LOCAL L^p NORMS OF SCHRÖDINGER EIGENFUNCTIONS ON $\mathbb{S}^2$

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Abstract. On the canonical 2-sphere and for Schrödinger eigenfunctions, we obtain a simple geometric criterion on the potential under which we can improve, near a given point and for every $p \neq 6$, Sogge's estimates by a power of the eigenvalue. This criterion can be formulated in terms of the critical points of the Radon transform of the potential and it is independent of the choice of eigenfunctions.

1. Introduction

The purpose of this work is to study high frequency asymptotics of eigenfunctions to the Schrödinger operator on the 2-sphere

$$\mathbb{S}^2 := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\right\}. \tag{1}$$

We endow $\mathbb{S}^2$ with the Riemannian metric g_0 induced by the Euclidean metric on $\mathbb{R}^3$. In that geometric context and given an element $V \in \mathcal{C}^\infty(\mathbb{S}^2, \mathbb{R})$, there exists an orthonormal basis [58, Th. 14.7] of $L^2(\mathbb{S}^2, dv_{g_0})$ made of solutions to

$$-\Delta_{g_0}\psi_\lambda + V\psi_\lambda = \lambda^2\psi_\lambda, \quad \lambda \in \mathbb{C}, \tag{2}$$

where Δ_{g_0} is the Laplace-Beltrami operator and dv_{g_0} is the Riemannian volume, both induced by g_0. By elliptic regularity, solutions to (2) are smooth [58, § 14.3] and a classical Theorem of Sogge [36] states that, for every $2 \leq p \leq +\infty$, there exists $C_p > 0$ such that, for any solution (ψ_λ, λ) to (2),

$$\|\psi_\lambda\|_{L^p(\mathbb{S}^2)} \leq C_p(1 + |\lambda|)^{\sigma_0(p)}\|\psi_\lambda\|_{L^2(\mathbb{S}^2)}, \tag{3}$$

where[1]

$$\sigma_0(p) := \max\left\{\frac{1}{4} - \frac{1}{2p}, \frac{1}{2} - \frac{2}{p}\right\}.$$

The critical exponent for which both quantities in the maximum coincide is given by $p_c = 6$. In the case where $V \equiv 0$, these upper bounds are optimal using appropriate sequences of spherical harmonics [38]. However, for generic sequences [50, 55, 12] or for families satisfying certain extra invariance properties [11], these bounds can drastically be improved when $V \equiv 0$.

Our aim is to show that the presence of a potential allows to improve (3) away from certain critical geodesics and for *any* sequence of eigenfunctions. In order to state our results, we introduce the space of oriented closed geodesics $G(\mathbb{S}^2)$ of the sphere. By identifying each oriented closed geodesic with an oriented plane of $\mathbb{R}^3$, $G(\mathbb{S}^2)$ is diffeomorphic to $\mathbb{S}^2$.

[1] The case $p = \infty$ is a consequence of the local Weyl law [27].

Through this identification, $G(\mathbb{S}^2) \simeq \mathbb{S}^2$ is endowed with the symplectic structure induced by the one on the cotangent bundle $T^*\mathbb{S}^2$ [2, p. 58]. We also define the Radon transform of the potential V:

$$\mathcal{R}(V) : \gamma \in G(\mathbb{S}^2) \mapsto \frac{1}{2\pi} \int_0^{2\pi} V(\gamma(s))ds \in \mathbb{R},$$

which belongs to $\mathcal{C}^\infty(G(\mathbb{S}^2))$. Thanks to the symplectic structure on $G(\mathbb{S}^2)$, one can define its Hamiltonian vector field $X_{\langle V \rangle}$. We denote its critical points by

$$\text{Crit}(\mathcal{R}(V)) := \left\{\gamma \in G(\mathbb{S}^2) : D_\gamma \mathcal{R}(V) = 0\right\} = \left\{\gamma \in G(\mathbb{S}^2) : X_{\langle V \rangle}(\gamma) = 0\right\}.$$

Observe that $\mathcal{R}(V)$ is always an even function on $G(\mathbb{S}^2)$. In particular, it can be identified with a function on $\mathbb{R}P^2$ and it has thus at least 6 critical points on $G(\mathbb{S}^2)$ by Morse inequalities. In fact, Guillemin showed [22] that

$$\mathcal{R} : \mathcal{C}^\infty_{\text{even}}(\mathbb{S}^2) \to \mathcal{C}^\infty_{\text{even}}(G(\mathbb{S}^2))$$

is an isomorphism. As a corollary, for a generic choice of V in the $\mathcal{C}^\infty$-topology, $\text{Crit}(\mathcal{R}(V))$ is a finite set. Finally, given $x_0 \in \mathbb{S}^2$, we set

$$\Gamma_{x_0} := \left\{\gamma \in G(\mathbb{S}^2) : \ x_0 \in \gamma\right\}.$$

Our main result reads as follows

Theorem 1.1. *Let $x_0 \in \mathbb{S}^2$ such that*

$$\text{Crit}(\mathcal{R}(V)) \cap \Gamma_{x_0} = \emptyset, \tag{4}$$

and

$$\mathcal{R}(V)|_{\Gamma_{x_0}} \text{ is a Morse function.} \tag{5}$$

Then, there exists $r_0 > 0$ such that, for every $2 \leq p \leq +\infty$, one can find $C_{x_0,p} > 0$ so that, for any solution (ψ_λ, λ) to (2),

$$\|\psi_\lambda\|_{L^p(B_{r_0}(x_0))} \leq C_{x_0,p}(\log(2 + |\lambda|))^{\varepsilon(p)}(1 + |\lambda|)^{\sigma_0(p)-\delta(p)}\|\psi_\lambda\|_{L^2(\mathbb{S}^2)},$$

where $B_{r_0}(x_0)$ is the closed (geodesic) ball of radius r_0 centered at x_0 and where, for $4 < p \leq \infty$,

$$\delta(p) := \frac{1}{18}\left|1 - \frac{6}{p}\right|, \quad \varepsilon(p) = 0$$

and, for $2 \leq p \leq 4$

$$\delta(p) := \frac{1}{18}\left(1 - \frac{2}{p}\right), \quad \varepsilon(p) := 2\left(1 - \frac{2}{p}\right).$$

Remark 1.2. Given a point x_0, we note that (4) and (5) are satisfied for an open and dense subset $\mathcal{U}_{x_0}$ of potentials in $\mathcal{C}^\infty(\mathbb{S}^2, \mathbb{R})$ (endowed with its natural Fréchet topology). Assumption (5) implies that the Hamiltonian vector field is transverse to Γ_{x_0} except at finitely many points. Combined with (4), one has that, at the points where $X_{\langle V \rangle}(\gamma)$ is tangent to Γ_{x_0}, the tangency is of order 1. See Remark 4.5 for an interpretation of these assumptions in terms of Lagrangian tori.

Remark 1.3. A direct Corollary of Theorem 1.1 is that, if K is a compact subset of $\mathbb{S}^2$ such that, for every $x_0 \in K$, (4) and (5) hold, then, for any solution (ψ_λ, λ) to (2),

$$\|\psi_\lambda\|_{L^p(K)} \leq C_{K,p} \log(2+|\lambda|)^{\varepsilon(p)}(1+|\lambda|)^{\sigma_0(p)-\delta(p)}\|\psi_\lambda\|_{L^2(\mathbb{S}^2)}. \tag{6}$$

Yet, our main result does not allow to take $K = \mathbb{S}^2$ as $\mathrm{Crit}(\mathcal{R}(V))$ cannot be empty.

This Theorem yields a *local* improvement for $p \neq 6$ over Sogge's upper bounds near certain points of $\mathbb{S}^2$ which are *independent* of the sequence $(\psi_\lambda)_\lambda$ under consideration. The condition on these points are of purely dynamical nature and they depend on the subprincipal symbol of our operator. It may happen that Sogge's upper bounds are saturated for these operators but this can only occur away from points x_0 verifying (4) and (5). The critical case $p_c = 6$ could maybe be treated using similar ideas and the methods of Blair and Sogge to handle this exponent on nonpositively curved surfaces [40, 6]. Yet, this would probably require a much more delicate analysis than the one presented in this article.

Our hypothesis (4) and (5) are reminiscent from assumptions that appear when studying joint eigenfunctions of quantum completely integrable systems – see [47, §1] for a definition. For instance, the critical points involved in hypothesis (4) were used to obtain lower bounds by Toth in [46] and by Toth-Zelditch in [47, 48]. Similarly, assumption (5) was recently used by Galkowski–Toth [21] and by Tacy [45] to study the growth of L^∞-norms of joint eigenfunctions. The main differences with these last references are that we handle every $p \neq 6$ and that we consider here eigenfunctions of the *single* operator $-\Delta_{g_0} + V$. In fact recall from [24, Lemma 1] (see also [52]) that there exists a unitary pseudodifferential operator $\mathcal{U}$ of order 0 such that

$$\mathcal{U}^{-1}\left(-\Delta_{g_0} + V\right)\mathcal{U} = -\Delta_{g_0} + V^\sharp, \tag{7}$$

where $[\Delta_{g_0}, V^\sharp] = 0$ and where the principal symbol of $V^\sharp$ is $\mathcal{R}(V)$. In other words, $-\Delta_{g_0} + V$ is the sum of two commuting pseudodifferential operators $\widehat{H}_1 := \mathcal{U}\Delta_{g_0}\mathcal{U}^{-1}$ and $\widehat{H}_2 := \mathcal{U}V^\sharp\mathcal{U}^{-1}$. In particular, it is a quantum completely integrable operator in the sense of [47, §1] whenever $X_{\langle V\rangle}$ does not vanish on a dense and open subset of finite complexity (say outside finitely many points). Hence, upper bounds on L^p norms of solutions to (2) which are joint eigenfunctions of $(\widehat{H}_1, \widehat{H}_2)$ would follow from the results in [21, 45] in the range $p > 6$. However, in Theorem 1.1, we only suppose $p \neq 6$ and we do not make any assumption on the fact that ψ_λ is a joint eigenfunction[2] of $(\widehat{H}_1, \widehat{H}_2)$ which makes the analysis slightly more delicate. Despite that, Theorem 1.1 shows that there is room for (weaker) polynomial improvements on (3) even for such eigenfunctions and even for $p < 6$. In [44], Tacy obtained better estimates up to $p = 2$ but she made stronger assumptions than ours on the sequence of eigenfunctions. Indeed, when restricted to our framework, the main result from this reference applies to sequences of joint eigenfunctions that concentrate away from the critical points of $\mathcal{R}(V)|_{\Gamma_{x_0}}$.

[2] We are not aware of a geometric criterion ensuring that all eigenfunctions are the image of joint eigenfunctions. This could also be achieved by showing the simplicity of the spectrum of $-\Delta_{g_0} + V$ but we could not find in the literature a reference showing this (at least for a generic V).

1.1. **Earlier and related results.** The upper bounds (3) are in fact valid in the general framework of compact Riemannian surfaces and, up to modifying the exponent $\sigma_0(p)$, they remain true in higher dimensions [36]. Trying to improve them using the geometry of the manifold has been a classical topic in global harmonic analysis over the last thirty years.

