THE QUANTUM LOSCHMIDT ECHO ON FLAT TORI

GABRIEL RIVIÈRE AND HENRIK UEBERSCHÄR

ABSTRACT. The Quantum Loschmidt Echo is a measurement of the sensitivity of a quantum system to perturbations of the Hamiltonian. In the case of the standard 2-torus, we derive some explicit formulae for this quantity in the transition regime where it is expected to decay in the semiclassical limit. The expression involves both a two-microlocal defect measure of the initial data and the form of the perturbation. As an application, we exhibit a non-concentration criterium on the sequence of initial data under which one does not observe a macroscopic decay of the Quantum Loschmidt Echo. We also apply our results to several examples of physically relevant initial data such as coherent states and plane waves.

1. INTRODUCTION

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the standard[1] torus. Motivated by the fact that the quantum evolution is unitary and thus cannot be sensitive to perturbations of initial conditions, Peres suggested to study the sensitivity of the Schrödinger equation to perturbations of the Hamiltonian [23]. More precisely, he proposed to compare the dynamics induced by the following two semiclassical Schrödinger equations:

$$i\hbar\partial_t u_\hbar = -\frac{\hbar^2\Delta u_\hbar}{2}, \quad u_\hbar(t=0) = \psi_\hbar, \tag{1}$$

and

$$i\hbar\partial_t u_\hbar^\epsilon = -\frac{\hbar^2\Delta u_\hbar^\epsilon}{2} + \epsilon_\hbar V u_\hbar^\epsilon, \quad u_\hbar^\epsilon(t=0) = \psi_\hbar, \tag{2}$$

where V belongs to $\mathcal{C}^\infty(\mathbb{T}^2, \mathbb{R})$, $(\psi_\hbar)_{\hbar\to 0^+}$ is a normalized sequence in $L^2(\mathbb{T}^2)$ and $(\epsilon_\hbar)_{\hbar\to 0^+}$ satisfies

$$\lim_{\hbar\to 0^+} \epsilon_\hbar = 0.$$

In order to measure the difference between the two evolved systems, Peres used the so-called notion of quantum fidelity,

$$\mathcal{E}_{\hbar,\epsilon}(t) := \left|\langle u_\hbar^\epsilon(t), u_\hbar(t)\rangle_{L^2(\mathbb{T}^2)}\right|^2. \tag{3}$$

Here, $u_\hbar(t)$ represents the solution to (1) at time t and $u_\hbar^\epsilon(t)$ the solution to (2) at time t, both of them having the same initial condition which is normalized in $L^2(\mathbb{T}^2)$. This fidelity

Date: May 10, 2019.

[1]We consider the standard torus for simplicity and our analysis would extend to $\mathbb{R}^2/\Gamma$ with $\Gamma = a\mathbb{Z}\oplus b\mathbb{Z}$ with $a, b > 0$.

between pure states can be rewritten as

$$\mathcal{E}_{\hbar,\epsilon}(t) = \left| \left\langle \psi_\hbar, e^{\frac{it\widehat{P}_\epsilon(\hbar)}{\hbar}} e^{-\frac{it\widehat{P}_0(\hbar)}{\hbar}} \psi_\hbar \right\rangle_{L^2(\mathbb{T}^2)} \right|^2,$$

where we have set $\widehat{P}_0(\hbar) := -\frac{\hbar^2\Delta}{2}$, and $\widehat{P}_\epsilon(\hbar) := -\frac{\hbar^2\Delta}{2} + \epsilon_\hbar V$. From the physical point of view, this means that the quantum state $\psi_\hbar$ has been evolved under the Schrödinger flow of $\widehat{P}_0(\hbar)$ up to time t, then evolved back up to time $-t$ by a slightly perturbed quantum Hamiltonian $\widehat{P}_\epsilon(\hbar)$ and compared with the initial state $\psi_\hbar$. Under this form, it is now often referred to as the Quantum Loschmidt Echo in the physics literature. Peres predicted that this quantity should decay for *any* quantum system and that the rate of decay should depend on the dynamical properties of the underlying classical Hamiltonian. He motivated his conjecture with numerical simulations and with the following formal asymptotic expansion:

$$\mathcal{E}_{\hbar,\epsilon}(t) \simeq 1 - \frac{\epsilon_\hbar^2 t^2}{\hbar^2}\left(\langle \psi_\hbar, V^2\psi_\hbar\rangle - \langle \psi_\hbar, V\psi_\hbar\rangle^2\right) + \dots. \quad (4)$$

Hence, for *any* type of Hamiltonian, he deduced that, for short times $t \ll \tau_\hbar^c := \frac{\hbar}{\epsilon_\hbar}$, the quantity $\mathcal{E}_{\hbar,\epsilon}(t)$ should follow some quadratic decay. This first approximation just relies on the fact that the quantum fidelity can be well approximated by the Taylor expansion. After that, the Quantum Loschmidt Echo is expected to continue its decay at a rate that will now also depend on the dynamical features of the classical Hamiltonian, namely chaotic vs. integrable. This kind of general behaviour seems to be commonly accepted in the literature on the subject – see for example [14, Sect. 2.1.1] or [15, Sect. 2.3.1]. Note that the rates of decay (after this short time regime) depend in a subtle manner on the parameters $\hbar$ and $\epsilon_\hbar$ but also on the choice of initial data and of V. For much larger time scales, we refer for instance to paragraphs 2.3.1 and 2.3.2 in [15] for a brief review of the different possible regimes and references to the literature – see also [16]. Observe that $\epsilon_\hbar$ also implicitly depends on $\hbar$ in these different regimes, as it is often compared with the size of the mean level spacing, which is $\hbar^2$ in dimension 2. We shall not discuss these questions here and we will mostly focus on the case of the transition regime $t \approx \tau_\hbar^c$ for which one should already observe some decay of the Quantum Loschmidt Echo. We emphasize that we are mainly concerned with the case of the free Schrödinger evolution on the 2-torus which is the simplest example of an *integrable* system. In this dynamical framework, the situation is known to be quite subtle and many phenomena may occur – e.g. see [8] and the references therein.

One of the main consequences of our analysis will be to exhibit large classes of semiclassical initial data for which we do not have any *macroscopic decay* of the Quantum Loschmidt Echo in this transition regime – see section 4 below. By macroscopic decay, we roughly mean that the Quantum Loschmidt Echo would give something < 1 in the semiclassical limit. More precisely, for any time $t\tau_\hbar^c$ with $t \in \mathbb{R}$ fixed, we will verify that the Quantum Loschmidt Echo will tend to 1 in the semiclassical limit for large families of initial data. Note that this does not exclude the possibility that $\mathcal{E}_{\hbar,\epsilon}(t)$ is strictly less than

1 but rather that the deviation from 1 is asymptotically small in the semiclassical regime without trying to be quantitative in the size of the deviation. Our strategy is to apply to this physical problem the 2-microlocal techniques recently developed by Anantharaman and Macià for the study of controllability and of semiclassical measures on flat tori [18, 4] – see also [2, 1, 3, 20] for related results on integrable systems. In particular, our analysis will be highly dependent on the *integrable structure of our Hamiltonian.*

Note also that our results will be valid for a certain regime of small perturbations[2]:

$$\hbar^2 \ll \epsilon_\hbar \ll \hbar. \tag{5}$$

In fact, the case of stronger perturbations $\epsilon_\hbar \geq \hbar$ can be easily treated for any compact Riemannian manifold – see section 3. Here, we manage to deal with smaller perturbations due to the specific structure of the torus. The requirement $\hbar^2 \ll \epsilon_\hbar$ comes from the fact that we will analyse the properties of our semiclassical states in regions of phase space whose size will be of order $\epsilon_\hbar \hbar^{-1}$, hence $\gg \hbar$ thanks to (5). This will allow us to deal with symbols that are still amenable to pseudodifferential calculus [27]. It is plausible that the case $\epsilon_\hbar \sim \hbar^2$ can also be handled by microlocal techniques but the situation would be more subtle. Our results do not a priori extend to other types of quantum systems, where different phenomena may occur. Finally, we emphasize that, even if the Quantum Loschmidt Echo is now a rather well studied and understood quantity in the physics literature, much less seems to be known from the mathematical perspective. For recent mathematical results we refer the reader to [5, 7] for $\mathbb{R}^d$, to [10, 6] for general compact manifolds, to [9, 24] for negatively curved surfaces and to [19] for Zoll manifolds.

2. Main results

We will now state our results more precisely. First of all, we emphasize that we will deal with semiclassical sequences of initial data. More precisely, in the following, we shall always suppose that $(\psi_\hbar)_{0<\hbar\leq 1}$ is a family of initial data which is *normalized* in $L^2(\mathbb{T}^2)$ and which satisfies

$$\lim_{\delta\to 0^+} \limsup_{\hbar\to 0^+} \left\| \mathbf{1}_{[0,\delta]}(-\hbar^2\Delta)\psi_\hbar \right\|_{L^2(\mathbb{T}^2)} = 0, \tag{6}$$

and

$$\lim_{R\to +\infty} \limsup_{\hbar\to 0^+} \left\| \mathbf{1}_{[R,+\infty[}(-\hbar^2\Delta)\psi_\hbar \right\|_{L^2(\mathbb{T}^2)} = 0. \tag{7}$$

Equivalently, the sequence of initial data *oscillates* at the frequency $\hbar^{-1}$. Here, the semiclassical parameter $\hbar \to 0^+$ is the one appearing in the Schrödinger equations (1) and (2). Our main result is to establish an explicit formula for the Quantum Loschmidt Echo at the critical time scale

$$\tau_\hbar^c := \frac{\hbar}{\epsilon_\hbar}, \tag{8}$$

[2]In all the article, we say that $f_1(\hbar) \ll f_2(\hbar)$ for two sequences $f_1(\hbar), f_2(\hbar) > 0$ if $\lim_{\hbar\to 0^+} f_1(\hbar) f_2(\hbar)^{-1} = 0$.

in terms of the initial conditions $(\psi_\hbar)_{0<\hbar\leq 1}$. Due to (5), this time scale tends to $+\infty$ in the semiclassical limit and it is always much smaller than the Heisenberg time $\hbar^{-1}$ from the physics literature. Recall that, in dimension 2, the mean level spacing δE between quantum states is of order $\hbar^2$, and, by the uncertainty principle, the corresponding scale of times δt is of order $\hbar^{-1}$.

In order to state our main result, we need to fix some conventions. We denote by $\mathcal{L}_1$ the family of all primitive rank 1 sublattices of $\mathbb{Z}^2$. Recall that a sublattice Λ is said to be primitive if $\langle\Lambda\rangle\cap\mathbb{Z}^2=\Lambda$, where $\langle\Lambda\rangle$ is the subspace of the momentum space $\mathbb{R}^2$ spanned by Λ. Moreover, it is of rank 1 if $\langle\Lambda\rangle$ is one dimensional. Any such lattice is generated by an element $\vec{v}_\Lambda$ of $\mathbb{Z}^2$ such that $\mathbb{Z}\vec{v}_\Lambda=\Lambda$. Denote then by $\vec{v}_\Lambda^\perp$ the lattice vector which is directly orthogonal to $\vec{v}_\Lambda$ and which has the same length $L_\Lambda:=\|\vec{v}_\Lambda\|=\|\vec{v}_\Lambda^\perp\|$. We then introduce two Hamiltonian functions associated with Λ:

$$\forall\xi\in\mathbb{R}^2,\ H_\Lambda(\xi):=\frac{1}{L_\Lambda}\langle\xi,\vec{v}_\Lambda\rangle\text{ and }H_\Lambda^\perp(\xi):=\frac{1}{L_\Lambda}\langle\xi,\vec{v}_\Lambda^\perp\rangle.$$

This defines a completely integrable system and the flow corresponding to $H_\Lambda^\perp$ is defined by

$$\varphi^t_{H_\Lambda^\perp}(x,\xi):=\left(x+t\frac{\vec{v}_\Lambda^\perp}{L_\Lambda},\xi\right). \tag{9}$$

Note that this flow is L_Λ-periodic. Using these conventions, we can define the following map

$$\mathbf{F}_{\Lambda,\hbar}:c\in\mathcal{C}^\infty_c(\mathbb{T}^2\times\mathbb{R})\mapsto\langle\psi_\hbar,\mathrm{Op}_\hbar\left(c_{\Lambda,\hbar}\right)\psi_\hbar\rangle,$$

where

$$c_{\Lambda,\hbar}(x,\xi):=\frac{1}{L_\Lambda}\int_0^{L_\Lambda}c\circ\varphi^t_{H_\Lambda^\perp}\left(x,\frac{\hbar H_\Lambda(\xi)}{\epsilon_\hbar}\right)dt,$$

and where $\mathrm{Op}_\hbar$ is the standard quantization – see appendix A. Observe that $c_{\Lambda,\hbar}$ has only Fourier coefficients (in the x variable) along Λ. Roughly speaking, these quantities measure the concentration of the initial data in an $\frac{\epsilon_\hbar}{\hbar}$-neighborhood of $\Lambda^\perp:=\mathbb{R}\vec{v}_\Lambda^\perp$ in momentum space. From our assumption (5), the quantity $\frac{\epsilon_\hbar}{\hbar}$ goes to 0, but not faster than $\hbar$. Up to an extraction, we can suppose that, for every $\Lambda\in\mathcal{L}_1$, there exists a finite positive measure[3] $\mathbf{F}^0_\Lambda$ such that, for every $c\in\mathcal{C}^\infty_c(\mathbb{T}^2\times\mathbb{R})$,

$$\lim_{\hbar\to0^+}\langle\mathbf{F}_{\Lambda,\hbar},c\rangle=\langle\mathbf{F}^0_\Lambda,c\rangle. \tag{10}$$

Hence, the measure $\mathbf{F}^0_\Lambda$ describes the part of the mass which is asymptotically concentrated along $\Lambda^\perp$. These are rescaled versions of the so-called semiclassical measures [13, 27]. It turns out that the asymptotic properties of the Quantum Loschmidt Echo are in fact related to these quantities:

[3] We refer to paragraph 6 for further precisions on the regularity of these objects.

Theorem 2.1. *Suppose that* (5) *holds, and that we are given a sequence of normalized initial data* $(\psi_\hbar)_{\hbar\to 0^+}$ *satisfying* (6) *and* (7) *which generates a unique family* $(\mathbf{F}^0_\Lambda)_{\Lambda\in\mathcal{L}_1}$. *Then, for every* $t\in\mathbb{R}$, *one has*

$$\begin{aligned}\lim_{\hbar\to 0^+}\langle u^\epsilon_\hbar(t\tau^c_\hbar), u_\hbar(t\tau^c_\hbar)\rangle &= e^{it\int_{\mathbb{T}^2}V}\left(1-\sum_{\Lambda\in\mathcal{L}_1}\langle \mathbf{F}^0_\Lambda,1\rangle\right)\\ &+\sum_{\Lambda\in\mathcal{L}_1}\int_{\mathbb{T}^2\times\mathbb{R}} e^{i\int_0^t \mathcal{I}_\Lambda(V)\left(x+s\frac{\eta\vec{v}_\Lambda}{L_\Lambda}\right)ds}\mathbf{F}^0_\Lambda(dx,d\eta),\end{aligned}$$

where $u_\hbar(t')$ *(resp.* $u^\epsilon_\hbar(t')$*) is the solution at time* t' *to* (1) *(resp.* (2)*).*

This Theorem provides an explicit formula for the Quantum Loschmidt Echo at the transition regime $\tau^c_\hbar$. It is in some sense slightly more precise as we compute the overlap $\langle u^\epsilon_\hbar(t\tau^c_\hbar), u_\hbar(t\tau^c_\hbar)\rangle$ and not only its modulus. If $\mathbf{F}^0_\Lambda$ identically vanishes for any Λ in $\mathcal{L}_1$, then this Theorem shows

$$\lim_{\hbar\to 0^+}|\langle u^\epsilon_\hbar(t\tau^c_\hbar), u_\hbar(t\tau^c_\hbar)\rangle|^2=1.$$

The assumption $\mathbf{F}^0_\Lambda\equiv 0$ for every $\Lambda\in\mathcal{L}_1$ exactly means that the initial data do not concentrate too fast near the invariant tori where the geodesic flow is periodic. Again, this does not mean that the Quantum Loschmidt Echo does not decay but it rather states that the deviation from 1 is asymptotically small. In section 4, we will apply this result to standard families of initial data such as coherent states and plane waves. In that manner, we will illustrate the many possibilities for the behaviour of the Quantum Loschmidt Echo for integrable systems.

Note that a similar statement was obtained by Macià and the first author in the case of *degenerate* integrable systems, like the geodesic flow on the sphere [19, Sect. 5]. In that framework, the case of smaller perturbations could be treated up to the scales $\epsilon_\hbar\gg\hbar^3$. Compared with that reference, the situation here is more complicated from a dynamical point of view as there are directions where the Hamiltonian flow is periodic (with a period that depends on the direction) and other ones where the geodesic flow fills the torus. In order to deal with these different behaviours, the main additional ingredient compared with [19] will be to introduce two-microlocal objects as in [18, 4, 1], namely $(\mathbf{F}^0_\Lambda)_{\Lambda\in\mathcal{L}_1}$. These quantities will capture the properties of the Schrödinger evolution near directions where the classical Hamiltonian flow is periodic while, away from these directions, we will be able to use equidistribution of the geodesic flow. Finally, we emphasize that the regime of perturbations we consider here is the same as in [20] which describes the structure of semiclassical measures for the Schrödinger equation (2). However, the two problems require in fact a second microlocalization at different scales and they yield propagation laws given by different two-microlocal quantities.

Organization of the article. Section 3 is devoted to the simpler case of strong perturbations $\epsilon_\hbar\geq\hbar$. In section 4, we start by applying Theorem 2.1 to families of relevant initial data. In section 5, we introduce families of distributions on the cotangent bundle $T^*\mathbb{T}^2$ that are close to the so-called Wigner distributions. Yet, as they are of slightly different nature, we review some of their basic properties such as invariance under the geodesic flow following the classical arguments from [13, 17, 27]. We also relate them to the Quantum Loschmidt Echo at the critical time scale, and we reduce the problem to an analysis of their restriction to rational directions. Then, in section 6, we define the two-microlocal framework needed to analyze the behaviour along rational directions. The proof of Theorem 2.1 is given in section 7. Finally, appendix A provides a short toolbox of semiclassical analysis on $\mathbb{T}^2$.

