is $\mathcal{O}(h)$ on $[x_+(\lambda, h), \sqrt{\lambda}]$, so that the contribution of this interval is $\mathcal{O}(h^3)$. Applying (4.60) to $\lambda = \lambda_k(h)$ and using (4.52), we obtain

$$(4.61) \quad \frac{I_h(\lambda_k(h))}{h} = \frac{S^*(\lambda_k(h))}{h} + \mathcal{O}(h^{1+\delta/2}).$$

Moreover,

$$S^{*'}(E) = -\frac{1}{2} \operatorname{arcosh}(E^{-1/2}) = \mathcal{O}(\sqrt{1 - E}), \quad E \uparrow 1.$$

Since $\lambda_k(h) - e_k(h) = \mathcal{O}(h)$, the mean value theorem gives

$$S^*(\lambda_k(h)) - S^*(e_k(h)) = S^{*'}(\xi_k)(\lambda_k(h) - e_k(h)),$$

for some ξ_k between $e_k(h)$ and $\lambda_k(h)$. Since $1 - \xi_k \asymp h^\delta$, it follows that

$$\frac{S^*(\lambda_k(h)) - S^*(e_k(h))}{h} = \mathcal{O}(h^{\delta/2}).$$

Combining this with (4.58) and (4.61), we obtain

$$(4.62) \quad \frac{4}{3}s_k(h)^{3/2} = \frac{2S^*(e_k(h))}{h} + \mathcal{O}(h^{\delta/2}).$$

Finally, inserting (4.62) into (4.55) yields

$$(4.63) \quad \lambda_k(h) = e_k(h) + \frac{2h}{\pi} \exp\left(-\frac{2S^*(e_k(h))}{h}\right) \left(1 + \mathcal{O}\left(h^{\min\{\delta/2, 1-3\delta/2\}}\right)\right),$$

uniformly for $t \in [t_0, t_1]$ in (4.51). Thus, in the matching regime

$$4kh = 1 - th^\delta, \quad t \in [t_0, t_1], \quad 0 < \delta < \frac{2}{3},$$

the Airy asymptotics agree with the subcritical tunneling formula. In particular, we recover both the exponential action and the prefactor $2h/\pi$ in Theorem 1.2.

4.4.3. Supercritical matching. We now consider the supercritical matching regime

$$(4.64) \quad 4kh = 1 + th^\delta, \quad t \in [t_0, t_1], \quad 0 < \delta < \frac{2}{3}.$$

By Proposition 4.1,

$$\lambda_k(h) = e_k(h) + \mathcal{O}(h^{3\delta/2}).$$

Since

$$e_k(h) = 1 + th^\delta + (2\nu - 2)h,$$

we obtain, uniformly for $t \in [t_0, t_1]$,

$$(4.65) \quad \lambda_k(h) - 1 = th^\delta \left(1 + \mathcal{O}\left(h^{\min\{1-\delta, \delta/2\}}\right)\right).$$

As on the subcritical side, we set

$$\kappa_k(h) = \frac{\lambda_k(h)}{4h}, \quad s_k(h) = \kappa_k(h)^{2/3} \widehat{\zeta}_k(h).$$

The corresponding expansion of the Liouville–Green variable gives

$$(4.66) \quad s_k(h) = -\frac{t}{2^{2/3}} h^{\delta-2/3} \left(1 + \mathcal{O}\left(h^{\min\{1-\delta, \delta/2\}}\right)\right) \longrightarrow -\infty.$$

We may therefore use the oscillatory Airy expansions (4.49)–(4.50) from Subsection 4.4.1. Moreover, by (4.41) and (4.66),

$$(4.67) \quad r(-s_k(h))^{1/2} = \mathcal{O}(h^{1+\delta/2}).$$

The quantization condition (4.41) can equivalently be written as

$$\sin(\pi a_k)(\text{Bi}(s_k(h)) + r \text{Bi}'(s_k(h))) + \cos(\pi a_k)(\text{Ai}(s_k(h)) + r \text{Ai}'(s_k(h))) = 0.$$

Using the oscillatory Airy expansions from Subsection 4.4.1 and (4.67), we obtain easily

$$(4.68) \quad \sin\left(\pi a_k + \frac{2}{3}(-s_k(h))^{3/2} + \frac{\pi}{4}\right) = \mathcal{O}\left(h^{1-3\delta/2}\right).$$

Using the same labeling of the k -th zero as in Proposition 4.3, we obtain

$$\pi a_k = (1 - k)\pi - \frac{2}{3}(-s_k(h))^{3/2} - \frac{\pi}{4} + \mathcal{O}\left(h^{1-3\delta/2}\right).$$