- **Flat tori.** In the case of flat tori and where $V \equiv 0$, this was achieved by Cooke [17] and Zygmund [57] in dimension 2 while the higher dimensional case was pursued by Bourgain [8] and by Bourgain-Demeter [10]. In that case, one can use the arithmetic structure of the torus to get polynomial improvements over (3). See also [51] for the case of Schrödinger operators on 2-dimensional tori. To the best of the author's knowledge, flat tori are the only geometric framework where one can get global polynomial improvements without any further assumptions on the sequence of eigenfunctions (see below for the case of joint eigenfunctions).
- **Negatively curved manifolds.** Another important class of examples where one expects improvements are negatively curved manifolds. For $p = \infty$, Bérard showed how to get logarithmic improvements [1]. This logarithmic gain was extended to the range $p > p_c$ by Hassell and Tacy [25] and to manifolds without conjugate points by Bonthonneau [7]. Still on negatively curved manifolds and for $p \leq p_c$, we obtained together with Hezari a logarithmic gain along generic sequences of eigenfunctions [26]. In a series of works related to Kakeya-Nikodym norms [40, 5, 6], Blair and Sogge proved logarithmic gains (with a slightly worst exponent) in this geometric context without any restriction on the sequence of eigenfunctions.
- **Arithmetic eigenfunctions.** A natural way to look for improvements over (3) is to consider families of eigenfunctions that verify extra symmetries, for instance joint eigenfunctions of the Laplacian and of a family of commuting operators. In the case of a compact arithmetic surface, Iwaniec and Sarnak considered joint eigenfunctions of the Laplacian and of Hecke operators. For such sequences of eigenfunctions, they proved a polynomial improvement in the case of the L^∞-norm [28]. In the case of the sphere, Brooks and Le Masson considered the related problem of joint eigenfunctions of Δ_{g_0} and the averaging operator for a finitely-generated free algebraic subgroup of $SO(3)$ [11]. For such eigenfunctions, they obtained the same logarithmic improvement as Hassell and Tacy in the negatively curved case. On a rank r symmetric space of dimension n, Sarnak improved the bound on the L^∞-norm by a polynomial factor for eigenfunctions of the full ring of differential operators [33]. This was generalized to the case of L^p-norms by Marshall [31].
- **Completely integrable systems.** Another context (closely related to ours) is the case of *joint* eigenfunctions of a quantum completely integrable system. Toth and Zelditch proved that such eigenfunctions cannot have their L^p norms uniformly bounded except in the case of flat tori [47, 48]. See [56, Ch. 11] for a detailed discussion on joint eigenfunctions of quantum completely integrable systems. More recently, Galkowski and Toth obtained polynomial improvements on the L^∞-bound for joint eigenfunctions of a quantum completely integrable systems [21] and Tacy

proved improved Sogge's bounds for joint eigenfunctions of general families of semiclassical pseudodifferential operators [44, 45].

- **Local improvements.** Sogge and Zelditch considered the problem from a more local perspective as we are doing here. They proved that, if, for a given point x_0 on a Riemannian manifold (M, g), the set of covectors $\xi \in S^*_{x_0}M$ that come back to x_0 in finite time has zero measure, then one can improve locally near x_0 the upper bound on the L^∞-norm by a $o(1)$ term [42]. This was based on improvements on the remainder in the local Weyl law. See also [32] for earlier related results of Safarov. This result was later extended by Sogge, Toth and Zelditch under the weaker assumptions that the set of recurrent co-vectors at x_0 has 0-measure[3] [41]. We also refer to [43] for further developments of this approach when the metric is analytic and to [56, Ch. 10] for a detailed review. Related to these works, Galkowski and Toth showed how to relate precisely the growth of the L^∞-norm near a point x_0 to the semiclassical measure restricted to the (geodesic) flow-out of the fiber $S^*_{x_0}M$ [20] – see also [19]. More precisely, they proved that, if the n-dimensional Hausdorff measure of the support of this restriction is 0, then one can get a $o(1)$-improvement on the growth of L^∞-norm near x_0.
- **Using Gaussian beams.** This local approach was further improved by Canzani-Galkowski in a series of work using Gaussian beams [13, 14]. In [14, Th. 1], they showed how to use this notion in order to give quantitative and at most logarithmic improvements on the growth of L^p-norms near a point x_0 when the conjugate points to x_0 do not pass too close to x_0. Among other things, they recover in that manner the results of Bérard, Hassell-Tacy and Bonthonneau on manifolds without conjugate points. Besides that, they manage to deduce from their main results local improvements near x_0 on the growth of L^p norms (for $p > p_c$) under quantitative assumptions on the geodesics passing through the point x_0 as in the works of Sogge, Toth and Zelditch. Finally, they also applied their main results to certain integrable (non-periodic) geometries on $\mathbb{S}^2$ and obtain logarithmic improvements away from certain critical points when $p = \infty$ [13, Th. 5]. As in our framework, their result holds for the eigenfunctions of a single operator.

1.2. **Strategy of proof.** In the range $p > 6$, the proof is based on an argument to study the growth of L^p norms that was used by Hezari and the author in [26] and further improved by Sogge in [39]. It consists in relating the growth of L^p-norms to the growth of

$$\int_{B_r(x)} |\psi_\lambda(y)|^2 dv_{g_0}(y) \tag{8}$$

as $\lambda \to +\infty$ and $r \to 0^+$ (in a way that depends on λ). For $2 \leq p < 6$, we rather make use of results due to Blair and Sogge [37, 3, 4] to control L^p-norms in terms of Kakeya-Nikodym averages around closed geodesics. See also [9] for earlier related results of Bourgain. Then,

[3] We emphasize that Theorem 1.1 considers the somehow opposite case where the set of recurrent vectors has full measure. Despite that, we are able to get local polynomial improvements using the periodicity of the geodesic flow and the presence of a subprincipal symbol.

we obtain rough bounds on these averages in terms of (8). The results from these references are briefly recalled (and adapted to Schrödinger eigenfunctions) in Sections 2 and 3.

Up to smoothing the characteristic function of the balls, these local quantities can be interepreted in terms of Wigner distributions (or microlocal lifts). In particular, as was for instance observed by Shnirelman in his seminal work on quantum ergodicity [34, 35], these distributions verify an almost invariance property by the geodesic flow. See for instance [35, Lemma 2, Eq. (10)]. This yields an upper bound of order $\mathcal{O}(r)$ on (8) at least if r does not go too fast to 0 (say $r \gg \lambda^{-\frac{1}{2}}$). This is valid in a quite general framework. Yet, this is not sufficient to get an improvement over Sogge's upper bound. In order to implement this approach, one needs to have upper bounds of order $\mathcal{O}(r^{1+\alpha})$ for some $\alpha > 0$, or at least $\mathcal{O}(\delta(r)r)$ with $\delta(r) \to 0$ as $r \to 0^+$.

As pointed out by Sarnak in [33], a natural manner to look for improvements over Sogge's upper bounds is to consider operators commuting with the Laplacian and to study the L^p norm of joint eigenfunctions. These joint eigenfunctions enjoy more symmetries which may lead to improvements. This was for instance the strategy followed in [28, 11, 31, 20, 44, 45]. Here, we are not a priori in this situation as we consider eigenfunctions of the *single* operator $-\Delta_{g_0} + V$ – see the discussion following Theorem 1.1. However, the periodicity of the geodesic flow and the presence of the potential imply the existence of an extra invariance property besides the one by the geodesic flow. More precisely, in [29, 30], together with Macià, we showed that Schrödinger eigenfunctions satisfy an extra invariance property by the Hamiltonian flow of $\mathcal{R}(V)$ which is reminiscent from the properties of joint eigenfunctions. This was achieved using Weinstein averaging method [52]. Using this extra property, we will be able to get an upper bound of order $\mathcal{O}(r^{\frac{3}{2}})$ on (8) up to scales $r \approx \lambda^{-\frac{2}{9}}$ near points verifying (4) and (5). This will be the content of Section 4. This additional invariance will be the reason for the polynomial improvement of Theorem 1.1. As we shall see in our proof[4], the reason for being limited to $p \neq 6$ comes from this exponent 3/2 and, in dimension 2, any bound on (8) of order $\mathcal{O}(r^{1+\alpha})$ with $\alpha > 1/2$ would give a local improvement over Sogge's upper bound (3) even for $p = 6$ (using the arguments of § 2).

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2. Reduction to L^2 localized estimates for $p > 6$

In this section, we revisit an argument due to Sogge[5] in order to relate L^p estimates to localized L^2-estimates in small balls. This argument will allow us to get our upper bounds

[4]See for instance (18).

[5]See also [26] for earlier related arguments of Hezari and the author using semiclassical methods [58, §10].

in the range $6 < p \leq \infty$. The proof given in [39] was for Laplace eigenfunctions and we verify that it can be adapted to Schrödinger eigenfunctions.

Remark 2.1. Due to our L^2-localized estimates in Section 4, we could as well work only with $p = \infty$ and conclude by interpolation with the case $p = 6$ in (3). Yet, we write things down for general p in order to identify the quantitative improvements one would need to reach the case $p = 6$. See Equation (18) below.

Let ψ_λ be a solution to (2) that we suppose to be L^2-normalized. In the following, we suppose that λ^2 is large enough so that we can pick $\lambda > 0$. We write

$$(\sqrt{-\Delta_{g_0}} - \lambda)\psi_\lambda = -(\sqrt{-\Delta_{g_0}} + \lambda)^{-1} V \psi_\lambda. \tag{9}$$

In particular, one has

$$(\sqrt{-\Delta_{g_0}} - \lambda)\psi_\lambda = \mathcal{O}_{L^2}(\lambda^{-1}). \tag{10}$$

Following [39, §2] and for $j \in \mathbb{Z}_+$, we denote by E_j the spectral projector onto the eigenspace of $\sqrt{-\Delta_{g_0}}$ with eigenvalue $\lambda_j := \sqrt{j(j+1)}$. We also fix a nonnegative $\rho \in \mathcal{S}(\mathbb{R})$ satisfying

$$\rho(0) = 1 \quad \text{and} \quad \operatorname{supp}(\hat{\rho}) \subset [-1, 1], \tag{11}$$

where $\hat{\rho}$ is the Fourier transform of ρ. For $\lambda > 0$ and $0 < r \leq 1$, setting

$$T_{\lambda,r} := \frac{1}{\pi} \int_{-\infty}^{+\infty} r^{-1} \hat{\rho}(r^{-1}t) e^{it\lambda} \cos(t\sqrt{-\Delta_{g_0}}) dt,$$

one finds

$$T_{\lambda,r} = \rho\left(r\left(\lambda - \sqrt{-\Delta_{g_0}}\right)\right) + \rho\left(r\left(\lambda + \sqrt{-\Delta_{g_0}}\right)\right).$$

The main result of [39, Eq. (3.1)] is that, for every $p > 2$ and for every $f \in L^2(\mathbb{S}^2)$,

$$\|T_{\lambda,r} f\|_{L^p(\mathbb{S}^2)} \leq C_p r^{-\frac{1}{2}} \lambda^{\sigma_0(p)} \|f\|_{L^2(\mathbb{S}^2)}, \quad \lambda \geq 1, \quad \lambda^{-1} \leq r \leq \frac{\pi}{2}, \tag{12}$$

where the constant C_p is uniform for (λ, r) in the above range. Recall now from Huygens principle that the Schwartz kernel $\cos(t\sqrt{-\Delta_{g_0}})(x, y)$ vanishes if the geodesic distance between x and y is $> t$. In particular, the Shwartz kernel $T_{\lambda,r}(x, y)$ of $T_{\lambda,r}$ vanishes if $d_{g_0}(x, y) > r$ thanks to our assumptions (11) on the support of ρ. Gathering these informations, Sogge observed that, for every $p > 2$ and for every $f \in L^2(\mathbb{S}^2)$,

$$\|T_{\lambda,r} f\|_{L^p(B_r(x_0))} \leq C_p r^{-\frac{1}{2}} \lambda^{\sigma_0(p)} \|f\|_{L^2(B_{2r}(x_0))}, \quad \lambda \geq 1, \quad \lambda^{-1} \leq r \leq \frac{\pi}{2}, \tag{13}$$

where the constant C_p is uniform for (λ, r) in the above range and for $x_0 \in \mathbb{S}^2$. Fix now some compact subset K of $\mathbb{S}^2$. We can cover K by finitely many balls $(B_r(x_l))_{l=1,\ldots,N(r)}$ of radius r and centered at points inside K. We require that the number $N(r)$ is of order $\sim r^{-2}$ and that each point of K is contained in at most C_0 balls of the covering $(B_{2r}(x_l))_{l=1,\ldots,N(r)}$. Here $C_0 > 0$ is independent of r – see for instance [15, Lemma 2]. Recall that we have in mind to apply this result when $K = B_{r_0}(x_0)$ is a fixed ball. Hence, one has, for $2 < p < \infty$ and for f in $L^2(\mathbb{S}^2)$,

$$\begin{aligned}\|f\|^p_{L^p(K)} &\leq 2^{p-1}\left(\sum_{l=1}^{N(r)}\|T_{\lambda,r}f\|^p_{L^p(B_r(x_l))} + \|(T_{\lambda,r}-\text{Id})f\|^p_{L^p(\mathbb{S}^2)}\right)\\ &\leq C_p r^{-\frac{p}{2}}\lambda^{\sigma_0(p)p}\sum_{l=1}^{N(r)}\|f\|^p_{L^2(B_{2r}(x_l))} + C_p\|(T_{\lambda,r}-\text{Id})f\|^p_{L^p(\mathbb{S}^2)}\\ &\leq C_pC_0 r^{-\frac{p}{2}}\lambda^{\sigma_0(p)p}\left(\max_{1\leq l\leq N(r)}\left\{\|f\|^{p-2}_{L^2(B_{2r}(x_l))}\right\}\right)\|f\|^2_{L^2(\mathbb{S}^2)} + C_p\|(T_{\lambda,r}-\text{Id})f\|^p_{L^p(\mathbb{S}^2)}.\end{aligned}$$

