Université Lille 1, Sciences et Technologies
2012/2013 – Master degree in mathematical engineering
Refresher course in physics

Final exam.
October, 18^{th} 2012. **3 hours.**

Documents, cell phones and calculators are forbidden.

Each exercise should be done on a different sheet of paper.

Exercise 1

Consider a mass M and a finite string whose mass per unit of length is λ. Denote g the gravitational constant.

(1) Using these objects, recall how we derived the wave equation
$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = 0.$$
Express c in function of M, g and λ.

(2) In this model, we recall [1] that the energy of a solution $y(x,t)$ is given by
$$E(t) := \frac{1}{2} \int_0^L \left(\lambda \left(\frac{\partial y}{\partial t} \right)^2 + Mg \left(\frac{\partial y}{\partial x} \right)^2 \right) dx.$$
What is L in this model? What does each term in the sum represent?

(3) From this point of the problem, we will suppose that each piece dx of the string is submitted to a friction force of the form $-f_0 dx \vec{v}$, where $\vec{v}$ is the velocity of the piece dx and $f_0 > 0$ is constant. Show that the new equation is of the form
$$\frac{\partial^2 y}{\partial t^2} + 2a_0 c^2 \frac{\partial y}{\partial t} - c^2 \frac{\partial^2 y}{\partial x^2} = 0.$$
Express a_0 in function of λ, c and f_0.

(4) Show that $E(t)$ is a nonincreasing function for $y(x,t)$ solving this new equation.

(5) Show that $E(t) \geq E(0) e^{-4a_0 c^2 t}$ for $t \geq 0$.

(6) Suppose $y(x,t)$ is of the form $g(x)h(t)$. Fix t and show that g must be of the form
$$g(x) = C \sin\left(\frac{n\pi x}{L}\right),$$
for some integer $n \geq 0$.

1. You will not prove this formula.

(7) Fix an integer $n \geq 1$. Show that there exist some constants A_n, B_n, α_n and β_n such that

$$h_n(t) = A_n e^{\alpha_n t} + B_n e^{\beta_n t}.$$

(8) Show that the real parts of α_n and β_n are < 0 and comment this result.

(9) Fix $n \geq 1$. Suppose

$$y_n(x, t = 0) = y_n \sin\left(\frac{n\pi x}{L}\right) \text{ and } \frac{\partial y_n}{\partial t}(x, t = 0) = v_n \sin\left(\frac{n\pi x}{L}\right).$$

Show that

$$y_n(x,t) = e^{-a_0 c^2 t}\left(\frac{v_n + y_n a_0 c^2}{\sqrt{\gamma_n}}\mathrm{sh}(\sqrt{\gamma_n}t) + y_n \mathrm{ch}(\sqrt{\gamma_n}t)\right)\sin\left(\frac{n\pi x}{L}\right),$$

where you will express γ_n in function of a_0, c, n and L.

(10) *Bonus Question.* Show that there exist some constant $\gamma > 0$ and $C > 0$ independent of n such that

$$E(t) \leq Ce^{-\gamma t}E(0),$$

for any solution of the form $y_n(x,t)$. Conclude that $E(t) \leq Ce^{-\gamma t}E(0)$ for any solution $y(x,t)$ of the damped wave equation.

Exercise 2

We want to study two-dimensional flows of an incompressible irrotational and inviscid fluid filling the region $\Omega = \Omega_t$ defined by $0 < x < l, 0 < y < h(x,t)$. The surface $\Sigma = \Sigma_t$ of the liquid with equation $y = h(x,t)$ is a free surface.

For a regular function f, we denote $\dfrac{\partial f}{\partial t} = f_t$, $\dfrac{\partial f}{\partial x} = f_x$, $\dfrac{\partial f}{\partial y} = f_y$, $\dfrac{\partial^2 f}{\partial x^2} = f_{xx}$, $\dfrac{\partial^2 f}{\partial y^2} = f_{yy}$.
We recall that the (i,j) element of the matrix $\mathbf{u} \otimes \mathbf{u}$ is $u_i u_j$.

(1) For an inviscid fluid, explain why we can write the conservation of momentum in Euler equation

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \mathrm{div}(\rho\mathbf{u} \otimes \mathbf{u}) + \nabla p = \mathbf{F} \quad \text{in } \Omega_t \times I$$

as following :

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathrm{grad}\frac{|\mathbf{u}|^2}{2} + (\mathrm{rot}\,\mathbf{u}) \wedge \mathbf{u}\right) + \nabla p = \mathbf{F} \quad \text{in } \Omega_t \times I.$$

(2) For an incompressible irrotational and inviscid fluid, show that there exists a function $\phi = \phi(x,y,t)$ such that $\mathbf{u} = \nabla\phi$ and

$$\phi_{xx} + \phi_{yy} = 0.$$

(3) We assume that $\rho(x,y,t) = \rho_0 \in \mathbb{R}^+$ and the mass density of forces derives from a potential : $\dfrac{\mathbf{F}}{\rho_0} = -\nabla(gy)$. Deduce from the Euler equation the following relation (which is the Bernoulli equation) :

$$\phi_t + \frac{p}{\rho_0} + \frac{1}{2}(\phi_x^2 + \phi_y^2) + gy = const.$$

(4) We denote p_0 the pressure on the surface Σ_t. In order to determine the boundary condition satisfied by ϕ on the free surface Σ_t, show that we have

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + gh(x,t) = c(t) \quad \text{ on } \Sigma_t \times I.$$

Explain why we can replace $\phi(x,y,t)$ by $\phi(x,y,t) - \displaystyle\int_0^t c(s)ds$ and give the equation obtained.