Since

$$a_k = \frac{\nu}{2} - \frac{\lambda_k(h)}{4h} + \frac{1}{2},$$

we obtain

$$(4.69) \quad \lambda_k(h) = 4h \left(k + \frac{\nu}{2} - \frac{1}{4} \right) + \frac{8h}{3\pi} (-s_k(h))^{3/2} + \mathcal{O}(h^{2-3\delta/2}).$$

By (4.66),

$$(-s_k(h))^{3/2} = \frac{t^{3/2}}{2} h^{3\delta/2-1} \left(1 + \mathcal{O}(h^{\min\{1-\delta, \delta/2\}}) \right).$$

Substituting this into (4.69), and using $0 < \delta < 2/3$, we obtain

$$(4.70) \quad \lambda_k(h) = 4h \left(k + \frac{\nu}{2} - \frac{1}{4} \right) + \frac{4}{3\pi} t^{3/2} h^{3\delta/2} + o(h^{3\delta/2}),$$

uniformly for $t \in [t_0, t_1]$. This is precisely the threshold expansion of the supercritical Bohr–Sommerfeld formula. Indeed, setting

$$\rho_k = 4h \left(k + \frac{\nu}{2} - \frac{1}{4} \right) = 1 + th^\delta + (2\nu - 1)h,$$

and using

$$E_0(1 + \varepsilon) = 1 + \varepsilon + \frac{4}{3\pi} \varepsilon^{3/2} + O(\varepsilon^2), \quad \varepsilon \downarrow 0,$$

we recover (4.70). Hence the Airy asymptotics and the supercritical Bohr–Sommerfeld formula match in the regime (4.64).

5. UNIFORM GAP

We prove in this section the non-asymptotic inequality on the gap $\lambda_{k+1}(h) - \lambda_k(h)$.

Proof of Theorem 1.7. For simplicity, we drop h from the notation and write λ_k for $\lambda_k(h)$. We will prove that $\lambda_{k+1} > \lambda_k + 4h$.

For any $\mu \geq \lambda_1$, recall from (1.18) that

$$(5.1) \quad u_\mu(x) = x^{\nu+1/2} e^{-x^2/2h} M\left(\frac{\nu+1}{2} - \frac{\mu}{4h}, \nu+1, \frac{x^2}{h}\right)$$

is a solution of $(T_h - \mu)u = 0$ which is regular on $\overline{\mathbb{R}}_+$. In particular,

$$f_k = u_{\lambda_k} = x^{\nu+1/2} e^{-x^2/2h} M\left(\frac{\nu+1}{2} - \frac{\lambda_k}{4h}, \nu+1, \frac{x^2}{h}\right)$$

is an eigenfunction of T_h associated to the eigenvalue λ_k .

Now we take $\mu = \lambda_k + 4h$ and express u_μ in terms of f_k as

$$(5.2) \quad u_\mu = c^{-1} A f_k, \quad c = \frac{\nu+1}{2} + \frac{\lambda_k}{4h}, \quad A = \frac{x}{2} \frac{d}{dx} + \frac{\lambda_k + h}{4h} - \frac{x^2}{2h}.$$

This follows from a direct calculation of $A f_k$ and by using the following identity

$$z \frac{d}{dz} M(a, b, z) + (b - a - z) M(a, b, z) = (b - a) M(a - 1, b, z),$$

which results from combining (see [5, Eq. (13.3.15), (13.3.4), (13.3.1)])

$$\begin{aligned} \frac{d}{dz} M(a, b, z) &= \frac{a}{b} M(a+1, b+1, z), \\ z M(a+1, b+1, z) &= b M(a+1, b, z) - b M(a, b, z), \\ a M(a+1, b, z) &= (b-a) M(a-1, b, z) + (2a-b+z) M(a, b, z). \end{aligned}$$

It is worth noting that in light of the identity

$$(T_h - \lambda_k - 4h)A = (A+1)(T_h - \lambda_k),$$

the operator A is akin to a creation operator for T_h .

Note that u_μ does not satisfy the Dirichlet boundary condition at $x = 1$ since by (5.2),

$$u_\mu(1) = \frac{1}{2c} f'_k(1) \neq 0.$$

Actually, since f_k is positive near $x = 0$ and it has $k - 1$ simple zeros in $(0, 1)$, the sign of $u_\mu(1)$ is the same as $(-1)^k$. Thus, $\lambda_{k+1} \neq \mu$. We will prove that $\lambda_{k+1} > \mu$. We argue by contradiction and suppose that $\lambda_{k+1} < \mu$. We claim that

(5.3) If f_k does not vanish on an open interval $I \subset (0, 1)$, then u_μ has at most one zero in I .