Conventions. All along the article, we use, for every u and v in $L^2(\mathbb{T}^2)$,

$$\langle u, v\rangle_{L^2(\mathbb{T}^2)} := \int_{\mathbb{T}^2} \overline{u}(x)v(x)dx.$$

We will work with smooth (or regular enough) test functions defined on different spaces. In order to facilitate the reading, we will use the following conventions. The symbol $a(x,\xi)$ will be used to designate functions on $T^*\mathbb{T}^2 \simeq \mathbb{T}^2\times\mathbb{R}^2$, $b(x,\xi,\eta)$ for functions on $T^*\mathbb{T}^2\times\mathbb{R} \simeq T^*\mathbb{T}^2\times\langle\Lambda\rangle$ (or more generally $T^*\mathbb{T}^2\times\widehat{\mathbb{R}}$), $c(x,\eta)$ for functions on $\mathbb{T}^2\times\mathbb{R}$ (or more generally $\mathbb{T}^2\times\widehat{\mathbb{R}}$) and $f(x)$ for functions on $\mathbb{T}^2$. In some places we will also need to introduce functions depending on an extra time variable $t\in\mathbb{R}$ and we will use the following conventions $\tilde{a}(t,x,\xi)$, $\tilde{b}(t,x,\xi,\eta)$, etc. For every such function, we will write its Fourier decomposition in the x variables. For instance, given a regular enough function b defined on $T^*\mathbb{T}^2\times\mathbb{R}$, we write

$$b(x,\xi,\eta) = \sum_{k\in\mathbb{Z}^2}\widehat{b}(k,\xi,\eta)e^{2i\pi k.x}.$$

Then, given a sublattice $\Lambda\in\mathcal{L}_1$, we will write

$$\mathcal{I}_\Lambda(b)(x,\xi,\eta) = \frac{1}{L_\Lambda}\int_0^{L_\Lambda} b\left(x+s\frac{\vec{v}_\Lambda^\perp}{L_\Lambda},\xi,\eta\right)ds = \sum_{k\in\Lambda}\widehat{b}(k,\xi,\eta)e^{2i\pi k.x}. \tag{11}$$

The same can obviously be defined for functions $a(x,\xi)$, $c(x,\eta)$ or $f(x)$. Given a function $b(x,\xi,\eta)$ defined on the space $T^*\mathbb{T}^2\times\mathbb{R}$ (or $c(x,\eta)$ on $\mathbb{T}^2\times\mathbb{R}$), we will identify it with a function on $T^*\mathbb{T}^2$ by the following operations. Let $\Lambda\in\mathcal{L}_1$ and $\delta>0$, we define

$$D_{\Lambda,\delta}(b)(x,\xi) = b\left(x,\xi,\frac{H_\Lambda(\xi)}{\delta}\right). \tag{12}$$

The definition can naturally be extended to functions c defined on $\mathbb{T}^2\times\mathbb{R}$. As an illustration, with these conventions, we can rewrite

$$\forall c\in\mathcal{C}_c^\infty(\mathbb{T}^2\times\mathbb{R}),\ \langle\mathbf{F}_{\Lambda,\hbar},c\rangle = \left\langle\psi_\hbar,\mathrm{Op}_\hbar\left(D_{\Lambda,\epsilon_\hbar\hbar^{-1}}(\mathcal{I}_\Lambda(c))\right)\psi_\hbar\right\rangle.$$

Finally, we will use the following convention. Given a Radon measure μ defined on the space Y, we will write $\mu(y)$ to designate the measure and to remind the reader that it depends on the variable y. When we will integrate a function g against μ, we will write $\int_Y g(y)\mu(dy)$ but it does not mean that μ is absolutely continuous with respect to some natural Lebesgue measure on the space Y.

Acknowledgements. Part of this work was carried out when the second author was a postdoc of the Labex CEMPI program (ANR-11-LABX-0007-01). The first author is also partially supported by the Agence Nationale de la Recherche through the Labex CEMPI and the ANR project GERASIC (ANR-13-BS01-0007-01). The authors warmly thank Rémi Dubertrand for an interesting discussion related to the Quantum Loschmidt Echo for integrable systems. The first author is also grateful to Fabricio Macià for many explanations about his works [18, 4, 1]. Finally, we express our deep gratitude to the referee and to the editors for their suggestions, their detailed comments and their useful feedbacks.

3. The case of strong perturbations: $\epsilon_\hbar \geq \hbar$

All along the article, we will focus on the case where $\epsilon_\hbar$ satisfies (5). In particular, we will suppose that $\epsilon_\hbar \ll \hbar$ which ensures that certain invariance properties are satisfied – see e.g. Lemma 5.2 below. Before doing so, we will briefly explain what can be said when the strength of the perturbation is stronger, i.e.

$$\hbar \leq \epsilon_\hbar \leq 1.$$

In that case, one has $\tau_\hbar^c = \frac{\hbar}{\epsilon_\hbar} \leq 1$ and we are in a scale of times where semiclassical rules for pseudodifferential operators apply. Precisely, we prove:

Theorem 3.1. *Denote $u_\hbar(t')$ (resp. $u_\hbar^\epsilon(t')$) the solution to* (1) *(resp.* (2)*) with initial condition $\psi_\hbar$ satisfying* (7) *and such that, for every $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^2)$,*

$$\lim_{\hbar\to 0^+} \langle \psi_\hbar, \mathrm{Op}_\hbar(a)\psi_\hbar\rangle = \langle F_0, a\rangle.$$

Then, one has

(1) *if $\epsilon_\hbar = \delta\hbar$ with $\delta > 0$ fixed, then, for every $t \in \mathbb{R}$,*

$$\lim_{\hbar\to 0^+} \left|\langle u_\hbar^\epsilon(t\tau_\hbar^c), u_\hbar(t\tau_\hbar^c)\rangle_{L^2(\mathbb{T}^2)}\right|^2 = \left|\langle F_0, e^{i\delta\int_0^{\frac{t}{\delta}} V\circ\varphi^s ds}\rangle\right|^2,$$

(2) *if $\hbar \ll \epsilon_\hbar \leq 1$, then, for every $t \in \mathbb{R}$,*

$$\lim_{\hbar\to 0^+} \left|\langle u_\hbar^\epsilon(t\tau_\hbar^c), u_\hbar(t\tau_\hbar^c)\rangle_{L^2(\mathbb{T}^2)}\right|^2 = \left|\langle F_0, e^{itV}\rangle\right|^2.$$

The existence of F_0 is guaranteed up to an extraction by the Calderón-Vaillancourt Theorem – see appendix. The theory of semiclassical measures [27, Ch. 5] shows that F_0 is a probability measure carried on $T^*\mathbb{T}^2$. Note also that, as $\delta \to 0^+$, one has

$$\delta \int_0^{\frac{t}{\delta}} V \circ \varphi^s ds \to tV^*(x,\xi),$$

where $V^*(.,\xi)$ is a smooth function on $\mathbb{T}^2$ for every $\xi \in \mathbb{R}^2$ which can be expressed in terms of the functions $\mathcal{I}_\Lambda(V)$ with Λ depending on ξ. Hence, in the limit of small perturbation and for a fixed time, we see that the Birkhoff average of the potential appears in the description of the Quantum Loschmidt Echo. Our main Theorem will make this more precise when δ goes to 0 in the semiclassical limit.

Proof. Here, we focus on the case of the 2-dimensional torus as we shall do in the rest of the article. Yet, the proof we give can be adapted verbatim to any smooth compact Riemannian manifold (M, g) while this will not be the case for smaller perturbations. We can write

$$\langle u_\hbar^\epsilon(t\tau_\hbar^c), u_\hbar(t\tau_\hbar^c)\rangle = \left\langle \psi_\hbar, e^{\frac{it\tau_\hbar^c \widehat{P}_\epsilon(\hbar)}{\hbar}} e^{-\frac{it\tau_\hbar^c \widehat{P}_0(\hbar)}{\hbar}} \psi_\hbar \right\rangle.$$

Compared with the rest of the article, we have here $\tau_\hbar^c = \frac{\hbar}{\epsilon_\hbar} \leq 1$. In particular, we can apply the Egorov Theorem to determine the value of $F(t)$ where the only difference with the classical case is that we have $e^{\frac{it\tau_\hbar^c \widehat{P}_\epsilon(\hbar)}{\hbar}}$ on one side and $e^{-\frac{it\tau_\hbar^c \widehat{P}_0(\hbar)}{\hbar}}$ on the other. Yet, the classical proof can be adapted to encompass this case[4] [25, p. 71-72]. More precisely, we have

$$e^{\frac{it\tau_\hbar^c \widehat{P}_\epsilon(\hbar)}{\hbar}} e^{-\frac{it\tau_\hbar^c \widehat{P}_0(\hbar)}{\hbar}} = \mathrm{Op}_\hbar\left(e^{i\frac{\epsilon_\hbar}{\hbar}\int_0^{t\tau_\hbar^c} V\circ\varphi^s ds}\right) + o_{L^2\to L^2}(1).$$

This yields

- if $\epsilon_\hbar = \delta\hbar$, then one has
$$\lim_{\hbar\to 0^+} \langle u_\hbar^\epsilon(t\tau_\hbar^c), u_\hbar(t\tau_\hbar^c)\rangle = \left\langle F_0, e^{i\delta\int_0^{\frac{t}{\delta}} V\circ\varphi^s ds} \right\rangle.$$
- if $\hbar \ll \epsilon_\hbar \leq 1$, then one has
$$\lim_{\hbar\to 0^+} \langle u_\hbar^\epsilon(t\tau_\hbar^c), u_\hbar(t\tau_\hbar^c)\rangle = \left\langle F_0, e^{itV} \right\rangle.$$

□

4. Examples of initial data

The purpose of this section is to apply our results to specific examples of initial data that are frequently discussed in the physics literature: plane waves and coherent states (or superposition of such states). The goal of these examples is to illustrate the variety of behaviour that may occur. For most of the cases, we shall verify that the Quantum Loschmidt Echo is in fact asymptotically equal to 1 in the transition regime. Recall that one expects a decay for any type of quantum system. Still, in certain specific cases, where the quantum state concentrates along periodic trajectories in phase space, we may observe different phenomena, e.g. polynomial decay for certain families of plane waves or revivals for a superposition of coherent states.

In order to describe these various phenomena, we have to calculate the distributions $\mathbf{F}_\Lambda^0$ for all primitive rank 1 sublattices Λ. Then, we shall apply the time evolution formula given by Theorem 2.1 in order to derive explicit expressions for the Quantum Loschmidt Echo for these sequences of initial data. It turns out that these distributions will be nontrivial only if the corresponding sequence of wave vectors , when projected on $\mathbb{S}^1$, converges to a rational direction at a certain rate. In other words, the sequence of initial data is concentrated near the tori of periodic orbits of the unperturbed Hamiltonian flow. In some sense, our series

[4] The proof in that reference is given for V of the form iW with W real valued but it can be adapted to treat the selfadjoint case which is actually simpler to deal with.

of examples illustrates that, for most initial data, we do not observe a macroscopic decay for times of order $\tau_\hbar^c$.

In order to state our results, we define the set of "rational" unit vectors as

$$\mathbb{S}^1_{\mathbb{Q}} := \{\vec{v}_\Lambda / L_\Lambda \mid \Lambda \in \mathcal{L}_1\}.$$

Again, this corresponds to the directions where the classical Hamiltonian flow is periodic. We also introduce the convenient notation $\vec{v}_\Lambda/L_\Lambda = (\cos\alpha_\Lambda, \sin\alpha_\Lambda)$ for some unique $\alpha_\Lambda \in [0, 2\pi)$.

4.1. **Plane waves.** Let us consider the initial data $\psi_\hbar(x) = e^{2\pi i k x} =: e_k(x)$ for a sequence of lattice vectors $k \in \mathbb{Z}^2$, $\|k\| \to \infty$. In that case, we choose

$$\hbar = \hbar_k = \|k\|^{-1}.$$

In this entire paragraph, we suppose that there exists $\vec{v} \in \mathbb{S}^1$ such that $k/\|k\| \to \vec{v}$ as $\|k\| \to \infty$. According to Appendix A, one has, for $c \in C_c^\infty(\mathbb{T}^2 \times \mathbb{R})$ and for $\Lambda \in \mathcal{L}_1$,

$$\langle \mathbf{F}_{\Lambda,\hbar}, c\rangle = \langle \psi_\hbar, \mathrm{Op}_\hbar(D_{\Lambda,\epsilon_\hbar \hbar^{-1}}(\mathcal{I}_\Lambda(c)))\psi_\hbar\rangle = \widehat{c}(0, 2\pi\hbar^2\epsilon_\hbar^{-1} H_\Lambda(k)), \tag{13}$$

where $c(x,\eta) = \sum_{l\in\mathbb{Z}^2} \widehat{c}(l,\eta) e^{2i\pi l.x}$ and where we refer to (11) and (12) for our different conventions.

4.1.1. *Rational directions.* We start with the case, where the limit vector $\vec{v}$ belongs to $\mathbb{S}^1_{\mathbb{Q}}$. In that case, the following holds:

Proposition 4.1. *Let $(k := n(\hbar)\vec{v}_{\Lambda_0} + m(\hbar)\vec{v}_{\Lambda_0}^{\perp})_{\hbar\to 0^+}$ for some $\Lambda_0 \in \mathcal{L}_1$ and with $|m(\hbar)| \ll 1/\hbar$. In particular, $\vec{v} = \vec{v}_{\Lambda_0}/L_{\Lambda_0}$. Then, the following holds:*

(1) *If $2\pi m(\hbar)\hbar^2\epsilon_\hbar^{-1} \to \omega \in \mathbb{R}$, then*

$$\mathbf{F}^0_\Lambda(x,\eta) = \begin{cases} \delta_{\omega L_{\Lambda_0}}(\eta) & \text{if } \langle\Lambda\rangle = \Lambda_0^\perp, \\ 0 & \text{otherwise.} \end{cases}$$

(2) *If $2\pi|m(\hbar)|\hbar^2\epsilon_\hbar^{-1} \to +\infty$, we have*

$$\forall \Lambda \in \mathcal{L}_1,\ \mathbf{F}^0_\Lambda = 0.$$

Proof. We have $H_\Lambda(k) = mL_{\Lambda_0}$ if $\langle\Lambda\rangle = \Lambda_0^\perp$. Therefore, in that case, the term $\hbar^2\epsilon_\hbar^{-1}H_\Lambda(k)$ appearing in (13) may remain bounded as $\hbar \to 0$. If $2\pi m\hbar^2\epsilon_\hbar^{-1} \to \omega$, we obtain, for every $a \in C_c^\infty(\mathbb{T}^2 \times \mathbb{R})$,

$$\langle \mathbf{F}_{\Lambda,\hbar}, c\rangle = \widehat{c}(0, 2\pi\hbar^2\epsilon_\hbar^{-1}H_\Lambda(k)) \to \widehat{c}(0, \omega L_{\Lambda_0}) = \int_{\mathbb{T}^2\times\mathbb{R}} c(x,\eta)\delta_{\omega L_{\Lambda_0}}(d\eta)dx.$$

On the other hand, if $m\hbar^2\epsilon_\hbar^{-1} \to \infty$, then, one finds

$$\langle \mathbf{F}_{\Lambda,\hbar}, c\rangle = \widehat{c}(0, 2\pi\hbar^2\epsilon_\hbar^{-1} m L_{\Lambda_0}) = 0,$$

for $\hbar$ small enough since c is compactly supported. It remains to discuss the case where $\langle\Lambda\rangle \neq \Lambda_0^\perp$. In that case, we have

$$\hbar^2\epsilon_\hbar^{-1}|H_\Lambda(k)| \gtrsim \hbar^2\epsilon_\hbar^{-1}|n(\hbar)| \to \infty,$$

since $\|k\| = \hbar^{-1}$, $m(\hbar) = o(\hbar^{-1})$ and $\epsilon_\hbar \ll \hbar$. Hence, we can apply (13) one more time to conclude. $\square$

4.1.2. *Irrational directions.* Let us now consider the case where $\vec{v}$ belongs to $\mathbb{S}^1 \setminus \mathbb{S}^1_{\mathbb{Q}}$:

Proposition 4.2. *Let $k = (m(\hbar), n(\hbar))_{\hbar\to 0^+}$ be a sequence of lattice points in $\mathbb{Z}^2$ such that*

$$\vec{v} = (\cos\alpha, \sin\alpha) \notin \mathbb{S}^1_{\mathbb{Q}}.$$

Then, one has, for all $\Lambda \in \mathcal{L}_1$,

$$\mathbf{F}^0_\Lambda = 0.$$

Proof. For a given Λ, one has

$$\hbar H_\Lambda(k) = \hbar \langle k, \vec{v}_\Lambda / L_\Lambda \rangle \to (\cos\alpha\cos\alpha_\Lambda + \sin\alpha\sin\alpha_\Lambda).$$

Therefore, as $\hbar$ goes to 0,

$$\frac{\hbar^2}{\epsilon_\hbar} H_\Lambda(k) \sim \frac{\hbar}{\epsilon_\hbar}\cos(\alpha_\Lambda - \alpha),$$

and it follows that $\hbar^2\epsilon_\hbar^{-1}|H_\Lambda(k)| \to +\infty$ because $\hbar/\epsilon_\hbar \to +\infty$ and $\cos(\alpha_\Lambda - \alpha) \neq 0$, since $\alpha \notin \mathbb{S}^1_{\mathbb{Q}}$. Thanks to (13), this implies $\langle \mathbf{F}_{\Lambda,\hbar}, c\rangle = \widehat{c}(0, 2\pi\hbar^2\epsilon_\hbar^{-1}H_\Lambda(k)) \to 0$ as $\hbar \to 0$. Hence $\mathbf{F}^0_\Lambda = 0$ for all $\Lambda \in \mathcal{L}_1$. $\square$

4.1.3. *Superposition of two plane waves.* In this last paragraph, we consider a slightly different example. We will verify that more complicated measures may arise for a superposition of two plane waves. For instance, one has in the rational case:

Proposition 4.3. *Let $(k(\hbar))_{\hbar\to 0^+}$ and $(l(\hbar))_{\hbar\to 0^+}$ be two distinct sequences of lattice points of the form*

$$k(\hbar) = n_1(\hbar)\vec{v}_{\Lambda_1} + m_1(\hbar)\vec{v}^{\perp}_{\Lambda_1} \quad \textit{and} \quad l(\hbar) = n_2(\hbar)\vec{v}_{\Lambda_2} + m_2(\hbar)\vec{v}^{\perp}_{\Lambda_2},$$

where $|k(\hbar)| \sim \hbar^{-1}$, $|l(\hbar)| \sim \hbar^{-1}$, and, for $j = 1, 2$, $\Lambda_j \in \mathcal{L}_1$ with $\Lambda_1 \neq \Lambda_2$. Suppose also that, for $j \in \{1, 2\}$, there exists ω_j in $\mathbb{R}$ such that[5]

$$\lim_{\hbar\to 0^+} 2\pi m_j(\hbar)\hbar^2\epsilon_\hbar^{-1} = \omega_j.$$

Set

$$\psi_\hbar(x) = \frac{1}{\sqrt{2}}(e_k(x) + e_l(x)).$$

Then, the corresponding measure $\mathbf{F}^0_\Lambda$ satisfies the following:

$$\mathbf{F}^0_\Lambda(x,\eta) = \begin{cases} \frac{1}{2}\delta_{\omega_1 L_{\Lambda_1}}(\eta) & \textit{if } \langle\Lambda\rangle = \Lambda_1^\perp, \\ \frac{1}{2}\delta_{\omega_2 L_{\Lambda_2}}(\eta) & \textit{if } \langle\Lambda\rangle = \Lambda_2^\perp, \\ 0 & \textit{otherwise.} \end{cases}$$

This proposition just reflects the fact that two plane waves do not interact with each other in the semiclassical limit as soon as $\Lambda_1 \neq \Lambda_2$.

