We let $n \geq 1$ and we consider the $n$-dimensional torus $\mathbb{T}^n = \mathbb{R}^n / 2\pi \mathbb{Z}^n$. Given $k \in \mathbb{Z}^n$, we set

$$e_k(x) := \frac{e^{ik \cdot x}}{(2\pi)^{n/2}}.$$  

Given $u \in \mathcal{D}'(\mathbb{T}^n)$, we write its Fourier decomposition:

$$u(x) = \sum_{k \in \mathbb{Z}^n} \hat{u}(k) e_k(x),$$

with, for every $k \in \mathbb{Z}^n$, $\hat{u}(k) := \langle u, e_{-k} \rangle_{\mathcal{D}'(\mathbb{T}^n) \times C^\infty}$. Given $a(x, \xi)$ in $C^\infty(\mathbb{T}^n \times \mathbb{R}^n)$, we also set

$$\hat{a}(k, \xi) := \int_{\mathbb{T}^n} a(x, \xi) e^{-ik \cdot x} dx.$$  

For $\hbar > 0$, we define the Weyl quantization of $a$ by its action on the basis $(e_k)_{k \in \mathbb{Z}^n}$:

$$\text{Op}_w^{\hbar}(a) e_k(x) = \frac{1}{(2\pi)^{n/2}} \sum_{l \in \mathbb{Z}^n} c_l(x) \hat{a} \left( l - k, \frac{\hbar(k + l)}{2} \right).$$

(1) (a) Show that, if $a \in C^\infty(\mathbb{T}^n)$, then, for every $u$ in $C^\infty(\mathbb{T}^n)$,

$$\text{Op}_w^{\hbar}(a) u = a \times u.$$  

(b) Show that, if $a(x, \xi) = \xi_1$, then, for every $u$ in $C^\infty(\mathbb{T}^n)$,

$$\text{Op}_w^{\hbar}(a) u = -i\hbar \partial_{x_1} u.$$  

(c) Suppose that, for every $\alpha \in \mathbb{Z}^n$, $\|\partial^\alpha a\|_\infty < +\infty$. Using Plancherel equality, show that

$$\text{Op}_w^{\hbar}(a) : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$$

defines a bounded operator.

(d) Let $(k_m)_{m \geq 1}$ be a sequence in $\mathbb{Z}^n$ such that $h_m = \|k_m\|^{-1} \to 0$ and that $rac{k_m}{\|k_m\|} \to \theta$. Determine the accumulation point (in $\mathcal{D}'(\mathbb{T}^n \times \mathbb{R}^n)$) of the sequence

$$w_{h_m} : a \in C^\infty_c(\mathbb{T}^n \times \mathbb{R}^n) \mapsto \langle e_{k_m}, \text{Op}_w^{h_m}(a)e_{k_m} \rangle_{L^2}, \quad m \geq 1.$$  

