DELOCALIZATION OF SLOWLY DAMPED EIGENMODES ON ANOSOV MANIFOLDS

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Abstract. We look at the properties of high frequency eigenmodes for the damped wave equation on a compact manifold with an Anosov geodesic flow. We study eigenmodes with damping parameters which are asymptotically close enough to the real axis. We prove that such modes cannot be completely localized on subsets satisfying a condition of negative topological pressure. As an application, one can deduce the existence of a "strip" of logarithmic size without eigenvalues below the real axis under this dynamical assumption on the set of undamped trajectories.

1. Introduction

Let M be a smooth, compact, connected Riemannian manifold of dimension $d \geq 2$ and without boundary. We will be interested in the high frequency analysis of the damped wave equation,

$$\left(\partial_t^2 - \Delta + 2V(x)\partial_t\right) u(x,t) = 0,\ u(x,0) = u_0,\ \partial_t u(x,0) = u_1,$$

where Δ is the Laplace-Beltrami operator on M and $V \in \mathcal{C}^\infty(M, \mathbb{R}_+)$ is the *damping function*. This problem can be rewritten as

$$(-\imath\partial_t + \mathcal{A})\mathbf{u}(t) = 0, \tag{1}$$

where $\mathbf{u}(t) := (u(t), \imath\partial_t u(t))$ and

$$\mathcal{A} = \begin{pmatrix} 0 & \mathrm{Id} \\ -\Delta & -2\imath V \end{pmatrix}. \tag{2}$$

This operator generates a strongly continuous and uniformly bounded semigroup $\mathcal{U}(t) = e^{-\imath t\mathcal{A}}$ on $H^1(M) \times L^2(M)$ which solves (1) – e.g. [14], §2. Hence, it is quite natural to study the spectral properties of $\mathcal{A}$ in order to understand the behavior of the solutions of (1). For instance, in [17], Lebeau established several important relations (related to the decay of energy of solutions) between the evolution problem (1), the spectral properties of $\mathcal{A}$ and the properties of the geodesic flow $(g^t)_t$ on the unit cotangent bundle

$$S^*M := \left\{(x,\xi) \in T^*M : \|\xi\|_x^2 = 1\right\}.$$

Recall that the spectrum of this *nonselfadjoint* operator is a discrete subset of $\mathbb{C}$ made of countably many eigenvalues (τ_n) which satisfy $\lim_{n\to+\infty} \mathrm{Re}\ \tau_n = \pm\infty$. We underline that τ is an eigenvalue of $\mathcal{A}$ when there exists a non trivial function u in $L^2(M)$ such that

$$(-\Delta - \tau^2 - 2\imath\tau V)u = 0. \tag{3}$$

Hence, each eigenvalue τ can be associated with a normalized "eigenmode" u in $L^2(M)$ which gives raise to the following solution of the damped wave equation

$$v(t,x) = e^{-\imath t\tau}u(x).$$

We also underline that $\mathrm{Im}\ \tau_n \in [-2\|V\|_\infty, 0]$ for every n and that (τ, u) solves the eigenvalue problem (3) if and only if $(-\overline{\tau}, \overline{u})$ solves it [21]. Our main concern in the following will be to *understand the asymptotic properties of slowly damped eigenmodes.* Precisely, we consider sequences $(\tau_n, u_n)_n$ solving (3) with

$$\mathrm{Re}\ \tau_n \to +\infty \text{ and } \mathrm{Im}\ \tau_n \to 0.$$

In the case where V is not identically 0, Lebeau proved the existence of a constant $C>0$ such that for every $\tau\neq 0$ in the spectrum of $\mathcal{A}$, one has [17]

$$\text{Im }\tau\leq -\frac{1}{C}e^{-C|\tau|}. \tag{4}$$

Hence, eigenfrequencies cannot accumulate faster than exponentially on the real axis. Moreover, Lebeau also provided in [17] a geometric situation where this inequality is optimal. An important feature of this example is that it does not satisfy the so-called Geometric Control Condition:

$$\exists T_0>0\text{ such that }\forall\rho\in S^*M,\ \{g^t\rho:0\leq t\leq T_0\}\cap\{(x,\xi):V(x)>0\}\neq\emptyset. \tag{5}$$

