Université Lille 1, Sciences et Technologies
2011/2012 – Master degree in mathematical engineering
Refresher course in physics

Final exam.
October, 14^{th} 2011. **3 hours.**

Documents, cell phones and calculators are forbidden.

Each exercise should be done on a different sheet of paper.

Exercise 1

We study a one-dimensional, diatomic crystalline solid. In its equilibrium state, it is modeled as

- an infinite number of atoms of mass m_1 located at $x = 2na$ on the (Ox) axis (where $n \in \mathbb{Z}$) ;
- an infinite number of atoms of mass m_2 located at $x = (2n+1)a$ on the (Ox) axis (where $n \in \mathbb{Z}$) ;
- a spring of constant k and length a between each mass m_1 and m_2.

The constants k and a are identical for every spring. The masses can only move on the (Ox)-axis.

We denote $u_n(t)$ the displacement of the mass initially located in $2na$ and $v_n(t)$ the displacement of the mass initially located in $(2n+1)a$.

We make the assumption that $m_1 > m_2$.

(1) Show that the motion is described, for every n in $\mathbb{Z}$, by

$$m_1 \frac{d^2 u_n}{dt^2} = k(v_n + v_{n-1} - 2u_n) \text{ and } m_2 \frac{d^2 v_n}{dt^2} = k(u_{n+1} + u_n - 2v_n).$$

(2) We search for solutions of the form

$$u_n(t) = U e^{\imath(nKa - \omega t)} \text{ and } v_n(t) = V e^{\imath(nKa - \omega t)}.$$

Deduce the dispersion relation.

(3) Explain why we can restrict ourselves to $K \in [-\pi/a, \pi/a]$.

(4) Compute the values of ω^2 for $K = \pm\frac{\pi}{a}$.

(5) Show that for $Ka \ll 1$, there are two dispersion relations
$$\omega^2 = 2k\left(\frac{1}{m_1}+\frac{1}{m_2}\right) \text{ and } \omega^2 = \frac{k}{2(m_1+m_2)}K^2a^2.$$

(6) Represent the approximative shape of the dispersion relation $\omega = f(K) \geq 0$ for $K \in [-\pi/a, \pi/a]$.

(7) Describe the behavior of the atoms for $K = 0$.

(8) Show that there are two domains of pulsations for which there is no wave propagation.

(9) Suppose one of the atom in the chain is excited with such a pulsation. What happens ?

Exercise 2

We recall that "a set that moves with the flow" is a set $\Omega_t, t \in I \subset \mathbb{R}$, whose pre-image Ω_0 at the initial time remains constant. Using the assumptions introduced in the course, let us denote $\Omega_t = \Phi(\Omega_0)$ a set (here a surface or a volume) in the domain filled by the fluid.

(1) Show that the acceleration field $\mathbf{a}(\mathbf{x}, t)$ in Eulerian representation can be written as
$$\mathbf{a} = \frac{\partial \mathbf{u}}{\partial t} + \nabla(\frac{|\mathbf{u}|^2}{2}) + (\vec{\text{rot}}\,\mathbf{u}) \wedge \mathbf{u}.$$

(2) Let $V_t \subset \mathbb{R}^3$ be the domain filled by the fluid. Explain why we can write the Navier–Stokes equations for an incompressible and homogeneous fluid as following :
$$(NS)\left\{\begin{array}{ll} \bar{\rho}\left(\dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) - \mu\Delta\mathbf{u} + \nabla p = \mathbf{F} & \text{in } V_t \times I, \\ \text{div}\mathbf{u} = 0 & \text{in } V_t \times I, \end{array}\right.$$
where $\rho(\mathbf{x}, t) = \bar{\rho},\ \forall \mathbf{x} \in V_t, \forall t \in I$.

(3) The vorticity is the vector field $\vec{\omega} = \vec{\text{rot}}\,\mathbf{u}$. Taking the $\vec{\text{rot}}$ of the first (NS) equation, show that we can find the following vorticity equation :
$$\bar{\rho}\left(\frac{\partial \vec{\omega}}{\partial t} + \vec{\text{rot}}(\vec{\omega} \wedge \mathbf{u})\right) = \mu\Delta\vec{\omega} + \vec{\text{rot}}\mathbf{F} \quad \text{in } V_t \times I.$$

(4) Show that for an inviscid incompressible fluid for which the volume density of forces derives from a potential, the vorticity equation becames
$$\frac{\partial \vec{\omega}}{\partial t} + (\mathbf{u}\cdot\nabla)\vec{\omega} - (\vec{\omega}\cdot\nabla)\mathbf{u} = 0 \quad \text{in } V_t \times I.$$

(5) We admit the following equation, similar to Proposition 1 seen in the course, in which $\mathbf{n}$ is a unit normal to the surface Σ_t that moves with the flow :
$$\frac{d}{dt}\int_{\Sigma_t} \vec{\omega}\cdot\mathbf{n}\, d\gamma = \int_{\Sigma_t}\left[\frac{\partial \vec{\omega}}{\partial t} + \vec{\text{rot}}(\vec{\omega} \wedge \mathbf{u})\right]\cdot\mathbf{n}\, d\gamma.$$

Prove the following Kelvin's theorem :
Let us consider an inviscid incompressible fluid for which the volume density of forces derives from a potential. Then, the flux of the vorticity vector through a surface moving with the flow remains constant.

Exercise 3

(a) Write down the Maxwell equations in vacuum.

(b) Let $\vec{E}, \vec{B}$ be a solution to the Maxwell equations. Les S be a (not necessarily closed) surface and ∂S its boundary. Show that the circulation of E around ∂S equals the time rate of change of the magnetic flux through S (where the latter is by definition the flux of the magnetic field through S). This is called Faraday's law of magnetic induction. Explain how it implies that a changing magnetic field can produce a current in a wire.

(c) Let $\vec{E}_0, \vec{B}_0, \vec{e}, \vec{e}' \in \mathbb{R}^3$ and $v, v' > 0$. Consider

$$\vec{E}(\vec{r}, t) = \vec{E}_0 g(\vec{e} \cdot \vec{r} - vt), \quad \vec{B}(\vec{r}, t) = \vec{B}_0 h(\vec{e}' \cdot \vec{r} - v't),$$

where g and h are smooth functions on $\mathbb{R}$, $g(0) = h(0) = g'(0) = h'(0) = 1, g"(0) \neq 0$. Find the conditions on $\vec{E}_0, \vec{B}_0, \vec{e}, \vec{e}', v, v'$ so that $\vec{E}(\vec{r}, t)$, $\vec{B}(\vec{r}, t)$ are a solution of the Maxwell equations. Hint : you may find it convenient to introduce the Poynting vector $\vec{R}_0 = \vec{E}_0 \wedge \vec{B}_0$.

(d) Why are such solutions called traveling plane waves ? In which direction do they travel ?