Université Lille 1, Sciences et Technologies
2014/2015 – Master degree in mathematical engineering
Refresher course in modeling

Final exam.
October, 13^{th} 2014. **3 hours.**

Cell phones and calculators are forbidden.

Each exercise should be done on a different sheet of paper.

Exercise 1

(1) Describe a model where the following equation appears :

$$\partial_t N = \kappa \Delta N + g(x, N).$$

Explain precisely how one derive the previous equation and what are N, g and κ in your model.
In the following of the exercise, we make the assumption that $g(x, N) \equiv 0$.

(2) We consider the previous equation on a disk of radius $R > 0$ centered at 0. Show that the previous equation can be expressed in polar coordinates (r, θ) as follows :

$$\partial_t N = \kappa \left(\frac{\partial^2 N}{\partial r^2} + \frac{1}{r}\frac{\partial N}{\partial r} + \frac{1}{r^2}\frac{\partial N}{\partial \theta} \right).$$

In the following of the exercise, we suppose that we impose the limit condition

$$N(R, \theta, t) = N_0(\theta),$$

and that N *has reached a permanent regime where* N *does not depend on* t.

(3) Set $\psi(s, \theta) = N(e^s, \theta)$. Show that ψ satisfies

$$\frac{\partial^2 \psi}{\partial s^2} + \frac{\partial^2 \psi}{\partial \theta^2} = 0.$$

(4) Justify that any solution ψ of the previous equation satisfies

$$\psi(s, \theta) = \sum_{n \geq 0} \left(A_n(s) \cos(n\theta) + B_n(s) \sin(n\theta) \right).$$

(5) Conclude that any solution $N(r,\theta)$ is of the following form

$$N(r,\theta) = \sum_{n\geq 0} \left(a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)\right).$$

Express the coefficients a_n and b_n in terms of N_0.

Exercise 2

Consider the classical Hamiltonian

$$H(q,p) = \frac{1}{2}p^2 - Fq,$$

where $F > 0$ and $(q,p) \in \mathbb{R}^2$.

(i) Write, then solve, the corresponding Hamiltonian equations of motion. What kind of physical situation does this Hamiltonian describe?

We now consider the following one-dimensional Schrödinger equation ($y \in \mathbb{R}$)

$$i\partial_t \psi_t(y) = \frac{1}{2}(P^2\psi_t)(y) - F(Q\psi_t)(y), \qquad \psi_0(y) = \varphi(y),$$

where $F > 0$ and where P and Q were defined in the course. Note that we put $\hbar = 1$ to simplify the notation and that $\psi_t(y) = \psi(y,t)$. In what follows φ is assumed to be known, but not given explicity. Also, $\hat{\psi}_t$ denotes the Fourier transform of ψ_t.

(ii) Compute

$$\frac{d}{dt}\langle \psi_t, Q\psi_t\rangle, \quad \frac{d}{dt}\langle \psi_t, P\psi_t\rangle,$$

and then determine explicitly $\langle \psi_t, Q\psi_t\rangle$ and $\langle \psi_t, P\psi_t\rangle$ in terms of the initial condition. Compare with (i) and comment.

(iii) Use conservation of energy and the previous result to compute $\langle \psi_t, P^2\psi_t\rangle$ and ΔP_t. What can you conclude about the evolution of the momentum distribution of the particle from the information obtained so far?

(iv) Show that

$$i\partial_t \hat{\psi}_t(p) = \frac{1}{2}p^2\hat{\psi}(p) - Fi\partial_p \hat{\psi}_t(p).$$

(v) Introduce the function

$$f(p,t) := \hat{\psi}(p + Ft, t),$$

and show that $i\partial_t f(p,t) = \frac{1}{2}(p+Ft)^2 f(p,t)$. Solve this equation and then give a complete description of the evolution of the momentum probability distribution $|\hat{\psi}_t|^2(p)dp$ of the particle and link it to the result found in (ii)-(iii).

(vi) Give an expression for $\psi_t(y)$ in the form

$$\psi_t(y) = \int_{\mathbb{R}} \exp(-iS(p-Ft,t))\ \hat{\varphi}(p-Ft)\exp(ipy)\frac{dp}{\sqrt{2\pi}}.$$

Identify the (real-valued) function $S(p,t)$ explicitly.

(vii) Show that

$$\frac{d^2}{dt^2}\langle\psi_t, Q^2\psi_t\rangle = 4\langle\psi_t, H\psi_t\rangle + 6F\langle\psi_t, Q\psi_t\rangle.$$

Solve this equation and compute ΔQ_t. **Hint :** This exercice does not use (vi) but must be done in the same way as parts (ii) and (iii) above.

Exercise 3

The Boussinesq equations are an approximation of the general equations of Newtonian fluids. These equations govern the evolution of slightly compressible fluids when thermal phenomena are taken into account. The Boussinesq hypothesis reads :

The density ρ is assumed to be constant ($= \rho_0$) everywhere in the equations, except in the gravity force term. For the gravity forces, ρ is replaced by a linear function of the temperature.

We denote $\mathbf{u}(\mathbf{x},t)$ the velocity field, $\rho(\mathbf{x},t)$ the density, $p(\mathbf{x},t)$ the pressure and $T(\mathbf{x},t)$ the temperature in the domain $\Omega_\infty = \{(x_1,x_2), 0 < x_2 < L\}$, for $t \in [0, T_{max}]$. The equations of a viscous, homogeneous, incompressible fluid submitted to gravity and which conducts heat read :

$$\begin{cases} \mathrm{div}\mathbf{u} = 0, \\ \rho_0\left(\dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) - \mu\Delta\mathbf{u} + \nabla p = -\rho_0 g(T - T_*)\mathbf{e}_2, \\ C_V\left(\dfrac{\partial T}{\partial t} + (\mathbf{u}\cdot\nabla)T\right) - \kappa\Delta T = 0, \end{cases}$$

where $\mathbf{e}_2 = (0,1)^T$, $g = 9.81$, $\mu > 0, C_V > 0, \kappa > 0$ and T_* is the reference temperature.

(1) Explain each equation of this system.

(2) We assume that the temperature is maintained at $T = T_0$ at $x_2 = 0$ and at $T = T_1$ at $x_2 = L$. Compute the stationary solution satisfying $\mathbf{u}_s = 0$ and $T_s = \psi(x_2)$ (we call $p_s(x_1,x_2)$ the pressure).

(3) From now on, we set $\theta = T - T_s$, $P = p - p_s$. Rewrite the system of equations for the new variables $\mathbf{u}, P, \theta$.

(4) We assume that the solutions $(\mathbf{u}, P, \theta)$ of the new system are periodic with period 2π in x_1 and that $\mathbf{u}$ vanishes at $x_2 = 0$ and $x_2 = L$. We set $\Omega = (0,2\pi)\times(0,L)$. Show that we have :

$$\int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{u}]\mathbf{u}\,dx = 0, \quad \int_\Omega [(\mathbf{u}\cdot\nabla)\theta]\theta\,dx = 0, \quad \int_\Omega \nabla p\cdot\mathbf{u}\,dx = 0.$$

(5) Give an expression of the total energy

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\rho_0|\mathbf{u}|^2 + C_V\theta^2)\,dx$$

assuming that we have the homogeneous Dirichlet boundary conditions on $\mathbf{u}, P, \theta$.

(6) Show that the total energy decreases when $g = 0$ and $T_1 = T_0$.