Hence, one finds

$$\|f\|_{L^p(K)} \leq C_p'\left(r^{-\frac{1}{2}}\lambda^{\sigma_0(p)}\left(\max_{1\leq l\leq N(r)}\left\{\|f\|^{1-\frac{2}{p}}_{L^2(B_{2r}(x_l))}\right\}\right)\|f\|^2_{L^2(\mathbb{S}^2)} + \|(T_{\lambda,r}-\text{Id})f\|_{L^p(\mathbb{S}^2)}\right). \tag{14}$$

This upper bound is valid uniformly in the range $\lambda \geq 1$ and $\lambda^{-1} \leq r \leq \frac{\pi}{2}$. Similarly, in the case of the L^∞ norm, we would get

$$\|f\|_{L^\infty(K)} \leq Cr^{-\frac{1}{2}}\lambda^{\frac{1}{2}}\left(\max_{1\leq l\leq N(r)}\left\{\|f\|_{L^2(B_{2r}(x_l))}\right\}\right) + \|(T_{\lambda,r}-\text{Id})f\|_{L^\infty(\mathbb{S}^2)}. \tag{15}$$

Note that so far we did not use the eigenvalue equation (9) and this is valid for any f in $L^2(\mathbb{S}^2)$. We will now specify these results in the case where $f = \psi_\lambda$. We begin with the remainder term:

Proposition 2.2. *Let $2 < p \leq \infty$ and let $0 < \beta < 1$. Then, there exists a constant $C > 0$ such that, for any solution ψ_λ to (2) with $\lambda \geq 1$ and for any $\lambda^{-\beta} \leq r \leq \frac{\pi}{2}$, one has*

$$\|(T_{\lambda,r} - Id)\,\psi_\lambda\|_{L^p(\mathbb{S}^2)} \leq C(r\lambda)^{\sigma_0(p)}\|\psi_\lambda\|_{L^2(\mathbb{S}^2)},$$

Gathering this Proposition with our estimates (14) and (15) on $\|f\|_{L^p(K)}$, we find that, for $\lambda \geq 1$, $\lambda^{-\beta} \leq r \leq \frac{\pi}{2}$ (with $\beta < 1$), for any $2 < p \leq +\infty$ and for any L^2-normalized solution ψ_λ to (2),

$$\|\psi_\lambda\|_{L^p(K)} \leq C_p\left(r^{-\frac{1}{2}}\lambda^{\sigma_0(p)}\max_{1\leq l\leq N(r)}\left\{\|\psi_\lambda\|^{1-\frac{2}{p}}_{L^2(B_{2r}(x_l))}\right\} + (r\lambda)^{\sigma_0(p)}\right). \tag{16}$$

The involved constants $C_p > 0$ depend only on V, K, ρ, β and p. Hence, as in [26, 39], we have reduced the problem of estimating the L^p norm of Schrödinger eigenfunctions to determining bounds on L^2-localized norms,

$$\int_{B_{2r}(x_l)} |\psi_\lambda(x)|^2 dv_{g_0}(x), \tag{17}$$

as $\lambda \to +\infty$ with r verifying $\lambda^{-\beta} \leq r \leq \frac{\pi}{2}$. In particular, if, for some $0 < \alpha \leq 1$, we were able to bound (17) uniformly (in terms of λ) by $Cr^{1+\alpha}$, then we would be able to get an improved upper bound inside K of the form

$$\|\psi_\lambda\|_{L^p(K)} \leq C_{p,K}\left(r^{\frac{\alpha}{2}-\frac{1+\alpha}{p}}\lambda^{\sigma_0(p)} + (r\lambda)^{\sigma_0(p)}\right),$$

in the range

$$\frac{\alpha}{2} - \frac{1+\alpha}{p} > 0 \quad \Longleftrightarrow \quad p > 2\left(1+\frac{1}{\alpha}\right). \tag{18}$$

However, as explained in [39, §4], one cannot expect such improved bounds on the sphere when $V \equiv 0$ thanks to the example of the spherical harmonics. In section 4, we shall see how to get *locally* improved bounds on (17) when V does not identically vanish. Before going to this question, we give the proof of Proposition 2.2.

Proof. Considering a solution to (9) and letting $2 \leq p \leq +\infty$, one has

$$\begin{aligned} \|(T_{\lambda,r} - \mathrm{Id})\,\psi_\lambda\|_{L^p(\mathbb{S}^2)} &\leq \sum_{j\in\mathbb{Z}_+} \|E_j\,(T_{\lambda,r}-\mathrm{Id})\,E_j\psi_\lambda\|_{L^p(\mathbb{S}^2)} \\ &\leq \sum_{j\in\mathbb{Z}_+} \left(|\rho(r(\lambda-\lambda_j))-1| + |\rho(r(\lambda+\lambda_j))|\right)\|E_j(\psi_\lambda)\|_{L^p(\mathbb{S}^2)}. \end{aligned}$$

As ρ belongs to the Schwartz class, we find using Sogge's estimate (3) that, for every $N \geq 1$, there exists $C_N > 0$ such that, for $\lambda \geq 1$ and $r \geq \lambda^{-\beta}$,

$$\sum_{j\in\mathbb{Z}_+} |\rho(r(\lambda+\lambda_j))|\,\|E_j(\psi_\lambda)\|_{L^p(\mathbb{S}^2)} \leq C_N(1+r\lambda)^{-N}\|\psi_\lambda\|_{L^2(\mathbb{S}^2)}.$$

Using one more time Sogge's estimate, we deduce that

$$\|(T_{\lambda,r}-\mathrm{Id})\,\psi_\lambda\|_{L^p(\mathbb{S}^2)} \leq \sum_{j\in\mathbb{Z}_+} |\rho(r(\lambda-\lambda_j))-1|\,\lambda_j^{\sigma_0(p)}\,\|E_j(\psi_\lambda)\|_{L^2(\mathbb{S}^2)} + C_N(1+r\lambda)^{-N}\|\psi_\lambda\|_{L^2(\mathbb{S}^2)}. \tag{19}$$

We now fix some $\delta \geq r$ so that $\delta \leq r\lambda$ and we split the sum over $j \in \mathbb{Z}_+$ in two parts. On the one hand, we consider the j such that $|\lambda - \lambda_j| \leq \delta/r$ and on the other hand, the integers such that $|\lambda-\lambda_j| > \delta/r$. Recall that $\lambda_j^2 = j(j+1)$. Hence, the number of terms in the first sum is $\mathcal{O}(\delta/r)$ and one is left with

$$\begin{aligned} \|(T_{\lambda,r}-\mathrm{Id})\,\psi_\lambda\|_{L^p(\mathbb{S}^2)} &\leq \sum_{j\in\mathbb{Z}_+:|\lambda-\lambda_j|>\delta/r} |\rho(r(\lambda-\lambda_j))-1|\,\lambda_j^{\sigma_0(p)}\,\|E_j(\psi_\lambda)\|_{L^2(\mathbb{S}^2)} \\ &+ \left(C\frac{\delta^2}{r}\lambda^{\sigma_0(p)} + C_N(1+r\lambda)^{-N}\right)\|\psi_\lambda\|_{L^2(\mathbb{S}^2)}. \end{aligned}$$

For the remaining sum, we can finally make use of the eigenvalue equation (9). It implies the existence of some constant $C_{\rho,V} > 0$ depending only on ρ and V such that

$$\sum_{j\in\mathbb{Z}_+:|\lambda-\lambda_j|>\delta/r} |\rho(r(\lambda-\lambda_j))-1|\,\lambda_j^{\sigma_0(p)}\,\|E_j(\psi_\lambda)\|_{L^2(\mathbb{S}^2)} \leq C_{\rho,V}\sum_{j\in\mathbb{Z}_+:|\lambda-\lambda_j|>\delta/r}\frac{\lambda_j^{\sigma_0(p)}}{|\lambda^2-\lambda_j^2|}\|\psi_\lambda\|_{L^2(\mathbb{S}^2)}.$$

As $\sigma_0(p)$ varies between 0 (for $p=2$) and 1/2 (for $p=\infty$), this last quantity is finite and it remains to evaluate

$$\sum_{j\in\mathbb{Z}_+:|\lambda-\lambda_j|>\delta/r}\frac{\lambda_j^{\sigma_0(p)}}{|\lambda^2-\lambda_j^2|} \tag{20}$$

in terms of δ, r, λ and p. We now recall that, for $X>0$, one has $(1+X)^{\sigma_0(p)}\leq 1+X^{\sigma_0(p)}$ (as $\sigma_0(p)\leq 1/2$). Hence, one has

$$\begin{aligned}\sum_{j\in\mathbb{Z}_+:|\lambda-\lambda_j|>\delta/r}\frac{\lambda_j^{\sigma_0(p)}}{|\lambda^2-\lambda_j^2|} &\leq \sum_{j\in\mathbb{Z}_+:|\lambda-\lambda_j|>\delta/r}\frac{|\lambda-\lambda_j|^{\sigma_0(p)}}{|\lambda^2-\lambda_j^2|}+\sum_{j\in\mathbb{Z}_+:|\lambda-\lambda_j|>\delta/r}\frac{\lambda^{\sigma_0(p)}}{|\lambda^2-\lambda_j^2|}\\ &\leq 2\sum_{j\in\mathbb{Z}_+:|\lambda-\lambda_j|>\delta/r}\frac{\lambda^{-1+\frac{3}{2}\sigma_0(p)}}{|\lambda-\lambda_j|^{1+\frac{\sigma_0(p)}{2}}}\\ &\leq 2\lambda^{-\frac{1}{4}}\sum_{j\in\mathbb{Z}_+:|\lambda-\sqrt{j(j+1)}|>\delta/r}\frac{1}{|\lambda-\sqrt{j(j+1)}|^{1+\frac{\sigma_0(p)}{2}}}\\ &\leq C\lambda^{-\frac{1}{4}}\sum_{j\in\mathbb{Z}_+^*}j^{-1-\frac{\sigma_0(p)}{2}}.\end{aligned}$$

In summary, if we suppose that $r\geq\lambda^{-\beta}$ (for some $\beta<1$), we obtain the following upper bound

$$\|(T_{\lambda,r}-\mathrm{Id})\,\psi_\lambda\|_{L^p(\mathbb{S}^2)}\leq C\left(\frac{\delta^2}{r}\lambda^{\sigma_0(p)}+\lambda^{-\frac{1}{4}}\right)\|\psi_\lambda\|_{L^2(\mathbb{S}^2)},$$

where $C>0$ depends on ρ, V, β and p. Recall that we supposed $r\leq\delta\leq r\lambda$. Hence, as $0\leq\sigma(p)\leq\frac{1}{2}$, we can set $\delta=r^{\frac{1+\sigma_0(p)}{2}}$ provided $r\geq\lambda^{-\frac{2}{\sigma_0(p)+1}}$, which is ensured by our assumption $r\geq\lambda^{-\beta}$. Implementing this, we obtain the existence of a constant $C_{\rho,V,\beta,p}>0$ (depending on ρ, V, β and p) such that

$$\|(T_{\lambda,r}-\mathrm{Id})\,\psi_\lambda\|_{L^p(\mathbb{S}^2)}\leq C_{\rho,V,\beta,p}(r\lambda)^{\sigma_0(p)}\|\psi_\lambda\|_{L^2(\mathbb{S}^2)},$$

as long as $r\geq\lambda^{-\beta}$. □

Remark 2.3. In view of applications of our method to semiclassical problems, it is worth noting that the above arguments work as well for solutions to

$$-\Delta_{g_0}\psi_\lambda+\beta_\lambda V\psi_\lambda=\lambda^2\psi_\lambda,\quad \|\psi_\lambda\|_{L^2(\mathbb{S}^2)}=1, \tag{21}$$

where $(\beta_\lambda)_\lambda$ is a given nonnegative sequence that may tend to $+\infty$. In that case, the upper bound (16) becomes, for every $\epsilon>0$,

$$\|\psi_\lambda\|_{L^p(K)}\leq C_{p,\epsilon}\left(r^{-\frac{1}{2}}\lambda^{\sigma_0(p)}\max_{1\leq l\leq N(r)}\left\{\|\psi_\lambda\|_{L^2(B_{2r}(x_l))}^{1-\frac{2}{p}}\right\}+(r\lambda)^{\sigma_0(p)}+\beta_\lambda\lambda^{-1+\epsilon}\lambda^{\sigma_0(p)}\right). \tag{22}$$

The calculation is indeed exactly the same except for the upper bound on the size of the remainder in (20) that we need to improve. Hence, we have potentially improvements as long as[6] $\beta_\lambda \lambda^{-1+\epsilon} \to 0$.

