Université des Sciences et Technologies de Lille 1
2014/2015 – Master degree in Mathematical Engineering
Refresher Course in Physics – Semester 3

Waves, Propagation Phenomena

Preliminary remark

Verify that a complex valued function ψ is a solution of

$$\Delta\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

if and only $\mathrm{Re}(\psi)$ and $\mathrm{Im}(\psi)$ are.

Exercise 1

Consider a conducting and vibrating string. Denote λ its mass by unit of length. Suppose that the motion takes place only the y and z directions. Suppose that there is a constant density of current j in the string and that there is a constant magnetic field B on the x direction.

Recall that the Laplace force on a piece of string $d\vec{l}$ is given by

$$\vec{F} = j d\vec{l} \wedge \vec{B}.$$

(1) Determine the propagation equation.

(2) Let $(y(x,t), z(x,t)) = (y_0 e^{\imath(kx-\omega t)}, z_0 e^{\imath(kx-\omega t)})$ be a solution of this equation. Determine the dispersion relation.

(3) Express y_0 in terms of z_0.

Exercise 2

Consider the membrane of a drum that is initially in the plane (xOy) and that it is slightly perturbed on the z axis (and not on the x and y axis). Denote μ the mass per unit of surface.

We want to establish the equation satisfied by $z(x,y,t)$.

Assumptions. The strength of the tension force at a point M is denoted $T(M)$. If a small piece dl of the membrane is torn around M, then one must exerce a force $T(M)dl$ on each part to close the hole. This force is directed on the normal of dl and tangent to the membrane of the drum.

We make the assumption that the weight force on the membrane is negligible.

Consider a small piece $dS = dxdy$ of the membrane.

(1) Write the Newton's law and deduce the values of $\frac{\partial T}{\partial x}$ and $\frac{\partial T}{\partial y}$.

(2) Derive that $T(x, y)$ is a constant that we will denote T_0.

(3) Show that z satisfies

$$\mu\frac{\partial^2 z}{\partial t^2} = T_0\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right).$$

Exercise 3

Consider the model of the vibrating string studied in the course. We make the additional assumption that a friction force of the form

$$d\vec{F} = -\vec{v}f(x)dx$$

acts on every piece dx of the string. We will use the same notations as in the course.

(1) Determine the equation equation satisfied by this system (it is called the damped wave equation).

(2) Suppose $f(x) = f_0 > 0$ is a constant and $\tilde{y}(x, t) = \tilde{y}_0 e^{\imath(kx-\omega t)}$ solves the damped wave equation. Determine the "dispersion" relation between k and ω.

(3) Suppose k is real. Show that ω has to be of the form $a + \imath b$ with $b < 0$.

Exercise 4

Consider the Klein-Gordon equation

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2} - \frac{m^2c^2}{\hbar^2}\psi.$$

(1) Determine the dispersion relation associated to this equation.

(2) Compute the phase velocity v_ϕ and the group velocity v_g. Comment.

Exercise 5

We consider the setting of exercise 2.

(1) Under which assumption on k_x, k_y and ω, $z(x, y, t) = \sin(k_x x)\sin(k_y y)\cos(\omega t)$ is a solution of

$$\frac{\partial^2 z}{\partial t^2} = \frac{1}{c^2}\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right).$$

(2) Suppose the drum is rectangular and delimited by the lines $x = 0$, $x = a$, $y = 0$ and $y = b$. Determine all the possible frequencies associated to the above solutions.

(3) Compare with the one dimensional case.

Exercise 6

We look at a simplified model for the wave propagation into a monoatomic cristal.

Consider an infinite sequence of equal mass m indexed by n in $\mathbb{Z}$ (each mass representing an atom). We will suppose that two consecutive mass na and $(n+1)a$ are coupled by a spring of constant k (independent of n). We denote a the length of the spring.

All the mass stay on the same line and we denote $u_n(t)$ the displacement at time t of the mass standing at the postion n.