We postpone the proof of (5.3) and proceed with the derivation of the sought contradiction. By the Sturm-Liouville theory, we know that f_k has $k - 1$ distinct roots in $(0, 1)$. A root z of f_k is not a root of u_μ since by (5.2), $u_\mu(z) = \frac{z}{2c} f'_k(z) \neq 0$. As a consequence of (5.3), u_μ has at most k roots in $(0, 1)$.

At the same time, f_{k+1} has k distinct roots

$$0 < y_1 < \dots < y_k < 1.$$

Put $y_{k+1} = 1$. Thanks to the assumption $\lambda_{k+1} < \mu$, we get by the Sturm-Picone comparison theorem [16, Theorem B*] that u_μ has a root in every (y_i, y_{i+1}) , for $i = 1, \dots, k$, thereby obtaining k roots of u_μ in $(y_1, 1)$. Moreover, u_μ has a root in $(0, y_1)$. Indeed, suppose not. Since $M(a, b, 0) = 1$, both u_μ and f_{k+1} are positive near 0, hence both u_μ and f_{k+1} are positive on $(0, y_1)$, so that $f'_{k+1}(y_1) < 0$. Consider the Wronskian

$$W := u_\mu f'_{k+1} - u'_\mu f_{k+1}, \quad W' = \frac{\mu - \lambda_{k+1}}{h^2} u_\mu f_{k+1} > 0 \quad \text{on } (0, y_1).$$

In particular, W is increasing on $(0, y_1)$. Since $\lim_{x \rightarrow 0} W(x) = 0$, we get that $W(x) \geq 0$ on $(0, y_1]$. With $0 < \alpha < y_1$, it follows by integration on (α, y_1) that

$$W(y_1) = W(\alpha) + \int_\alpha^{y_1} W'(x) dx > 0.$$

However, since y_1 is a zero of f_{k+1} , we get directly that $W(y_1) = u_\mu(y_1) f'_{k+1}(y_1) \leq 0$, a contradiction.

Now, under the assumption $\lambda_{k+1} < \mu$, we have proved that u_μ has $k+1$ roots. This contradicts the previous conclusion that u_μ has at most k roots in $(0, 1)$.

It remains to prove the claim in (5.3).

Note that, on an interval I where f_k does not vanish, we have

$$A f_k = \frac{x}{2}(\phi + q) f_k, \quad \phi = \frac{f'_k}{f_k}, \quad q = \frac{\lambda_k + h}{2hx} - \frac{x}{h}.$$

In light of (5.2), u_μ and $g = \phi + q$ have the same zero set in I . We will prove that $g' < 0$ at a zero of g , which means that g has at most one zero in I .

Combining the identity

$$\phi' = \frac{x^2}{h^2} + \frac{\nu^2 - 1/4}{x^2} - \frac{\lambda_k}{h^2} - \phi^2,$$

with $q = -\phi$ at a zero of g , we obtain

$$g' = \frac{x^2}{h^2} + \frac{\nu^2 - 1/4}{x^2} - \frac{\lambda_k}{h^2} - q^2 + q' \quad \text{on } \{g = 0\}.$$

By a straightforward calculation

$$\frac{x^2}{h^2} + \frac{\nu^2 - 1/4}{x^2} - \frac{\lambda_k}{h^2} - q^2 + q' = \frac{1}{x^2} \left(\nu^2 - \left(\frac{\lambda_k}{2h} + 1 \right)^2 \right),$$

with

$$\lambda_k \geq \lambda_1 \geq e_1(h) = (2\nu + 2)h.$$

This proves that $g'(x) < 0$ if $g(x) = 0$. □

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APPENDIX A. ON THE BOHR–SOMMERFELD–LANGER INTERPRETATION OF THE GABUTTI–GATTESCHI EQUATION

As recalled in the introduction, the spectral condition can be reformulated in terms of the Whittaker function $M_{\kappa,\mu}$, with

$$(A.1) \quad \kappa = \frac{\lambda}{4h}, \quad \mu = \frac{\nu}{2}.$$

The spectral condition then takes the form

$$(A.2) \quad M_{\kappa,\mu}(1/h) = 0.$$

The Whittaker function $M_{\kappa,\mu}$ satisfies

$$(A.3) \quad w''(z) + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{\frac{1}{4} - \mu^2}{z^2} \right) w(z) = 0.$$

In the supercritical regime, where the energy stays in a compact subset of $(1, +\infty)$, we have

$$(A.4) \quad \kappa = \frac{\lambda}{4h} \longrightarrow +\infty \quad \text{as } h \rightarrow 0^+.$$

The purpose of this appendix is to explain the origin of the implicit relation (3.18) used in the Gabutti–Gatteschi asymptotics. In the semiclassical limit $\kappa \rightarrow +\infty$, the Whittaker equation can be reduced to a Bessel model by a suitable change of variable. This change of variable is determined by requiring that the classical actions of the two equations coincide. This is precisely the content of (3.18). This equality of actions will then allow us to recover the Bohr–Sommerfeld–Langer quantization rule of Theorem 1.4.