3. Reduction to L^2 localized estimates for $p < 6$ via Kakeya-Nikodym bounds

We now deal with the range $2 < p < 6$ which can also be reduced to estimating similar quantities. For such p, we can make use of the results of Blair and Sogge relating the growth of L^p norms for small p to Kakeya-Nikodym averages.

We let $0 \leq \chi \leq 1$ be a smooth cutoff function which is equal to 1 on $[-1, 1]$ and to 0 outside $[-2, 2]$. Given $x \in \mathbb{S}^2$, we denote by $\exp_x$ the exponential map induced by the metric g_0 and we set

$$\chi_{x,r}(y) := \chi\left(\frac{\| \exp_x^{-1}(y)\|}{r}\right) \in \mathcal{C}^\infty(\mathbb{S}^2).$$

This function is equal to 1 on $B_r(x)$ and to 0 outside $B_{2r}(x)$. We fix some $r_0 > 0$ and some $x_0 \in \mathbb{S}^2$. For any *normalized* solution to (2), one has

$$-\lambda^{-2}\Delta_{g_0}\psi_\lambda - \psi_\lambda = \lambda^{-2} V \psi_\lambda.$$

In particular, one can verify, using commutation rules for semiclassical pseudodifferential operators [58, § 4 and 14],

$$(-\lambda^{-2}\Delta_{g_0} - 1)^k \left(\chi_{x_0,r_0}\psi_\lambda\right) = \mathcal{O}(\lambda^{-k}), \quad k = 1, 2. \tag{23}$$

These two assumptions are exactly the ones needed to apply [4, Th. 1.1] in dimension 2. In order to formulate this result, we denote by $\tilde{G}(\mathbb{S}^2)$ the set of unit length geodesic segments in $\mathbb{S}^2$ and, for every $r > 0$ and for every $\gamma \in \tilde{G}(\mathbb{S}^2)$,

$$\mathcal{T}_r(\gamma) := \left\{x \in \mathbb{S}^2 : \ d_{g_0}(x, \gamma) \leq r\right\}.$$

With these conventions, the main result from [4] applied to $\chi_{x_0,r_0}\psi_\lambda$ tells us that, for $4 < p < 6$,

$$\|\psi_\lambda\|_{L^p(B_{r_0}(x_0))} \leq C_p \lambda^{\sigma_0(p)} \left(\sup_{\gamma\in\tilde{G}(\mathbb{S}^2)} \int_{B_{2r_0}(x_0)\cap \mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)} |\psi_\lambda(x)|^2 dv_{g_0}(x)\right)^{\frac{1}{2}\left(\frac{6}{p}-1\right)}, \tag{24}$$

and

$$\|\psi_\lambda\|_{L^4(B_{r_0}(x_0))} \leq C_p (\log\lambda) \lambda^{\frac{1}{8}} \left(\sup_{\gamma\in\tilde{G}(\mathbb{S}^2)} \int_{B_{2r_0}(x_0)\cap \mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)} |\psi_\lambda(x)|^2 dv_{g_0}(x)\right)^{\frac{1}{4}}, \tag{25}$$

[6]This can probably sligthly improved to replace the λ^ϵ by some logarithmic factor but we did not try to optimize that.

where the constants $C_p > 0$ depend only on p. Thanks to these results, it is sufficient to derive nontrivial upper bounds on the Kakeya-Nikodym averages

$$\int_{B_{2r_0}(x_0)\cap\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)} |\psi_\lambda(x)|^2 dv_{g_0}(x).$$

in order to improve locally Sogge's upper bounds (12) in the range $4 < p < 6$. By interpolation, it will automatically yields an improvement for $2 < p < 4$.

Finally, we can relate these quantities to the ones appearing in (17). Indeed, we can pick $0 < \beta < 1/2$ and we can cover $B_{2r_0}(x_0) \cap \mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)$ by a family of $2r_0r^{-1}$ balls of radius $r \geq \lambda^{-\beta}$ centered on a point of $\gamma \cap B_{2r_0}(x_0)$. Hence, one has

$$(26) \qquad \int_{B_{2r_0}(x_0)\cap\mathcal{T}_{\lambda^{-\frac{1}{2}}}(\gamma)} |\psi_\lambda(x)|^2 dv_{g_0}(x) \leq 4r_0r^{-1} \sup_{x\in\gamma\cap B_{2r_0}(x_0)} \left\{\int_{B_r(x)} |\psi_\lambda(y)|^2 dv_{g_0}(y)\right\},$$

which are exactly the quantities that appeared in Section 2. Hence, in both cases, we are reduced to estimating these localized L^2-estimates.

Remark 3.1. As in Remark 2.3, we can consider solutions to (21). One can verify that the assumption (23) is still verified as long as $0 \leq \beta_\lambda \leq \lambda$. Hence, (24) and (25) remain true in that generalized framework.

Remark 3.2. As we will only consider balls of radius $r \gg \lambda^{-\frac{1}{2}}$, the logarithmic factor appearing in (25) could probably be removed following [3].

4. L^2-localized estimates using invariance by the classical flows

Thanks to (16), (24), (25) and (26), we know that proving Theorem 1.1 amounts to control uniformly the following quantity

$$M_{B_{r_0}(x_0),\alpha,r}(\psi_\lambda) := \sup\left\{\frac{1}{r^{1+\alpha}}\int_{B_r(x)} |\psi_\lambda(y)|^2 dv_{g_0}(y) : x \in B_{r_0}(x_0)\right\},$$

with $0 < \alpha \leq 1$ and $\lambda^{-\beta} \leq r$ that goes to 0 as $\lambda \to +\infty$. The following Proposition answers this problem and it is the main new technical result of the article:

Proposition 4.1. *Let x_0 be a point in $\mathbb{S}^2$ verifying the assumption of Theorem 1.1. Then, there exist $r_0 > 0$ and $C_0 > 0$ such that, for any (ψ_λ, λ) solution to* (2),

$$\lambda^{-\frac{2}{9}} \leq r \leq \frac{\pi}{2} \quad\Longrightarrow\quad M_{B_{r_0}(x_0),\frac{1}{2},r}(\psi_\lambda) \leq C_0\|\psi_\lambda\|^2_{L^2(\mathbb{S}^2)}.$$

Implementing this bound in (16) and in (24), we find that, for $4 < p \leq \infty$ and for $\lambda > 0$,

$$\|\psi_\lambda\|_{L^p(B_{r_0}(x_0))} \leq C_{p,x_0}\lambda^{\sigma_0(p)-\frac{1}{18}\left|1-\frac{6}{p}\right|}\|\psi_\lambda\|_{L^2(\mathbb{S}^2)}.$$

Finally, for $p = 4$, we derive from (25) that, for $\lambda > 1$,

$$\|\psi_\lambda\|_{L^4(B_{r_0}(x_0))} \leq C_{4,x_0}(\log\lambda)\lambda^{\frac{1}{8}-\frac{1}{36}}\|\psi_\lambda\|_{L^2(\mathbb{S}^2)},$$

which also yields the result for $2 < p \leq 4$ by interpolation. Hence, in order to prove Theorem 1.1, we are left with the proof of Proposition 4.1 which will be the object of the rest of the article.

Coming back to Proposition 4.1, it is in fact sufficient to get an uniform upper bound on

$$\tilde{M}_{B_{r_0}(x_0),\alpha,r}(\psi_\lambda) := \sup\left\{\frac{1}{r^{1+\alpha}}\int_{\mathbb{S}^2}\chi_{x,r}(y)|\psi_\lambda(y)|^2 dv_{g_0}(y) : x \in B_{r_0}(x_0)\right\},$$

where we used the conventions of §3 for the function $\chi_{x,r}$. In order to get this uniform control, we will make use of the invariance properties of semiclassical Wigner distributions that we recently obtained with Macià [29, 30]. In order to make use of semiclassical methods [58], we set $h = \lambda^{-1}$ and $u_h = \psi_\lambda$. Hence, one has

$$-h^2\Delta_{g_0}u_h + h^2 V u_h = u_h, \quad \|u_h\|_{L^2(\mathbb{S}^2)} = 1. \tag{27}$$

Let now x be a point in $B_{r_0}(x_0)$ and $h^\beta \leq r \leq \frac{\pi}{4}$. In terms of pseudodifferential operators on $\mathbb{S}^2$ [58, §14.2], the quantity we are interested in can be rewritten as

$$\int_{\mathbb{S}^2}\chi_{x,r}(y)|u_h(y)|^2 dv_{g_0}(y) = \langle \mathrm{Op}_h\left(\chi_{x,r}\right)u_h, u_h\rangle_{L^2(\mathbb{S}^2)},$$

where Op_h is a semiclassical quantization [58, §14.2.3]. Note that, in order to have $\chi_{x,r}$ amenable to semiclassical pseudodifferential calculus [58, §4.4.1] (see also [18, §2.2, App.A] for the case of manifolds), we need to impose that

$$r \geq h^\beta \quad \text{and} \quad 0 \leq \beta < \frac{1}{2}. \tag{28}$$

We will now revisit the arguments of [29, 30] in that specific framework and show how they yield the expected result.

