SEMICLASSICAL BEHAVIOUR OF QUANTUM EIGENSTATES ON $\mathbb{T}^n$

We let $n \geq 1$ and we consider the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$. Given $k \in \mathbb{Z}^n$, we set

$$e_k(x) := \frac{e^{ik.x}}{(2\pi)^{\frac{n}{2}}}.$$

Given $u \in \mathcal{D}'(\mathbb{T}^n)$, we write its Fourier decomposition:

$$u(x) = \sum_{k\in\mathbb{Z}^n} \widehat{u}(k) e_k(x),$$

with, for every $k \in \mathbb{Z}^n$, $\widehat{u}(k) := \langle u, e_{-k}\rangle_{\mathcal{D}'\times\mathcal{C}^\infty}$. Given $a(x,\xi)$ in $\mathcal{C}^\infty(\mathbb{T}^n \times \mathbb{R}^n)$, we also set

$$a(x,\xi) = \sum_{k\in\mathbb{Z}^n} \widehat{a}(k,\xi) e_k(x),$$

where

$$\widehat{a}(k,\xi) := \int_{\mathbb{T}^n} a(x,\xi) e_{-k}(x) dx.$$

For $\hbar > 0$, we define the Weyl quantization of a by its action on the basis $(e_k)_{k\in\mathbb{Z}^n}$:

$$\operatorname{Op}_\hbar^w(a) e_k(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{l\in\mathbb{Z}^n} e_l(x) \widehat{a}\left(l-k, \frac{\hbar(k+l)}{2}\right).$$

(1) (a) Show that, if $a \in \mathcal{C}^\infty(\mathbb{T}^n)$, then, for every u in $\mathcal{C}^\infty(\mathbb{T}^n)$,

$$\operatorname{Op}_\hbar^w(a) u = a \times u.$$

(b) Show that, if $a(x,\xi) = \xi_1$, then, for every u in $\mathcal{C}^\infty(\mathbb{T}^n)$,

$$\operatorname{Op}_\hbar^w(a) u = -i\hbar\partial_{x_1} u.$$

(c) Suppose that, for every $\alpha \in \mathbb{Z}^n$, $\|\partial^\alpha a\|_\infty < +\infty$. Using Plancherel equality, show that

$$\operatorname{Op}_\hbar^w(a) : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$$

defines a bounded operator.

(d) Let $(k_m)_{m\geq 1}$ be a sequence in $\mathbb{Z}^n$ such that $\hbar_m = \|k_m\|^{-1} \to 0$ and that $\frac{k_m}{\|k_m\|} \to \theta$. Determine the accumulation point (in $\mathcal{D}'(\mathbb{T}^n \times \mathbb{R}^n)$) of the sequence

$$w_{\hbar_m} : a \in \mathcal{C}_c^\infty(\mathbb{T}^n \times \mathbb{R}^n) \mapsto \langle e_{k_m}, \operatorname{Op}_{\hbar_m}^w(a) e_{k_m}\rangle_{L^2}, \quad m \geq 1.$$

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(2) (a) Show that $\psi_\lambda \in \mathcal{D}'(\mathbb{T}^n)$ solves

$$-\Delta\psi_\lambda = \lambda^2\psi_\lambda, \quad \|\psi_\lambda\|_{L^2} = 1,$$

if and only if

$$\psi_\lambda = \sum_{\|k\|=\lambda} \widehat{\psi}_\lambda(k)e_k, \quad \sum_{\|k\|=\lambda} |\widehat{\psi}_\lambda(k)|^2 = 1.$$

(b) Show that, as $\lambda \to +\infty$,

$N(\lambda, 2\lambda) := |\{k \in \mathbb{Z}^n : \lambda < \|k\| \leq 2\lambda\}| \sim \mathrm{Vol}(B_n(0,1))(2^n - 1)\lambda^n$,

where $B_n(0,1) := \{x \in \mathbb{R}^n : \sum_j x_j^2 \leq 1\}$. *This is the Weyl law.*

(3) Let $a \in \mathcal{C}^\infty(\mathbb{T}^n \times \mathbb{R}^n)$ all of whose derivatives are bounded.

(a) Using Plancherel equality, show that

$$\left\|\mathrm{Op}^w_{\|k\|^{-1}}(a)e_k\right\|^2 = \frac{1}{(2\pi)^n}\sum_{l\in\mathbb{Z}^n}\left|\widehat{a}\left(l, \frac{k}{\|k\|}\right)\right|^2 + \mathcal{O}(\|k\|^{-1}).$$

(b) Show that, as $\lambda \to +\infty$

$$\frac{1}{N(\lambda,2\lambda)}\sum_{\lambda<\|k\|\leq 2\lambda}\left\|\mathrm{Op}^w_{\|k\|^{-1}}(a)e_k\right\|^2 \longrightarrow \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n\times\mathbb{S}^{n-1}}|a\left(x,\theta\right)|^2 dx d\sigma(\theta),$$

where σ is the normalized Lebesgue measure on the unit sphere $\mathbb{S}^{n-1} = \partial B_n(0,1)$.