We now rewrite (A.3) in semiclassical form in order to identify its principal part as $\kappa \rightarrow +\infty$. Introduce

$$(A.5) \quad z = \kappa X, \quad \gamma = \frac{2\mu}{\kappa}.$$

Then (A.3) becomes

$$(A.6) \quad \kappa^{-2} w_{XX} + \frac{1}{4X^2} (-X^2 + 4X - \gamma^2 + \kappa^{-2}) w = 0.$$

Thus, at leading semiclassical order, the equation reduces to

$$(A.7) \quad \kappa^{-2} w_{XX} + q_W(X; \gamma) w = 0, \quad q_W(X; \gamma) = \frac{-X^2 + 4X - \gamma^2}{4X^2}.$$

The turning points of the principal equation are the roots of

$$-X^2 + 4X - \gamma^2 = 0,$$

namely

$$\xi_1 = 2 - \sqrt{4 - \gamma^2}, \quad \xi_2 = 2 + \sqrt{4 - \gamma^2}.$$

The oscillatory region is

$$\xi_1 < X < \xi_2.$$

In this region, the classical momentum associated with the principal equation is

$$(A.8) \quad p_W(X; \gamma) = \sqrt{q_W(X; \gamma)} = \frac{1}{2X} \sqrt{(X - \xi_1)(\xi_2 - X)}.$$

The corresponding Whittaker action, measured from the left turning point ξ_1 , is therefore

$$(A.9) \quad S_W(X; \gamma) = \int_{\xi_1}^X p_W(t; \gamma) dt = \int_{\xi_1}^X \frac{1}{2t} \sqrt{(t - \xi_1)(\xi_2 - t)} dt.$$

Writing

$$R(X) = \sqrt{(X - \xi_1)(\xi_2 - X)},$$

a direct computation gives

$$(A.10) \quad \begin{aligned} S_W(X; \gamma) &= \frac{1}{2}R(X) - \frac{\gamma}{2} \arctan\left(\frac{2X - \gamma^2}{\gamma R(X)}\right) \\ &\quad - \arctan\left(\frac{2 - X}{R(X)}\right) + \frac{\pi}{2}\left(1 - \frac{\gamma}{2}\right). \end{aligned}$$

Thus, the right-hand side of (3.18) is precisely the Whittaker action.

We now compare the principal Whittaker equation (A.7) with a Bessel model. Consider

$$(A.11) \quad u''(r) + \left(1 + \frac{\frac{1}{4} - 4\mu^2}{r^2}\right) u(r) = 0.$$

Introducing

$$(A.12) \quad r = \kappa Y, \quad \gamma = \frac{2\mu}{\kappa},$$

we obtain

$$(A.13) \quad \kappa^{-2} u_{YY} + \left(1 - \frac{\gamma^2}{Y^2} + \frac{1}{4\kappa^2 Y^2}\right) u = 0.$$

Thus, at leading semiclassical order, the Bessel model reduces to

$$(A.14) \quad \kappa^{-2} u_{YY} + q_B(Y; \gamma) u = 0, \quad q_B(Y; \gamma) = 1 - \frac{\gamma^2}{Y^2}.$$

Thus, both principal equations have the same semiclassical parameter κ^{-1} . They depend on the parameter $\gamma = \frac{2\mu}{\kappa}$, although it enters the two principal potentials differently.

We now relate these equations by a Liouville change of variable, which is determined by requiring that the corresponding classical actions coincide.