(1) Show that, for every n, one has

$$m\frac{d^2u_n}{dt^2} = k(u_{n+1} + u_{n-1} - 2u_n).$$

(2) Justify the fact that it describes a propagation phenomenon.

(3) We search solutions of the form $u_n(t) = Ue^{\imath(nKa-\omega t)}$. Give an interpretation for U, K and ω.

(4) Show that

$$\omega^2 = \frac{4k}{m}\sin^2\left(\frac{Ka}{2}\right).$$

Draw ω in function of K.

(5) Prove that we can restrict ourselves to $-\frac{\pi}{a} \leq K \leq \frac{\pi}{a}$. What is the physical interpretation of the limit cases $K = \pm\frac{\pi}{a}$?

(6) Compute the group velocity. Comment the the limit cases $K = \pm\frac{\pi}{a}$.

(7) Describe what happens in the case $K = 0$.

Exercise 7 (Exam 2011)

We study a one-dimensional, diatomic crystalline solid. In its equilibrium state, it is modeled as

- an infinite number of atoms of mass m_1 located at $x = 2na$ on the (Ox) axis (where $n \in \mathbb{Z}$) ;
- an infinite number of atoms of mass m_2 located at $x = (2n+1)a$ on the (Ox) axis (where $n \in \mathbb{Z}$) ;
- a spring of constant k and length a between each mass m_1 and m_2.

The constants k and a are identical for every spring. The masses can only move on the (Ox)-axis.

We denote $u_n(t)$ the displacement of the mass initially located in $2na$ and $v_n(t)$ the displacement of the mass initially located in $(2n+1)a$.

We make the assumption that $m_1 > m_2$.

(1) Show that the motion is described, for every n in $\mathbb{Z}$, by
$$m_1\frac{d^2u_n}{dt^2} = k(v_n + v_{n-1} - 2u_n) \text{ and } m_2\frac{d^2v_n}{dt^2} = k(u_{n+1} + u_n - 2v_n).$$
(2) We search for solutions of the form
$$u_n(t) = Ue^{\imath(nKa-\omega t)} \text{ and } v_n(t) = Ve^{\imath(nKa-\omega t)}.$$
Deduce the dispersion relation.

(3) Explain why we can restrict ourselves to $K \in [-\pi/a, \pi/a]$.

(4) Compute the values of ω^2 for $K = \pm\frac{\pi}{a}$.

(5) Show that for $Ka \ll 1$, there are two dispersion relations
$$\omega^2 = 2k\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \text{ and } \omega^2 = \frac{k}{2(m_1+m_2)}K^2a^2.$$
(6) Represent the approximative shape of the dispersion relation $\omega = f(K) \geq 0$ for $K \in [-\pi/a, \pi/a]$.

(7) Describe the behavior of the atoms for $K = 0$.

(8) Show that there are two domains of pulsations for which there is no wave propagation.

(9) Suppose one of the atom in the chain is excited with such a pulsation. What happens ?

Exercise 8 (Exam 2012)

Consider a mass M and a finite string whose mass per unit of length is λ. Denote g the gravitational constant.

(1) Using these objects, recall how we derived the wave equation
$$\frac{\partial^2 y}{\partial t^2} - c^2\frac{\partial^2 y}{\partial x^2} = 0.$$
Express c in function of M, g and λ.

(2) Give the expression of the energy $E(t)$ of a solution $y(x,t)$.

(3) From this point of the problem, we will suppose that each piece dx of the string is submitted to a friction force of the form $-f_0 dx\vec{v}$, where $\vec{v}$ is the velocity of the piece dx and $f_0 > 0$ is constant. Show that the new equation is of the form
$$\frac{\partial^2 y}{\partial t^2} + 2a_0c^2\frac{\partial y}{\partial t} - c^2\frac{\partial^2 y}{\partial x^2} = 0.$$
Express a_0 in function of λ, c and f_0.