In fact, under this assumption, one can prove that there exists a constant $\gamma>0$ such that for every $\tau\neq 0$ in the spectrum of $\mathcal{A}$, one has [17, 33]

$$\text{Im }\tau\leq-\gamma<0.$$

It is then natural to understand how close to the real axis eigenfrequencies can be under the assumption that the Geometric Control Condition does not hold. For that purpose, we introduce the *set of undamped trajectories*[1]

$$\Lambda_V=\bigcap_{t\in\mathbb{R}}g^t\{(x,\xi)\in S^*M:V(x)=0\}. \tag{6}$$

In fact, even if inequality (4) is optimal, there may be some geometric assumptions on M or on Λ_V under which the accumulation is much slower than exponential. In recent works, many progresses have been made in understanding the spectral properties of $\mathcal{A}$ in different geometric situations, e.g. when Λ_V is arbitrary [33, 2, 31], when Λ_V is a closed geodesic [14, 8, 9] or when Λ_V satisfies a pressure condition [32, 21]. We will explain some of these results which are related to ours but before that, we will proceed to a semiclassical reformulation of this spectral problem as in [33], §1.

Semiclassical reformulation. Thanks to the different symmetries of our problem, we will only consider the limit Re $\tau\rightarrow+\infty$. Introduce then $0<\hbar\ll 1$. We will look at eigenfrequencies τ of order $\hbar^{-1}$ by setting

$$\tau=\frac{\sqrt{2z}}{\hbar},\ \text{where } z=\frac{1}{2}+\mathcal{O}(\hbar).$$

With this notation, studying the high frequency eigenmodes of the damped wave equation corresponds to look at sequences $(z(\hbar)=\frac{1}{2}+\mathcal{O}(\hbar))_{0<\hbar\ll 1}$ and $(\psi_\hbar)_{0<\hbar\ll1}$ in $L^2(M)$ satisfying[2]

$$(\mathcal{P}(\hbar,z)-z(\hbar))\psi_\hbar=0,\ \text{where } \mathcal{P}(\hbar,z):=-\frac{\hbar^2\Delta}{2}-\imath\hbar\sqrt{2z(\hbar)}V(x). \tag{7}$$

For every t in $\mathbb{R}$, we also introduce the quantum propagator associated to $\mathcal{P}(\hbar,z)$, i.e.

$$\mathcal{U}_\hbar^t:=\exp\left(-\frac{\imath t\mathcal{P}(\hbar,z)}{\hbar}\right). \tag{8}$$

After this semiclassical reduction, the question of the accumulation of the eigenfrequencies to the real axis can be translated in understanding how close to 0 the *quantum decay rate* $\frac{\text{Im } z(\hbar)}{\hbar}$ can be. For $\hbar$ small enough, introduce now

$$\Sigma_\hbar=\left\{z(\hbar):\exists\psi_\hbar\neq 0\in L^2(M),\ \mathcal{P}(\hbar,z)\psi_\hbar=z(\hbar)\psi_\hbar\right\}.$$

In [33], Sjöstrand proved several results on the distribution of this semiclassical spectrum. For instance, he showed that eigenvalues $z(\hbar)$ with Re $z(\hbar)$ in a small box around $1/2$ satisfies a Weyl's law in the semiclassical limit $\hbar\rightarrow0^+$. Moreover, he proved that, in such boxes, most of the imaginary parts $\frac{\text{Im } z(\hbar)}{\hbar}$ concentrate on the ergodic averages of V with respect to the geodesic flow. We refer the reader to [33] for the precise statements.

[1] By a compactness argument, one can verify that the Geometric Control Condition holds if and only if $\Lambda_V=\emptyset$.

[2] We underline that for simplicity of exposition, we only deal with operators of this form. However, our approach could be adapted to treat the case of more general families of nonselfadjoint operators like the ones considered in [33], §1.

In this semiclassical setting, Lebeau's result reads

$$\exists C>0 \text{ such that for } \hbar \text{ small enough, } \forall z(\hbar)\in\Sigma_\hbar,\ \frac{\text{Im } z(\hbar)}{\hbar}\leq -\frac{1}{C}e^{-\frac{C}{\hbar}}.$$

Finaly, under the Geometric Control Condition (5), one can prove the existence of $\gamma>0$ such that for $\hbar$ small enough, one has $\frac{\text{Im } z(\hbar)}{\hbar}\leq-\gamma$ [17, 33]: one says that there is a *spectral gap*.