(5) We remark that the normal velocity of Σ_t is the same as the normal velocity of the fluid. Recalling that $y = h(x,t)$ is the equation of the free surface Σ_t, deduce that the velocity of each point of Σ_t is $(0, h_t)$. Show that the vector $(-h_x, 1)$ is normal to Σ_t. Using $\mathbf{u} = \nabla\phi$, show that

$$h_t + \phi_x h_x - \phi_y = 0 \quad \text{ on } \Sigma_t \times I.$$

(6) Give the non penetration condition on the lower horizontal wall satisfied by ϕ and write the Cauchy problem.

Exercise 3 – Electromagnetism

(a) First a little warm-up exercice that has nothing to do with electromagnetism, but that will be of use in (g) below. Let $\vec{K}(M)$ be a vectorfield of the form $\vec{K}(M) = k(r)\vec{u}_r$, in spherical coordinates ; here k is a smooth function of $r \geq 0$. Show that, then, $\vec{\text{rot}}\vec{K} = 0$. Show also that $\text{div}\vec{K} = 0$ if and only if $k(r) = 0$, for all $r \geq 0$.

Hint : Since you probably don't know the expression for $\vec{\text{rot}}$ by heart in spherical coordinates, I suggest you do the computation in cartesian coordinates. For the divergence, let me help you out : in spherical coordinates, and with the usual notation,

$$\text{div}\vec{K} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 K_r\right) + \frac{1}{r\sin\theta}\frac{\partial \sin\theta K_\theta}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial K_\varphi}{\partial\varphi}.$$

Before going on, we first recall the Maxwell equations, in case you forgot what they look like :

$$\text{div}\vec{E}(M,t) = \frac{\rho(M,t)}{\epsilon_0} \qquad \text{div}\vec{B}(M,t) = 0$$

$$\vec{\text{rot}}\vec{E}(M,t) = -\frac{\partial\vec{B}}{\partial t}(M,t) \quad \vec{\text{rot}}\vec{B}(M,t) = \mu_0\left(\vec{j}(M,t) + \epsilon_0\tfrac{\partial\vec{E}}{\partial t}(M,t)\right).$$

(b) *A theoretical question.* Let $\vec{E}, \vec{B}$ be a solution to the Maxwell equations. Les S be a (not necessarily closed) surface and ∂S its boundary. Show that the circulation of E around ∂S equals the time rate of change of the magnetic flux through S (where the latter is by definition the flux of the magnetic field through S). This is called Faraday's law of magnetic induction. Explain how it implies that a changing magnetic field can produce a current in a wire.

(c) *One more theoretical question.* I gave a physical argument during the lectures implying that a charge density ρ and a current density $\vec{j}$ must satisfy the continuity equation

$$\frac{\partial \rho}{\partial t}(M,t) + \mathrm{div}\vec{j}(M,t) = 0.$$

Show that this equation is in fact *implied* by the Maxwell equations.

Now, something slightly more concrete. Let us consider in what follows a charge distribution $\rho(M,t)$, given by

$$\rho(M,t) = \frac{3\alpha}{4\pi t^3} Q_0 \exp(-\alpha(\frac{r}{t})^3),$$

where $Q_0 \in \mathbb{R}$, $\alpha > 0$ are given constants and $t > 0$.

(d) Compute the total charge present in space at any given time t. Compute the total charge present in a ball of radius R centered at the origin, at any given time t. What happens to this charge as $t \to +\infty$? And as $t \to 0$? Explain in simple intuitive terms what you think is happening.

We wish to compute $\vec{j}$.

(e) Assume $\vec{j}$ is of the form $\vec{j}(M,t) = \frac{1}{t^3} g(\frac{r}{t})\vec{u}_r$ and find the differential equation satisfied by g if $\vec{j}$ and ρ above satisfy the continuity equation. Then show that this differential equation is solved by

$$g(u) = \frac{9\alpha Q_0}{u^2} \int_0^u u' \left(1 - \alpha u'^3\right) \exp(-\alpha u'^3) \mathrm{d}u'.$$

Attention : One shall not try to compute this integral !

(f) How does the result in (d) fit the intuitive picture of what is happening to the charge that you developed in (c) ?

Having found ρ and $\vec{j}$, we now wish to compute the electric and magnetic fields $\vec{E}$ and $\vec{B}$ they generate. Let's see if we can do this.

(g) Symmetry arguments strongly suggest $\vec{E}$ and $\vec{B}$ should be spherically symetric. Show that, then, $\vec{B}(M) = 0$, for all M. Write $\vec{E}(M) = E(r,t)\vec{u}_r$ and establish the two partial differential equations satisfied by $E(r,t)$.