The Bessel potential $q_B(Y; \gamma)$ has a positive turning point at $Y = \gamma$, and the oscillatory region is $Y > \gamma$. In this region, the corresponding classical momentum is

$$(A.15) \quad p_B(Y; \gamma) = \sqrt{q_B(Y; \gamma)} = \frac{\sqrt{Y^2 - \gamma^2}}{Y}.$$

The Bessel action, measured from the turning point $Y = \gamma$, is therefore

$$(A.16) \quad S_B(Y; \gamma) = \int_{\gamma}^Y p_B(s; \gamma) ds = \int_{\gamma}^Y \frac{\sqrt{s^2 - \gamma^2}}{s} ds.$$

To recover the variable used in the Gabutti–Gatteschi equation, we now set

$$(A.17) \quad \zeta = Y^2.$$

With the change of variable $t = s^2$, equation (A.16) becomes

$$(A.18) \quad S_B(\zeta; \gamma) = \int_{\gamma^2}^{\zeta} \frac{1}{2t} \sqrt{t - \gamma^2} dt.$$

A direct computation gives

$$(A.19) \quad S_B(\zeta; \gamma) = \sqrt{\zeta - \gamma^2} - \gamma \arctan\left(\frac{\sqrt{\zeta - \gamma^2}}{\gamma}\right).$$

The new variable ζ in Dunster's Liouville transformation is defined by requiring equality of the two actions:

$$(A.20) \quad S_W(X; \gamma) = S_B(\zeta; \gamma).$$

This equality determines the change of variable $X \mapsto \zeta$, which transforms the principal Whittaker equation into the principal Bessel equation. It is exactly the implicit relation (3.18). Thus, the somewhat complicated equation used by Gabutti–Gatteschi simply expresses, in explicit form, the equality of the two classical actions.

We finally explain how the Bohr–Sommerfeld–Langer quantization appearing in Theorem 1.4 is encoded in this construction. For the k -th positive Whittaker zero, the Bessel variable is chosen as

$$(A.21) \quad \zeta_k = \left(\frac{j_{2\mu,k}}{\kappa} \right)^2,$$

where $j_{2\mu,k}$ denotes the k -th positive zero of the Bessel function $J_{2\mu}$. The corresponding Whittaker variable X_k is then determined by the action identity

$$(A.22) \quad S_W(X_k; \gamma) = S_B(\zeta_k; \gamma).$$

For large k , the classical McMahon expansion gives

$$(A.23) \quad j_{2\mu,k} = \pi \left(k + \mu - \frac{1}{4} \right) + \mathcal{O}(k^{-1}).$$

Therefore, in the semiclassical regime $h \rightarrow 0^+$ with $\rho_0 \leq 4kh \leq \rho_1$, the action identity (A.22), together with the McMahon expansion, naturally leads to a Bohr–Sommerfeld quantization condition. The rigorous asymptotic analysis carried out in Paragraph 3.2.5 shows that this procedure yields

$$(A.24) \quad I(E) = \pi h \left(k + \frac{\nu}{2} - \frac{1}{4} \right) + \mathcal{O}(h^2),$$

which is precisely the Bohr–Sommerfeld–Langer quantization rule appearing in Theorem 1.4.

APPENDIX B. ZEROS OF THE KUMMER FUNCTION WITH RESPECT TO THE FIRST PARAMETER

B.1. Introduction. In this appendix, we reformulate the spectral asymptotics obtained in the main text as asymptotic formulas for the zeros of the Kummer confluent hypergeometric function $M(a, b, z)$ with respect to its first parameter. Our purpose is to collect these formulas in a form that can be read independently of the spectral problem. To this end, we recall the standard definitions and relations that have already been introduced at various places in the paper.

We first recall the definition of the Kummer confluent hypergeometric function. For $a, z \in \mathbb{C}$ and $b \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$, it is defined by

$$(B.1) \quad M(a, b, z) = {}_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!},$$

where $(a)_n$ denotes the Pochhammer symbol,

$$(a)_0 = 1, \quad (a)_n = a(a+1) \cdots (a+n-1), \quad n \geq 1.$$

The series in (B.1) converges for every $z \in \mathbb{C}$ and defines an entire function of z . It also defines an entire function of the first parameter a , for fixed admissible b and z . We refer to [5, §13.2, Eq. (13.2.2)] for these standard definitions and properties.

We also recall the classical relation between the Kummer and Whittaker functions. The Whittaker function $M_{\kappa,\mu}$ is related to the Kummer function by

$$(B.2) \quad M_{\kappa,\mu}(z) = e^{-z/2} z^{\mu+1/2} M\left(\mu - \kappa + \frac{1}{2}, 2\mu + 1, z\right).$$

Equivalently, setting

$$\mu = \frac{b-1}{2}, \quad \kappa = \frac{b}{2} - a,$$

one obtains

$$M(a, b, z) = e^{z/2} z^{-b/2} M_{\frac{b}{2}-a, \frac{b-1}{2}}(z).$$

We refer to [5, §13.14, Eqs. (13.14.2) and (13.14.4)] for these identities and for the standard definitions and properties of the Whittaker functions. In particular, for fixed b and z , the zeros of $a \mapsto M(a, b, z)$ are in one-to-one correspondence with the zeros of

$$\kappa \mapsto M_{\kappa, (b-1)/2}(z), \quad \kappa = \frac{b}{2} - a.$$

Thus, the asymptotic formulas obtained below for the zeros of the Kummer function immediately yield corresponding formulas for the zeros of the Whittaker function with respect to its first parameter. For simplicity, we shall state our results only in terms of the Kummer function.