Chaotic dynamics. It is natural to ask whether the results above can be improved when the manifold M satisfies additional geometric properties. In this article, we will be interested in the specific case where the geodesic flow (g^t) on the unit cotangent bundle S^*M has the Anosov property (manifolds of negative curvature are the main example). This assumption implies that the dynamical system (S^*M, g^t) is *strongly chaotic* (e.g. ergodicity, mixing of the Liouville measure L on S^*M). Motivated by the properties of the semiclassical approximation, one can expect to exploit these chaotic dynamical properties to obtain more precise results on the distribution of eigenvalues – see e.g. [2, 31, 32, 21] for applications of this idea. In [31, 32, 21], it is proved that, under various assumptions on V, there exists a spectral gap below the real axis. Yet, to our knowledge, the existence of a spectral gap is not known for a general nontrivial V even in this chaotic setting.

Our precise aim in this article is to *describe the asymptotic distribution of eigenmodes for which* $\frac{\text{Im } z(\hbar)}{\hbar}\to 0$ *fast enough when the geodesic flow is chaotic.* We will prove that such eigenmodes must in a certain sense be partly delocalized on S^*M.

Semiclassical measures. In order to describe the asymptotic properties of these slowly damped eigenmodes, we will use the notion of semiclassical measures [7, 12]. Consider a sequence of normalized eigenmodes $(\psi_\hbar)_{\hbar\to0^+}$ satisfying

$$\mathcal{P}(\hbar,z)\psi_\hbar = z(\hbar)\psi_\hbar, \tag{9}$$

where $z(\hbar)\to 1/2$ and $\frac{\text{Im } z(\hbar)}{\hbar}\to 0$ as $\hbar$ tends to 0. If such modes exist, one must at least have $\Lambda_V\neq\emptyset$. For a given sequence $(\psi_\hbar)_{\hbar\to0^+}$, we introduce a family of distributions on the cotangent space T^*M, i.e.

$$\forall a\in\mathcal{C}^\infty_o(T^*M),\ \mu_{\psi_\hbar}(a):=\langle\psi_\hbar,\text{Op}_\hbar(a)\psi_\hbar\rangle_{L^2(M)}, \tag{10}$$

where $\text{Op}_\hbar(a)$ is a $\hbar$-pseudodifferential operator (see appendix A.1). This distribution tells us where the eigenfunction $\psi_\hbar$ is located on the phase space T^*M and one can try to describe the accumulation points of this sequence of distributions in order to understand the asymptotic localization of $\psi_\hbar$. Using results from semiclassical analysis [12] (Chapter 5), one can verify that any accumulation point μ (as $\hbar\to0$) of the sequence $(\mu_{\psi_\hbar})_\hbar$ is a probability measure μ with support in the unit cotangent bundle S^*M. moreover, μ satisfies, for every $t\geq0$,

$$\forall a\in\mathcal{C}^0(S^*M),\ \mu(a)=\mu\left(a\circ g^t e^{-2\int_0^t V\circ g^s ds}\right),$$

due to the fact that $\frac{\text{Im} z}{\hbar}\to0$. When specifying this relation with $a\equiv1$, one can verify that $\mu(V\circ g^t)=0$ for every t. In particular, as $V\geq0$, it implies that the measure μ is invariant under the geodesic flow.

We will call *semiclassical measure* any accumulation point μ (as $\hbar$ tends to 0) of a sequence of the form $(\mu_{\psi_\hbar})$, where $\psi_\hbar$ satisfies equation (7). We will denote

$$\mathcal{M}\left((\psi_\hbar)_{\hbar\to0^+}\right)$$

the set of semiclassical measures associated to the sequence $(\psi_\hbar)_{\hbar\to0^+}$. Under our assumption, it forms a subset of $\mathcal{M}(S^*M,g^t)$ which is the set of g^t-invariant probability measure on S^*M. Hence, $\mathcal{M}\left((\psi_\hbar)_{\hbar\to0^+}\right)$ is a subset of a natural family in ergodic theory: our precise goal is then to give ergodic properties on its elements. Finally, one can verify that any the support of any μ in $\mathcal{M}\left((\psi_\hbar)_{\hbar\to0^+}\right)$ is included in the *weakly undamped set*

$$\mathcal{N}_V:=\overline{\bigcup_{\mu\in\mathcal{M}(S^*M,g^t)}\{\text{supp}(\mu):\mu(V)=0\}},$$

which is a subset of Λ_V. It is explained in [32] that Λ_V and $\mathcal{N}_V$ could be different; yet, we would like to mention that, in our setting, the Geometric Condition Condition (5) is also equivalent to $\mathcal{N}_V = \emptyset$.

