Université des Sciences et Technologies de Lille 1
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Refresher Course in Physics – Semester 3

Solving linear ordinary differential equations of order 1 and 2

In this short text, we recall some elementary methods for solving linear Ordinary Differential Equations (ODE) of order 1 and 2. We do not claim for any originality. Above all, our aim is to provide a theoretical toolbox that will be useful (and hopefully sufficient) to solve the physical problems we will study during the lectures.

One can find these results in any standard reference on this subject or in general mathematics textbook for License degree. For instance, a classical and very complete textbook on general ODE is *Ordinary Differential Equations* by Vladimir I. Arnold.

These methods are crucial to solve a lot of classical equations in physics and we will extensively use them at several points of the lectures (e.g. when studying the harmonic oscillator, the heat equation, etc.).

For simplicity of exposition, all the functions we consider are complex valued and continuous on $\mathbb{R}$.

1. Scalar linear ODE of order 1

Suppose you want to solve the following equation :

$$\dot{y} = a(t)y + b(t),\ y(t=0) = y_0, \tag{1}$$

where a and b are continuous functions on $\mathbb{R}$. A first step is to look at the corresponding homegeneous equation

$$\dot{y} = a(t)y. \tag{2}$$

1.1. **General case.** We will look for solutions of the form $v(t) = u(t)e^{\int_0^t a(s)ds}$. One can verify that v is a solution of (2) if and only if $\dot{u} = 0$. Hence, any solution of (2) is of the form

$$v : t \mapsto Ce^{\int_0^t a(s)ds},$$

where C is a constant. We now want to determine the solution to equation (1). The first case is when you know some function ϕ satisfying $\dot{\phi} = a\phi + b$. In this case, the solution to equation (1) is given by

$$y(t) = (y_0 - \phi(0))e^{\int_0^t a(s)ds} + \phi(t).$$

If there is no obvious function satisfying ϕ satisfying $\dot{\phi} = a\phi + b$, we search for such ϕ of the form

$$\phi(t) = u(t)e^{\int_0^t a(s)ds}.$$

One can verify that ϕ satisfies $\dot{\phi} = a\phi + b$ if and only if $\dot{u} = b(t)e^{-\int_0^t a(s)ds}$. Hence, the solution to equation (1) is given by

$$y(t) = y_0 e^{\int_0^t a(s)ds} + \int_0^t b(\tau)e^{\int_\tau^t a(s)ds}d\tau. \tag{3}$$

1.2. **Particular case : $a(t)$ is a constant function.** In the case where $a(t) = a$ is a constant, the general solution to (1) is given by

$$y(t) = y_0 e^{at} + \int_0^t b(\tau)e^{a(t-\tau)}d\tau. \tag{4}$$

In the case where $b(t) = e^{\beta t}P(t)$ for some polynom P, it can be more easy to search directly for a particular solution of the form $\phi(t) = Q(t)e^{\beta t}$. One can show that such a solution exists with $\deg P = \deg Q$ if $\beta \neq a$ and with $\deg Q = \deg P + 1$ if $\beta = a$. One can find Q by identification and then the solution to (1) is given by

$$y(t) = (y_0 - Q(0))e^{at} + Q(t)e^{\beta t}.$$

Example. $a(t) = -1$ and $b(t) = 2e^t t^2$. We search a particular solution of the form $\phi(t) = Q(t)e^{\beta t}$ with $\deg Q = 2$. We find that $2Q + \dot{Q} = 2t^2$. Hence, by identification, one finds

$$Q(t) = t^2 - t + \frac{1}{2}.$$

In this case, the solution to (1) is given by

$$y(t) = \left(y_0 - \frac{1}{2}\right)e^{-t} + \left(t^2 - t + \frac{1}{2}\right)e^t.$$

2. Scalar linear ODE of order 2

Suppose you want to solve the following equation :

$$\ddot{y} = a(t)\dot{y} + b(t)y + c(t),\ y(t=0) = y_0,\ \dot{y}(t=0) = v_0, \tag{5}$$

where a, b and c are continuous functions on $\mathbb{R}$. A first step is to look at the corresponding homegeneous equation

$$\ddot{y} = a(t)\dot{y} + b(t)y. \tag{6}$$

2.1. **General case.** Suppose you are given two solutions v_1 and v_2 to equation (6) which satisfies

$$v_1(t_0)v_2'(t_0) - v_1'(t_0)v_2(t_0) \neq 0, \tag{7}$$

for some $t_0 \in \mathbb{R}$.