[5]Note that this implies $m_j(\hbar) = o(\hbar^{-1})$.

Proof. Using appendix A, we find that

$$\begin{aligned}\langle \mathbf{F}_{\Lambda,\hbar}, c\rangle &= \sum_{m\in\{l,k\}} \frac{1}{\sqrt{2}} \sum_{n\in\{l,k\}:m-n\in\Lambda} \frac{1}{\sqrt{2}} \widehat{c}\left(n-m, 2\pi\frac{\hbar^2}{\epsilon_\hbar} H_\Lambda(m)\right) \\ &= \frac{1}{2}\widehat{c}(0, 2\pi\hbar^2\epsilon_\hbar^{-1} H_\Lambda(k)) + \frac{1}{2}\widehat{c}(0, 2\pi\hbar^2\epsilon_\hbar^{-1} H_\Lambda(l)) + \mathcal{O}_N(\hbar^N),\end{aligned} \tag{14}$$

where we used the fact that $\|k-l\| \sim \hbar^{-1}$ and $|\widehat{c}(k-l, \hbar^2\epsilon_\hbar^{-1} H_\Lambda(k))| \lesssim_N \|k-l\|^{-N} \sim \hbar^N$. So, if $\langle\Lambda\rangle = \Lambda_1^\perp$, then, for all $c \in \mathcal{C}_c^\infty(\mathbb{T}^2 \times \mathbb{R})$, we have by the same argument as in the proof of Proposition 4.1

$$\langle \mathbf{F}_{\Lambda,\hbar}, c\rangle \sim \frac{1}{2}\widehat{c}(0, 2\pi\hbar^2\epsilon_\hbar^{-1} H_\Lambda(k)) \to \frac{1}{2}\widehat{c}(0, \omega_1 L_{\Lambda_1})$$

as $\hbar \to 0$. Similarly, if $\langle\Lambda\rangle = \Lambda_2^\perp$, then we have

$$\langle \mathbf{F}_{\Lambda,\hbar}, c\rangle \to \frac{1}{2}\widehat{c}(0, \omega_2 L_{\Lambda_2}).$$

Otherwise, one can make use of the fact that c is compactly supported in η to deduce that $\langle \mathbf{F}_{\Lambda,\hbar}, c\rangle \to 0$. □

4.2. **Limit of the Quantum Loschmidt Echo for plane waves.** Now that we have computed the limit measures associated with the initial data, we can apply Theorem 2.1 in order to derive an explicit formula for the Quantum Loschmidt Echo:

Proposition 4.4. *Suppose that* (5) *is satisfied. Then, for sequences of plane waves, the following hold:*

(1) *If $(\psi_\hbar)_{\hbar\to 0^+}$ verifies the assumptions of Proposition 4.2 or of part (2) of Proposition 4.1, then*
$$\lim_{\hbar\to 0^+} |\langle u_\hbar^\epsilon(t\tau_\hbar^c), u_\hbar(t\tau_\hbar^c)\rangle|^2 = 1.$$

(2) *If $(\psi_\hbar)_{\hbar\to 0^+}$ verifies the assumptions of part (1) of Proposition 4.1, then*
$$\lim_{\hbar\to 0^+} |\langle u_\hbar^\epsilon(t\tau_\hbar^c), u_\hbar(t\tau_\hbar^c)\rangle|^2 = \left|\int_{\mathbb{T}^2} e^{i\int_0^t \mathcal{I}_{\Lambda_0^\perp}(V)(x+s\omega\vec{v}_{\Lambda_0}^\perp)ds} dx\right|^2,$$

(3) *If $(\psi_\hbar)_{\hbar\to 0^+}$ verifies the assumptions of Proposition 4.3, then*
$$\lim_{\hbar\to 0^+} |\langle u_\hbar^\epsilon(t\tau_\hbar^c), u_\hbar(t\tau_\hbar^c)\rangle|^2 = \left|\frac{1}{2}\sum_{j=1}^2 \int_{\mathbb{T}^2} e^{i\int_0^t \mathcal{I}_{\Lambda_j^\perp}(V)(x+s\omega_j\vec{v}_{\Lambda_j}^\perp)ds} dx\right|^2.$$

In all the statements, we used the conventions of Theorem 2.1.

The proof of this Proposition is a direct application of Theorem 2.1 combined with the above Propositions which computed $\mathbf{F}_\Lambda^0$. Note that the last part of the Proposition could be generalized to a finite superposition of plane waves by similar calculations. In the case where $\omega_j = 0$ (or $\omega = 0$ in part (2)), we can observe that the integral reduces to

$$\int_{\mathbb{T}^2} e^{it\mathcal{I}_{\Lambda^\perp}(V)(x)} dx,$$

which can be viewed as an integral on the 1-dimensional torus $\mathbb{S}_{\Lambda^\perp}=\mathbb{R}/(L_\Lambda\mathbb{Z})$. In particular, if $\mathcal{I}_{\Lambda^\perp}(V)$ has only finitely many critical points (which are nondegenerate) as a function on the 1-dimensional torus, then an application of stationary phase asymptotics states that this integral is of order $t^{-1/2}$ as $t\to+\infty$ which shows that the Quantum Loschmidt Echo becomes small for times $t\tau_\hbar^c$ with t large. On the other hand, plane waves associated with an irrational limit vector $\vec{v}$ do not give rise to any macroscopic decay of the Quantum Loschmidt Echo. As a last comment, let us observe that, when $\omega\neq 0$ in part (2) of the Proposition, one has

$$\int_0^t \mathcal{I}_{\Lambda_0^\perp}(V)(x+s\omega\vec{v}_{\Lambda_0}^\perp)ds=\frac{[t\omega]}{\omega}\widehat{V}_0+\frac{1}{\omega}\int_{[t\omega]}^{t\omega}\mathcal{I}_{\Lambda_0^\perp}(V)(x+s\vec{v}_{\Lambda_0}^\perp)ds,$$

where $[t\omega]$ is the unique integer satisfying $[t\omega]\leq t\omega<[t\omega]+1$. Hence, the limit of the fidelity distrbution can be rewritten as

$$\lim_{\hbar\to 0^+}|\langle u_\hbar^\epsilon(t\tau_\hbar^c),u_\hbar(t\tau_\hbar^c)\rangle|^2=\left|\int_{\mathbb{T}^2}e^{i\frac{1}{\omega}\int_{[t\omega]}^{t\omega}\mathcal{I}_{\Lambda_0^\perp}(V)(x+s\vec{v}_{\Lambda_0}^\perp)ds}dx\right|^2,$$

which is a periodic function of t.

4.3. **Coherent states.** We first define a notion of (generalized) coherent state following for instance [1]. Let $\varphi\in\mathcal{C}_c^\infty(\mathbb{R}^2)$ with[6] $\operatorname{supp}\varphi\subset(-\frac{1}{2},\frac{1}{2})^2$, $\|\varphi\|_2=1$. We define a coherent state on $\mathbb{R}^2$ by

$$\varphi_\hbar(x)=\hbar^{-1/2}\varphi\left(\frac{x-x_0}{\sqrt{\hbar}}\right)e^{2\pi i\frac{\xi_0}{\hbar}\cdot x},$$

and we periodize it to obtain a coherent state on the torus, $\psi_\hbar\in\mathcal{C}^\infty(\mathbb{T}^2)$,

$$\psi_\hbar(x)=\sum_{l\in\mathbb{Z}^2}\varphi_\hbar(x+l). \tag{15}$$

These semiclassical states concentrate at (x_0,ξ_0) as $\hbar\to 0^+$. Note that the terminology "coherent state" is somewhat abusive from the physical perspective as our states do not a priori minimize an uncertainty principle. Yet, they share some similarities in the sense that they are localized in a ball of radius $\sqrt{\hbar}$ around (x_0,ξ_0). In the following, we make the assumption that $\xi_0\neq 0$ in order to verify property (6). We can verify that the L^2 norm of $\psi_\hbar$ is equal to 1 for $\hbar$ small enough as the supports of the translates do not overlap in the semiclassical limit.

The Fourier decomposition of $\psi_\hbar$ can be easily expressed in terms of the Fourier transform of φ. Indeed, for k in $\mathbb{Z}^2$, one has

$$\widehat{\psi_\hbar}(k)=\hbar^{1/2}e^{-2\pi i(k-\frac{\xi_0}{\hbar})\cdot x_0}\widehat{\varphi}\left(\hbar^{1/2}\left(k-\frac{\xi_0}{\hbar}\right)\right), \tag{16}$$

where $\widehat{\varphi}$ denotes the Fourier transform on $\mathbb{R}^2$. Recall that $\|\widehat{\varphi}\|_{L^2(\mathbb{R}^2)}=\|\varphi\|_{L^2(\mathbb{R}^2)}=1$.

[6]Note that the arguments could be extended to deal with the more classical case where $\varphi=\exp(-\|x\|^2/2)$.

Remark 4.5. Let us make the following useful observation that we shall use at several stages of our argument, namely that $\widehat{\psi_\hbar}$ is supported in a ball $B(\xi_0/\hbar, r_\hbar)$. We fix a sequence of radii $r_\hbar \gg \hbar^{-1/2}$ and, using (16), one has

$$\sum_{\|m-\xi_0\hbar^{-1}\|\geq r_\hbar} |\widehat{\psi_\hbar}(m)|^2 \leq \hbar \sum_{\|m-\xi_0\hbar^{-1}\|\geq r_\hbar} \left|\widehat{\varphi}\left(\hbar^{1/2}\left(m-\frac{\xi_0}{\hbar}\right)\right)\right|^2.$$

As $\widehat{\varphi}$ is rapidly decaying, there exists $C>0$ such that

$$\left|\widehat{\varphi}\left(\hbar^{1/2}\left(m-\frac{\xi_0}{\hbar}\right)\right)\right| \leq C\left(1+\left\|\hbar^{1/2}\left(m-\frac{\xi_0}{\hbar}\right)\right\|^2\right)^{-2},$$

from which we can infer

$$\begin{aligned} \sum_{\|m-\xi_0\hbar^{-1}\|\geq r_\hbar} |\widehat{\psi_\hbar}(m)|^2 &\leq C\hbar \int_{\|y-\xi_0\hbar^{-1}\|\geq r_\hbar} \left(1+\left\|\hbar^{1/2}\left(y-\frac{\xi_0}{\hbar}\right)\right\|^2\right)^{-2} dy \\ &\leq C\int_{\|y'\|\geq r_\hbar \hbar^{\frac{1}{2}}} \left(1+\|y'\|^2\right)^{-2} dy' \to 0. \end{aligned} \tag{17}$$

The same calculation shows that

$$\sum_{\|m-\xi_0\hbar^{-1}\|\geq r_\hbar} |\widehat{\psi_\hbar}(m)| = o_{\hbar\to 0}(\hbar^{-1/2}).$$

4.3.1. *A preliminary reduction of* $\mathbf{F}_{\Lambda,\hbar}$. We aim at computing the limit distribution derived from the sequence $(\mathbf{F}_{\Lambda,\hbar})_{\hbar\to 0^+}$. For that purpose, we start with a general computation valid for any regime of perturbations. For c in $\mathcal{C}_c^\infty(\mathbb{T}^2\times\mathbb{R})$, write

$$\begin{aligned} \langle \mathbf{F}_{\Lambda,\hbar}, c\rangle &= \langle \psi_\hbar, \mathrm{Op}_\hbar(D_{\Lambda,\epsilon_\hbar\hbar^{-1}}(\mathcal{I}_\Lambda(c)))\psi_\hbar\rangle \\ &= \sum_{\substack{m\in\mathbb{Z}^2 \\ 2\pi\hbar^2\epsilon_\hbar^{-1}H_\Lambda(m)\in\mathrm{supp}_\eta(c)}} \overline{\widehat{\psi_\hbar}(m)} \sum_{n\in\mathbb{Z}^2: m-n\in\Lambda} \widehat{\psi_\hbar}(n)\widehat{c}\left(m-n, 2\pi\frac{\hbar^2}{\epsilon_\hbar}H_\Lambda(m)\right). \end{aligned}$$

Let us now compute the sum over n in $\mathbb{Z}^2$ by observing that $\psi_\hbar$ vanishes surely outside a ball $B(x_0,\hbar^{\frac{3}{8}})$. This implies, as $\hbar\to 0$,

(18)
$$\begin{aligned}\sum_{n\in\mathbb{Z}^2:m-n\in\Lambda}\widehat{\psi_\hbar}(n)\widehat{c}\left(m-n,2\pi\frac{\hbar^2}{\epsilon_\hbar}H_\Lambda(m)\right) &= \int_{\mathbb{T}^2}\mathcal{I}_\Lambda(c)\left(x,2\pi\frac{\hbar^2}{\epsilon_\hbar}H_\Lambda(m)\right)e_{-m}(x)\overline{\psi_\hbar(x)}dx\\ &= \int_{B(x_0,\hbar^{\frac{3}{8}})}\mathcal{I}_\Lambda(c)\left(x,2\pi\frac{\hbar^2}{\epsilon_\hbar}H_\Lambda(m)\right)e_{-m}(x)\overline{\psi_\hbar(x)}dx\\ &= \mathcal{I}_\Lambda(c)\left(x_0,2\pi\frac{\hbar^2}{\epsilon_\hbar}H_\Lambda(m)\right)\widehat{\psi_\hbar}(m)\\ &\quad+\mathcal{O}_c\left(\hbar^{\frac{3}{8}}\int_{B(x_0,\hbar^{\frac{3}{8}})}|\psi_\hbar(x)|dx\right)\\ &= \mathcal{I}_\Lambda(c)\left(x_0,2\pi\frac{\hbar^2}{\epsilon_\hbar}H_\Lambda(m)\right)\widehat{\psi_\hbar}(m)+\mathcal{O}_c(\hbar^{\frac{3}{4}}),\end{aligned}$$

where we used the bound

$$\int_{B(x_0,\hbar^{\frac{3}{8}})}|\psi_\hbar(x)|dx\leq \mathrm{Vol}(B(x_0,\hbar^{\frac{3}{8}}))^{1/2}\|\psi_\hbar\|_2=\mathcal{O}(\hbar^{\frac{3}{8}}).$$

In particular, we have

$$|\langle \mathbf{F}_{\Lambda,\hbar},c\rangle|\lesssim\|c\|_\infty\sum_{\substack{m\in\mathbb{Z}^2\\ 2\pi\hbar^2\epsilon_\hbar^{-1}H_\Lambda(m)\in\mathrm{supp}_\eta(c)}}|\widehat{\psi_\hbar}(m)|^2+\mathcal{O}_c(\hbar^{\frac{3}{4}})\sum_{\substack{m\in\mathbb{Z}^2\\ 2\pi\hbar^2\epsilon_\hbar^{-1}H_\Lambda(m)\in\mathrm{supp}_\eta(c)}}|\widehat{\psi_\hbar}(m)|. \tag{19}$$

We are now in a position to compute $\mathbf{F}^0_\Lambda$. For that purpose, we shall distinguish different regimes depending on the relative size of $\epsilon_\hbar$ and $\hbar$

4.3.2. *Small perturbations $\hbar^2\ll\epsilon_\hbar\ll\hbar^{3/2}$.* We start with the case of small perturbations. In that case, one has:

Proposition 4.6. *Suppose that $\hbar^2\ll\epsilon_\hbar\ll\hbar^{3/2}$. Then, for the sequence of initial data defined by* (15) *with $\xi_0\neq 0$ and for every $\Lambda\in\mathcal{L}_1$, we have $\mathbf{F}^0_\Lambda=0$.*

Proof. We fix c in $\mathcal{C}^\infty_c(\mathbb{T}^2\times\mathbb{R})$ and we want to compute $\langle\mathbf{F}^0_\Lambda,c\rangle$. The idea is as follows. In order to compute this distribution, we need, on the one hand, to evaluate the mass of the coherent state in a cylinder of size $\hbar\ll\epsilon_\hbar/\hbar\ll\hbar^{1/2}$ around $\langle\Lambda\rangle$ in momentum space. On the other hand, the coherent state is "uniformly" located in a ball of size $\sqrt{\hbar}$ around (x_0,ξ_0). Hence, its mass inside the cylinder under consideration will tend to 0 as $\hbar\to 0^+$.