2. Main results

Motivated by questions concerning the Quantum Unique Ergodicity Conjecture[3], Anantharaman studied the Kolmogorov-Sinai entropy of semiclassical measures in the case of eigenfunctions of the Laplacian on Anosov manifolds [1] – see also [5, 19] for earlier results due to Bourgain and Lindenstrauss in an arithmetic setting. Precisely, her result concerns the selfadjoint case $V \equiv 0$. It roughly says that, in this context, eigenmodes must be partly delocalized on S^*M (for instance, they cannot concentrate only on closed geodesics). In this article, we will prove similar results in the non selfadjoint setting $V \geq 0$.

Before giving details, we would like to recall that the Kolmogorov-Sinai entropy $h_{KS}(\mu, g)$ is a nonnegative quantity associated to an invariant probability μ in $\mathcal{M}(S^*M, g^t)$ – see [34] or section 3 for a brief reminder). This quantity characterizes what the measure perceives of the complexity of the geodesic flow. For instance, if μ is carried by a closed orbit of the geodesic flow, then $h_{KS}(\mu, g) = 0$. On the other hand, if $\mu = L$, the measure has a "good understanding" of the complexity of the dynamic and so it has a large entropy. Moreover, entropy is affine with respect to the ergodic decomposition of a measure μ [11]. In fact, thanks to the Birkhoff Ergodic Theorem, one knows that, for μ almost every ρ in S^*M,

$$\frac{1}{T}\int_0^T \delta_{g^s\rho} ds \rightharpoonup \mu_\rho, \text{ as } T \to +\infty,$$

where δ_w is the Dirac measure in $w \in S^*M$. The measure μ_ρ is ergodic and one has the ergodic decomposition $\mu = \int_{S^*M} \mu_\rho d\mu(\rho)$. Then, the Kolmogorov-Sinai entropy satisfies

$$h_{KS}(\mu, g) = \int_{S^*M} h_{KS}(\mu_\rho, g) d\mu(\rho). \tag{11}$$

2.1. **Main result.** We can now state our main result which is the following:

Theorem 2.1. *Suppose (S^*M, g^t) satisfies the Anosov property. Let P_0 be a positive constant. There exist $c_0(P_0) > 0$ and $C(P_0) > 0$ depending only on P_0, on V and on M such that if*

- *$(\psi_\hbar)_{\hbar\to 0^+}$ is a sequence of eigenmodes satisfying (9) with*
$$\forall 0 < \hbar \leq \hbar_0,\ z(\hbar) \in \left[\frac{1}{2} - \hbar, \frac{1}{2} + \hbar\right] + \imath\left[-C(P_0)\frac{\hbar}{|\log \hbar|}, +\infty\right];$$
- *μ is in $\mathcal{M}\left((\psi_\hbar)_{\hbar\to 0^+}\right)$,*

then, one has

$$\mu\left(\left\{\rho \in S^*M : h_{KS}(\mu_\rho, g) \geq -\frac{1}{2}\int_{S^*M} \log J^u d\mu_\rho - P_0\right\}\right) \geq c_0(P_0),$$

where $J^u(\rho)$ is the unstable Jacobian, i.e. $J^u(\rho) := \left|\det\left(d_{g^1\rho} g^{-1}_{|E^u(g^1\rho)|}\right)\right|$.

We will recall in section 3 basic facts on entropy and Anosov systems (in particular the definition of J^u [16]). We underline that, for any invariant probability measure $\nu \in \mathcal{M}(S^*M, g^t)$, the quantity $-\int_{S^*M} \log J^u d\nu$ is positive.

[3]We refer the reader to [20, 30, 35] for recent reviews on these questions.