The zeros of the Kummer function with respect to its argument z have been extensively studied. We refer to [5, §13.9] and the references therein for results concerning their number and location, as well as asymptotic formulas for large zeros. In particular, for fixed a and b , the asymptotic distribution of the large z -zeros is described in [5, §13.9, Eq. (13.9.9)].

Much less seems to be known when the zeros are considered with respect to the first parameter a . For fixed b and z , the classical large-index asymptotics of the a -zeros is given in [5, §13.9, Eq. (13.9.10)]. In our indexing convention (see below), the large-index expansion in [5, Eq. (13.9.10)] reads

$$(B.3) \quad a_k(z) = -\frac{\pi^2}{4z} \left(k^2 + \left(b - \frac{3}{2} \right) k \right) + \mathcal{O}(1), \quad k \rightarrow +\infty,$$

for fixed b and z . This asymptotic regime, however, does not describe the behavior of the low-lying a -zeros when z becomes large.

For fixed $k \geq 0$, a much more precise description in the limit $z \rightarrow +\infty$ follows from the asymptotics of Gannot [10], recalled in (1.4). Translated into the present notation, his result gives, for fixed b and $k \geq 0$,

$$(B.4) \quad a_k(z) = -k - \frac{z^{2k+b}}{k! \Gamma(k+b)} e^{-z} (1 + \mathcal{O}(z^{-1})), \quad z \rightarrow +\infty.$$

Thus the k -th a -zero is exponentially close to the negative integer $-k$.

The purpose of the following subsections is to extend this picture beyond the fixed-index regime and to describe the a -zeros uniformly when k is allowed to grow with z , both in the subcritical and supercritical regimes. Before stating our results, let us clarify the indexing convention used throughout this appendix. The eigenvalues studied in this paper are indexed by $k \geq 1$, whereas the negative zeros of the Kummer function are conventionally indexed starting from $k = 0$; their negativity will be justified below. Consequently, there is a shift of one between the two indexings: the k -th Kummer zero $a_k(z)$ corresponds to the $(k+1)$ -st eigenvalue.

Throughout the remainder of this appendix, we assume that $b \geq 1$ is fixed, in accordance with the range of parameters considered in our spectral analysis. For $z > 0$, all the zeros of $a \mapsto M(a, b, z)$ are real and simple. Indeed, by (1.19), they are in one-to-one correspondence with the simple eigenvalues of the self-adjoint Sturm–Liouville problem considered above. More precisely, setting $h = z^{-1}$ and $b = 1 + \nu$, with $\nu \geq 0$, this correspondence is given by

$$(B.5) \quad a_k(z) = \frac{b}{2} - \frac{z}{4} \lambda_{k+1}(z^{-1}).$$

Since $\lambda_{k+1}(h) \geq e_{k+1}(h)$ and

$$(B.6) \quad e_{k+1}(z^{-1}) = \frac{4k + 2b}{z},$$

we obtain

$$(B.7) \quad a_k(z) \leq -k.$$

Moreover, $a_0(z) \neq 0$, since $M(0, b, z) = 1$. Hence (B.7) implies that all the zeros are strictly negative (see also [22, Theorem 1.6]). With the indexing convention introduced above, we therefore write

$$(B.8) \quad \cdots < a_2(z) < a_1(z) < a_0(z) < 0.$$

Furthermore, Theorem 1.7 yields a non-asymptotic separation bound for consecutive a -zeros: for every fixed $b \geq 1$ and every $z > 0$,

$$a_k(z) - a_{k+1}(z) > 1, \quad k \geq 0.$$

Indeed, the correspondence (B.5) gives

$$a_k(z) - a_{k+1}(z) = \frac{z}{4} (\lambda_{k+2}(z^{-1}) - \lambda_{k+1}(z^{-1})) > 1.$$

B.2. The subcritical regime.

B.2.1. The low-energy regime. As a consequence of Theorem 1.1 and the correspondence (B.5), we obtain the following asymptotics for the Kummer zeros. As $z \rightarrow +\infty$,

$$(B.9) \quad a_k(z) = -k - \frac{z^{2k+b}}{k! \Gamma(k+b)} e^{-z} (1 + o(1)),$$

uniformly with respect to the integers $k \geq 0$ satisfying $k+1 \leq z^\gamma$, $0 < \gamma < 1/2$. In particular, (B.9) applies to every fixed $k \geq 0$ as $z \rightarrow +\infty$.