We start with the case where $\xi_0\notin\Lambda^\perp$. Then, there exists a constant $0<C_\Lambda(\xi_0)\leq 1$ such that

$$\inf\left\{\left\|m-\frac{\xi_0}{\hbar}\right\|:m\in\mathbb{Z}^2,\ 2\pi\hbar^2\epsilon_\hbar^{-1}H_\Lambda(m)\in\mathrm{supp}_\eta(c)\right\}\geq\frac{C_\Lambda(\xi_0)}{\hbar}. \tag{20}$$

Indeed, suppose for a contradiction that there exist $\delta_\hbar \to 0$ and a sequence of lattice points $m_\hbar \in \mathbb{Z}^2$ such that $\|m_\hbar - \frac{\xi_0}{\hbar}\| \le \delta_\hbar \hbar^{-1}$ and such that $2\pi\hbar^2\epsilon_\hbar^{-1} H_\Lambda(m_\hbar) \in \mathrm{supp}_\eta(c)$. Decompose $m_\hbar$ as $m_\hbar = s_\hbar \xi_0 + t_\hbar \xi_0^\perp$. It follows that $|s_\hbar - \hbar^{-1}|, |t_\hbar| \le \delta_\hbar \hbar^{-1} \|\xi_0\|^{-1}$ which implies $|s_\hbar| \ge (1 - \delta_\hbar \|\xi_0\|^{-1})\hbar^{-1}$. Now write

$$H_\Lambda(m_\hbar) = s_\hbar \left\langle \frac{\vec{v}_\Lambda}{L_\Lambda}, \xi_0 \right\rangle + t_\hbar \left\langle \frac{\vec{v}_\Lambda}{L_\Lambda}, \xi_0^\perp \right\rangle = \frac{1}{\hbar} \left\langle \frac{\vec{v}_\Lambda}{L_\Lambda}, \xi_0 \right\rangle + o\left(\hbar^{-1}\right).$$

Then, we use that $|H_\Lambda(m_\hbar)| \lesssim \hbar^{-2}\epsilon_\hbar \ll \hbar^{-1}$, which yields the contradiction as $\xi_0 \notin \Lambda^\perp$. In view of the estimate (19), all that remains to be shown is

$$\sum_{m : \|m - \xi_0 \hbar^{-1}\| \ge C_\Lambda(\xi_0)\hbar^{-1}} |\widehat{\psi}_\hbar(m)|^2 \to 0 \quad \text{and} \quad \sum_{m : \|m - \xi_0 \hbar^{-1}\| \ge C_\Lambda(\xi_0)\hbar^{-1}} |\widehat{\psi}_\hbar(m)| = o(\hbar^{-\frac{3}{4}}),$$

as $\hbar \to 0$. This is exactly the content of Remark 4.5. It follows that $\mathbf{F}_\Lambda^0 = 0$ for every $\Lambda \in \mathcal{L}_1$ such that $\xi_0 \notin \Lambda^\perp$. Note that this part of the argument is in fact valid for any $\hbar^2 \ll \epsilon_\hbar \ll \hbar$.

On the other hand, suppose that $\xi_0 \in \Lambda^\perp$. We shall use (19) one more time. Yet, we have to argue in a slightly different manner and to make use of the fact that $\epsilon_\hbar \ll \hbar^{\frac{3}{2}}$, equivalently $\hbar^{-\frac{3}{2}}\epsilon_\hbar \to 0$. Using (16), one has

$$\sum_{\substack{m \in \mathbb{Z}^2 \\ 2\pi\hbar^2\epsilon_\hbar^{-1} H_\Lambda(m) \in \mathrm{supp}_\eta(c)}} |\widehat{\psi_\hbar}(m)|^2 \le \hbar \sum_{\substack{m \in \mathbb{Z}^2 \\ |H_\Lambda(m)| \le C_c \hbar^{-2}\epsilon_\hbar}} \left| \widehat{\varphi}\left(\hbar^{1/2}\left(m - \frac{\xi_0}{\hbar}\right)\right)\right|^2,$$

for some $C_c > 0$ that depends only on c. Recall from our assumptions that $\hbar^{-2}\epsilon_\hbar \to +\infty$. As $\widehat{\varphi}$ is rapidly decaying, one knows that there exists $C > 0$ such that

$$\left| \widehat{\varphi}\left(\hbar^{1/2}\left(m - \frac{\xi_0}{\hbar}\right)\right)\right| \le C \left(1 + \left\|\hbar^{1/2}\left(m - \frac{\xi_0}{\hbar}\right)\right\|^2\right)^{-1}.$$

Hence, implementing this bound in (19) and using that $\xi_0 \in \Lambda^\perp$ yields

(21)

$$\begin{aligned}\sum_{\substack{m \in \mathbb{Z}^2 \\ 2\pi\hbar^2\epsilon_\hbar^{-1} H_\Lambda(m) \in \mathrm{supp}_\eta(c)}} |\widehat{\psi_\hbar}(m)|^2 &\le C^2 \hbar \int_{|H_\Lambda(y - \xi_0\hbar^{-1})| \le C_c \hbar^{-2}\epsilon_\hbar} \left(1 + \left\|\hbar^{1/2}\left(y - \frac{\xi_0}{\hbar}\right)\right\|^2\right)^{-2} dy \\ &\le C^2 \int_{|H_\Lambda(y')| \le C_c \hbar^{-\frac{3}{2}}\epsilon_\hbar} \left(1 + \|y'\|^2\right)^{-2} dy' \to 0, \text{ as } \hbar \to 0^+.\end{aligned}$$

A similar calculation shows that

$$\sum_{\substack{m \in \mathbb{Z}^2 \\ 2\pi\hbar^2\epsilon_\hbar^{-1} H_\Lambda(m) \in \mathrm{supp}_\eta(c)}} |\widehat{\psi_\hbar}(m)| = \mathcal{O}(\hbar^{-1/2}), \text{ as } \hbar \to 0^+.$$

Indeed, following the strategy explained at the beginning of the proof, equation (16) yields

(22)
$$\begin{aligned}\sum_{\substack{m\in\mathbb{Z}^2\\ \hbar^2\epsilon_\hbar^{-1}H_\Lambda(m)\in\text{supp}_\eta(c)}}|\widehat{\psi_\hbar}(m)| &\leq \hbar^{1/2}\sum_{\substack{m\in\mathbb{Z}^2\\ |H_\Lambda(m-\xi_0\hbar^{-1})|\leq C_c\hbar^{-2}\epsilon_\hbar}}\left|\widehat{\varphi}\left(\hbar^{1/2}\left(m-\frac{\xi_0}{\hbar}\right)\right)\right|\\
&\leq C'\hbar^{1/2}\int_{|H_\Lambda(y-\xi_0\hbar^{-1})|\leq C_c\hbar^{-2}\epsilon_\hbar}\left(1+\left\|\hbar^{1/2}\left(y-\frac{\xi_0}{\hbar}\right)\right\|^2\right)^{-2}dy\\
&\leq C'\hbar^{-1/2}\int_{|H_\Lambda(y')|\leq C_c\hbar^{-\frac{3}{2}}\epsilon_\hbar}\left(1+\|y'\|^2\right)^{-2}dy'\\
&\leq C'\hbar^{-1/2}\int_{\mathbb{R}^2}\left(1+\|y'\|^2\right)^{-2}dy'\\
&=\mathcal{O}(\hbar^{-1/2}),\ \text{as}\ \hbar\to 0^+.\end{aligned}$$

Gathering these upper bounds implies that the limit distribution $\mathbf{F}^0_\Lambda$ is zero for any Λ. Along the way, we also note that the upper bound (22) did not rely on the fact that $\epsilon_\hbar \ll \hbar^{\frac{3}{2}}$. □

4.3.3. *Critical perturbations $\epsilon_\hbar = \hbar^{3/2}$.*

Proposition 4.7. *Suppose that $\epsilon_\hbar = \hbar^{3/2}$ and that the sequence of initial data is given by* (15) *with $\xi_0 \neq 0$. Then, we have*

$$\mathbf{F}^0_\Lambda(x,\eta) = \begin{cases}\delta_{x_0}(x)\mu(\eta) & \textit{if } \xi_0\in\Lambda^\perp,\\ \\ 0 & \textit{otherwise.}\end{cases}$$

where

$$\int_{\mathbb{R}} c(\eta)\mu(d\eta) = \int_{\mathbb{R}^2} c(2\pi H_\Lambda(\xi))|\widehat{\varphi}(\xi)|^2 d\xi.$$

Proof. The first part of the proof of Proposition 4.6 applies to the case where $\xi_0 \notin \Lambda^\perp$ and $\epsilon_\hbar \geq \hbar^{\frac{3}{2}}$. Hence, it remains to treat the case where $\xi_0 \in \Lambda^\perp$. In that case, equation (18) yields

$$\langle \mathbf{F}_{\Lambda,\hbar}, c\rangle = \sum_{\substack{m\in\mathbb{Z}^2\\ 2\pi\hbar^{\frac{1}{2}}H_\Lambda(m)\in\text{supp}_\eta(c)}} \mathcal{I}_\Lambda(c)\left(x_0, 2\pi\hbar^{\frac{1}{2}}H_\Lambda(m)\right)|\widehat{\psi_\hbar}(m)|^2 + \mathcal{O}_c(\hbar^{\frac{3}{4}})|\widehat{\psi_\hbar}(m)|.$$

The argument of Remark 4.5 shows that

$$\lim_{R\to+\infty}\limsup_{\hbar\to 0^+}\sum_{\|m-\xi_0\hbar^{-1}\|\geq R\hbar^{-1/2}}|\widehat{\psi_\hbar}(m)|^2 = 0.$$

Thus,

$$\langle \mathbf{F}_{\Lambda,\hbar}, c\rangle = \sum_{\|m-\xi_0\hbar^{-1}\|\leq R\hbar^{-1/2}} \left(\mathcal{I}_\Lambda(c)\left(x_0, 2\pi\hbar^{\frac{1}{2}} H_\Lambda(m)\right)|\widehat{\psi_\hbar}(m)|^2 + \mathcal{O}_c(\hbar^{\frac{3}{4}})|\widehat{\psi_\hbar}(m)|\right) + r(R,\hbar),$$

where $\lim_{R\to+\infty}\limsup_{\hbar\to 0^+} r(R,\hbar) = 0$. We start by estimating the first part of the sum which is equal to

$$\hbar \sum_{\|m-\xi_0\hbar^{-1}\|\leq R\hbar^{-1/2}} \mathcal{I}_\Lambda(c)\left(x_0, 2\pi\hbar^{\frac{1}{2}} H_\Lambda\left(m - \frac{\xi_0}{\hbar}\right)\right)\left|\widehat{\varphi}\left(\hbar^{1/2}\left(m - \frac{\xi_0}{\hbar}\right)\right)\right|^2,$$

where we used that $\xi_0 \in \Lambda^\perp$. Letting $\hbar \to 0^+$, one finds

(23)

$$\begin{aligned}\langle \mathbf{F}_{\Lambda,\hbar}, c\rangle &= \int_{\|\xi\|\leq R} |\widehat{\varphi}(\xi)|^2 \mathcal{I}_\Lambda(c)(x_0, 2\pi H_\Lambda(\xi))d\xi + \mathcal{O}_c(\hbar^{\frac{3}{4}}) \sum_{\substack{m\in\mathbb{Z}^2 \\ 2\pi\hbar^{\frac{1}{2}} H_\Lambda(m)\in \mathrm{supp}_\eta(c)}} |\widehat{\psi_\hbar}(m)| + r(R,\hbar) \\ &= \int_{\|\xi\|\leq R} |\widehat{\varphi}(\xi)|^2 \mathcal{I}_\Lambda(c)(x_0, 2\pi H_\Lambda(\xi))d\xi + \mathcal{O}_c(\hbar^{\frac{1}{4}}) + r(R,\hbar),\end{aligned}$$

where we used estimate (22) in the last line. The result follows by taking $\hbar \to 0^+$ and then $R \to +\infty$. □

4.3.4. *Large perturbations $\hbar^{3/2} \ll \epsilon_\hbar \ll \hbar$.* In this regime, we have $\hbar^{-1/2} \ll \epsilon_\hbar \hbar^{-2} \ll \hbar^{-1}$ and the following holds:

Proposition 4.8. *Suppose that $\hbar^{3/2} \ll \epsilon_\hbar \ll \hbar$ and that the sequence of initial data is given by* (15) *with $\xi_0 \neq 0$. Then, we have*

$$\mathbf{F}^0_\Lambda(x,\eta) = \begin{cases} \delta_{x_0}(x)\delta_0(\eta) & \text{if } \xi_0 \in \Lambda^\perp, \\ 0 & \text{otherwise.}\end{cases}$$

In particular, as it was already the case in Proposition 4.7, if $\xi_0/\|\xi_0\| \notin \mathbb{S}^1_{\mathbb{Q}}$, then $\mathbf{F}^0_\Lambda = 0$ for every $\Lambda \in \mathcal{L}_1$.

Proof. Compared with the proof of Proposition 4.6, we are now evaluating the mass of the coherent state inside a cylinder whose is of order $\hbar^{\frac{1}{2}} \ll \epsilon_\hbar/\hbar \ll 1$. In particular, the size of the cylinder is much larger than the size $\hbar^{\frac{1}{2}}$ of the ball where the coherent state is located. Hence, for $\xi_0 \in \Lambda^\perp$, the distribution $\mathbf{F}^0_\Lambda$ will only capture a Dirac mass at 0 in momentum space. For other Λ, it will give 0 as we are away from the location of ξ_0.

Hence, arguing as in the proof of Proposition 4.6, we can verify that $\mathbf{F}^0_\Lambda = 0$ for every $\Lambda \in \mathcal{L}_1$ such that $\xi_0 \notin \Lambda^\perp$. Hence, as above, we just need to discuss the case where $\xi_0 \in \Lambda^\perp$.

From (18), one knows that

$$\langle \mathbf{F}_{\Lambda,\hbar}, c\rangle = \sum_{\substack{m\in\mathbb{Z}^2\\ 2\pi\hbar^2\epsilon_\hbar^{-1}H_\Lambda(m)\in\text{supp}_\eta(c)}} \mathcal{I}_\Lambda(c)\left(x_0, 2\pi\frac{\hbar^2}{\epsilon_\hbar}H_\Lambda(m)\right)|\widehat{\psi_\hbar}(m)|^2 + \mathcal{O}_c(\hbar^{\frac{3}{4}})|\widehat{\psi_\hbar}(m)|.$$

We fix a sequence of radii $r_\hbar$ such that $\epsilon_\hbar\hbar^{-2} \gg r_\hbar \gg \hbar^{-1/2}$. The argument of Remark 4.5 allows to show that

$$\begin{aligned}\langle \mathbf{F}_{\Lambda,\hbar}, c\rangle &= \sum_{m:\|m-\xi_0\hbar^{-1}\|\leq r_\hbar} \mathcal{I}_\Lambda(c)\left(x_0, 2\pi\frac{\hbar^2}{\epsilon_\hbar}H_\Lambda(m)\right)|\widehat{\psi_\hbar}(m)|^2\\ &+ \mathcal{O}_c(\hbar^{\frac{3}{4}}) \sum_{\substack{m\in\mathbb{Z}^2\\ 2\pi\hbar^2\epsilon_\hbar^{-1}H_\Lambda(m)\in\text{supp}_\eta(c)}} |\widehat{\psi_\hbar}(m)| + o(1).\end{aligned}$$

Hence, using Remark 4.5 one more time, the fact that $\|\psi_\hbar\| \to 1$ and the fact that $\xi_0 \in \Lambda^\perp$, one has

$$\begin{aligned}\langle \mathbf{F}_{\Lambda,\hbar}, c\rangle &= \mathcal{I}_\Lambda(c)\,(x_0,0) + \mathcal{O}_c(\hbar^{\frac{3}{4}}) \sum_{\substack{m\in\mathbb{Z}^2\\ 2\pi\hbar^2\epsilon_\hbar^{-1}H_\Lambda(m)\in\text{supp}_\eta(c)}} |\widehat{\psi_\hbar}(m)| + o(1)\\ &= \mathcal{I}_\Lambda(c)\,(x_0,0) + o(1),\end{aligned} \tag{24}$$

where we used estimate (22) in the last line. □

4.3.5. *Superpositions of two coherent states.* Let us now consider the initial data

$$\psi_\hbar = \frac{1}{\sqrt{2}}(\psi_\hbar^{(x_0,\xi_0)} + \psi_\hbar^{(y_0,\eta_0)}),$$

where $\psi_\hbar^{(x_0,\xi_0)}$ (resp. $\psi_\hbar^{(y_0,\eta_0)}$) denotes a coherent state centered at (x_0,ξ_0) (resp. (y_0,η_0)) with $\xi_0,\eta_0 \neq 0$. We will also suppose for the sake of simplicity that $x_0 \neq y_0$ (but not necessarily $\xi_0 \neq \eta_0$). The case $x_0 = y_0$ could be treated in a similar manner but it would require slightly more work. As the case $x_0 \neq y_0$ already displays interesting features regarding the question of the Quantum Loschmidt Echo, we limit ourselves to this case. The key observation is that we have

$$\langle \mathbf{F}_{\Lambda,\hbar}, c\rangle = \frac{1}{2}\left\langle \mathbf{F}_{\Lambda,\hbar}^{(x_0,\xi_0)}, c\right\rangle + \frac{1}{2}\left\langle \mathbf{F}_{\Lambda,\hbar}^{(y_0,\eta_0)}, c\right\rangle + o(1). \tag{25}$$

Hence, the calculation reduces to the analysis of a single coherent state as it was performed in the above Propositions. To see this, we simply have to show that the off-diagonal terms are small, i.e. as $\hbar \to 0^+$:

$$r_\hbar^\Lambda := \left\langle \psi_\hbar^{(x_0,\xi_0)}, \text{Op}_\hbar(D_{\Lambda,\epsilon_\hbar\hbar^{-1}}(\mathcal{I}_\Lambda(c)))\psi_\hbar^{(y_0,\eta_0)}\right\rangle = o(1).$$

For that purpose, we write that

$$r_\hbar^\Lambda = \sum_{k\in\mathbb{Z}^2} e^{2i\pi\frac{\xi_0.k}{\hbar}}\left\langle \varphi_\hbar^{x_0+k,\xi_0}, \text{Op}_\hbar\left(D_{\Lambda,\epsilon_\hbar\hbar^{-1}}(\mathcal{I}_\Lambda(c))\right)\varphi_\hbar^{y_0,\eta_0}\right\rangle_{L^2(\mathbb{R}^2)}.$$

We will now estimate each term in the sum by making use of the fact that $x_0 \neq y_0$:

$$\begin{aligned} r_\hbar^\Lambda(k) &:= \left\langle \varphi_\hbar^{x_0+k,\xi_0}, \mathrm{Op}_\hbar\left(D_{\Lambda,\epsilon_\hbar\hbar^{-1}}(\mathcal{I}_\Lambda(c))\right)\varphi_\hbar^{y_0,\eta_0}\right\rangle_{L^2(\mathbb{R}^2)} \\ &= \frac{1}{(2\pi\hbar)^2}\int_{\mathbb{R}^6} e^{\frac{i}{\hbar}\langle x-y,\xi\rangle}\overline{\varphi_\hbar^{x_0+k,\xi_0}(x)}\mathcal{I}_\Lambda(c)(x,\hbar\epsilon_\hbar^{-1}H_\Lambda(\xi))\varphi_\hbar^{y_0,\eta_0}(y)dxdyd\xi \\ &= \frac{e^{i\alpha(\hbar)}}{(2\pi)^2}\int_{\mathbb{R}^6} e^{\frac{i}{\sqrt{\hbar}}\theta^{(k)}_{x_0,\xi_0,y_0,\eta_0}(x,y,\xi)}e^{\frac{i}{\hbar}\langle x_0+k-y_0,\xi\rangle}\overline{\varphi(x)}\mathcal{I}_\Lambda(c)(x_0+\sqrt{\hbar}x,\hbar\epsilon_\hbar^{-1}H_\Lambda(\xi))\varphi(y)dxdyd\xi, \end{aligned}$$

where $\alpha(\hbar)$ is some real number depending on (x_0,ξ_0,y_0,η_0), and where

$$\theta^{(k)}_{x_0,\xi_0,y_0,\eta_0}(x,y,\xi) := -\langle x, 2\pi\xi_0\rangle + \langle y, 2\pi\eta_0\rangle + \langle x-y,\xi\rangle.$$

Note that we have identified x_0 and y_0 with elements in $[0,1)^2$. We can now use the fact that

$$\frac{\hbar(x_0+k-y_0).\partial_\xi}{i\|x_0+k-y_0\|^2}\left(e^{\frac{i}{\hbar}\langle x_0+k-y_0,\xi\rangle}\right) = e^{\frac{i}{\hbar}\langle x_0+k-y_0,\xi\rangle},$$

and integrate by parts. In that manner, we find that, for every $N\geq 1$ and for every k in $\mathbb{Z}^2$,

$$r_\hbar^\Lambda(k) = \|x_0+k-y_0\|^{-N}\mathcal{O}_N\left(\hbar^{\frac{N}{2}} + (\hbar^2\epsilon_\hbar^{-1})^N\right),$$

which allows to conclude.