2.2. **Comments.** The lower bound that appears in our Theorem is a natural dynamical quantity. In fact, for any invariant probability measure ν in $\mathcal{M}(S^*M, g^t)$, one has the Ruelle-Margulis upper bound [29], i.e.

$$h_{KS}(\nu, g) \leq -\int_{S^*M} \log J^u d\nu, \tag{12}$$

with equality if and only if ν is the Liouville measure on S^*M [18]. Thus, if one has $\frac{\text{Im}(z(\hbar))}{\hbar} = o\left(|\log \hbar|^{-1}\right)$, our result states that a semiclassical measure of our problem must have ergodic components which are close to be "half delocalized".

A direct consequence of Theorem 2.1 can be expressed in terms of topological pressure – see section 3 for a brief reminder. We recall that, for a compact and g^t-invariant subset K of S^*M, the variational principle [24] relates this dynamical quantity to the entropy as follows

$$P_{top}\left(K, g^t, \frac{1}{2}\log J^u\right) = \sup_{\nu \in \mathcal{M}(S^*M, g^t)} \left\{ h_{KS}(\nu, g) + \frac{1}{2}\int_K \log J^u d\nu : \nu(K) = 1 \right\}.$$

Thus, if K is a subset of S^*M which satisfies $P_{top}(K, g^t, \log J^u/2) < 0$, then Theorem 2.1 implies that $\mu(K) < 1$ if $\frac{\text{Im}(z(\hbar))}{\hbar} = o\left(|\log \hbar|^{-1}\right)$. In other words, Theorem 2.1 implies that eigenmodes cannot put all their weights on subsets of negative topological pressure. In some sense, proposition 4.3 (which is the key result of this article) is another way to formulate this fact in a discrete setting. In the case of *surfaces*, such a condition on the topological pressure holds if K has Hausdorff dimension < 2 [4]. This condition on the topological pressure already appeared in [22] where it was used to establish a spectral gap for resonances – see also [31, 32, 21] in the case of the damped wave equation.

In the case $V \equiv 0$, Anantharaman proved that Theorem 2.1 holds [1] if we replace the quantity $-\int_{S^*M} \log J^u d\mu_\rho$ by

$$\Lambda_{\min} = \inf_{\nu \in \mathcal{M}(S^*M, g^t)} \left\{ -\int_{S^*M} \log J^u d\nu \right\} > 0.$$

In particular, our result improves this earlier result of Anantharaman in the selfadjoint case $V \equiv 0$. Yet, our main interest here was to show that these entropic properties remain true for slowly damped eigenmodes in the high frequency limit of (1). In particular, our result shows that semiclassical measures of such modes cannot be carried only by closed orbits of the geodesic flow.

We underline that Anantharaman proved her result in the general setting of quasimodes satisfying $\|(-\hbar^2\Delta - 1)\psi_\hbar\| = \mathcal{O}(\hbar/|\log \hbar|)$ – in our setting, the sequences of states satisfy a priori only $\|(-\hbar^2\Delta - 1)\psi_\hbar\| = \mathcal{O}(\hbar)$. For simplicity of exposition, we only treat the case of eigenmodes satisfying (9) (in principle, the case of quasimodes could also be derived combining our inputs to the strategy in [1]). An interesting extension of our main Theorem would also be to understand if the entropic results in [3, 26] could also be adapted in this non selfadjoint situtation.

Our assumption on the rate of convergence of $\frac{\text{Im}(z(\hbar))}{\hbar}$ is a very strong assumption. Except in the case $V \equiv 0$, it is not clear if it can be satisfied by a sequence of eigenvalues. In a subsequent work [27], we will describe (weaker) properties that can be derived in the case where we only suppose $\frac{\text{Im}(z(\hbar))}{\hbar} \to \beta$.

Let us mention an interesting consequence of our main Theorem:

Corollary 2.2. *Suppose (S^*M, g^t) satisfies the Anosov property. Let $V \geq 0$ be a smooth function on M such that $\mathcal{N}_V$ is a nonempty subset satisfying*

$$P_{top}\left(\mathcal{N}_V, g^t, \frac{1}{2}\log J^u\right) < 0, \tag{13}$$

where $P_{top}\left(\mathcal{N}_V, g^t, \frac{1}{2}\log J^u\right)$ is the topological pressure of $\mathcal{N}_V$ with respect to $\frac{1}{2}\log J^u$.

