Université des Sciences et Technologies de Lille 1
2014/2015 – Master degree in Mathematical Engineering
Refresher Course in Physics – Semester 3

Vectorial analysis (and a few words on the heat equation)

Except if we mention the contrary, we will work in the $3D$ euclidean space.

Exercise 1

Compute the volumes of the following sets :

(1) $A = \{(x, y, z) : \ 0 \le z \le a, \ 0 \le \sqrt{x^2 + y^2} \le b(z)\}$, where $b(z)$ is a function of z ;
(2) $A = \{(x, y, z) : \ a^2 \le x^2 + y^2 + z^2 \le b^2\}$, where $a, b \ge 0$;
(3) $A = \left\{(x, y, z) : \ x^2 + y^2 \le ax, \ x^2 + y^2 + z^2 \le a^2\right\}$.

Exercise 2

Give an explicit expression for the operators div, Δ and $\vec{\text{rot}}$ in cylindrical and spherical coordinates. Give also an expression for Δ for general (q_1, q_2, q_3) coordinates.

Exercise 3

Suppose that f and g are scalar fields and that $\vec{A}$ and $\vec{B}$ are vector fields. Verify that the following properties hold :

(1) $\vec{\text{grad}}(fg) = g\vec{\text{grad}}(f) + f\vec{\text{grad}}(g)$;
(2) $\text{div}(f\vec{A}) = f\text{div}(\vec{A}) + \vec{\text{grad}}(f).\vec{A}$;
(3) $\vec{\text{rot}}(f\vec{A}) = f\vec{\text{rot}}(\vec{A}) + \vec{\text{grad}}(f) \wedge \vec{A}$;
(4) $\text{div}(\vec{\text{rot}}\vec{A}) = 0$;
(5) $\vec{\text{rot}}(\vec{\text{grad}}f) = \vec{0}$;
(6) $\text{div}(\vec{A} \wedge \vec{B}) = \vec{B}.\vec{\text{rot}}(\vec{A}) - \vec{A}.\vec{\text{rot}}(\vec{B})$.

Exercise 4

(1) Give examples of scalar fields f such that $\Delta f = 0$.
(2) We now use spherical coordinates. Suppose there exists α and $F(\theta, \phi)$ such that $f = r^\alpha F(\theta, \phi)$ satisfies $\Delta f = 0$. Show that there exists $\beta \ne \alpha$ such that
$$\Delta\left(r^\beta F(\theta, \phi)\right) = 0.$$
(3) Using the previous question, give more examples of scalar fields f such that $\Delta f = 0$.

Exercise 5

Suppose $\vec{E}$ is a vector field. Prove the following properties :

(1) A potential V satisfying $\vec{E} = -\vec{\text{grad}}\ V$ is uniquely determined up to a constant.

(2) If $\vec{E}$ a vector field with conservative circulation satisfies a cylindrical symmetry with respect to the z-axis, then $\vec{E}.\vec{u}_\theta = 0$.

(3) If $\vec{E}$ satisfies a spherical symmetry, then it has conservative circulation.

(4) Suppose $\vec{E} = \vec{\text{rot}}\ \vec{A} = \vec{\text{rot}}\ \vec{A'}$. There exists V such that $\vec{A} = \vec{A'} - \vec{\text{grad}}\ V$.

(5) If a vector field $\vec{E}$ with conservative flux satisfies a cylindrical symmetry with respect to the z-axis, then $\vec{E}.\vec{u}_r = 0$.

(6) If a vector field $\vec{E}$ with conservative flux satisfies a spherical symmetry, then $\vec{E} = 0$.

Exercise 6

Consider S a surface in the plane (xOy). Suppose it has an oriented and closed boundary Γ (that does not intersect itself). Denote $\vec{E} = -y\vec{u}_x + x\vec{u}_y$ in euclidean coordinates.

(1) Show that the area of S is given by $\frac{1}{2}\int_\Gamma \vec{E}.d\vec{M}$.

(2) Compute the area of the surface delimited by the $(0x)$ axis and the curve $\gamma : t \mapsto (a(t - \sin t), a(1 - \cos t))$, where $0 \leq t \leq 2\pi$.

(3) Compute the area of $S = \{(r, \theta) :\ 0 \leq r \leq a(1 + \cos\theta)\}$.

Exercise 7

Let V be a vector field. Denote

$$\langle V\rangle = \frac{V(h,0,0) + V(-h,0,0) + V(0,h,0) + V(0,-h,0) + V(0,0,h) + V(0,0,-h)}{6}.$$

(1) Show that

$$\langle V\rangle \approx V(0) + \frac{h^2}{6}\Delta V(0).$$

(2) How can you interpret $\Delta V(0)$ regarding this approximation ?

(3) In electrostatics, if there is a density of charges ρ, then the associated potential V satisfies the Poisson equation

$$\Delta V = -\frac{\rho}{\epsilon_0}.$$

Interpret the qualitative meaning of this equation.

Exercise 8

Consider an infinite solid on the half space $x \geq 0$. Suppose that this solid is of constant thermal conductivity and capacity and that it is initially at a temperature T_0. For $t > 0$, we impose $T = T_1$ on the plane $x = 0$.

(1) What is the equation (with the limit conditions) satisfied by $\theta(x,t) = \frac{T-T_1}{T_0-T_1}$?

(2) Suppose θ is of the form $f\left(\frac{x}{2\sqrt{Dt}}\right)$, where $D = \frac{\kappa}{c_v}$. Determine the equation satisfied by f.

(3) Show that $f'(u) = Ae^{-u^2}$ and that T is of the form

$$T(x,t) = T_1 + (T_0 - T_1)\frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{2\sqrt{Dt}}} e^{-u^2}du.$$

(4) Comment this solution for small t.

Exercise 9

Consider a sphere of radius R which has constant thermal conductivity and capacity. At $t = 0$, the sphere is at temperature T_0 and for $t > 0$, we impose $T = T_1$ on the surface of the sphere.

(1) Justify that T is only a function of r (in spherical coordinates) and give the equation satisfied by T.

(2) Set $\psi(r,t) = r(T - T_1)$. What is the equation satisfied by ψ? What are the limit conditions for ψ at $t = 0$, $r = 0$ and $r = R$?

(3) Suppose $\psi(r,t)$ is of the form $f(r)g(t)$. Fix $t = t_0 > 0$ and prove that $\psi(r,t) = g(t)\sin\left(\frac{n\pi r}{R}\right)$ for some constant A and some integer n.

(4) Show that $g(t)$ is of the form $E_n e^{-\frac{n^2\pi^2}{R^2}Dt}$ and give an expression for D.

(5) Justify that a general solution $\psi(r,t)$ of our problem is of the form

$$\sum_{n\geq 1} E_n e^{-\frac{n^2\pi^2}{R^2}Dt}\sin\left(\frac{n\pi r}{R}\right).$$

Explain how one can find the values of E_n for $n \geq 1$.

(6) Comment the evolution of temperature for $t > 0$.