(c) Deduce that, as $\hbar \to 0^+$,

$$\frac{1}{N(\hbar)}\sum_{0<\hbar\|k\|\leq 1}\left\|\mathrm{Op}^w_{\|k\|^{-1}}(a)e_k\right\|^2 \longrightarrow \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n\times\mathbb{S}^{n-1}}|a\left(x,\theta\right)|^2 dx d\sigma(\theta),$$

where

$$N(\hbar) := |\{k \in \mathbb{Z}^n : 0 < \hbar\|k\| \leq 1\}|.$$

This is a microlocal Weyl law.

(4) Let a be an element in $\mathcal{C}^0(\mathbb{T}^n \times \mathbb{S}^{n-1})$.

(a) Show that, for every x in $\mathbb{T}^n$ and for σ-a.e. θ in $\mathbb{S}^{n-1}$, one has

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T a(x+t\theta,\theta)dt = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} a(x,\theta)dx.$$

Indication. Start with the case where a is trigonometric polynomial.

(b) Deduce that, if a does not depend on the variable θ, then

$$\lim_{T\to+\infty}\int_{\mathbb{T}^n\times\mathbb{S}^{n-1}}\left|\frac{1}{T}\int_0^T a(x+t\theta)dt - \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} a(x)dx\right|^2 dx d\sigma(\theta) = 0.$$

(5) Let a be an element in $\mathcal{C}^\infty(\mathbb{T}^n)$. Show that, for every $t \in \mathbb{R}$,

$$e^{-\frac{it\hbar\Delta}{2}} a e^{\frac{it\hbar\Delta}{2}} = \mathrm{Op}^w_\hbar\left(a_t\right),$$

where $a_t(x,\xi) = a(x+t\xi)$.

(6) Let $(\psi_j)_{j\geq 1}$ be an orthonormal basis of $L^2(\mathbb{T}^n)$ such that, for every $j \geq 1$,

$$-\Delta\psi_j = \lambda_j^2\psi_j,$$

with $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_j \leq \lambda_{j+1} \leq \ldots$. For $j > 1$, we set $\hbar_j := \lambda_j^{-1}$. Let a in $\mathcal{C}^\infty(\mathbb{T}^n)$ such that $\int_{\mathbb{T}^n} a dx = 0$.

(a) Show that, for every $T > 0$

$$V_\hbar(a) := \frac{1}{N(\hbar)} \sum_{0<\hbar\lambda_j\leq 1} |\langle\psi_j, a\psi_j\rangle|^2 \leq \frac{1}{N(\hbar)} \sum_{0<\hbar\lambda_j\leq 1} \left\|\mathrm{Op}^w_{\hbar_j}(\langle a\rangle_T)\psi_j\right\|^2,$$

where $\langle a\rangle_T(x,\xi) = \frac{1}{T}\int_0^T a(x+t\xi)dt$.

(b) Deduce that, for every $T > 0$,

$$V_\hbar(a) \leq \frac{1}{N(\hbar)} \sum_{0<\hbar\|k\|\leq 1} \left\|\mathrm{Op}^w_{\|k\|^{-1}}(\langle a\rangle_T)e_k\right\|^2.$$

(c) Conclude that

$$\lim_{\hbar\to 0} V_\hbar(a) = 0.$$

(7) Let a in $\mathcal{C}^\infty(\mathbb{T}^n)$. Deduce from the previous question that there exists $S \subset \mathbb{Z}_+^*$ such that

$$\lim_{N\to+\infty} \frac{|\{1 \leq j \leq N : j \in S\}|}{N} = 1,$$

and

$$\lim_{j\to+\infty, j\in S} \int_{\mathbb{T}^n} a(x)|\psi_j(x)|^2 dx = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} a(x)dx.$$

(8) Show that S can be chosen such that

$$\lim_{N\to+\infty} \frac{|\{1 \leq j \leq N : j \in S\}|}{N} = 1,$$

and, for every a in $\mathcal{C}^0(\mathbb{T}^n)$,

$$\lim_{j\to+\infty, j\in S} \int_{\mathbb{T}^n} a(x)|\psi_j(x)|^2 dx = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} a(x)dx.$$

(9) Is it necessary to extract a subsequence?