Then, there exists a positive constant C and $\hbar_C > 0$ such that for every $0 < \hbar \leq \hbar_C$,

$$\Sigma_\hbar \cap \left(\left[\frac{1}{2} - \hbar, \frac{1}{2} + \hbar\right] + \imath \left[-\frac{C\hbar}{|\log \hbar|}, +\infty \right] \right) = \emptyset.$$

This corollary follows from the above observations on the measure of subsets of negative topological pressure and from the fact that $\mu(\mathcal{N}_V) = 1$. It establishes the presence of an inverse logarithmic strip without eigenvalues in the case where $\mathcal{N}_V$ satisfies a condition of negative topological pressure. In the appendix of [9], Christianson obtains a similar result in the case where $\mathcal{N}_V$ is a (single) closed hyperbolic geodesic with the notable difference that he does not make any global assumption on the geodesic flow. In fact, in the appendix of a subsequent work [27], we prove with S. Nonnenmacher that one can remove the the global assumption on the geodesic flow in the previous corollary. Yet, the results in [27] do not allow to recover the results of Theorem 2.1. In [31, 32, 21], spectral gaps were obtained in the case of Anosov geodesic flow under a topological pressure condition or under an assumption on the amplitude of the damping function V. As conjectured in [21], it seems natural (even if it is not proved yet) to expect that, for Anosov geodesic flows, a spectral gap (and not an inverse logarithmic gap) could be obtained under condition (13).

2.3. **Some words about the proof.** The general strategy of our proof follows the one from [1]. Without getting into the details of the proof, we would like to mention what are the main differences which allows the improvements presented above. In this reference, the general strategy was to use *hyperbolic dispersive estimates* (40) in order to prove that eigenmodes cannot concentrate entirely on subsets of small topological entropy (meaning $< \frac{\Lambda_{\min}}{2}$). Regarding the estimates (40), it was natural to expect that the results of Anantharaman could be improved using "thermodynamical quantities" like topological pressure. This allows to take more into account the variations of the unstable jacobian J^u and to improve slightly the entropic lower bounds from [1]. Our first input is to introduce these quantities and to prove that eigenmodes cannot concentrate entirely on subsets of small topological pressure with respect to $\frac{1}{2}\log J^u$ (meaning < 0). Translated in a discrete setting, it is exactly the statement of proposition 4.3 which is the main result of this article.

An additional difficulty we have to face here is that we want to extend the results to the eigenmodes of a non selfadjoint operator. We would like now to illustrate the kind of difficulties created by this generalization. Thanks to the long time Egorov property [6] (see also paragraph A.3), one can verify that, for every a in $\mathcal{C}_c^\infty(T^*M)$, there exists $\kappa > 0$ such that

$$\forall\ 0 \leq t \leq \kappa|\log\hbar|,\ \mu_{\psi_\hbar}(a) = e^{-\frac{2t\,\mathrm{Im}\ z(\hbar)}{\hbar}} \left\langle \psi_\hbar, \mathrm{Op}_\hbar\left(a\circ g^t e^{-2\int_0^t V\circ g^s ds}\right)\psi_\hbar\right\rangle + \mathcal{O}(\hbar^\nu),$$

where $\nu > 0$ and the constant in the remainder is uniform for $0 \leq t \leq \kappa|\log\hbar|$. Hence, in the case $V \equiv 0$, the distibution $\mu_{\psi_\hbar}$ is invariant[4] under g^t modulo small error terms which are uniform for logarithmic times in $\hbar$. This invariance property for logarithmic times was extensively used in [1] under various forms. In the case where V is non trivial, we did not find a simple equivalent of this "pseudo-invariance" property for long times in $\hbar$. However, instead of it, we make a simple observation that we will use at different steps of our proof – e.g. in proposition 6.3. In fact, one can remark that the quantization procedure is "almost positive" (see paragraph A.2). As $V \geq 0$, there exists, for any a in $\mathcal{C}_c^\infty(T^*M, \mathbb{R}_+)$, $\kappa > 0$ such that

$$\forall\ 0 \leq t \leq \kappa|\log\hbar|,\ \mu_{\psi_\hbar}(a) \leq e^{-\frac{2t\,\mathrm{Im}\ z(\hbar)}{\hbar}} \mu_{\psi_\hbar}(a\circ g^t) + \mathcal{O}(\hbar^\nu).$$