B.2.2. The higher-energy subcritical regime. We now turn to the regime described by Theorem 1.2. Let $0 < \rho^* < 1$, and let $\ell : (0, 1] \rightarrow \mathbb{R}_+$ satisfy

$$(B.10) \quad \ell(h) \rightarrow 0, \quad h^{-1} \ell(h) \rightarrow +\infty \quad (h \rightarrow 0^+).$$

Then, as $z \rightarrow +\infty$,

$$(B.11) \quad a_k(z) = -k - \frac{1}{2\pi} \exp(-2z S_{k,z}) (1 + o(1)),$$

uniformly with respect to the integers $k \geq 0$ satisfying

$$(B.12) \quad \ell(z^{-1}) \leq \frac{4k+2b}{z} \leq \rho^*,$$

where

$$(B.13) \quad S_{k,z} = \int_{\sqrt{(4k+2b)/z}}^1 \sqrt{s^2 - \frac{4k+2b}{z}} ds.$$

Remark B.1. The low-energy asymptotic formula (B.9) matches the subcritical asymptotic formula (B.11) in their common range of validity. More precisely, in the regime

$$k \rightarrow +\infty, \quad \frac{k^2}{z} \rightarrow 0,$$

the two asymptotic formulas for the exponentially small displacement of $a_k(z)$ from the negative integer $-k$ are equivalent.

B.3. The transition regime. We now consider the transition regime between the subcritical and supercritical asymptotics. For $k \geq 0$, set

$$(B.14) \quad \tau_{k,z} := \frac{z^{2/3}}{2^{2/3}} \left(1 - \frac{4(k+1)}{z} \right).$$

Let $T > 0$ and assume that $|\tau_{k,z}| \leq T$. Let $\theta : \mathbb{R} \rightarrow \mathbb{R}$ be the continuous phase defined by

$$(B.15) \quad \text{Ai}(t) = A(t) \sin \theta(t), \quad \text{Bi}(t) = A(t) \cos \theta(t), \quad A(t) = (\text{Ai}(t)^2 + \text{Bi}(t)^2)^{1/2},$$

with the determination chosen so that $\theta(t) \rightarrow 0$ as $t \rightarrow +\infty$. Then, uniformly for $|\tau_{k,z}| \leq T$,

$$(B.16) \quad \begin{aligned} a_k(z) = & -k - \frac{\theta(\tau_{k,z})}{\pi} \\ & + \frac{z^{-1/3}}{\pi 2^{2/3}} \left(2b - 4 + \frac{4}{\pi} \theta(\tau_{k,z}) \right) \theta'(\tau_{k,z}) + \mathcal{O}(z^{-2/3}), \text{ as } z \rightarrow +\infty. \end{aligned}$$

B.4. The supercritical regime. Let $1 < \rho_0 < \rho_1 < +\infty$. In view of the shift in the indexing described above, for $k \geq 0$ we set

$$(B.17) \quad \rho = \frac{4(k+1)}{z},$$

and assume that $\rho_0 \leq \rho \leq \rho_1$. We recall the function E_0 appearing in the supercritical spectral asymptotics. For $\rho > 1$, $E_0(\rho) > 1$ is uniquely determined by

$$I(E_0(\rho)) = \frac{\pi\rho}{4}, \quad I(E) = \frac{1}{2}\sqrt{E-1} + \frac{E}{2} \arcsin(E^{-1/2}).$$

Then, as $z \rightarrow +\infty$,

$$(B.18) \quad a_k(z) = -\frac{z}{4}E_0(\rho) + \frac{b}{2} - \frac{\pi(2b-3)}{8 \arcsin(E_0(\rho)^{-1/2})} + \mathcal{O}(z^{-1}),$$

uniformly with respect to $\rho \in [\rho_0, \rho_1]$. Equivalently, setting

$$(B.19) \quad \vartheta(\rho) = \arcsin(E_0(\rho)^{-1/2}) \in \left(0, \frac{\pi}{2}\right),$$

we have

$$(B.20) \quad E_0(\rho) = \frac{1}{\sin^2 \vartheta(\rho)},$$

where $\vartheta(\rho)$ is the unique solution of

$$(B.21) \quad \vartheta + \sin \vartheta \cos \vartheta = \frac{\pi\rho}{2} \sin^2 \vartheta.$$

Hence (B.18) can also be written as

$$(B.22) \quad a_k(z) = -\frac{z}{4 \sin^2 \vartheta(\rho)} + \frac{b}{2} - \frac{\pi(2b-3)}{8 \vartheta(\rho)} + \mathcal{O}(z^{-1}),$$

uniformly with respect to $\rho \in [\rho_0, \rho_1]$.