4.1. **Spectral cutoff.** We fix some smooth cutoff function $0 \leq \chi_0 \leq 1$ which is equal to 1 on the interval $[1/2, 2]$ and to 0 outside $[1/4, 4]$. Thanks to (27), one has

$$\langle \mathrm{Op}_h\left(\chi_{x,r}\right)u_h, u_h\rangle_{L^2(\mathbb{S}^2)} = \left\langle \mathrm{Op}_h\left(\chi_{x,r}\right)\chi_0(-h^2\Delta_{g_0} + h^2 V)u_h, u_h\right\rangle_{L^2(\mathbb{S}^2)}.$$

According to [58, Th. 14.9], $\chi_0(-h^2\Delta_{g_0}+h^2V)$ is a semiclassical pseudodifferential operator in the class $\Psi^{-\infty}(\mathbb{S}^2)$ with principal symbol equal to $\chi_0(\|\eta\|^2_{g_0^*(y)})$. Hence, the composition rule for pseudodifferential operators [58, Th. 4.18 and 14.1] implies that

$$\langle \mathrm{Op}_h\left(\chi_{x,r}\right)u_h, u_h\rangle_{L^2(\mathbb{S}^2)} = \left\langle \mathrm{Op}_h\left(\chi_{x,r}(y)\chi_0(\|\eta\|^2)\right)u_h, u_h\right\rangle_{L^2(\mathbb{S}^2)} + \mathcal{O}(h^{1-2\beta}),$$

where the constant in the remainder is uniform for $x \in \mathbb{S}^2$ and $r \geq h^\beta$. In the following, we set

$$a_{x,r}(y,\eta) := \chi_{x,r}(y)\chi_0(\|\eta\|^2_{g_0^*(y)}).$$

4.2. **Applying the evolution by the free Schrödinger flow.** We write

$$-\Delta_{g_0} = A^2 - \frac{1}{4}, \tag{29}$$

where A is a selfadjoint pseudodifferential operator of order 1 with principal symbol $\|\eta\|_{g_0^*(y)}$ and satisfying

$$e^{2i\pi A} = -\mathrm{Id}. \tag{30}$$

Equivalently, one has $A = \sqrt{\frac{1}{4} - \Delta_{g_0}}$. The eigenvalue equation (27) can be rewritten as

$$\left(A^2 - \frac{1}{h^2}\right) u_h = \left(\frac{1}{4} - V\right) u_h \quad \Longrightarrow \quad \left(A - \frac{1}{h}\right) u_h = \mathcal{O}_{L^2}(h).$$

In particular, one has

$$e^{is\left(A-\frac{1}{h}\right)}u_h = u_h + \int_0^s e^{i\tau\left(A-\frac{1}{h}\right)}\left(A - \frac{1}{h}\right) u_h d\tau = u_h + \mathcal{O}_{L^2}(|s|h). \tag{31}$$

This leads to
(32)

$$\int_{\mathbb{S}^2} \chi_{x,r}(y)|u_h(y)|^2 dv_{g_0}(y) = \left\langle \left(\frac{1}{2\pi}\int_0^{2\pi} e^{isA}\operatorname{Op}_h\left(a_{x,r}\right)e^{-isA}ds\right) u_h, u_h \right\rangle_{L^2(\mathbb{S}^2)} + \mathcal{O}(h^{1-2\beta})$$

In the following, given a in $\mathcal{C}^\infty_c(T^*\mathbb{S}^2 \setminus \underline{0})$, we set, by analogy with the Radon transfom,

$$\mathcal{R}_{\mathrm{qu}}(\operatorname{Op}_h(a)) := \frac{1}{2\pi}\int_0^{2\pi} e^{isA}\operatorname{Op}_h(a)e^{-isA}ds.$$

According to Remark 4.2 below, the Egorov Theorem allows to relate the operator $\mathcal{R}_{\mathrm{qu}}(\operatorname{Op}_h(a_{x,r}))$ to the classical average by the geodesic flow:

$$\mathcal{R}_{\mathrm{qu}}(\operatorname{Op}_h(a_{x,r})) = \operatorname{Op}_h\left(\frac{1}{2\pi}\int_0^{2\pi} a_{x,r}\circ\varphi_0^t dt\right) + \mathcal{O}_{L^2\to L^2}(h^{1-2\beta}), \tag{33}$$

where the constant in the remainder is uniform for $x \in \mathbb{S}^2$ and $r \geq h^\beta$ and where φ_0^t is the Hamiltonian flow associated with the Hamiltonian function[7] $H_0(y,\eta) := \|\eta\|_{g_0(y)}$. Given a in $\mathcal{C}^\infty_c(T^*\mathbb{S}^2 \setminus \underline{0})$, we set

$$\mathcal{R}_{\mathrm{cl}}(a) := \frac{1}{2\pi}\int_0^{2\pi} a\circ\varphi_0^t dt.$$

Remark 4.2. Let us briefly remind how to prove (33). This is standard [18, App. A.3] and we just need to pay attention to our class of symbols. First, we write, for every $s, t \in [0, 2\pi]$,

$$\frac{d}{ds}\left(e^{isA}\operatorname{Op}_h(a_{x,r}\circ\varphi_0^{t-s})e^{-isA}\right) = e^{isA}\left(\frac{i}{h}\left[hA, \operatorname{Op}_h(a_{x,r}\circ\varphi_0^{t-s})\right] - \operatorname{Op}_h\left(\{H_0, a_{x,r}\circ\varphi_0^{t-s}\}\right)\right)e^{-isA}.$$

We now let χ_1 be a smooth function which is equal to 1 in a neighborhood of $[1/4, 4]$ and to 0 outside $[1/8, 8]$. In particular, $\chi_1(H_0^2)$ is equal to 1 on the support of $a_{x,r}$. Combining

[7]This is just a reparametrization of the standard geodesic flow.

this with the composition rules for pseudodifferential operators with exotic symbols on manifolds [18, Lemma A.6], we know that, for every $\tau\in[0,2\pi]$,

$$\begin{aligned}\mathrm{Op}_h(a_{x,r}\circ\varphi_0^\tau) &= \mathrm{Op}_h(a_{x,r}\circ\varphi_0^\tau)\,\mathrm{Op}_h(\chi_1(H_0^2))+\mathcal{O}_{L^2\to L^2}(h^2)\\ &= \mathrm{Op}_h(\chi_1(H_0^2))\,\mathrm{Op}_h(a_{x,r}\circ\varphi_0^\tau)+\mathcal{O}_{L^2\to L^2}(h^2).\end{aligned}$$

We can also remark using the composition rules for pseudodifferential operators that

$$hA\,\mathrm{Op}_h(\chi_1(H_0^2))=\mathrm{Op}_h(\chi_1(H_0^2))hA+h\,\mathrm{Op}_h(r)+\mathcal{O}_{L^2\to L^2}(h^2),$$

where r is a smooth compactly supported function that depends in a multilinear way of the derivatives of order ≥ 1 of the function $\chi_1(H_0^2)$. Thus its support does not intersect the support of $a_{x,r}$. In particular, using the composition rule [18, Lemma A.6] one more time and the support properties of $a_{x,r}$, one has $\mathrm{Op}_h(a_{x,r})\,\mathrm{Op}_h(r)=\mathcal{O}_{L^2\to L^2}(h^2)$. Hence, after integration over the interval $[0,2\pi]$ and applying the Calderón-Vaillancourt Theorem, one finds

$$\begin{aligned}\mathcal{R}_{\mathrm{qu}}(\mathrm{Op}_h(a_{x,r})) &= \mathrm{Op}_h\left(\frac{1}{2\pi}\int_0^{2\pi}a_{x,r}\circ\varphi_0^t dt\right)+\mathcal{O}_{L^2\to L^2}(h)\\ &+ \frac{1}{2\pi}\int_0^{2\pi}\int_0^t\left(\frac{i}{h}\left[hA\,\mathrm{Op}_h(\chi_1(H_0^2)),\mathrm{Op}_h(a_{x,r}\circ\varphi_0^{t-s})\right]\right)dsdt\\ &- \frac{1}{2\pi}\int_0^{2\pi}\int_0^t\mathrm{Op}_h\left(\{H_0,a_{x,r}\circ\varphi_0^{t-s}\}\right)dsdt.\end{aligned}$$

As all our pseudodifferential operators are microlocally supported in a compact[8] set of $T^*\mathbb{S}^2$, we can again apply the composition rule for exotic symbols on a compact manifold as stated in [18, Lemma A.6]. Thus, we can conclude that (33) holds. Inspecting carefully the argument, we can in fact conclude that

$$\mathcal{R}_{\mathrm{qu}}(\mathrm{Op}_h(a_{x,r}))=\mathrm{Op}_h(\tilde{a}_{x,r})+\mathcal{O}_{L^2\to L^2}(h^2),\tag{34}$$

where the constant in the remainder is uniform for $x\in\mathbb{S}^2$ and $r\geq h^\beta$ and $\tilde{a}_{x,r}$ is a symbol in the class $S_\beta^{\mathrm{comp}}(T^*\mathbb{S}^2)$ as defined in [18, §2.2]. This symbol is equal to $\mathcal{R}_{\mathrm{cl}}(a_{x,r})$ modulo $h^{1-2\beta}S_\beta^{\mathrm{comp}}(T^*\mathbb{S}^2)$ and its support is contained in the support of $\mathcal{R}_{\mathrm{cl}}(a_{x,r})$.

Remark 4.3. The arguments used from the beginning of this Section would work as well for the following semiclassical problem:

$$-h^2\Delta_{g_0}u_h+\varepsilon_h V u_h=u_h,\quad \|u_h\|_{L^2(\mathbb{S}^2)}=1,$$

where $\varepsilon_h\to 0$ fast enough. More precisely, the above proofs only require $h^{-1}\varepsilon_h\to 0$ in order to have a small remainder in (31). In this case, this would yield the bound

$$\int_{\mathbb{S}^2}\chi_{x,r}(y)|u_h(y)|^2dv_{g_0}(y)=\left\langle\left(\frac{1}{2\pi}\int_0^{2\pi}e^{-isA}\,\mathrm{Op}_h\left(a_{x,r}\right)e^{isA}ds\right)u_h,u_h\right\rangle_{L^2(\mathbb{S}^2)}+\mathcal{O}(h^{1-2\beta})+\mathcal{O}(h^{-1}\varepsilon_h).$$

[8]This was the main reason for inserting the pseudodifferential cutoff $\mathrm{Op}_h(\chi_1(H_0^2))$.

The argument from [29] would allow to remove this extra remainder $\mathcal{O}(h^{-1}\varepsilon_h)$ and to handle the case $\varepsilon_h \to 0^+$. Yet, as this kind of condition on the size of the potential already appeared in Remarks 2.3 and 3.1, we do not pursue this here.

4.3. **Weinstein averaging method.** Following Weinstein [52], one can use (30) to obtain the following exact commutation relation:

$$[\mathcal{R}_{\mathrm{qu}}(\mathrm{Op}_h(a_{x,r})), A] = 0.$$

In particular, thanks to (29), one has

$$[\mathcal{R}_{\mathrm{qu}}(\mathrm{Op}_h(a_{x,r})), \Delta_{g_0}] = 0. \tag{35}$$

Using (27), this implies that

$$\langle [V, \mathcal{R}_{\mathrm{qu}}(\mathrm{Op}_h(a_{x,r}))]\, u_h, u_h \rangle_{L^2(\mathbb{S}^2)} = 0.$$

Thanks to (34), this can be rewritten as

$$\langle [V, \mathrm{Op}_h(\tilde{a}_{x,r})]\, u_h, u_h \rangle_{L^2(\mathbb{S}^2)} = \mathcal{O}(h^2).$$

As in Remark 4.2, we can insert pseudodifferential cutoffs and we find

$$\left\langle \left[V\, \mathrm{Op}_h(\chi_1(H_0^2)), \mathrm{Op}_h(\tilde{a}_{x,r})\right] u_h, u_h \right\rangle_{L^2(\mathbb{S}^2)} = \mathcal{O}(h^2).$$

Hence, thanks to the composition rule for pseudodifferential operators [18, Lemma A.6] with exotic symbols, we get

$$\langle \mathrm{Op}_h\left(\{V, \mathcal{R}_{\mathrm{cl}}(a_{x,r})\}\right) u_h, u_h \rangle_{L^2(\mathbb{S}^2)} = \mathcal{O}(h^{1-3\beta}),$$

where the constant in the remainder is uniform for $x \in \mathbb{S}^2$ and $r \geq h^\beta$. Observe that the extra loss in $\mathcal{O}(h^{1-3\beta})$ (compared with $\mathcal{O}(h^{1-2\beta})$) comes from the subprincipal term in $\tilde{a}_{x,r}$. Applying the argument of paragraph 4.2 one more time, we find that

$$\left\langle \mathrm{Op}_h\left(\frac{1}{2\pi}\int_0^{2\pi} \{V, \mathcal{R}_{\mathrm{cl}}(a_{x,r})\} \circ \varphi_0^t dt\right) u_h, u_h \right\rangle_{L^2(\mathbb{S}^2)} = \mathcal{O}(h^{1-3\beta}),$$

from which we infer

$$\langle \mathrm{Op}_h\left(\{\mathcal{R}_{\mathrm{cl}}(V), \mathcal{R}_{\mathrm{cl}}(a_{x,r})\}\right) u_h, u_h \rangle_{L^2(\mathbb{S}^2)} = \mathcal{O}(h^{1-3\beta}),$$

with the constant in the remainder enjoying the same uniformity property as before. Here V is identified with its pullback on $T^*\mathbb{S}^2 \setminus \underline{0}$ via the canonical projection $\Pi(y, \eta) = y$.