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(2) (a) Show that $\psi_\lambda \in \mathcal{D}'(\mathbb{T}^n)$ solves
$$-\Delta \psi_\lambda = \lambda^2 \psi_\lambda, \quad \|\psi_\lambda\|_{L^2} = 1,$$
if and only if
$$\psi_\lambda = \sum_{|k| = \lambda} \hat{\psi}_\lambda(k) e_k, \quad \sum_{|k| = \lambda} |\hat{\psi}_\lambda(k)|^2 = 1.$$
(b) Show that, as $\lambda \to +\infty$,
$$N(\lambda, 2\lambda) := |\{k \in \mathbb{Z}^n : \lambda < \|k\| \leq 2\lambda\}| \sim \text{Vol}(B_n(0, 1))(2^n - 1)\lambda^n,$$
where $B_n(0, 1) := \{x \in \mathbb{R}^n : \sum_j x_j^2 \leq 1\}$. This is the Weyl law.
(3) Let $a \in C^\infty(\mathbb{T}^n \times \mathbb{R}^n)$ all of whose derivatives are bounded.
(a) Using Plancherel equality, show that
$$\|\text{Op}_w^{\lambda-1}(a) e_k\|^2 = \frac{1}{(2\pi)^n} \sum_{l \in \mathbb{Z}^n} \left| \hat{a} \left( l, \frac{k}{\|k\|} \right) \right|^2 + \mathcal{O}(\|k\|^{-1}).$$
(b) Show that, as $\lambda \to +\infty$
$$\frac{1}{N(\lambda, 2\lambda)} \sum_{\lambda < \|k\| \leq 2\lambda} \|\text{Op}_w^{\lambda-1}(a) e_k\|^2 \longrightarrow \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n \times \mathbb{S}^{n-1}} |a(x, \theta)|^2 dx d\sigma(\theta),$$
where $\sigma$ is the normalized Lebesgue measure on the unit sphere $\mathbb{S}^{n-1} = \partial B_n(0, 1)$.
(c) Deduce that, as $\hbar \to 0^+$,
$$\frac{1}{N(\hbar)} \sum_{0 < \hbar \|k\| \leq 1} \|\text{Op}_w^{\lambda-1}(a) e_k\|^2 \longrightarrow \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n \times \mathbb{S}^{n-1}} |a(x, \theta)|^2 dx d\sigma(\theta),$$
where
$$N(\hbar) := |\{k \in \mathbb{Z}^n : 0 < \hbar \|k\| \leq 1\}|.$$
This is a microlocal Weyl law.
(4) Let $a$ be an element in $C^0(\mathbb{T}^n \times \mathbb{S}^{n-1})$.
(a) Show that, for every $x$ in $\mathbb{T}^n$ and for $\sigma$-a.e. $\theta$ in $\mathbb{S}^{n-1}$, one has
$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T a(x + t\theta, \theta) dt = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} a(x, \theta) dx.$$
Indication. Start with the case where $a$ is trigonometric polynomial.
(b) Deduce that, if $a$ does not depend on the variable $\theta$, then
$$\lim_{T \to +\infty} \int_{\mathbb{T}^n \times \mathbb{S}^{n-1}} \left| \frac{1}{T} \int_0^T a(x + t\theta) dt \right|^2 dx d\sigma(\theta) = 0.$$
(5) Let $a$ be an element in $C^\infty(\mathbb{T}^n)$. Show that, for every $t \in \mathbb{R}$,
$$e^{-\frac{i\hbar}{2} \Delta} ae^{\frac{i\hbar}{2} \Delta} = \text{Op}_w^\hbar(a_t),$$
where $a_t(x, \xi) = a(x + t\xi)$. 
(6) Let \((\psi_j)_{j \geq 1}\) be an orthonormal basis of \(L^2(\mathbb{T}^n)\) such that, for every \(j \geq 1\),
\[-\Delta \psi_j = \lambda_j^2 \psi_j,
\]
with \(0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_j \leq \lambda_{j+1} \leq \ldots\). For \(j > 1\), we set \(h_j := \lambda_j^{-1}\). Let \(a\) in \(C^\infty(\mathbb{T}^n)\) such that \(\int_{\mathbb{T}^n} adx = 0\).
(a) Show that, for every \(T > 0\)
\[V_h(a) := \frac{1}{N(h)} \sum_{0 < h \lambda_j \leq 1} |\langle \psi_j, a \psi_j \rangle|^2 \leq \frac{1}{N(h)} \sum_{0 < h \lambda_j \leq 1} \left\| \text{Op}_w^w(h_j \langle a \rangle_T) \psi_j \right\|^2,
\]
where \(\langle a \rangle_T(x, \xi) = \frac{1}{T} \int_0^T a(x + t\xi) dt\).
(b) Deduce that, for every \(T > 0\),
\[V_h(a) \leq \frac{1}{N(h)} \sum_{0 < h \|k\| \leq 1} \left\| \text{Op}_w^{w-1}(\langle a \rangle_T) e_k \right\|^2.
\]
(c) Conclude that
\[\lim_{h \to 0} V_h(a) = 0.
\]
(7) Let \(a\) in \(C^\infty(\mathbb{T}^n)\). Deduce from the previous question that there exists \(S \subset \mathbb{Z}_+^*\) such that
\[\lim_{N \to +\infty} \frac{|\{1 \leq j \leq N : j \in S\}|}{N} = 1,
\]
and
\[\lim_{j \to +\infty, j \in S} \int_{\mathbb{T}^n} a(x)|\psi_j(x)|^2 dx = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} a(x) dx.
\]
(8) Show that \(S\) can be chosen such that
\[\lim_{N \to +\infty} \frac{|\{1 \leq j \leq N : j \in S\}|}{N} = 1,
\]
and, for every \(a\) in \(C^0(\mathbb{T}^n)\),
\[\lim_{j \to +\infty, j \in S} \int_{\mathbb{T}^n} a(x)|\psi_j(x)|^2 dx = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} a(x) dx.
\]
(9) Is it necessary to extract a subsequence?