REFERENCES

- [1] M. Baur and T. Weidl. Eigenvalues of the magnetic Dirichlet Laplacian with constant magnetic field on disks in the strong field limit, *Anal. Math. Phys.* 15 (2025), no. 1, Paper No. 9.
- [2] K. Beauchard, L. Miller and M. Morancey. 2D Grushin-type equations: minimal time and null controllable data, *J. Differential Equations* **259** (2015), no. 11, 5813–5845.
- [3] C. Bolley. Modélisation du champ de retard à la condensation d'un supraconducteur par un problème de bifurcation. *ESAIM: Mathematical Modelling and Numerical Analysis - (1992) Volume: 26, Issue: 2*, page 235-287
- [4] C. Bolley and B. Helffer. An application of semi-classical analysis to the asymptotic study of the supercooling field of a superconducting material. *Ann. Inst. H. Poincaré (Section Physique Théorique)* 58 (2), p. 169-233 (1993).
- [5] NIST Digital Library of Mathematical Functions. F. W. J. Olver et al., eds. <https://dlmf.nist.gov/>.
- [6] T. M. Dunster. Uniform asymptotic expansions for Whittaker's confluent hypergeometric functions. *SIAM J. Math. Anal.* **20** (1989), no. 3, 744–760.
- [7] T. M. Dunster. Whittaker functions with one or both parameters large: simplified uniform asymptotic expansions involving Bessel and Airy functions. *arXiv:2608.15374* (2026).
- [8] P. Exner, P. Šťovíček and P. Vytřas. Generalized boundary conditions for the Aharonov-Bohm effect combined with a homogeneous magnetic field, *J. Math. Phys.* **43** (2002), no. 5, 2151–2168.
- [9] B. Gabutti and L. Gatteschi. New asymptotics for the zeros of Whittaker's functions. *Numerical Algorithms* **28** (2001), 159–170.
- [10] O. Gannot. Effects of confinement for single-well potentials, *J. Math. Phys.* 57 (2016), no. 2, 023510.
- [11] E. Harrell. Double wells. *Comm. Math. Phys.* 75, 239-261 (1980).

- [12] B. Helffer and C. Léna. Eigenvalues of the Neumann magnetic Laplacian in the unit disk, *J. Math. Phys.* **66** (2025), no. 8, Paper No. 081513, 23 pp.
- [13] B. Helffer. Semi-classical analysis for the Schrödinger operator and applications. *Lecture Notes in Mathematics*, 1336, Springer, Berlin, 1988.
- [14] B. Helffer, and D. Robert. Puits de potentiel généralisés et asymptotique semi-classique. *Ann. Inst. Henri Poincaré, Phys. Théor.* **41**, 291-331 (1984).
- [15] B. Helffer, and J. Sjöstrand. Multiple wells in the semi-classical limit. I. *Commun. Partial Differ. Equations* **9**, 337-408 (1984).
- [16] D. B. Hinton. Sturm’s 1836 oscillation results evolution of the theory, in *Sturm-Liouville theory*, 1–27, Birkhäuser, Basel, 2005.
- [17] A. Kachmar and G. Miranda. The magnetic Laplacian on the disc for strong magnetic fields. *J. Math. Anal. Appl.* **546** (2025), no. 2, Paper No. 129261.
- [18] W. Magnus, F. Oberhettinger and R.P. Soni. Formulas and theorems for the special functions of mathematical physics, 3rd enlarged ed, *Grundlehren der Mathematischen Wissenschaften*, Volume 52, Springer, (1966).
- [19] F. W. J. Olver. *Asymptotics and Special Functions*. Academic Press, New York, 1974.
- [20] J. Sjöstrand. Semi-excited states in nondegenerate potential wells. *Asymptotic Anal.* **6** (1992), no. 1, 29–43.
- [21] G. Szegő. *Orthogonal Polynomials*. American Mathematical Society Colloquium Publications, vol. 23, 4th ed., Providence, RI, 1975.
- [22] R. Vanlaere. Spectral asymptotics for a class of singular Sturm–Liouville operators with applications to magnetic Laplacian and a -zeros of Kummer functions. *J. Differential Equations* **475** (2026), Art. No. 114450.
- [23] R. Vanlaere, Observability properties of the singular Grushin equation, arXiv:2602.08044 (2026).

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