Let us now denote by $\varphi^t_{\langle V \rangle}$ the Hamiltonian flow induced by $\mathcal{R}_{\mathrm{cl}}(V)$. As $\mathcal{R}_{\mathrm{cl}}(V)$ and H_0 Poisson commute, one has $\varphi_0^t \circ \varphi^s_{\langle V \rangle} = \varphi^s_{\langle V \rangle} \circ \varphi_0^t$ for every t and s in $\mathbb{R}$. We note that all the above argument would work as well if we replace $a_{x,r}$ by $a_{x,r} \circ \varphi^\tau_{\langle V \rangle}$ and the remainder would remain uniform in τ (and in (x, r)) provided that τ remains on a bounded interval. Hence, one has, uniformly for $\tau \in [-\tau_0, \tau_0]$, $x \in \mathbb{S}^2$ and $r \geq h^\beta$,

$$\left\langle \mathrm{Op}_h\left(\left\{\mathcal{R}_{\mathrm{cl}}(V), \mathcal{R}_{\mathrm{cl}}(a_{x,r}) \circ \varphi^\tau_{\langle V \rangle}\right\}\right) u_h, u_h \right\rangle_{L^2(\mathbb{S}^2)} = \mathcal{O}(h^{1-3\beta}). \tag{36}$$

We integrate this expression between 0 and τ:

$$\left\langle \mathrm{Op}_h\left(\mathcal{R}_{\mathrm{cl}}(a_{x,r}) \circ \varphi^\tau_{\langle V \rangle}\right) u_h, u_h \right\rangle_{L^2(\mathbb{S}^2)} = \langle \mathrm{Op}_h\left(\mathcal{R}_{\mathrm{cl}}(a_{x,r})\right) u_h, u_h \rangle_{L^2(\mathbb{S}^2)} + \mathcal{O}(h^{1-3\beta}).$$

Combining this with (32), we find

(37)
$$\int_{\mathbb{S}^2}\chi_{x,r}(y)|u_h(y)|^2dv_{g_0}(y)=\left\langle \mathrm{Op}_h\left(\frac{1}{2\tau_0}\int_{-\tau_0}^{\tau_0}\mathcal{R}_{\mathrm{cl}}(a_{x,r})\circ\varphi^{\tau}_{\langle V\rangle}d\tau\right)u_h,u_h\right\rangle_{L^2(\mathbb{S}^2)}+\mathcal{O}(h^{1-3\beta}),$$

where the constant in the remainder is uniform for x in K and $r\geq h^{\beta}$.

4.4. **Applying Calderón-Vaillancourt Theorem.** We are now in position to apply the Calderón-Vaillancourt Theorem [58, Th. 5.1] which tells us that

$$\left\|\mathrm{Op}_h\left(\frac{1}{2\tau_0}\int_{-\tau_0}^{\tau_0}\mathcal{R}_{\mathrm{cl}}(a_{x,r})\circ\varphi^{\tau}_{\langle V\rangle}d\tau\right)\right\|_{L^2\to L^2}\leq C\left\|\frac{1}{2\tau_0}\int_{-\tau_0}^{\tau_0}\mathcal{R}_{\mathrm{cl}}(a_{x,r})\circ\varphi^{\tau}_{\langle V\rangle}d\tau\right\|_{L^\infty(T^*\mathbb{S}^2)}+\mathcal{O}(h^{1-3\beta}),$$

where C_0 is some universal constant and where the constant in the remainder is one more time uniform for x in $\mathbb{S}^2$ and $r\geq h^{\beta}$. Together with (37), we finally get

$$\int_{\mathbb{S}^2}\chi_{x,r}(y)|u_h(y)|^2dv_{g_0}(y)\leq C\left\|\frac{1}{2\tau_0}\int_{-\tau_0}^{\tau_0}\mathcal{R}_{\mathrm{cl}}(a_{x,r})\circ\varphi^{\tau}_{\langle V\rangle}d\tau\right\|_{L^\infty(T^*\mathbb{S}^2)}+\mathcal{O}(h^{1-3\beta}).$$

From the construction of $a_{x,r}$, one can in fact reduce to the unit cotangent bundle:

(38)
$$\int_{\mathbb{S}^2}\chi_{x,r}(y)|u_h(y)|^2dv_{g_0}(y)\leq C\left\|\frac{1}{4\pi\tau_0}\int_{-\tau_0}^{\tau_0}\int_0^{2\pi}\chi_{x,r}\circ\varphi_0^t\circ\varphi^{\tau}_{\langle V\rangle}dtd\tau\right\|_{L^\infty(S^*\mathbb{S}^2)}+\mathcal{O}(h^{1-3\beta}),$$

where we identify $\chi_{x,r}$ with its pullback on $S^*\mathbb{S}^2$.

In order to facilitate the discussion, we shall work on the space of geodesic $G(\mathbb{S}^2)\simeq\mathbb{S}^2$. With the induced symplectic form on $\mathbb{S}^2$, $\varphi^{\tau}_{\langle V\rangle}$ can be viewed as the Hamiltonian flow of $\mathcal{R}(V)$ on $\mathbb{S}^2$. Hence, what we are aiming at is an upper bound on

$$0\leq\frac{1}{2\tau_0}\int_{-\tau_0}^{\tau_0}\mathcal{R}(\chi_{x,r})\circ\varphi^{\tau}_{\langle V\rangle}(\gamma)d\tau,$$

when $\gamma\in G(\mathbb{S}^2)\simeq\mathbb{S}^2$ and when $r\ll\tau_0$. It is in fact sufficient to find an upper bound on

$$\frac{1}{2\tau_0}\int_{-\tau_0}^{\tau_0}\mathcal{R}(\mathbf{1}_{B_{2r}(x)})\circ\varphi^{\tau}_{\langle V\rangle}(\gamma)d\tau,$$

where $\mathbf{1}_{B_{2r}(x)}$ is the characteristic function of the geodesic ball of radius $2r$ centered at x. The function $\mathcal{R}(\mathbf{1}_{B_{2r}(x)})$ is supported in a neighborhood of width $4r$ of $\Gamma_x\subset G(\mathbb{S}^2)$ and it is bounded from above by $4r$. Hence,

$$\forall\gamma\in G(\mathbb{S}^2),\quad 0\leq\frac{1}{2\tau_0}\int_{-\tau_0}^{\tau_0}\mathcal{R}(\mathbf{1}_{B_{2r}(x)})\circ\varphi^{\tau}_{\langle V\rangle}(\gamma)d\tau\leq 4r. \tag{39}$$

Remark 4.4. In the case of semiclassical Schrödinger operators as in Remark 4.3, the argument would work similarly and we would also obtain the bound (38) for this semiclassical problem (up to the already extra remainder $\mathcal{O}(h^{-1}\varepsilon_h)$ that apeared in this Remark).

4.5. **Flow lines of $\varphi^t_{\langle V\rangle}$ near Γ_{x_0}.** So far we did not use our assumptions on V or on the point x_0. They will now be used to get an improvement of order $r^{1/2}$ on the upper bound (39) when $x\in B_{r_0}(x_0)$. To that aim, we now fix x_0 satisfying the assumption of the Theorem and we will analyze the flow lines of $\varphi^t_{\langle V\rangle}$ near a given point γ_0 of Γ_{x_0}.

Without loss of generality, we may suppose that x_0 is the north pole, i.e. with coordinates $(0,0,1)$ in the representation (1). Then, for every $x\in B_{\epsilon_0}(x_0)$, Γ_x is a great circle of the sphere lying in the annulus

$$\mathcal{A}_{\epsilon_0}:=\left\{(x_1,x_2,x_3)\in\mathbb{R}^3: x_1^2+x_2^2+x_3^2=1,\ |x_3|\leq\sin\epsilon_0\right\}.$$

Similarly, the function $\mathcal{R}(\mathbf{1}_{B_{2r}(x)})$ is supported on an annulus of width $2|\sin(2r)|$ around Γ_x and it takes the value $4r$ on this annulus. In particular, if $\tau_0>0$ and $r_1>0$ are chosen small enough, then, for every $x\in B_{\epsilon_0}(x_0)$ and for every $0<r<r_1$, the support of

$$\frac{1}{2\tau_0}\int_{-\tau_0}^{\tau_0}\mathcal{R}(\mathbf{1}_{B_{2r}(x)})\circ\varphi^\tau_{\langle V\rangle}d\tau \tag{40}$$

is contained in the annulus $\mathcal{A}_{2\epsilon_0}$. Hence, once we have fixed $x\in B_{\epsilon_0}(x_0)$, we just need to study the value of this function inside such an annulus. More precisely, we want to show that this is of order $\mathcal{O}(r^{3/2})$ uniformly for γ in this annulus.

Let $\gamma_0\in\Gamma_{x_0}$ and let us prove this uppper bound in a neighborhood of a fixed γ_0. Without loss of generality, we can suppose that, in spherical coordinates (ϕ,θ), one has $\gamma_0=(\pi/2,0)$. The vector field $X_{\langle V\rangle}$ can be written in this system of coordinates:

$$X_{\langle V\rangle}(\phi,\theta)=-\frac{1}{\sin\phi}\frac{\partial\mathcal{R}(V)}{\partial\theta}\partial_\phi+\frac{\partial\mathcal{R}(V)}{\partial\phi}\partial_\theta.$$

We need to distinguish two situations:

(1) $X_{\langle V\rangle}(\gamma_0)\notin T_{\gamma_0}\Gamma_{x_0}$ which means that $\frac{\partial\mathcal{R}(V)}{\partial\theta}(\pi/2,0)\neq0$;

(2) $X_{\langle V\rangle}(\gamma_0)\in T_{\gamma_0}\Gamma_{x_0}$ which means that $\frac{\partial\mathcal{R}(V)}{\partial\theta}(\pi/2,0)=0$. In that case, the hypothesis of Theorem 1.1 implies that $\frac{\partial\mathcal{R}(V)}{\partial\phi}(\pi/2,0)\neq0$ and $\frac{\partial^2\mathcal{R}(V)}{\partial\theta^2}(\pi/2,0)\neq0$

The Hamilton-Jacobi equations can be written as

$$\phi'(\tau)=-\frac{1}{\sin\phi(\tau)}\frac{\partial\mathcal{R}(V)}{\partial\theta}(\phi(\tau),\theta(\tau)),\quad\text{and}\quad\theta'(\tau)=\frac{\partial\mathcal{R}(V)}{\partial\phi}(\phi(\tau),\theta(\tau)).\tag{41}$$

4.5.1. *The transverse case.* Let us begin with the first situation which is slightly easier to handle. Witout loss of generality, we can suppose that $\frac{\partial\mathcal{R}(V)}{\partial\theta}(\pi/2,0)>0$ (the negative case is handled similarly). First, using spherical coordinates, we fix an open neighborhood $\mathcal{U}_{2\epsilon_0}:=(\pi/2-4\epsilon_0,\pi/2+4\epsilon_0)\times(-2\epsilon_0,2\epsilon_0)$ so that

$$\forall\gamma=(\phi,\theta)\in\mathcal{U}_{2\epsilon_0},\quad\frac{\partial\mathcal{R}(V)}{\partial\theta}(\phi,\theta)>\frac12\frac{\partial\mathcal{R}(V)}{\partial\theta}(\pi/2,0)=:a_0>0.\tag{42}$$

Up to decreasing the value τ_0, we can suppose without loss of generality that $\varphi^\tau_{\langle V\rangle}(\gamma)$ belongs to $\mathcal{U}_{2\epsilon_0}$ for every $|\tau|\leq\tau_0$ and for every $\gamma\in\mathcal{U}_{\epsilon_0}$. As already explained, the support of (40) is contained in $\mathcal{A}_{2\epsilon_0}$. For the moment, we will study locally its value inside $\mathcal{U}_{\epsilon_0} \subset \mathcal{A}_{2\epsilon_0}$. We now fix some γ in $\mathcal{U}_{\epsilon_0}$. In particular,

$$\forall |\tau| \leq \tau_0, \quad \frac{\partial \mathcal{R}(V)}{\partial \theta}\left(\varphi^\tau_{\langle V\rangle}(\gamma)\right) \geq a_0,$$

which implies thanks to (41) that $\phi'(\tau) < 0$ along this piece of trajectory. This yields the following upper bound along the orbit $\left(\varphi^\tau_{\langle V\rangle}(\gamma)\right)_{-\tau_0\leq\tau\leq\tau_0}$:

$$\phi(\tau_2) - \phi(\tau_1) \leq -\frac{a_0}{\cos(4\epsilon_0)}(\tau_2 - \tau_1) \iff \tau_2 - \tau_1 \leq \frac{\cos(4\epsilon_0)}{a_0}(\phi(\tau_1) - \phi(\tau_2)), \tag{43}$$

for every $-\tau_0 \leq \tau_1 \leq \tau_2 \leq \tau_0$.