4.4. **Limit of the Quantum Loschmidt Echo for coherent states.** Combining Theorem 2.1 with the above Propositions yields the following estimates for the evolution of the Quantum Loschmidt Echo:

Proposition 4.9. *Suppose that* (5) *is satisfied and that the sequence of initial data is given by* (15) *with* $\xi_0 \neq 0$. *Then, the following holds:*

(1) *If* $\epsilon_\hbar \gg \hbar^{\frac{3}{2}}$ *or* $\epsilon_\hbar \ll \hbar^{\frac{3}{2}}$, *then*

$$\lim_{\hbar\to 0^+} |\langle u_\hbar^\epsilon(t\tau_\hbar^c), u_\hbar(t\tau_\hbar^c)\rangle|^2 = 1.$$

(2) *If* $\epsilon_\hbar = \hbar^{\frac{3}{2}}$ *and* $\xi_0/\|\xi_0\| \notin \mathbb{S}^1_{\mathbb{Q}}$, *then*

$$\lim_{\hbar\to 0^+} |\langle u_\hbar^\epsilon(t\tau_\hbar^c), u_\hbar(t\tau_\hbar^c)\rangle|^2 = 1.$$

(3) *If* $\epsilon_\hbar = \hbar^{\frac{3}{2}}$ *and* $\xi_0 \in \Lambda^\perp$ *for some* $\Lambda \in \mathcal{L}_1$, *then*

$$\lim_{\hbar\to 0^+} |\langle u_\hbar^\epsilon(t\tau_\hbar^c), u_\hbar(t\tau_\hbar^c)\rangle|^2 = \left|\int_{\mathbb{R}^2}|\widehat{\varphi}(\xi)|^2 e^{i\int_0^t \mathcal{I}_\Lambda(V)\left(x_0+2\pi s H_\Lambda(\xi)\frac{\vec{v}_\Lambda}{L_\Lambda}\right)ds}d\xi\right|^2.$$

In all the statements, we used the conventions of Theorem 2.1.

In particular, this Proposition shows that, for most of the cases, we do not have any macroscopic decay of the Quantum Loschmidt Echo when the initial data are given by a sequence of coherent states. More precisely, the echo can only be < 1 if $\epsilon_\hbar = \hbar^{\frac{3}{2}}$. In the case where $\epsilon_\hbar \gg \hbar^{\frac{3}{2}}$, it is interesting to mention the case of a superposition of two coherent states pointing along rational directions:

Proposition 4.10. *Suppose that $\hbar^{\frac{3}{2}} \ll \epsilon_\hbar \ll \hbar$ and that we are given the sequence of initial data from paragraph 4.3.5 with $\xi_0 \in \Lambda_1^\perp$ and $\eta_0 \in \Lambda_2^\perp$. Then, the following holds:*

$$\lim_{\hbar\to 0^+} |\langle u_\hbar^\epsilon(t\tau_\hbar^c), u_\hbar(t\tau_\hbar^c)\rangle|^2 = \left|\cos\left(\frac{t(\mathcal{I}_{\Lambda_1}(V)(x_0) - \mathcal{I}_{\Lambda_2}(V)(y_0))}{2}\right)\right|^2,$$

where we used the conventions of Theorem 2.1.

This follows from Propostion 4.8 combined with paragraph 4.3.5 and Theorem 2.1. It shows that, in this particular case, the Quantum Loschmidt Echo is periodic in time for times scales of order $\tau_\hbar^c$. Finally, note that we do not necessarily suppose $\Lambda_1 \neq \Lambda_2$ here.

5. Semiclassical fidelity distributions

Even if we are primarily interested in the study of the Quantum Loschmidt Echo, we will in fact study some slightly more general quantities which may be of independent interest and which already appeared in [19]. We shall call these intermediary objects *semiclassical fidelity distributions.* Before defining them and relating them to the quantities of the introduction, let us first fix some conventions. Associated with the Schrödinger equations (1) and (2) are two semiclassical operators acting on $L^2(\mathbb{T}^2)$:

$$\widehat{P}_0(\hbar) := -\frac{\hbar^2\Delta}{2}, \text{ and } \widehat{P}_\epsilon(\hbar) := -\frac{\hbar^2\Delta}{2} + \epsilon_\hbar V.$$

We will always assume that assumption (5) on the size of the perturbation is satisfied. In order to define these semiclassical fidelity distributions, we fix two sequences of *normalized* initial data $(\psi_\hbar^1)_{0<\hbar\leq 1}$ and $(\psi_\hbar^2)_{0<\hbar\leq 1}$ satisfying the frequency assumptions (6) and (7). We then define the following *semiclassical fidelity distribution* on $T^*\mathbb{T}^2$:

$$\forall t \in \mathbb{R},\ F_\hbar(t) : a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^2) \mapsto \left\langle \psi_\hbar^1, e^{\frac{it\widehat{P}_\epsilon(\hbar)}{\hbar}} \operatorname{Op}_\hbar(a) e^{-\frac{it\widehat{P}_0(\hbar)}{\hbar}} \psi_\hbar^2 \right\rangle_{L^2(\mathbb{T}^2)},$$

where $\operatorname{Op}_\hbar(a)$ is the standard quantization defined in appendix A. The goal of this section is to describe the properties of their accumulation points. The main observation from this section is that the description of the Quantum Loschmidt Echo at the critical time scale $\tau_\hbar^c$ follows from the detailed analysis of these accumulation points – see paragraph 5.2 and Proposition 5.6.

5.1. **Extracting subsequences.**

Recall that we denote by $\tau_\hbar^c$ the critical time scale $\frac{\hbar}{\epsilon_\hbar}$. We first extract converging subsequences from these sequences of distributions. Let $\tilde{a}(t, x, \xi)$ be an element in $\mathcal{C}_c^\infty(\mathbb{R} \times T^*\mathbb{T}^2)$. From the Calderón-Vaillancourt Theorem A.3, one knows that

$$\left|\int_{\mathbb{R}} \langle F_\hbar(t\tau_\hbar^c), \tilde{a}(t)\rangle dt\right| \leq C \sum_{|\alpha|\leq D} \hbar^{\frac{|\alpha|}{2}} \int_{\mathbb{R}} \|\partial_{x,\xi}^\alpha \tilde{a}(t)\|_\infty dt, \tag{26}$$

for some universal positive constants C and D. In particular, this defines a sequence of bounded distributions on $\mathbb{R} \times T^*\mathbb{T}^2$. Thus, up to an extraction $\hbar_n \to 0^+$, there exists

$F(t,x,\xi)$ in $\mathcal{D}'(\mathbb{R}\times T^*\mathbb{T}^2)$ such that, for every $\tilde{a}$ in $\mathcal{C}_c^\infty(\mathbb{R}\times T^*\mathbb{T}^2)$, one has

$$\lim_{\hbar_n\to 0^+}\int_{\mathbb{R}}\langle F_{\hbar_n}(t\tau_{\hbar_n}^c),\tilde{a}(t)\rangle dt=\int_{\mathbb{R}\times T^*\mathbb{T}^2}\tilde{a}(t,x,\xi)F(dt,dx,d\xi).$$

Remark 5.1. In order to alleviate the notations, we shall write $\hbar\to 0^+$ instead of $\hbar_n\to 0^+$ which is a standard convention in semiclassical analysis.

From (26), one knows that

$$\left|\int_{\mathbb{R}\times T^*\mathbb{T}^2}\tilde{a}(t,x,\xi)F(dt,dx,d\xi)\right|\leq C\int_{\mathbb{R}}\|\tilde{a}(t)\|_{\mathcal{C}^0}dt.$$

Thus, for a.e. t in $\mathbb{R}$, $F(t)$ defines an element of the Banach space[7] $\mathcal{M}(T^*\mathbb{T}^2)$ of finite (complex) Radon measures on $T^*\mathbb{T}^2$. By an approximation argument, one can also verify that, for every θ in $L^1(\mathbb{R})$ and for every a in $\mathcal{C}_c^\infty(T^*\mathbb{T}^2)$, one has

$$\lim_{\hbar\to 0^+}\int_{\mathbb{R}}\theta(t)\langle F_\hbar(t\tau_\hbar^c),a\rangle dt=\int_{\mathbb{R}}\theta(t)\left(\int_{T^*\mathbb{T}^2}a(x,\xi)F(t,dx,d\xi)\right)dt.$$

Note that compared with the classical case of semiclassical measures [13, 17, 27], $F(t)$ is a priori only a *complex measure*. We also remark that, thanks to the frequency assumption (6), one has, for a.e. t in $\mathbb{R}$,

$$|F(t)|(\mathbb{T}^2\times\{0\})=0. \tag{27}$$

Up to another extraction, we can also suppose that there exists some finite (complex) Radon measure F_0 on $T^*\mathbb{T}^2$ such that, for every a in $\mathcal{C}_c^\infty(T^*\mathbb{T}^2)$,

$$\lim_{\hbar\to 0^+}\langle F_\hbar(0),a\rangle=\int_{T^*\mathbb{T}^2}a(x,\xi)F_0(dx,d\xi).$$

Observe that, by construction, $F(t)$ was only defined almost everywhere while here we extract a subsequence for the fixed time $t=0$.

From this point on, we fix the accumulation point $F(t)$ and we want to describe it in terms of t. Let us start with the following lemma:

Lemma 5.2 (Invariance by the geodesic flow)**.** *Denote by φ^s the geodesic flow, i.e.*

$$\forall(x,\xi)\in T^*\mathbb{T}^2,\ \forall s\in\mathbb{R},\ \varphi^s(x,\xi):=(x+s\xi,\xi).$$

Then, for every a in $\mathcal{C}_c^\infty(T^*\mathbb{T}^2)$ and for a.e. t in $\mathbb{R}$, one has

$$\forall s\in\mathbb{R},\ \int_{T^*\mathbb{T}^2}a\circ\varphi^s(x,\xi)F(t,dx,d\xi)=\int_{T^*\mathbb{T}^2}a(x,\xi)F(t,dx,d\xi).$$

This Lemma shows that, as for the case of semiclassical measures [17], the limit object we obtain is invariant by the geodesic flow. We emphasize that it is important here to have $\epsilon_\hbar\ll\hbar$.

[7]More generally, for a locally compact metric space X, we denote by $\mathcal{M}(X)$ the set of finite complex Radon measure on X.

Proof. As for the extraction argument, the proof of this Lemma is the same as for semi-classical measure. Yet, let us recall the proof as it is instructive regarding the upcoming proofs. We write

$$\begin{aligned}\frac{d}{dt}\langle F_\hbar(t\tau_\hbar^c),a\rangle &= \frac{i\tau_\hbar^c}{\hbar}\left\langle \psi_\hbar^1, e^{\frac{it\tau_\hbar^c\widehat{P}_\epsilon(\hbar)}{\hbar}}\left[\widehat{P}_0(\hbar),\mathrm{Op}_\hbar(a)\right]e^{-\frac{it\tau_\hbar^c\widehat{P}_0(\hbar)}{\hbar}}\psi_\hbar^2\right\rangle_{L^2(\mathbb{T}^2)}\\ &\quad+\frac{i\tau_\hbar^c\epsilon_\hbar}{\hbar}\left\langle \psi_\hbar^1, e^{\frac{it\tau_\hbar^c\widehat{P}_\epsilon(\hbar)}{\hbar}}V\,\mathrm{Op}_\hbar(a)e^{-\frac{it\tau_\hbar^c\widehat{P}_0(\hbar)}{\hbar}}\psi_\hbar^2\right\rangle_{L^2(\mathbb{T}^2)}\end{aligned}$$

As we use the standard quantization, we have that $V\,\mathrm{Op}_\hbar(a)=\mathrm{Op}_\hbar(Va)$ which is a bounded operator thanks to the Calderón-Vaillancourt Theorem A.3. Moreover, using the composition Theorem A.4 for pseudodifferential operators and the Calderón-Vaillancourt Theorem one more time, we have that $\left[\widehat{P}_0(\hbar),\mathrm{Op}_\hbar(a)\right]=\frac{\hbar}{i}\mathrm{Op}_\hbar(\xi.\partial_x a)+\mathcal{O}_{L^2\to L^2}(\hbar^2)$. Using our assumption (5) on the size of $\epsilon_\hbar$, we find that

$$\frac{d}{dt}\langle F_\hbar(t\tau_\hbar^c),a\rangle = i\tau_\hbar^c\langle F_\hbar(t\tau_\hbar^c),\xi.\partial_x a\rangle + o(\tau_\hbar^c).$$

Integrating this relation against θ in $\mathcal{C}_c^\infty(\mathbb{R})$, we find that

$$\frac{i}{\tau_\hbar}\int_{\mathbb{R}}\theta'(t)\langle F_\hbar(t\tau_\hbar^c),a\rangle dt = \int_{\mathbb{R}}\theta(t)\langle F_\hbar(t\tau_\hbar^c),\xi.\partial_x a\rangle dt + o(1),$$

which concludes the proof by letting $\hbar\to 0^+$. □

Finally, we note that, up to another extraction, we can suppose that, for a.e. t in $\mathbb{R}$, there exists $\nu(t)\in\mathcal{M}(\mathbb{T}^2)$ such that, for every θ in $L^1(\mathbb{R})$ and for every[8] $f\in\mathcal{C}^\infty(\mathbb{T}^2)$,

$$\lim_{\hbar\to 0^+}\int_{\mathbb{R}}\theta(t)\langle F_\hbar(t\tau_\hbar^c),f\rangle dt = \int_{\mathbb{R}}\theta(t)\left(\int_{T^*\mathbb{T}^2} f(x)\nu(t,dx)\right)dt.$$

Thanks to the frequency assumption (7), we know that there is no escape of mass at infinity. Therefore, one has

$$\nu(t,x)=\int_{\mathbb{R}^2}F(t,x,d\xi). \tag{28}$$

5.2. **Time evolution.** Let us now discuss the relation of these fidelity distributions with the quantities appearing in the introduction. There, we were mostly interested in the case where $a=1$. In that particular case, we have that

$$\frac{d}{dt}\left(e^{-it\int_{\mathbb{T}^2}V}\langle F_\hbar(t\tau_\hbar^c),1\rangle\right) = ie^{-it\int_{\mathbb{T}^2}V}\left\langle F_\hbar(t\tau_\hbar^c),\left(V-\int_{\mathbb{T}^2}V\right)\right\rangle,$$

or equivalently, for every $t\in\mathbb{R}$

$$\langle F_\hbar(t\tau_\hbar^c),1\rangle = e^{it\int_{\mathbb{T}^2}V}\langle F_\hbar(0),1\rangle + i\int_0^t e^{i(t-t')\int_{\mathbb{T}^2}V}\left\langle F_\hbar(t'\tau_\hbar^c),\left(V-\int_{\mathbb{T}^2}V\right)\right\rangle dt'.$$

[8]Note that it is not compactly supported in ξ.

Letting $\hbar \to 0^+$, we find that, for every $t \in \mathbb{R}$,
(29)

$$\lim_{\hbar\to 0^+} \langle F_\hbar(t\tau_\hbar^c), 1\rangle = e^{it\int_{\mathbb{T}^2} V}\langle F_0, 1\rangle + i\int_0^t e^{i(t-t')\int_{\mathbb{T}^2} V}\left(\int_{T^*\mathbb{T}^2}\left(V(x) - \int_{\mathbb{T}^2} V\right) F(t', dx, d\xi)\right) dt',$$

or equivalently

$$\lim_{\hbar\to 0^+} \langle F_\hbar(t\tau_\hbar^c), 1\rangle = e^{it\int_{\mathbb{T}^2} V}\langle F_0, 1\rangle + i\int_0^t e^{i(t-t')\int_{\mathbb{T}^2} V}\left(\int_{\mathbb{T}^2}\left(V(x) - \int_{\mathbb{T}^2} V\right) \nu(t', dx)\right) dt'.$$

Note that, in the particular case where $\psi_\hbar^1 = \psi_\hbar^2 = \psi_\hbar$, the first term of the right hand side is equal to 1. Hence, **describing the Quantum Loschmidt echo at the critical time scale boils down to the description of the fidelity distribution** $F(t)$ (more precisely of its pushforward $\nu(t)$ on $\mathbb{T}^2$) in terms of t and of the initial data.

Remark 5.3. Note that, up to this point, our analysis did not really used the fact that we are on the torus and it could be adapted to deal with more general Riemannian manifolds.