(4) Show that $E(t)$ is a nonincreasing function for $y(x,t)$ solving this new equation.

(5) Show that $E(t) \geq E(0)e^{-4a_0c^2t}$ for $t \geq 0$.

(6) Suppose $y(x,t)$ is of the form $g(x)h(t)$. Fix t and show that g must be of the form
$$g(x) = C\sin\left(\frac{n\pi x}{L}\right),$$
for some integer $n \geq 0$. What does L represent in our model ?

(7) Fix an integer $n \geq 1$. Show that there exist some constants A_n, B_n, α_n and β_n such that
$$h_n(t) = A_n e^{\alpha_n t} + B_n e^{\beta_n t}.$$

(8) Show that the real parts of α_n and β_n are < 0 and comment this result.

(9) Fix $n \geq 1$. Suppose
$$y_n(x, t=0) = y_n \sin\left(\frac{n\pi x}{L}\right) \text{ and } \frac{\partial y_n}{\partial t}(x, t=0) = v_n \sin\left(\frac{n\pi x}{L}\right).$$
Show that
$$y_n(x,t) = e^{-a_0 c^2 t}\left(\frac{v_n + y_n a_0 c^2}{\sqrt{\gamma_n}}\mathrm{sh}(\sqrt{\gamma_n}t) + y_n \mathrm{ch}(\sqrt{\gamma_n}t)\right)\sin\left(\frac{n\pi x}{L}\right),$$
where you will express γ_n in function of a_0, c, n and L.

(10) Show that there exist some constant $\gamma > 0$ and $C > 0$ independent of n such that
$$E(t) \leq Ce^{-\gamma t}E(0),$$
for any solution of the form $y_n(x,t)$. Conclude that $E(t) \leq Ce^{-\gamma t}E(0)$ for any solution $y(x,t)$ of the damped wave equation.

Exercise 9 (Exam 2013)

Consider the membrane of a *circular drum* which is centered at O in the (xOy)-plane. The radius of the drum is R and its mass per unit of surface is μ.

We neglect the friction forces and the force due do to gravitation. The only force that we take into account is the tension force. We make the assumption that the tension force on a small piece of drum dl located in $M = (x,y)$ is of strength
$$dF = T(x,y)dl,$$
where $T(x,y)$ does not depend on the time t. Moreover, the direction of the force is orthogonal to dl. We want to describe the motion of the membrane on the z-direction, i.e. describe
$$z(x,y,t),$$
which depends only on the variables (x,y,t). We will denote by (r,θ) the polar coordinates in the (xOy)-plane.

Finally, for any integer $n \in \mathbb{Z}$, we introduce the Bessel equation
$$u^2h''(u) + uh'(u) + (u^2 - n^2)h(u) = 0.$$
We will admit that this equation has a solution J_n which is well defined in $u = 0$.

(1) Write the Newton's laws on the (x, y, z) directions for a small piece of drum $dS = dxdy$.

(2) Deduce that $T(x, y)$ is a constant that we will denote T_0.

(3) Deduce that

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right),$$

and express c in terms of the physical quantities of the problem.

(4) What are the two limit conditions in $r = R$?

(5) We search for a solution of the form $z = \psi(r, \theta) \cos(\omega t)$. What property is satisfied by such a wave ?

(6) Show that

$$r \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) + \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\omega^2 r^2}{c^2} \psi = 0.$$

(7) What property satisfies $\psi(r, \theta)$ as a function of θ. Deduce that $\psi(r, \theta)$ can be written

$$\psi(r, \theta) = A_0(r) + \sum_{n=1}^{+\infty} A_n(r) \cos(n\theta) + A_{-n}(r) \sin(n\theta).$$

(8) Express A_k in terms of the Bessel function J_k.

(9) What are the restrictions on the values of ω ? For which physical reason these restrictions appear ?