Recall now that the function in (40) is defined by averaging $\mathcal{R}(\mathbf{1}_{B_{2r}(x)})$ for some $x \in B_{\epsilon_0}(x_0)$ and some $0 < r < r_1$. In spherical coordinates, x can be written (ϕ_x, θ_x) where $0 \leq \phi_x \leq \epsilon_0$ and $0 \leq \theta_x \leq 2\pi$. Hence, using our identification $G(\mathbb{S}^2) \simeq \mathbb{S}^2$, $\mathcal{R}(\mathbf{1}_{B_{2r}(x)})$ is $4r$ times the characteristic function of the annulus of width $4r$ centered at Γ_x,

$$\mathcal{A}_{2r}(x) = \left\{(\phi,\theta) :\ \phi - \arccos\left(-\cos(\theta-\theta_x)\sin(\phi_x)\right) \in [-2r, 2r],\ 0 \leq \theta \leq 2\pi\right\}.$$

The boundary of this annulus is given by

$$\partial\mathcal{A}_{2r}(x) = \left\{\left(\arccos\left(-\cos(\theta-\theta_x)\sin(\phi_x)\right) \pm 2r, \theta\right) :\ 0 \leq \theta \leq 2\pi\right\}$$

and it is oriented thanks to the natural orientation on $\mathbb{S}^2$. Using now that $\mathcal{R}(V)$ is of class $\mathcal{C}^1$ and (42), we know that, up to decreasing the value of ϵ_0 (and thus of τ_0 and r_1), the vector field $X_{\langle V\rangle}$ is uniformly (negatively) transverse to $\partial\mathcal{A}_{2r}(x) \cap \mathcal{U}_{2\epsilon_0}$ for every $x \in B_{\epsilon_0}(x_0)$ and for every $0 < r < r_1$. In particular, given $\gamma \in \mathcal{U}_{\epsilon_0}$, the set

$$\left\{\tau \in [-\tau_0, \tau_0] : \varphi^\tau_{\langle V\rangle}(\gamma) \in \mathcal{A}_{2r}(x)\right\}$$

is an interval that we denote by $I_{x,r}(\gamma)$. Hence,

$$0 \leq \frac{1}{2\tau_0}\int_{-\tau_0}^{\tau_0} \mathcal{R}(\mathbf{1}_{B_{2r}(x)}) \circ \varphi^\tau_{\langle V\rangle}(\gamma)d\tau \leq \frac{2r|I_{x,r}(\gamma)|}{\tau_0},$$

and it remains to determine an upper bound on the size of this interval in terms of r. Thanks to the upper bound (43), the length of the interval is bounded by the maximal variation of ϕ along the orbit of γ inside $\mathcal{A}_{2r}(x)$. If we denote the interval $I_{x,r}(\gamma)$ by $[\tau_1, \tau_2]$, then

$$\phi(\tau_1) - \phi(\tau_2) \leq 4r + \left|\arccos\left(-\cos(\theta(\tau_1)-\theta_x)\sin(\phi_x)\right) - \arccos\left(-\cos(\theta(\tau_2)-\theta_x)\sin(\phi_x)\right)\right|.$$

As $\phi_x \in [-\epsilon_0, \epsilon_0]$ (with $\epsilon_0 > 0$ small), this yields an upper bound of the form

$$\phi(\tau_1) - \phi(\tau_2) \leq 4r + C\sin(\epsilon_0)|\tau_2 - \tau_1|,$$

where $C > 0$ is some uniform constant. Combined with (43), it gives us

$$0 \leq |I_{x,r}(\gamma)| = \tau_2 - \tau_1 \leq \frac{4r}{1 - C\sin(\epsilon_0)},$$

and then, for every $x\in B_{\epsilon_0}(x_0)$ and every $0\leq r\leq r_1$,

$$\forall\gamma\in\mathcal{U}_{\epsilon_0},\ 0\leq\frac{1}{2\tau_0}\int_{-\tau_0}^{\tau_0}\mathcal{R}(\mathbf{1}_{B_{2r}(x)})\circ\varphi_{\langle V\rangle}^{\tau}(\gamma)d\tau\leq\frac{8r^2}{\tau_0(1-C\sin(\epsilon_0))}. \tag{44}$$

This shows the expected upper bound in the neighborhood $\mathcal{U}_\epsilon(\gamma_0):=\mathcal{U}_{\epsilon_0}$ of γ_0 when $X_{\langle V\rangle}(\gamma_0)$ is transverse to Γ_{x_0}.

4.5.2. *The tangent case.* We now deal with the slightly more delicate case where $X_{\langle V\rangle}(\gamma_0)$ is tangent to Γ_{x_0} where $\frac{\partial\mathcal{R}(V)}{\partial\theta}(\pi/2,0)=0$. Thanks to (4), we can again without loss of generality assume that $\frac{\partial\mathcal{R}(V)}{\partial\phi}(\pi/2,0)>0$, and suppose that

$$\forall\gamma=(\phi,\theta)\in\mathcal{U}_{2\epsilon_0},\quad\frac{\partial\mathcal{R}(V)}{\partial\phi}(\phi,\theta)>\frac{1}{2}\frac{\partial\mathcal{R}(V)}{\partial\phi}(\pi/2,0)=:a_0>0. \tag{45}$$

Moreover, thanks to hypothesis (5), the critical point at 0 of the map $\theta\mapsto\mathcal{R}(V)(\pi/2,\theta)$ is nondegenerate. In particular, without loss of generality and up to decreasing the value of ϵ_0, there exists $b_0>0$ such that

$$\forall\gamma=(\phi,\theta)\in\mathcal{U}_{2\epsilon_0},\quad\frac{\partial^2\mathcal{R}(V)}{\partial\theta^2}(\phi,\theta)>\frac{1}{2}\frac{\partial^2\mathcal{R}(V)}{\partial\theta^2}(\pi/2,0)=:b_0>0. \tag{46}$$

We now fix $\gamma\in\mathcal{U}_{\epsilon_0}\subset\mathcal{A}_{2\epsilon_0}$ and, as before, we can suppose that, for every $|\tau|\leq\tau_0$,

$$\frac{\partial\mathcal{R}(V)}{\partial\phi}\left(\varphi_{\langle V\rangle}^{\tau}(\gamma)\right)\geq a_0\quad\text{and}\quad\frac{\partial^2\mathcal{R}(V)}{\partial\theta^2}\left(\varphi_{\langle V\rangle}^{\tau}(\gamma)\right)\geq b_0.$$

As in the transverse case, one has

$$0\leq\frac{1}{2\tau_0}\int_{-\tau_0}^{\tau_0}\mathcal{R}(\mathbf{1}_{B_{2r}(x)})\circ\varphi_{\langle V\rangle}^{\tau}(\gamma)d\tau\leq\frac{2r|I_{x,r}(\gamma)|}{\tau_0},$$

where

$$I_{x,r}(\gamma):=\left\{\tau\in[-\tau_0,\tau_0]:\varphi_{\langle V\rangle}^{\tau}(\gamma)\in\mathcal{A}_{2r}(x)\right\}.$$

The main difference with the above case is that this set is not an interval in general. Yet, we can note that, along the trajectory of γ, the vector $(\phi'(\tau),\theta'(\tau))$ is nonvanishing thanks to (45). Moreover, it is tangent to $\partial\mathcal{A}_{r'}(x)$ (for some $r'<2\epsilon_0$) if and only if

$$F(\tau):=\phi'(\tau)-\theta'(t)\frac{\sin(\theta(\tau)-\theta_x)\sin(\phi_x)}{\sqrt{1-\cos^2(\theta(\tau)-\theta_x)\sin^2\phi_x}}=0.$$

We can observe that

$$\begin{aligned}F'(\tau)\ &=\ -\frac{1}{\sin\phi(\tau)}\frac{\partial\mathcal{R}(V)}{\partial\theta}(\phi(\tau),\theta(\tau))\frac{\partial^2\mathcal{R}(V)}{\partial\theta\partial\phi}(\phi(\tau),\theta(\tau))\\&\quad-\ \frac{1}{\sin\phi(\tau)}\frac{\partial\mathcal{R}(V)}{\partial\phi}(\phi(\tau),\theta(\tau))\frac{\partial^2\mathcal{R}(V)}{\partial\theta^2}(\phi(\tau),\theta(\tau))+\mathcal{O}(\epsilon_0),\end{aligned}$$

where the constant in the remainder is uniformly bounded for $\tau\in[-\tau_0,\tau_0]$ and $\gamma\in\mathcal{U}_{\epsilon_0}$. Thus, as $\partial_\theta\mathcal{R}(V)(\pi/2,0)=0$, we can suppose that, up to decreasing the value of $\epsilon_0>0$, $|F'(\tau)|\geq a_0b_0/2$. In particular, F is monotone and it vanishes at most at one point inside

$[-\tau_0,\tau_0]$. As a consequence, the set $I_{x,r}(\gamma)$ is the union of at most two disjoint intervals inside $[-\tau_0,\tau_0]$ that we denote by $[\tau_1,\tau_2]$ and $[\tau_3,\tau_4]$. Moreover, $X_{\langle V\rangle}(\varphi^{\tau}_{\langle V\rangle}(\gamma))$ is tangent to $\partial A_{2r}(x)$ at most at one point inside $[\tau_1,\tau_2]\cup[\tau_3,\tau_4]$.

It now remains to bound the length of these two intervals in terms of r. To that aim, we observe that, for $\tau\in[\tau_1,\tau_2]\cup[\tau_3,\tau_4]$, one can find $r(\tau)\in[-2r,2r]$ such that

$$\phi(\tau)=r(\tau)+\arccos\left(-\cos(\theta(\tau)-\theta_x)\sin\phi_x\right).$$

Given now $\tau,\tau'\in[\tau_1,\tau_2]\cup[\tau_3,\tau_4]$, one finds

$$r(\tau)-r(\tau')=F(\tau')(\tau-\tau')+\frac{F'(\tau')}{2}(\tau-\tau')^2+\mathcal{O}((\tau-\tau')^3),$$

where the constant in the remainder can be made uniform in terms of r, γ, τ and τ'. We use this equality to find an upper bound on the length of $[\tau_1,\tau_2]$. The other interval (if non empty) is handled simlarly. Recall from the above calculation that $F'(\tau)\leq -a_0b_0/2$ for every $\tau\in[-\tau_0,\tau_0]$. We have to distinguish three cases:

- $F(\tau_1)\leq 0$. In that case, we take $\tau'=\tau_1$ and $\tau=\tau_2$ and we find
$$r(\tau_2)-r(\tau_1)\leq-\frac{a_0b_0}{4}(\tau_2-\tau_1)^2+\mathcal{O}((\tau_2-\tau_1)^3).$$
From this, we can deduce that $|\tau_2-\tau_1|\leq\frac{32r^{1/2}}{a_0b_0}$.
- $F(\tau_1)\geq 0$ and $F(\tau_2)\geq 0$. In that case, we take $\tau'=\tau_2$ and $\tau=\tau_1$ and we find
$$r(\tau_1)-r(\tau_2)\leq-\frac{a_0b_0}{4}(\tau_2-\tau_1)^2+\mathcal{O}((\tau_2-\tau_1)^3).$$
Again, we deduce an upper bound of order $\mathcal{O}(r^{\frac{1}{2}})$.
- $F(\tau_1)>0$ and $F(\tau_2)<0$. In that case, one can find some $\tau_0\in[\tau_1,\tau_2]$ such that $F(\tau_0)=0$. Then, we apply the above inequality twice to get

$$r(\tau_2)-r(\tau_0)=\frac{F'(\tau_0)}{2}(\tau_2-\tau_0)^2+\mathcal{O}((\tau_2-\tau_0)^3)\text{ and }r(\tau_1)-r(\tau_0)=\frac{F'(\tau_0)}{2}(\tau_1-\tau_0)^2+\mathcal{O}((\tau_1-\tau_0)^3).$$

Combining the two equalities, we find that $|\tau_2-\tau_1|=\mathcal{O}(r^{\frac{1}{2}})$.