5.3. **Decomposition of phase space.** In order to describe $F(t)$ in terms of t, we will first exploit its invariance under the geodesic flow in order to decompose it into infinitely many pieces indexed by the family $\mathcal{L}$ of primitive sublattices of $\mathbb{Z}^2$. We follow here the presentation of [4]. Recall that a sublattice Λ is said to be primitive if $\langle\Lambda\rangle \cap \mathbb{Z}^2 = \Lambda$, where $\langle\Lambda\rangle$ is the subspace of $\mathbb{R}^2$ spanned by Λ. For Λ in $\mathcal{L}$, we introduce

$$\Lambda^\perp := \left\{\xi \in \mathbb{R}^2 : \xi.k = 0,\ \forall k \in \Lambda\right\}.$$

To every fixed covector $\xi \in \mathbb{R}^2$, we also associate the sublattice

$$\Lambda_\xi := \left\{k \in \mathbb{Z}^2 : k.\xi = 0\right\},$$

and, for every $0 \le j \le 2$, we denote by $\Omega_j \subset \mathbb{R}^2$ the following subsets of covectors

$$\Omega_0 := \{0\},\ \Omega_1 := \left\{\xi \in \mathbb{R}^2 : \mathrm{rk}\Lambda_\xi = 1\right\},\ \text{and } \Omega_2 = \mathbb{R}^2 - (\Omega_0 \cup \Omega_1).$$

The fact that $\xi \in \Omega_j$ is equivalent to say that the orbit $\{\varphi^s(x,\xi)\}$ fills a torus of dimension j. We also define

$$R_\Lambda := \Lambda^\perp \cap \Omega_{2-\mathrm{rk}(\Lambda)}.$$

Note that, for Λ of rank 1, we have $R_\Lambda = \Lambda^\perp - \{0\}$, and that we have the following partition of $\mathbb{R}^2$ indexed by the primitive sublattices of $\mathbb{Z}^2$:

$$\mathbb{R}^2 = \bigsqcup_{\Lambda\in\mathcal{L}} R_\Lambda.$$

Let us now decompose $F(t)$ according to this partition of $T^*\mathbb{T}^2$. In fact, the above discussion and the fact that $F(t)$ is a Radon measure allow to write two natural decompositions of $F(t)$:

$$F(t) = \sum_{\Lambda\in\mathcal{L}} F(t)\rceil_{\mathbb{T}^2\times R_\Lambda}, \tag{30}$$

and its Fourier decomposition

$$F(t,x,\xi)=\sum_{k\in\mathbb{Z}^2}\widehat{F}(t,k,\xi)e^{2i\pi k.x}.$$

For $\Lambda\in\mathcal{L}$, we denote by $\mathcal{I}_\Lambda(F(t))$ the distribution

$$\mathcal{I}_\Lambda(F(t)):=\sum_{k\in\Lambda}\widehat{F}(t,k,\xi)e^{2i\pi k.x}.$$

Note that this is consistent with the conventions we have introduced in section 2. The following result holds (see section 2 in [4]):

Proposition 5.4. *For a.e. $t\in\mathbb{R}$, one has*

1. *for every $\Lambda\in\mathcal{L}$, the distribution $\mathcal{I}_\Lambda(F(t))$ is a finite complex Radon measure on $T^*\mathbb{T}^2$;*
2. *every term in (30) is a finite complex Radon measure invariant by φ^s and*

$$F(t)\rceil_{\mathbb{T}^2\times R_\Lambda}=\mathcal{I}_\Lambda(F(t))\rceil_{\mathbb{T}^2\times R_\Lambda}. \tag{31}$$

Finally, property (31) is equivalent to the fact that $F(t)\rceil_{\mathbb{T}^2\times R_\Lambda}$ is invariant by the translations:

$$\tau^v:(x,\xi)\mapsto(x+v,\xi),\ \textit{for every } v\in\Lambda^\perp.$$

Remark 5.5. The proof in [4] was given for finite positive measure but it can be adapted verbatim to fit our framework where we have to deal with finite complex Radon measures.

To summarize, we can decompose the distribution we are interested in as follows:

$$F(t)=\mathcal{I}_0(F(t))\rceil_{\mathbb{T}^2\times\Omega_2}+\sum_{\Lambda:\text{rk}(\Lambda)=1}\mathcal{I}_\Lambda(F(t))\rceil_{\mathbb{T}^2\times\Lambda^\perp-\{0\}}. \tag{32}$$

Note that we do not have any term associated with $\Lambda=\mathbb{Z}^2$ thanks to the frequency assumption (6) – see (27). Recall from (29) that we are interested in determining $\langle F(t),V-\int_{\mathbb{T}^2}V\rangle$ or more precisely

$$\int_0^t e^{-it'\int_{\mathbb{T}^2}V}\left\langle F(t'),V-\int_{\mathbb{T}^2}V\right\rangle dt'=\sum_{\Lambda:\text{rk}(\Lambda)=1}\int_0^t e^{-it'\int_{\mathbb{T}^2}V}\left\langle F(t'),\mathcal{I}_\Lambda(V)-\int_{\mathbb{T}^2}V\right\rangle dt'.$$

Therefore, applying (32), we have shown the following which is **the main observation of this section**:

Proposition 5.6. *Using the above conventions, we have*

$$\begin{aligned}\lim_{\hbar\to0^+}\langle F_\hbar(t\tau_\hbar^c),1\rangle &= e^{it\int_{\mathbb{T}^2}V}\langle F_0,1\rangle\\ &+ i\sum_{\Lambda:\text{rk}(\Lambda)=1}\int_0^t e^{i(t-t')\int_{\mathbb{T}^2}V}\left\langle \mathcal{I}_\Lambda(F(t'))\rceil_{\mathbb{T}^2\times\Lambda^\perp-\{0\}},\mathcal{I}_\Lambda(V)-\int_{\mathbb{T}^2}V\right\rangle dt'.\end{aligned}$$

Hence, this formula combined with (29) allows to reduce the problem to analyzing the fidelity distribution along the submanifolds $\mathbb{T}^2 \times \Lambda^\perp$ (for every rank 1 sublattice). In order to this, we will proceed to a *second microlocalization along these submanifolds* following the strategies from [18, 4, 1].

6. Set-up of the two-microlocal tools

We will introduce two-microlocal objects in order to proceed to the analysis of the fidelity distribution near the submanifolds $\mathbb{T}^2 \times \Lambda^\perp \subset T^*\mathbb{T}^2$. For that purpose, we will make use of the tools developped in [18, 4, 1] that we will briefly review in this section using the conventions from [20]. We note that the main differences with these references are the choice of rescaling for the second microlocalization, and the nature of the propagation relation that comes out of the analysis. For instance, the potential appears in the Hamiltonian dynamics induced along Λ in reference [20] while here it will play a role as a phase factor – see Proposition 7.2.

To proceed with our analysis, we fix Λ a primitive sublattice of rank 1 and we denote by $\widehat{\mathbb{R}}$ the compactified space $\mathbb{R} \cup \{\pm\infty\}$. Then, we introduce an auxiliary distribution, for every $b \in \mathcal{C}_c^\infty(T^*\mathbb{T}^2 \times \widehat{\mathbb{R}})$,

$$\langle F_{\Lambda,\hbar}(t\tau_\hbar^c), b\rangle := \left\langle \psi_\hbar^1, e^{\frac{it\tau_\hbar^c \widehat{P}_\epsilon(\hbar)}{\hbar}} \operatorname{Op}_\hbar\left(D_{\Lambda,\epsilon_\hbar \hbar^{-1}}(\mathcal{I}_\Lambda(b))\right) e^{-\frac{it\tau_\hbar^c \widehat{P}_0(\hbar)}{\hbar}} \psi_\hbar^2 \right\rangle_{L^2(\mathbb{T}^2)}.$$

Note that these quantities are slightly more general than the ones introduced in (10) as they depend on (t, ξ) and as they extend to $\pm\infty$. We shall compare with definition (10) in paragraph 6.3. One of the reasons for working with the compactified space $\widehat{\mathbb{R}}$ is to encompass the case where the function is independent of η that was introduced previously.

Remark 6.1. Recall that semiclassical measures involving a second microlocalization primarly appeared in [21, 22, 11, 12] outside the context of integrable systems discussed in the above references.

The purpose of this section is to study the properties of these distributions and their relation to the fidelity distributions we have already defined. We proceed in several stages. First, we recall how one can extract converging subsequences and we explain how to decompose the limit distribution into two components (paragraph 6.1): the "compact" component and the one at infinity. After that, we show some invariance properties of the limiting distributions and relate these quantities to the ones we are primarly interested in (paragraph 6.2).

Remark 6.2. We point out the following useful observations. First, if $b(x, \xi, \eta) = a(x, \xi)$ is an element in $\mathcal{C}_c^\infty(T^*\mathbb{T}^2)$, we recover the fidelity distribution we have introduced before. Hence, this new distribution should be understood as a generalization of $F_\hbar(t\tau_\hbar^c)$ which captures some informations on the distribution near $\mathbb{T}^2 \times \Lambda^\perp$. We also note that this quantity is well defined as the standard quantization allows us to consider any observable which has bounded derivatives in the x variables – see appendix A. Finally, with a small abuse of notations, we emphasize that

$$\mathrm{Op}_\hbar\left(D_{\Lambda,\epsilon_\hbar\hbar^{-1}}(\mathcal{I}_\Lambda(b))\right) := \mathrm{Op}_\hbar\left(b\left(x,\xi,\frac{\hbar H_\Lambda(\xi)}{\epsilon_\hbar}\right)\right) = \mathrm{Op}_{\hbar^2\epsilon_\hbar^{-1}}\left(b\left(x,\frac{\epsilon_\hbar\xi}{\hbar},H_\Lambda(\xi)\right)\right),$$

where we remind that $\hbar^2 \ll \epsilon_\hbar \ll \hbar$. In particular, under this form, we get an $\hbar^2\epsilon_\hbar^{-1}$-pseudodifferential operator with a nice symbol (meaning with no blow up of the derivatives as $\hbar \to 0$). In the following, we will often make use of this observation to estimate the remainder terms in our semiclassical formulas.

6.1. **Extracting converging subsequences.** As for the semiclassical fidelity distributions, we would like to extract subsequences $\hbar_n \to 0$ such that $F_{\Lambda,\hbar_n}(t\tau^c_{\hbar_n})$ converges in a certain weak sense. For that purpose, we shall follow the more or less standard procedures of [13, 17, 27] in the case of semiclassical measures. We denote by

$$\mathcal{B} := \mathcal{C}^0_0(\mathbb{R}^2 \times \widehat{\mathbb{R}}, \mathcal{C}^3(\mathbb{T}^2)),$$

the space of continuous function on $\mathbb{R}^2 \times \widehat{\mathbb{R}}$ with values in $\mathcal{C}^3(\mathbb{T}^2)$ and which tends to 0 at infinity[9]. We endow this space with its natural topology of Banach space. According to the Calderón-Vaillancourt Theorem from the appendix, one knows that, for every $\tilde{b} \in L^1(\mathbb{R}, \mathcal{B})$, one has

$$\int_{\mathbb{R}} |\langle F_{\Lambda,\hbar}(t\tau_\hbar), \tilde{b}(t)\rangle| dt \leq C \int_{\mathbb{R}} \|\tilde{b}(t)\|_{\mathcal{B}} dt. \tag{33}$$

In other words, the map $t \mapsto F_{\Lambda,\hbar}(t\tau_\hbar)$ defines a bounded sequence in $L^1(\mathbb{R}, \mathcal{B})'$ endowed with its weak-$\star$ topology. Hence, after extracting a subsequence, one finds that there exists for a.e. t in $\mathbb{R}$ some element $F_\Lambda(t)$ in $\mathcal{B}'$ such that, for every $\tilde{b}$ in $L^1(\mathbb{R}, \mathcal{B})$, one has

$$\lim_{\hbar\to 0^+} \int_{\mathbb{R}\times T^*\mathbb{T}^2\times\widehat{\mathbb{R}}} \tilde{b}(t,x,\xi,\eta) F_{\Lambda,\hbar}(t\tau_\hbar, dx, d\xi, d\eta) dt = \int_{\mathbb{R}} \left(\int_{T^*\mathbb{T}^2\times\widehat{\mathbb{R}}} \tilde{b}(t,x,\xi,\eta) F_\Lambda(t, dx, d\xi, d\eta)\right) dt.$$

Remark 6.3. Here, we used the first part of the Calderón-Vaillancourt. If we had used the second part, it would just have changed slightly the Banach space involved in our argument.

By a density argument, one can find that, for every θ in $L^1(\mathbb{R})$ and for every b in $\mathcal{C}^\infty_c(T^*\mathbb{T}^2 \times \widehat{\mathbb{R}})$,

$$\lim_{\hbar\to 0^+} \int_{\mathbb{R}} \theta(t) \langle F_{\Lambda,\hbar}(t\tau_\hbar), b\rangle dt = \int_{\mathbb{R}} \theta(t)\langle F_\Lambda(t), b\rangle dt.$$

These limiting functionals are related to $F(t)$ in the following manner:

$$F(t) = \int_{\widehat{\mathbb{R}}} F_\Lambda(t, d\eta). \tag{34}$$

The main objective of the rest of the article is to describe the properties of $F_\Lambda(t)$ in function of t and of the initial data. Regarding this aim, it is convenient to split these two-microlocal distributions in two parts: the "compact" one and the one at "infinity" (in the $\widehat{\mathbb{R}}$ variable).

[9]Here, infinity means in the variables corresponding to $\mathbb{R}^2$ as $\widehat{\mathbb{R}}$ is compact.

Before doing that, we observe that, up to another extraction, we can also suppose that there exists F_Λ^0 in $\mathcal{B}'$ such that, for every b in $\mathcal{C}_c^\infty(T^*\mathbb{T}^2 \times \widehat{\mathbb{R}})$,

$$\lim_{\hbar\to 0^+} \langle F_{\Lambda,\hbar}(0), b\rangle = \langle F_\Lambda^0, b\rangle.$$

Therefore, we would like to relate $F_\Lambda(t)$ to F_Λ^0. Recall that $F_\Lambda(t)$ is defined almost everywhere and we will derive an explicit expression for it in terms of F_Λ^0 by showing that it verifies a certain transport equation. Finally, up to some diagonal extraction argument, we can suppose that these differerent linear functionals converge for any primitive sublattice Λ of rank 1 along the same subsequence.

At this point, $F_\Lambda(t)$ is *only an element in the dual of* $\mathcal{B}$ *and not a priori a complex Radon measure*. Yet, this can be overcome via the application of the second part of the Calderón-Vaillancourt Theorem A.3. In fact, thanks to this Theorem, one knows that, for every θ in $L^1(\mathbb{R})$ and for every $b \in \mathcal{C}_c^\infty(T^*\mathbb{T}^2 \times \widehat{\mathbb{R}})$, one has

$$\left|\int_{\mathbb{R}} \theta(t)\langle F_{\Lambda,\hbar}(t\tau_\hbar), b\rangle dt\right| \leq C\int_{\mathbb{R}} |\theta(t)| \|b\|_{\mathcal{C}^0} dt + \mathcal{O}(\hbar^2\epsilon_\hbar^{-1}).$$

Hence, passing to the limit $\hbar \to 0^+$, this gives

$$\left|\int_{\mathbb{R}} \theta(t)\langle F_\Lambda(t), b\rangle dt\right| \leq C\int_{\mathbb{R}} |\theta(t)| \|b\|_{\mathcal{C}^0} dt.$$

We emphasize that we crucially use here that $\hbar^2 \ll \epsilon_\hbar$, which yields comfortable simplifications compared with [4, 1] where $\epsilon_\hbar = \hbar^2$ (and thus the limit objects are not a priori measures). In particular, we find that, for a.e. t in $\mathbb{R}$, $F_\Lambda(t)$ belongs to $\mathcal{M}(T^*\mathbb{T}^2 \times \widehat{\mathbb{R}})$. Thus, we can split $F_\Lambda(t)$ as

$$F_\Lambda(t) = \tilde{F}_\Lambda(t) + \tilde{F}^\Lambda(t), \tag{35}$$

where

$$\tilde{F}_\Lambda(t) := F_\Lambda(t)\rceil_{\mathbb{T}^2\times\mathbb{R}^2\times\mathbb{R}}, \text{ and } \tilde{F}^\Lambda(t) := F_\Lambda(t)\rceil_{\mathbb{T}^2\times\mathbb{R}^2\times\{\pm\infty\}}.$$

Finally, we can split similarly F_Λ^0 in two parts that we denote by $\tilde{F}_\Lambda^0$ and $\tilde{F}^{\Lambda,0}$.

6.2. **First invariance properties.** Recall from Proposition 5.6 that we aim at describing

$$\mathcal{I}_\Lambda(F(t))\rceil_{\mathbb{T}^2\times\Lambda^\perp-\{0\}}.$$

Thanks to (34), this is related to the two-microlocal quantities we have introduced as follows:

$$\mathcal{I}_\Lambda(F(t))\rceil_{\mathbb{T}^2\times\Lambda^\perp-\{0\}} = \int_{\mathbb{R}} \mathcal{I}_\Lambda(\tilde{F}_\Lambda)(t,d\eta)\rceil_{\mathbb{T}^2\times\Lambda^\perp-\{0\}} + \int_{\{\pm\infty\}} \mathcal{I}_\Lambda(\tilde{F}^\Lambda)(t,d\eta)\rceil_{\mathbb{T}^2\times\Lambda^\perp-\{0\}}. \tag{36}$$

As a first step, we will show the following result concerning the part at infinity which is the analogue in our context of [4, Th. 13(ii)]:

Lemma 6.4. *Let* Λ *be rank* 1 *primitive sublattice. Then, for every* k *in* $\Lambda - \{0\}$, *for every* g *in* $\mathcal{C}_c^\infty(\mathbb{R}^2 \times \widehat{\mathbb{R}})$ *and for a.e.* t *in* $\mathbb{R}$,

$$\langle \tilde{F}^\Lambda(t), g(\xi,\eta)e^{-2i\pi k.x}\rangle = 0.$$

Before proving this Lemma, observe that it allows us to rewrite the quantities appearing in Proposition 5.6 as

(37)
$$\left\langle \mathcal{I}_\Lambda(F(t'))\rceil_{\mathbb{T}^2\times\Lambda^\perp-\{0\}}, \mathcal{I}_\Lambda(V) - \int_{\mathbb{T}^2} V \right\rangle = \left\langle \int_{\mathbb{R}} \mathcal{I}_\Lambda(\tilde{F}_\Lambda(t', d\eta))\rceil_{\mathbb{T}^2\times\Lambda^\perp-\{0\}}, \mathcal{I}_\Lambda(V) - \int_{\mathbb{T}^2} V \right\rangle,$$

for a.e. t in $\mathbb{R}$. Recall from Proposition 5.6 that this is exactly what remains to be computed if we want to find an expression for the Quantum Loschmidt Echo at the critical time scale $\tau_\hbar^c$. Hence, our analysis boils down to the description of the "compact" part $\tilde{F}_\Lambda(t)$ of $F_\Lambda(t)$.