Example. In the case $a = 0$ and $b = -1$, one can take $v_1 = \cos t$ and $v_2 = \sin t$.

It is then possible to solve equation (5) as follows. First, if you know a particular function ϕ satisfying $\ddot{\phi} = a(t)\dot{\phi} + b(t)\phi + c(t)$, then the solution to equation (5) is given by

$$y(t) = \alpha v_1 + \beta v_2 + \phi,$$

where

$$\alpha v_1(0) + \beta v_2(0) + \phi(0) = x_0 \text{ and } \alpha \dot{v}_1(0) + \beta \dot{v}_2(0) + \dot{\phi}(0) = v_0.$$

In general, it is not easy to have an obvious function ϕ satisfying $\ddot{\phi} = a(t)\dot{\phi} + b(t)\phi + c(t)$ and one can verify that such a ϕ can be chosen of the following form

$$\phi(t) = \alpha(t)v_1(t) + \beta(t)v_2(t),$$

where α and β satisfy

$$\dot{\alpha} v_1 + \dot{\beta} v_2 = 0 \text{ and } \dot{\alpha}\dot{v}_1 + \dot{\beta}\dot{v}_2 = c.$$

Example. In the case $a = 0$ and $b = -1$, we have to solve the equation

$$\dot{\alpha}\cos t + \dot{\beta}\sin t = 0 \text{ and } -\dot{\alpha}\cos t + \dot{\beta}\sin t = c.$$

We find that

$$\dot{\alpha} = -c(t)\sin t \text{ and } \dot{\beta} = c(t)\cos t.$$

Hence, in this case, the solution to (5) is given by

$$y(t) = y_0 \cos t + v_0 \sin(t) + \int_0^t c(\tau)\sin(t-\tau)d\tau.$$

2.2. **Simplifications when there exists a nonvanishing solution.** Suppose you know a nonvanishing solution v_0 to equation (6). By nonvanishing, we mean that, for every t, $v_0(t) \neq 0$. Then, one can verify that $y = v_0 z$ satisfies

$$\ddot{y} = a(t)\dot{y} + b(t)y + c(t),$$

if and only if

$$\ddot{z} + \frac{2\dot{v}_0 + av_0}{v_0}\dot{z} = \frac{c}{v_0}.$$

Thus, it remains to solve an ODE of order 1 to find $\dot{z}$.

2.3. **Particular case : $a(t)$ and $b(t)$ are constant functions.** Suppose $a(t) = a$ and $b(t) = b$ are constant functions and $c(t)$ is a continuous function on $\mathbb{R}$.

One can search for solutions v_1 and v_2 to equation (6) which are of the form e^{rt}. Such solutions exist if and only if

$$r^2 - ar - b = 0. \tag{8}$$

In the case where (8) has two different roots r_1 and r_2, one can verify that

$$v_1(t) = e^{r_1 t} \text{ and } v_2(t) = e^{r_2 t}$$

satisfy condition (7) for $t_0 = 0$. In the case where there is only one root r_0, one can take

$$v_1(t) = e^{r_0 t} \text{ and } v_2(t) = te^{r_0 t}$$

which also satisfy condition (7) for $t_0 = 0$. Thus, in both case, one can apply the strategy from paragraph 2.1.

If we make the extra assumption that $c(t) = e^{\beta t}P(t)$ (with P a polynom), one can show that there exists a function $\phi(t)$ of the form $e^{\beta t}Q(t)$ (with Q a polynom) which satisfies

$$\ddot{\phi} = a\dot{\phi} + b\phi + c(t).$$

Moreover, one can take

- $\deg P = \deg Q$ if β is not a root of (8) ;
- $\deg P + 1 = \deg Q$ if β is a simple root of (8) ;
- $\deg P + 2 = \deg Q$ if β is a double root of (8).