Gathering these bounds, we find that, for every $x\in B_{\epsilon_0}(x_0)$ and for every $r\leq r_1$,

$$\forall\gamma\in\mathcal{U}_{\epsilon_0},\ 0\leq\frac{1}{2\tau_0}\int_{-\tau_0}^{\tau_0}\mathcal{R}(\mathbf{1}_{B_{2r}(x)})\circ\varphi^{\tau}_{\langle V\rangle}(\gamma)d\tau\leq Cr^{\frac{3}{2}}.\tag{47}$$

4.5.3. *The conclusion.* By compactness, one can find $\gamma_1,\dots,\gamma_N$ in Γ_{x_0} and $\epsilon_1,\dots\epsilon_N>0$ such that $\cup_{j=1}^N\mathcal{U}_{\epsilon_j}(\gamma_j)$ covers Γ_{x_0}. We take $\epsilon_0:=\min\{\epsilon_j:1\leq j\leq N\}$ so that $\mathcal{A}_{2\epsilon_0}\subset\cup_{j=1}^N\mathcal{U}_{\epsilon_j}(\gamma_j)$. In particular, given any $x\in B_{\epsilon_0}(x_0)$ and any $r<r_1$ (with r_1 chosen small enough to handle each neighborhood $\mathcal{U}_{\epsilon_j}(\gamma_j)$), the support of the map

$$\gamma\mapsto\frac{1}{2\tau_0}\int_{-\tau_0}^{\tau_0}\mathcal{R}(\mathbf{1}_{B_{2r}(x)})\circ\varphi^{\tau}_{\langle V\rangle}(\gamma)d\tau$$

is contained in $\cup_{j=1}^N \mathcal{U}_{\epsilon_j}(\gamma_j)$. Thus, applying (44) and (47) to (38), we obtain, for any normalized solution u_h to (27),

$$\int_{\mathbb{S}^2} \chi_{x,r}(y)|u_h(y)|^2 d\upsilon_{g_0}(y) = \mathcal{O}(r^{\frac{3}{2}}) + \mathcal{O}(h^{1-3\beta}),$$

where the constant can be made uniform for $x \in B_{\epsilon_0}(x_0)$ and $r \geq h^\beta$. Taking $\beta = \frac{2}{9}$ yields Proposition 4.1.

Remark 4.5. The analysis of the vector field performed here is related to the analysis in [29, § 4]. In that reference, we showed with Macià that the semiclassical measures of $-\Delta_{g_0} + V$ can be decomposed as a convex combination of the Haar measures carried by the Lagrangian tori of the completely integrable system $(H_0, \mathcal{R}_{\text{cl}}(V))$. For 2-dimensional tori, the projection of the Haar measure on $\mathbb{S}^2$ is absolutely continuous [29, Th. 4.3] with some eventual blow-up of the density at some points which are often called caustics [29, Lemma 4.6]. This regularity of the projection is exactly the property we have been using here in a somewhat refined way to get our bounds $\mathcal{O}(r^{1+\alpha})$. The bound (44) ($\alpha = 1$) corresponds to points of these 2-dimensional Lagrangian tori where the projection is regular while (47) ($\alpha = 1/2$) corresponds to these caustics.

5. Final comments

5.1. **Relaxing assumption** (5)**.** Up to some extra work, assumption (5) could certainly be relaxed. For instance, one could require instead that the critical points are of finite order i.e. the derivative does not vanish at a certain order which may be larger than 2. We would then end up with some upper bound of order $\mathcal{O}(r^{1+\alpha})$ for some $0 < \alpha \leq 1/2$ related to the order of vanishing at the critical points of $\mathcal{R}(V)|_{\Gamma_{x_0}}$. This would give slightly worst upper bound on the growth of L^p-norms but it would allow to take larger compact subsets K in (6).

5.2. **Relaxing assumption** (4)**.** A priori, it does not seem possible to remove assumption (4) from the hypothesis of Proposition 4.1. Indeed, if there exists $\gamma_0 \in \Gamma_{x_0}$ such that $X_{\langle V\rangle}(\gamma_0) = 0$, then the value of (40) at γ_0 will be equal to $4r$ and it will prevent us from drawing the same conclusion using our argument.

5.3. **The range** $p > 6$**.** In this range, it is plausible that the methods from [20, 19, 13, 14] allow to handle these critical geodesics. Indeed, suppose that there exist a point $x_0 \in \mathbb{S}^2$ and a sequence $(\psi_{\lambda_k})_{k\geq 1}$ of normalized solutions to (2) verifying $\lambda_k \to +\infty$ and

$$\lim_{k\to+\infty} |\psi_{\lambda_k}(x_0)|\lambda_k^{-\frac{1}{2}} \neq 0. \tag{48}$$

Up to extracting a subsequence, we can suppose that $(\psi_{\lambda_k})_{k\geq 1}$ has a single semiclassical measure μ [58, Ch. 5]. Recall that it is a probability measure carried by $S^*\mathbb{S}^2$ which is invariant by the geodesic flow φ_0^t. In particular, it induces a measure $\tilde{\mu}$ on $G(\mathbb{S}^2)$. Then, we can consider $\tilde{\mu}_{x_0} = \tilde{\mu}|_{\Gamma_{x_0}}$. This measure can be decomposed into three parts: the absolutely continuous component, the singular continuous one and the pure point one. According to the results of Galkowski and Toth in [20], property (48) implies that the absolutely continuous part is not identically 0. Combined with [29, Prop. 2.3], this implies that $\mathcal{R}(V)|_{\Gamma_{x_0}}$ has infinitely many critical points. In other words, if $\mathcal{R}(V)|_{\Gamma_{x_0}}$ has finitely many critical points, then, for any sequence $(\psi_{\lambda_k})_{k\geq 1}$ of normalized solutions to (2), one has

$$|\psi_{\lambda_k}(x_0)| = o\left(\lambda_k^{\frac{1}{2}}\right),$$

which improves the remainder from the local Weyl law at x_0 without imposing (4). Compared with Theorem 1.1, this is of course not quantitative. If one is able to combine the quantitative arguments of Canzani and Galkowski [13, 14] with the extra invariance by the flow of $X_{\langle V\rangle}$ [29], then this may give rise to improvements on Sogge's upper bounds (3) in the range $p > 6$ under weaker geometric assumptions than the ones appearing in Theorem 1.1. Recall from the introduction that, thanks to the conjugation formula (7), eigenfunctions of $-\Delta_{g_0} + V$ which are the image under $\mathcal{U}$ of joint eigenfunctions for $(-\Delta_{g_0}, V^\sharp)$ enjoy improved L^p estimates near x_0 (for $p > 6$) under appropriate assumptions on the critical points of $\mathcal{R}(V)|_{\Gamma_{x_0}}$ [21, 45]. In particular, if the spectrum of $-\Delta_{g_0} + V$ is simple, then all eigenfunctions of $-\Delta_{g_0} + V$ will be the image of joint eigenfunctions.

5.4. **The case of odd potentials.** In [30], it was shown that one can uncover extra-invariance properties of semiclassical measures even if $\mathcal{R}(V)$ identically vanishes (meaning that V is an odd function, e.g. $V(x_1, x_2, x_3) = x_3$). In principle, the above arguments could be adapted following the lines of this reference, up to some extra technical work. In that case, the role of $\mathcal{R}(V)$ would be played by the function

$$\mathcal{R}^{(2)}(V) = \mathcal{R}(V^2) - \frac{1}{2\pi}\int_0^{2\pi}\int_0^t \{V\circ\varphi_0^t, V\circ\varphi_0^s\}dsdt.$$

See also [23, 49] for earlier related results on spectral asymptotics of Schrödinger operators.

5.5. **Semiclassical operators.** In Remarks 2.3 and 3.1, we observed that our bounds on L^p norms are valid more generally for solutions to

$$-h^2\Delta_{g_0}u_h + \varepsilon_h V u_h = u_h, \quad \|u_h\|_{L^2(\mathbb{S}^2)} = 1.$$

Even if it was maybe not optimal, for $p > 6$, we needed to impose $\varepsilon_h \leq h^{1+\epsilon}$ for some positive ϵ while for $4 \leq p < 6$, we only required $\varepsilon_h \leq h$. Thanks to Remarks 4.3 and 4.4, this yields the following bounds on L^p norms. For $p = \infty$, one has

$$\|u_h\|_{L^\infty(B_{r_0}(x_0))} \leq C_{\infty,x_0} h^{-\frac{1}{2}}\left(h^{\frac{1}{18}} + h^{\frac{\epsilon}{4}}\right),$$

which yields a polynomial improvement over the usual bound. In the range $4 < p < 6$, we get similarly, for any $r \geq h^{\frac{2}{9}}$,

$$\|u_h\|_{L^p(B_{r_0}(x_0))} \leq C_{p,x_0} h^{-\sigma_0(p)}\left(h^{\frac{1}{9}} + (rh)^{-1}\varepsilon_h\right)^{\frac{1}{2}\left(\frac{6}{p}-1\right)},$$

while for $p = 4$, we end up with

$$\|u_h\|_{L^4(B_{r_0}(x_0))} \leq C_{4,x_0}|\log h| h^{-\frac{1}{8}}\left(h^{\frac{1}{9}} + (rh)^{-1}\varepsilon_h\right)^{\frac{1}{4}},$$

In these last two cases, it yields improvements over Sogge's upper bound as soon as $h^{-1}\varepsilon_h \to 0$. Note that in every cases, ε_h may go to 0 very fast. For instance, one may have $\varepsilon_h \ll h^2$.

5.6. **The case of Zoll surfaces.** Following the lines of [29], we could adapt the results to Laplace eigenfunctions,

$$-\Delta_g \psi_\lambda = \lambda^2 \psi_\lambda,$$

where g is a $C_{2\pi}$ (or Zoll) metric on $\mathbb{S}^2$, i.e. all of whose geodesics are closed, simple and of length 2π. See [2] for a detailed review on this geometric assumption. In that case, it is known [16] that

$$\sqrt{-\Delta_g} = A + \frac{\alpha}{4} + Q,$$

where Q is a pseudodifferential operator of order -1, α is the Maslov index of the closed trajectories and $\mathrm{Sp}(A) \subset \mathbb{Z}_+$. Combining the above proof with the arguments from [29, §3.1], we will end up with the same quantities as in (38) except that $\mathcal{R}(V)$ will be replaced by some function $q_0(x, \xi)$ (related to the principal symbol of Q). An exact expression for q_0 was given by Zelditch in [53, 54] and it involves curvature terms of the metric. Under the geometric assumptions of Theorem 1.1 on the point x_0 but with q_0 replacing $\mathcal{R}(V)$, we could obtain improved L^p-bounds near x_0. Yet, the expression of q_0 being a little bit involved, this condition is harder to verify.

5.7. **The higher dimensional case.** For the sake of simplicity, we restricted ourselves to the 2-dimensional case but the extra invariance property by the flow of $X_{\langle V\rangle}$ remains true in higher dimensions $n \geq 3$ [29, Prop. 2.3]. Thus, modulo some extra work and some appropriate assumptions on $X_{\langle V\rangle}|_{\Gamma_{x_0}}$, one should be able to obtain localized L^2-estimates as in Proposition 4.1 but maybe for smaller values of α. Then, in the range $p_c = \frac{2(n+1)}{n-1} < p \leq +\infty$, this can be transferred into L^p bounds using that (16) remains true for $p = \infty$ in dimension $n \geq 3$ [39, Eq.(3.3)]. Similarly, for $p < p_c$, the Kakeya-Nikodym bounds of Section 3 remains true up to $p > \frac{2(n+2)}{n}$ and they can again be roughly bounded by the L^2-localized norms appearing in Proposition 4.1. Yet, we are not aware of an analogue of Guillemin's Theorem [22] showing that $\mathcal{R}$ is an isomorphism when restricted to the appropriate spaces of smooth functions on $\mathbb{S}^n$ and $G(\mathbb{S}^n)$ and hence making the condition on x_0 easy to verify.

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