Proof. Let χ be a smooth cutoff function on $\mathbb{R}$ which is equal to 1 near 0 and to 0 outside a slightly bigger neighborhood of 0. We fix g in $\mathcal{C}_c^\infty(\mathbb{R}^2 \times \widehat{\mathbb{R}})$ and $R > 0$. Then, we define

$$g^R(\xi, \eta) = \left(1 - \chi\left(\frac{\eta}{R}\right)\right) g(\xi, \eta).$$

Let Λ be a primitive rank one sublattice and $k \in \Lambda - \{0\}$. Recall that we use the standard quantization (see appendix A for a brief reminder). Hence, we have the identity

$$\begin{aligned} &\frac{d}{dt}\left\langle F_{\Lambda,\hbar}(t\tau_\hbar^c), \frac{g^R(\xi,\eta)e^{-2i\pi k.x}}{\eta}\right\rangle \\ (38)\quad &= \frac{i\tau_\hbar^c}{\hbar}\left\langle \psi_\hbar^1, e^{\frac{it\tau_\hbar^c \widehat{P}_\epsilon(\hbar)}{\hbar}} \left[-\frac{\hbar^2\Delta}{2}, \mathrm{Op}_\hbar\left(\frac{\epsilon_\hbar}{\hbar H_\Lambda(\xi)} g^R\left(\xi, \frac{\hbar H_\Lambda(\xi)}{\epsilon_\hbar}\right) e^{-2i\pi k.x}\right)\right] e^{-\frac{it\tau_\hbar^c \widehat{P}_0(\hbar)}{\hbar}} \psi_\hbar^2\right\rangle \\ &+ i\left\langle \psi_\hbar^1, e^{\frac{it\tau_\hbar^c \widehat{P}_\epsilon(\hbar)}{\hbar}} \mathrm{Op}_\hbar\left(\frac{\epsilon_\hbar}{\hbar H_\Lambda(\xi)} g^R\left(\xi, \frac{\hbar H_\Lambda(\xi)}{\epsilon_\hbar}\right) e^{-2i\pi k.x} V(x)\right) e^{-\frac{it\tau_\hbar^c \widehat{P}_0(\hbar)}{\hbar}} \psi_\hbar^2\right\rangle. \end{aligned}$$

For the first term on the right hand side, we have, using the composition formula for the standard quantization (see appendix A), that

$$\begin{aligned} &\left[-\frac{\hbar^2\Delta}{2}, \mathrm{Op}_\hbar\left(\frac{\epsilon_\hbar}{\hbar H_\Lambda(\xi)} g^R\left(\xi, \frac{\hbar H_\Lambda(\xi)}{\epsilon_\hbar}\right) e^{-2i\pi k.x}\right)\right] \\ &= \mathrm{Op}_\hbar\left(g^R\left(\xi, \frac{\hbar H_\Lambda(\xi)}{\epsilon_\hbar}\right) e^{-2i\pi k.x}\left(-2\epsilon_\hbar \pi H_\Lambda(k) - \frac{2\pi^2\hbar^2\epsilon_\hbar H_\Lambda(k)^2}{\hbar H_\Lambda(\xi)}\right)\right), \end{aligned}$$

where we used the fact that k belongs to Λ. Thanks to the Calderón-Vaillancourt Theorem A.3, we can deduce that

$$\begin{aligned} &\left[-\frac{\hbar^2\Delta}{2}, \mathrm{Op}_\hbar\left(\frac{\epsilon_\hbar}{\hbar H_\Lambda(\xi)} g^R\left(\xi, \frac{\hbar H_\Lambda(\xi)}{\epsilon_\hbar}\right) e^{-2i\pi k.x}\right)\right] \\ &\quad = -2\epsilon_\hbar \pi H_\Lambda(k)\,\mathrm{Op}_\hbar\left(g^R\left(\xi, \frac{\hbar H_\Lambda(\xi)}{\epsilon_\hbar}\right) e^{-2i\pi k.x}\right) + \mathcal{O}_{L^2\to L^2}(\hbar^2 R^{-1}). \end{aligned}$$

Similarly, we find using the Calderón-Vaillancourt Theorem that

$$\mathrm{Op}_\hbar\left(\frac{\epsilon_\hbar}{\hbar H_\Lambda(\xi)} g^R\left(\xi, \frac{\hbar H_\Lambda(\xi)}{\epsilon_\hbar}\right) e^{-2i\pi k.x} V(x)\right) = \mathcal{O}_{L^2\to L^2}(R^{-1}).$$

Implementing these two equalities in (38), we find that

$$\begin{aligned}(39)\qquad &\frac{d}{dt}\left\langle F_{\Lambda,\hbar}(t\tau_\hbar^c), \frac{g^R(\xi,\eta)e^{-2i\pi k.x}}{\eta}\right\rangle \\ &= -2i\pi H_\Lambda(k)\left\langle F_{\Lambda,\hbar}(t\tau_\hbar), g^R(\xi,\eta)e^{-2i\pi k.x}\right\rangle + \mathcal{O}\left(R^{-1}\left(\tau_\hbar^c\hbar+1\right)\right).\end{aligned}$$

Recall that $\tau_\hbar^c = \frac{\hbar}{\epsilon_\hbar} \leq \frac{1}{\hbar}$. Hence, the remainder in this equality is of order $\mathcal{O}(R^{-1})$. Another application of the Calderón-Vaillancourt Theorem yields

$$\left\langle F_{\Lambda,\hbar}(t\tau_\hbar^c), \frac{g^R(\xi,\eta)e^{-2i\pi k.x}}{\eta}\right\rangle = \mathcal{O}(R^{-1}).$$

Thus, if we fix θ in $\mathcal{C}_c^\infty(\mathbb{R})$, we find after integrating by parts in (39) that

$$\int_{\mathbb{R}} \theta(t)\left\langle F_{\Lambda,\hbar}(t\tau_\hbar^c), g^R(\xi,\eta)e^{-2i\pi k.x}\right\rangle dt = \mathcal{O}(R^{-1}).$$

Finally, if we let $\hbar$ go to 0 and R to $+\infty$ (in this order), we find

$$\int_{\mathbb{R}} \theta(t)\left\langle F_\Lambda(t)\rceil_{T^*\mathbb{T}^2\times\{\pm\infty\}}, g(\xi,\eta)e^{-2i\pi k.x}\right\rangle dt = 0.$$

This is valid for any θ in $\mathcal{C}_c^\infty(\mathbb{R})$ and thus concludes the proof of the Lemma. □

We conclude this section by showing that $F_\Lambda(t)$ is also invariant under the geodesic flow

Lemma 6.5. *Let Λ be a rank 1 primitive sublattice. Then, for every b in $\mathcal{C}_c^\infty(T^*\mathbb{T}^2\times\widehat{\mathbb{R}})$ and for a.e. t in $\mathbb{R}$, one has*

$$\langle F_\Lambda(t), \xi.\partial_x b\rangle = 0.$$

Equivalently, $F_\Lambda(t)$ is invariant by the geodesic flow on $T^*\mathbb{T}^2$.

Proof. Let b be an element in $\mathcal{C}_c^\infty(T^*\mathbb{T}^2\times\widehat{\mathbb{R}})$. Recall that we use the standard quantization and we have the identity

$$\begin{aligned}(40)\qquad &\frac{d}{dt}\langle F_{\Lambda,\hbar}(t\tau_\hbar^c), b\rangle \\ &= \frac{i\tau_\hbar^c}{\hbar}\left\langle \psi_\hbar^1, e^{\frac{it\tau_\hbar^c\widehat{P}_\epsilon(\hbar)}{\hbar}}\left[-\frac{\hbar^2\Delta}{2}, \mathrm{Op}_\hbar\left(D_{\Lambda,\epsilon_\hbar\hbar^{-1}}(\mathcal{I}_\Lambda(b))\right)\right]e^{\frac{it\tau_\hbar^c\widehat{P}_0(\hbar)}{\hbar}}\psi_\hbar^2\right\rangle \\ &\quad + i\left\langle \psi_\hbar^1, e^{\frac{it\tau_\hbar^c\widehat{P}_\epsilon(\hbar)}{\hbar}}\mathrm{Op}_\hbar\left(D_{\Lambda,\epsilon_\hbar\hbar^{-1}}(\mathcal{I}_\Lambda(b))V\right)e^{\frac{it\tau_\hbar^c\widehat{P}_0(\hbar)}{\hbar}}\psi_\hbar^2\right\rangle.\end{aligned}$$

Also, observe from the composition formula that

$$\left[-\frac{\hbar^2\Delta}{2}, \mathrm{Op}_\hbar\left(D_{\Lambda,\epsilon_\hbar\hbar^{-1}}(\mathcal{I}_\Lambda(b))\right)\right] = \mathrm{Op}_\hbar\left(\left(\frac{\hbar}{i}\xi\cdot\partial_x - \frac{\hbar^2}{2}\Delta_x\right)b\left(x,\xi,\frac{\hbar H_\Lambda(\xi)}{\epsilon_\hbar}\right)\right).$$

Hence, applying the Calderón-Vaillancourt Theorem (which only involves derivatives in the x variables here) implies that the first term on the RHS dominates, as $\hbar\to 0^+$. We find that

$$\frac{d}{dt}\langle F_\hbar(t\tau_\hbar^c), b\rangle = \tau_\hbar^c\langle F_\hbar(t\tau_\hbar^c), \xi.\partial_x b\rangle + \mathcal{O}(1) = \tau_\hbar^c\langle F_\hbar(t\tau_\hbar^c), (\xi.\partial_x b)\rangle + \mathcal{O}(1).$$

□

6.3. **Relation with $\mathbf{F}^0_\Lambda$.** In this section, we introduced a quantity $\tilde{F}^0_\Lambda$ which is slightly different from the one we considered in (10). Recall that $\mathbf{F}^0_\Lambda$ depended only on the (x,η) variables (while $\tilde{F}^0_\Lambda$ also depends on ξ) and that it was not supported at $\eta=\pm\infty$ (which is also the case for $\tilde{F}^0_\Lambda$). Yet, both quantities are related thanks to the frequency assumption (7). In fact, this hypothesis implies that, for $R>0$ and for c in $\mathcal{C}^\infty_c(\mathbb{T}^2\times\mathbb{R})$,

$$\begin{aligned}\langle\psi_\hbar,\mathrm{Op}_\hbar\left(D_{\Lambda,\epsilon_\hbar\hbar^{-1}}(\mathcal{I}_\Lambda(c))\right)\psi_\hbar\rangle &= \left\langle\psi_\hbar,\mathrm{Op}_\hbar\left(D_{\Lambda,\epsilon_\hbar\hbar^{-1}}(\mathcal{I}_\Lambda(c))\right)\chi(-\hbar^2\Delta/R)\psi_\hbar\right\rangle\\ &+ r(R,\hbar),\end{aligned}$$

where χ is a smooth and compactly supported function equal to 1 near 0 and where $\lim_{R\rightarrow+\infty}\limsup_{\hbar\rightarrow 0^+} r(R,\hbar)=0$. Applying the composition rules for pseudodifferential operators and letting $\hbar\rightarrow 0$, we find that

$$\langle\mathbf{F}^0_\Lambda,c\rangle=\langle\mathcal{I}_\Lambda(\tilde{F}^0_\Lambda),c(x,\eta)\chi(\|\xi\|^2/R)\rangle+o(1),$$

as $R\rightarrow+\infty$. From the dominated convergence Theorem, we conclude that

$$\mathbf{F}^0_\Lambda(x,\eta)=\int_{\mathbb{R}^2}\mathcal{I}_\Lambda(\tilde{F}^0_\Lambda)(x,d\xi,\eta). \tag{41}$$

7. Proof of Theorem 2.1

Thanks to Proposition 5.6 and to (37), it now remains to determine $\mathcal{I}_\Lambda(\tilde{F}_\Lambda(t))$ in terms of t and of $\tilde{F}^0_\Lambda$ in order to conclude the proof of Theorem 2.1. This will be the main purpose of this section. After deriving the exact expression for $\mathcal{I}_\Lambda(\tilde{F}_\Lambda(t))$, we will explain how to prove Theorem 2.1 in paragraph 7.3.

7.1. **Preliminary remarks.** Rather than $\tilde{F}_\Lambda(t)$, we are interested in the restriction of $\mathcal{I}_\Lambda(\tilde{F}_\Lambda(t))$ to $\mathbb{T}^2\times\Lambda^\perp-\{0\}\times\mathbb{R}$. Yet, this distinction is essentially irrelevant due to the following observation:

Lemma 7.1. *Let Λ be a rank 1 primitive sublattice. For every $b\in\mathcal{C}^\infty_c(T^*\mathbb{T}^2\times\mathbb{R})$ whose support does not intersect $\mathbb{T}^2\times\Lambda^\perp\times\mathbb{R}$, and for a.e. t in $\mathbb{R}$, one has*

$$\langle F_\Lambda(t),b\rangle=0.$$

From this lemma, we deduce that

$$\tilde{F}_\Lambda(t)=\tilde{F}_\Lambda(t)\rceil_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{R}}=\tilde{F}_\Lambda(t)\rceil_{\mathbb{T}^2\times\Lambda^\perp-\{0\}\times\mathbb{R}},$$

where the second equality follows from the frequency assumptions (6). Recall that $\tilde{F}_\Lambda(t)$ is an invariant complex Radon measure. Hence, thanks to Proposition 5.4 and to Lemma (6.5), we can write

$$\tilde{F}_\Lambda(t)=\mathcal{I}_\Lambda(\tilde{F}_\Lambda(t))\rceil_{\mathbb{T}^2\times\Lambda^\perp-\{0\}\times\mathbb{R}}. \tag{42}$$

Proof. Let b be an element in $\mathcal{C}_c^\infty(T^*\mathbb{T}^2\times\mathbb{R})$ whose support does not intersect $\mathbb{T}^2\times\Lambda^\perp\times\mathbb{R}$. It is sufficient to observe that

$$\langle F_{\Lambda,\hbar}(t\tau_\hbar^c), b\rangle = \left\langle \psi_\hbar^1, e^{\frac{it\tau_\hbar^c\widehat{P}_\epsilon(\hbar)}{\hbar}}\operatorname{Op}_\hbar\left(b\left(x,\xi,\frac{\hbar H_\Lambda(\xi)}{\epsilon_\hbar}\right)\right)e^{\frac{it\tau_\hbar^c\widehat{P}_\epsilon(\hbar)}{\hbar}}\psi_\hbar^2\right\rangle$$

is equal to 0 for $\hbar$ small enough thanks to our support assumption on a. □

7.2. **Propagation formulas for $\tilde{F}_\Lambda(t)$.** We can now express $\tilde{F}_\Lambda(t)$ in terms of t and of the initial data:

Proposition 7.2. *Let Λ be a rank one primitive sublattice. Then $t\mapsto\tilde{F}_\Lambda(t)$ is continuous and one has, for every b in $\mathcal{C}_c^\infty(T^*\mathbb{T}^2\times\mathbb{R})$*

$$\langle\tilde{F}_\Lambda(t),\mathcal{I}_\Lambda(a)\rangle = \left\langle\mathcal{I}_\Lambda(\tilde{F}_\Lambda^0)\left(x-t\frac{\eta\vec{v}_\Lambda}{L_\Lambda},\xi,\eta\right)e^{i\int_0^t\mathcal{I}_\Lambda(V)\left(x-s\frac{\eta\vec{v}_\Lambda}{L_\Lambda}\right)ds},\mathcal{I}_\Lambda(b)\right\rangle$$

Proof. Let $b(x,\xi,\eta)$ be an element in $\mathcal{C}_c^\infty(T^*\mathbb{T}^2\times\mathbb{R})$. In order to derive our equation, we start by differentiating the map $t\mapsto\langle F_{\Lambda,\hbar}(t\tau_\hbar^c),\mathcal{I}_\Lambda(b)\rangle$. Recalling that we are using the standard quantization, we find that

$$\begin{aligned}&\frac{d}{dt}\langle F_{\Lambda,\hbar}(t\tau_\hbar^c),\mathcal{I}_\Lambda(b)\rangle\\ =&\frac{i\tau_\hbar^c}{\hbar}\left\langle\psi_\hbar^1,e^{\frac{it\tau_\hbar^c\widehat{P}_\epsilon(\hbar)}{\hbar}}\left[-\frac{\hbar^2\Delta}{2},\operatorname{Op}_\hbar\left(D_{\Lambda,\epsilon_\hbar\hbar^{-1}}(\mathcal{I}_\Lambda(b))\right)\right]e^{\frac{it\tau_\hbar^c\widehat{P}_0(\hbar)}{\hbar}}\psi_\hbar^2\right\rangle\\ &+i\left\langle\psi_\hbar^1,e^{\frac{it\tau_\hbar^c\widehat{P}_\epsilon(\hbar)}{\hbar}}\operatorname{Op}_\hbar\left(D_{\Lambda,\epsilon_\hbar\hbar^{-1}}(\mathcal{I}_\Lambda(b))V\right)e^{\frac{it\tau_\hbar^c\widehat{P}_0(\hbar)}{\hbar}}\psi_\hbar^2\right\rangle.\end{aligned}\tag{43}$$

From the commutation formula for pseudodifferential operators (see Theorem A.4), we know that

$$\begin{aligned}&\left[-\frac{\hbar^2\Delta}{2},\operatorname{Op}_\hbar\left(D_{\Lambda,\epsilon_\hbar\hbar^{-1}}(\mathcal{I}_\Lambda(b))\right)\right]\\ =&\frac{\epsilon_\hbar}{i}\operatorname{Op}_\hbar\left(\left(\frac{\hbar H_\Lambda(\xi)}{\epsilon_\hbar}\frac{\vec{v}_\Lambda}{L_\Lambda}\cdot\partial_x+\frac{\hbar^2}{2i\epsilon_\hbar}\left(\frac{\vec{v}_\Lambda}{L_\Lambda}\cdot\partial_x\right)^2\right)\mathcal{I}_\Lambda(b)\left(x,\xi,\frac{\hbar}{\epsilon_\hbar H_\Lambda(\xi)}\right)\right),\end{aligned}$$

where we used that $\mathcal{I}_\Lambda(b)$ has only Fourier coefficients along Λ. Combining this formula with (43), we find that

$$\frac{d}{dt}\langle F_{\Lambda,\hbar}(t\tau_\hbar^c),\mathcal{I}_\Lambda(b)\rangle=\left\langle F_{\Lambda,\hbar}(t\tau_\hbar^c),\left(\eta\frac{\vec{v}_\Lambda}{L_\Lambda}\cdot\partial_x+\frac{\hbar^2}{2i\epsilon_\hbar}\left(\frac{\vec{v}_\Lambda}{L_\Lambda}\cdot\partial_x\right)^2+iV\right)\mathcal{I}_\Lambda(b)\right\rangle.$$

This can be rewritten as

$$\frac{1}{i}\frac{d}{dt}\langle F_{\Lambda,\hbar}(t\tau_\hbar^c),\mathcal{I}_\Lambda(b)\rangle=\left\langle F_{\Lambda,\hbar}(t\tau_\hbar^c),\left(\eta\frac{\vec{v}_\Lambda}{iL_\Lambda}\cdot\partial_x-\frac{\hbar^2}{2\epsilon_\hbar}\left(\frac{\vec{v}_\Lambda}{L_\Lambda}\cdot\partial_x\right)^2+V\right)\mathcal{I}_\Lambda(b)\right\rangle.$$

We can now integrate this relation against a smooth function θ in $\mathcal{C}_c^\infty(\mathbb{R})$. We use that $\epsilon_\hbar \gg \hbar^2$ and we find that, for every θ in $\mathcal{C}_c^\infty(\mathbb{R})$,

$$-\int_{\mathbb{R}} \theta'(t)\langle \tilde{F}_\Lambda(t), \mathcal{I}_\Lambda(b)\rangle dt = \int_{\mathbb{R}} \theta(t) \left\langle \tilde{F}_\Lambda(t), \left(\eta \frac{\vec{v}_\Lambda}{L_\Lambda} \cdot \partial_x + iV\right) \mathcal{I}_\Lambda(b) \right\rangle dt.$$

From Lemma 6.5 and as $\tilde{F}_\Lambda(t)$ is a finite complex Radon measure supported in $\mathbb{T}^2 \times \Lambda^\perp - \{0\} \times \mathbb{R}$, we find that this is equivalent to

$$-\int_{\mathbb{R}} \theta'(t)\langle \tilde{F}_\Lambda(t), \mathcal{I}_\Lambda(b)\rangle dt = \int_{\mathbb{R}} \theta(t) \left\langle \tilde{F}_\Lambda(t), \left(\eta \frac{\vec{v}_\Lambda}{L_\Lambda} \cdot \partial_x + i\mathcal{I}_\Lambda(V)\right) \mathcal{I}_\Lambda(b) \right\rangle dt,$$

which implies Proposition 7.2. □

7.3. **Conclusion.** Recall that we are primarly interested in describing the quantum fidelity distribution at the critical time scale $\tau_\hbar^c = \frac{\hbar}{\epsilon_\hbar}$. From Proposition 5.6 and (42), one has

$$\begin{aligned}\langle F(t), 1\rangle &= e^{it\int_{\mathbb{T}^2} V}\langle F(0), 1\rangle \\ &+ ie^{it\int_{\mathbb{T}^2} V} \int_0^t e^{-is\int_{\mathbb{T}^2} V} \sum_{\Lambda:\mathrm{rk}(\Lambda)=1} \left\langle \int_{\mathbb{R}^2\times\mathbb{R}} \mathcal{I}_\Lambda(\tilde{F}_\Lambda)(s, d\xi, d\eta), \mathcal{I}_\Lambda(V) - \int_{\mathbb{T}^2} V \right\rangle ds.\end{aligned}$$

Thanks to Proposition 7.2, this can be rewritten as

$$\begin{aligned}\langle F(t), 1\rangle &= e^{it\int_{\mathbb{T}^2} V}\langle F(0), 1\rangle \\ &+ e^{it\int_{\mathbb{T}^2} V} \int_0^t \sum_{\Lambda:\mathrm{rk}(\Lambda)=1} \left\langle \int_{\mathbb{R}^2\times\mathbb{R}} \mathcal{I}_\Lambda(\tilde{F}_\Lambda^0), \frac{d}{ds}\left(e^{i\int_0^s \left(\mathcal{I}_\Lambda(V)\left(x+s'\frac{\eta\vec{v}_\Lambda}{L_\Lambda}\right) - \int_{\mathbb{T}^2} V dx\right) ds'}\right)\right\rangle ds,\end{aligned}$$

which yields

$$\begin{aligned}\langle F(t), 1\rangle &= e^{it\int_{\mathbb{T}^2} V}\left(\langle F(0), 1\rangle - \sum_{\Lambda:\mathrm{rk}(\Lambda)=1} \langle \tilde{F}_\Lambda^0, 1\rangle\right) \\ &+ \sum_{\Lambda:\mathrm{rk}(\Lambda)=1} \int_{T^*\mathbb{T}^2\times\mathbb{R}} e^{i\int_0^t \mathcal{I}_\Lambda(V)\left(x+s\frac{\eta\vec{v}_\Lambda}{L_\Lambda}\right)ds} \mathcal{I}_\Lambda(\tilde{F}_\Lambda^0)(dx, d\xi, d\eta).\end{aligned}$$

Suppose now that we take $\psi_\hbar^1 = \psi_\hbar^2 = \psi_\hbar$ for the sequence of initial data. We are now in the framework of the introduction and we can make use of (41) to write

$$\begin{aligned}\langle F(t), 1\rangle &= e^{it\int_{\mathbb{T}^2} V}\left(\langle F(0), 1\rangle - \sum_{\Lambda:\mathrm{rk}(\Lambda)=1} \langle \mathbf{F}_\Lambda^0, 1\rangle\right) \\ &+ \sum_{\Lambda:\mathrm{rk}(\Lambda)=1} \int_{T^*\mathbb{T}^2\times\mathbb{R}} e^{i\int_0^t \mathcal{I}_\Lambda(V)\left(x+s\frac{\eta\vec{v}_\Lambda}{L_\Lambda}\right)ds} \mathbf{F}_\Lambda^0(dx, d\eta),\end{aligned}$$

which is exactly the content of Theorem 2.1 as $F(0) = 1$ in that case.

Appendix A. Background on semiclassical analysis

In this appendix, we give a brief reminder on semiclassical analysis and we refer to [27] (mainly Chapters 1 to 5) for a more detailed exposition. Given $\hbar > 0$ and a in $\mathcal{S}(\mathbb{R}^{2d})$ (the Schwartz class), one can define the standard quantization of a as follows:

$$\forall u \in \mathcal{S}(\mathbb{R}^d),\ \operatorname{Op}_\hbar(a)u(x) := \frac{1}{(2\pi\hbar)^d}\iint_{\mathbb{R}^{2d}} e^{\frac{i}{\hbar}\langle x-y,\xi\rangle} a\,(x,\xi)\,u(y)dyd\xi.$$

Remark A.1. We could use other quantization procedures like the Weyl's one [27]. The advantage of this quantization is that it has a simple action on trigonometric polynomials and that it behaves nicely with respect to multiplication by $V(x)$.

This definition can be extended to any observable a with uniformly bounded derivatives, i.e. such that for every $\alpha \in \mathbb{N}^{2d}$, there exists $C_\alpha > 0$ such that $\sup_{x,\xi}|\partial^\alpha a(x,\xi)| \leq C_\alpha$. More generally, we will use the convention, for every $m \in \mathbb{R}$ and every $k \in \mathbb{Z}$,

$$S^{m,k} := \left\{(a_\hbar(x,\xi))_{0<\hbar\leq 1}:\ \forall(\alpha,\beta)\in\mathbb{N}^d\times\mathbb{N}^d,\ \sup_{(x,\xi)\in\mathbb{R}^{2d};0<\hbar\leq 1}|\hbar^k\langle\xi\rangle^{-m}\partial_x^\alpha\partial_\xi^\beta a_\hbar(x,\xi)| < +\infty\right\},$$

where $\langle\xi\rangle := (1+\|\xi\|^2)^{1/2}$. For such symbols, $\operatorname{Op}_\hbar(a)$ defines a continuous operator $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$.

Remark A.2. We also note that we have the following relation that we use at different stages of our proof:

$$\forall\delta > 0,\ \forall a \in S^{m,k}, \operatorname{Op}_\hbar(a(x,\xi)) := \operatorname{Op}_{\hbar\delta^{-1}}(a(x,\delta\xi)). \tag{44}$$

Among the above symbols, we distinguish the family of $\mathbb{Z}^d$-periodic symbols that we denote by $S^{m,k}_{per}$. Note that any a in $\mathcal{C}^\infty(T^*\mathbb{T}^d)$ (with bounded derivatives) defines an element in $S^{0,0}_{per}$. According to Th. 4.19 in [27], for any $a \in S^{m,k}_{per}$, the operator $\operatorname{Op}_\hbar(a)$ maps trigonometric polynomials into a smooth $\mathbb{Z}^d$-periodic function, and more generally any smooth $\mathbb{Z}^d$-periodic function into a smooth $\mathbb{Z}^d$-periodic function. Thus, for every a in $S^{m,k}_{per}$, $\operatorname{Op}_\hbar(a)$ acts by duality on the space of distributions $\mathcal{D}'(\mathbb{T}^d)$. An important feature of this quantization procedure is that it defines a bounded operator on $L^2(\mathbb{T}^d)$:

Theorem A.3 (Calderón-Vaillancourt)**.** *There exists a constant $C_d > 0$ and an integer $D > 0$ such that, for every a in $S^{0,0}_{per}$, one has, for every $0 < \hbar \leq 1$,*

$$\|\operatorname{Op}_\hbar(a)\|_{L^2(\mathbb{T}^d)\to L^2(\mathbb{T}^d)} \leq C_d \sum_{|\alpha|\leq d+1}\|\partial_x^\alpha a\|_\infty,$$

and

$$\|\operatorname{Op}_\hbar(a)\|_{L^2(\mathbb{T}^d)\to L^2(\mathbb{T}^d)} \leq C_d \sum_{|\alpha|\leq D}\hbar^{\frac{|\alpha|}{2}}\|\partial^\alpha a\|_\infty.$$

The second part of the Theorem is in fact the "standard" Calderón-Vaillancourt Theorem whose proof can be found[10] in [27, Th. 5.5] while the first part can be found in [26].

[10]The proof in this reference is given for the Weyl quantization but the argument can be adapted for the standard quantization.

Proof. We recall the proof of the first part of the Theorem which is maybe less known [26] and we refer to [27, Chap. 5] for the second part of the Theorem. The first part follows from the fact that, for every k in $\mathbb{Z}^d$, one has

$$(\mathrm{Op}_\hbar(a)\, e_k)(x) = a(x, 2\pi\hbar k)e_k(x),$$

where $e_k(x) := e^{2i\pi k.x}$. The proof of this fact is given in chapter 4 of [27, Th. 4.19]. Once we have observed this, we can write, for every trigonometric polynomial u in $L^2(\mathbb{T}^d)$, its Fourier decomposition $u = \sum_{k\in\mathbb{Z}^d} \widehat{u}_k e_k$, and

$$\mathrm{Op}_\hbar(a)\, u = \sum_{k,l\in\mathbb{Z}^d} \widehat{u}_k \widehat{a}(l, 2\pi\hbar k)e_{k+l} = \sum_{p\in\mathbb{Z}^d} e_p \sum_{k\in\mathbb{Z}^d} \widehat{u}_k \widehat{a}(p-k, 2\pi\hbar k),$$

where $a(x,\xi) = \sum_{l\in\mathbb{Z}^d} \widehat{a}(l,\xi)e_l(x)$. Applying Plancherel equality, we get

$$\|\mathrm{Op}_\hbar(a)\, u\|^2_{L^2(\mathbb{T}^d)} = \sum_{p\in\mathbb{Z}^d} \left| \sum_{k\in\mathbb{Z}^d} \widehat{u}_k \widehat{a}(p-k, 2\pi\hbar k)\right|^2 .$$

Thanks to Cauchy-Schwarz inequality, one has

$$\|\mathrm{Op}_\hbar(a)\, u\|^2_{L^2(\mathbb{T}^d)} \leq \sum_{p\in\mathbb{Z}^d} \left(\sum_{k\in\mathbb{Z}^d} |\widehat{u}_k|^2 |\widehat{a}(p-k, 2\pi\hbar k)|\right) \left(\sum_{k'\in\mathbb{Z}^d} |\widehat{a}(p-k', 2\pi\hbar k')|\right).$$

This implies that

$$\|\mathrm{Op}_\hbar(a)\|^2_{L^2(\mathbb{T}^d)\to L^2(\mathbb{T}^d)} \leq \sup_{p\in\mathbb{Z}^d} \left(\sum_{k'\in\mathbb{Z}^d} |\widehat{a}(p-k', 2\pi\hbar k')|\right) \times \sup_{k\in\mathbb{Z}^d} \left(\sum_{p\in\mathbb{Z}^d} |\widehat{a}(p-k, 2\pi\hbar k)|\right),$$

which concludes the proof of the lemma. □

Another important feature of this quantization procedure is the composition formula:

Theorem A.4 (Composition formula)**.** *Let $a_1 \in S^{m_1,k_1}$ and $a_2 \in S^{m_2,k_2}$. Then, one has, for any $0 < \hbar \leq 1$*

$$\mathrm{Op}_\hbar(a_1) \circ \mathrm{Op}_\hbar(a_2) = \mathrm{Op}_\hbar(a_1 \sharp_\hbar a_2),$$

in the sense of operators from $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$, where $a_1\sharp_\hbar a_2$ has uniformly bounded derivatives, and, for every $N \geq 0$

$$a_1\sharp_\hbar a_2 \sim \sum_{k=0}^{N} \frac{1}{k!}\left(\frac{\hbar}{i} D\right)^k (a_1, a_2) + \mathcal{O}(\hbar^{N+1}),$$

where $D(a_1,a_2)(x,\xi) = (\partial_y.\partial_\xi)(a_1(x,\xi)a_2(y,\nu))\rceil_{x=y,\xi=\nu}$.

We refer to chapter 4 of [27] for a detailed proof of this result. We observe that for $N = 0$, the coefficient is given by the symbol $a_1 a_2$. As before, we can restrict this result to the case of periodic symbols, and we can check that the composition formula remains valid for operators acting on $\mathcal{C}^\infty(\mathbb{T}^d)$.

Remark A.5. We note that we have in fact the following useful property. If $a_1(\xi)$ is a polynomial in ξ of order $\leq N$, one has, the exact formula:

$$a_1 \sharp_\hbar a_2 - a_2 \sharp_\hbar a_1 = a_1 \sharp_\hbar a_2 = \sum_{k=0}^{N} \frac{1}{k!} \left(\frac{\hbar}{i} D \right)^k (a_1, a_2).$$

References

[1] N. Anantharaman, C. Fermanian-Kammerer, F. Macià *Semiclassical completely integrable systems: long-time dynamics and observability via two-microlocal Wigner measures*, Amer. J. Math. **137** (2015), 577–638

[2] N. Anantharaman, M. Léautaud, *Sharp polynomial decay rates for the damped wave equation on the torus*, Anal. PDE **7** (2014), 159–214

[3] N. Anantharaman, M. Léautaud, F. Macià, *Wigner measures and observability for the Schrödinger equation on the disk*, Invent. Math. **206** (2016), 485–599

[4] N. Anantharaman, F. Macià *Semiclassical measures for the Schrödinger equation on the torus*, J. Eur. Math. Soc. (JEMS) **16** (2014), 1253–1288

[5] J. Bolte, T. Schwaibold *Stability of wave packet dynamics under perturbations*, Phys. Rev. E73 (2006) 026223.

[6] Y. Canzani, D. Jakobson, J. Toth *On the distribution of perturbations of propagated Schrödinger eigenfunctions*, J. of Spectral Theory **4** (2014), 283–307

[7] M. Combescure, D. Robert *A phase-space study of the quantum Loschmidt Echo in the semiclassical limit*, Ann. H. Poincaré **8** (2007), 91–108

[8] R. Dubertrand, A. Goussev *Origin of the exponential decay of the Loschmidt echo in integrable systems*, Phys. Rev. E **89** (2014), 022915

[9] S. Eswarathasan, G. Rivière *Perturbation of the semiclassical Schrödinger equation on negatively curved surfaces*, J. Inst. Math. Jussieu **16** (2017), 787–835

[10] S. Eswarathasan, J. Toth *Average pointwise bounds for deformations of Schrodinger eigenfunctions* , Ann. H. Poincaré **14** (2012), 611–637

[11] C. Fermanian-Kammerer *Mesures semi–classiques 2-microlocales*, C. R. Acad. Sci. Paris Ser. I Math., **331** (2000), 515–518

[12] C. Fermanian-Kammerer, P. Gérard *Mesures semi-classiques et croisement de modes*, Bull. Soc. Math. France **130** (2002), 123–168

[13] P. Gérard *Mesures semi-classiques et ondes de Bloch*, Sem. EDP (Polytechnique) 1990–1991, Exp. 16 (1991)

[14] T. Gorin, T. Prosen, T.H. Seligman, M. Zdinaric *Dynamics of Loschmidt echoes and fidelity decay*, Physics Reports **435** (2006) 33–156

[15] A. Goussev, R.A. Jalabert, H.M. Pastawski, D. Wisniacki *Loschmidt Echo*, Scholarpedia 7(8), 11687, arXiv:1206.6348 (2012)

[16] P. Jacquod, C. Petitjean *Decoherence, Entanglement and Irreversibility in Quantum Dynamical Systems with Few Degrees of Freedom*, Adv. Phys. **58**, 67–196 (2009)

[17] F. Macià *Semiclassical measures and the Schrödinger flow on Riemannian manifolds*, Nonlinearity **22** (2009), 1003–1020

[18] F. Macià *High-frequency propagation for the Schrödinger equation on the torus*, Jour. Funct. Analysis **258** (2010), 933–955

[19] F. Macià, G. Rivière *Concentration and non concentration for the Schrödinger evolution on Zoll manifolds*, Comm. In Math. Phys. **345** (2016), 1019–1054

[20] F. Macià, G. Rivière, *Two-microlocal regularity of quasimodes on the torus*, Analysis and PDE **11** (2018), 2111–2136

[21] L. Miller *Propagation d'ondes semi–classiques à travers une interface et mesures 2–microlocales*, PhD thesis, Ecole polytechnique, Palaiseau (1996)
[22] F. Nier *A semiclassical picture of quantum scattering*, Ann. Sci. ENS **29** (1996), 149–183
[23] A. Peres *Stability of quantum motion in chaotic and regular systems*, Phys. Rev. A **30** (1984), 1610–1615
[24] G. Rivière *Long time dynamics of the perturbed Schrödinger equation on negatively curved surfaces*, Ann. H. Poincaré **17** (2016), 1955–1999
[25] J. Royer *Analyse haute fréquence de l'équation de Helmholtz dissipative*, Thèse Université de Nantes tel–00578423 (2010)
[26] M. Ruzhansky, V. Turunen *Pseudodifferential operators and symmetries*, Birkhäuser Verlag, Basel Boston Berlin (2010)
[27] M. Zworski *Semiclassical analysis*, Graduate Studies in Mathematics **138**, AMS (2012)

Laboratoire Paul Painlevé (U.M.R. CNRS 8524), U.F.R. de Mathématiques, Université Lille 1, 59655 Villeneuve d'Ascq Cedex, France
E-mail address: gabriel.riviere@math.univ-lille1.fr

Institut de Mathématiques de Jussieu (U.M.R. CNRS 7586), U.F.R. de Mathématiques, Sorbonne Université, 75252 Paris Cedex 05, France
E-mail address: henrik.ueberschar@imj-prg.fr