Under the assumption that $\frac{\mathrm{Im}\ z(\hbar)}{\hbar} \geq -C|\log\hbar|^{-1}$, one can then verify that the distribution satisfies $\mu_{\psi_\hbar}(a) \leq e^{2C\kappa}\mu_{\psi_\hbar}(a\circ g^t) + \mathcal{O}(\hbar^\nu)$ with an uniform remainder for $0 \leq t \leq \kappa|\log\hbar|$. Moreover, one can choose C small enough to have $e^{2C\kappa}$ arbitrarly close to 1. In this sense, the distribution $\mu_{\psi_\hbar}$ is subinvariant under the geodesic flow for logarithmic times (modulo small error terms) and this kind of property will be sufficient to prove our result.

2.4. **Organization of the article.** In section 3, we give a brief reminder on the dynamical systems concepts we will use in this article. In section 4, we proceed to a discretization of the manifold which allows to give a symbolic interpretation of the quantum system. The main result of this section is proposition 4.3 which shows that eigenmodes cannot concentrate entirely on subsets of small topological pressure. In section 5, we use this result to derive Theorem 2.1. Then, in section 6, we give the proof of several lemmas that we used to prove proposition 4.3. Finally, in

[4]In the case of quasi modes [1], this property remains true but with a worst remainder term.

the appendix, we give several results on semiclassical analysis related to our problem and that we used at different stages of the proof.

3. Background on dynamical systems

In this section, we draw a short review on Anosov flows and thermodynamical formalism. We refer the reader to the classical references on this subject for more details, e.g. [16, 34].

3.1. **Anosov flows.** In all this article, we make the assumption that the geodesic flow satisfies the Anosov property. It means that, for every $E>0$ and for every

$$\rho\in p_0^{-1}(\{E\}):=\left\{(x,\xi)\in T^*M: p_0(x,\xi)=\frac{\|\xi\|_x^2}{2}=E\right\},$$

one has the following decomposition [16]

$$T_\rho p_0^{-1}(\{E\})=\mathbb{R}X_{p_0}(\rho)\oplus E^u(\rho)\oplus E^s(\rho).$$

In the previous decomposition, $\mathbb{R}X_{p_0}(\rho)$ is the direction of the Hamiltonian vector field, $E^u(\rho)$ is the unstable space and $E^s(\rho)$ is the stable space. For every $E>0$, there exists a constant $C>0$ and $0<\lambda<1$ such that for every $t\geq 0$, one has

$$\forall v^u\in E^u(\rho),\ \|d_\rho g^{-t}v^u\|\leq C\lambda^t\|v^u\| \text{ and } \forall v^s\in E^s(\rho),\ \|d_\rho g^t v^s\|\leq C\lambda^t\|v^s\|.$$

Define now the unstable Jacobian at point $\rho\in S^*M$ and time $t\geq 0$

$$J_t^u(\rho):=\left|\det\left(d_{g^t\rho}g^{-t}_{|E^u(g^t\rho)}\right)\right|,$$

where the unstable spaces at ρ and $g^t\rho$ are equipped with the induced riemannian metric. This defines an Hölder continuous function on S^*M [16] (that can be extended to any energy layer $p_0^{-1}(\{E\})$). We underline that this quantity tends to 0 as t tends to infinity at an exponential rate. Moreover, it satisfies the following multiplicative property

$$J_{t+t'}^u(\rho)=J_t^u(g^{t'}\rho)J_{t'}^u(\rho).$$

In the following, we will use the notation $J^u(\rho)=J_1^u(\rho)$. Finally, we underline that these unstable Jacobian are related to the Lyapunov exponents [16], in the sense that, for a given $\mu\in\mathcal{M}(S^*M,g^t)$, one has, for μ almost every ρ,

$$\lim_{t\to+\infty}-\frac{1}{t}\log J_t^u(\rho)=\sum_{j=1}^{d-1}\chi_j^+(\rho),$$

where the $\chi_j^+(\rho)$ are the positive Lyapunov exponents at point ρ. We underline that this last quantity is also equal to $-\int_{S^*M}\log J^u d\mu_\rho$ for μ almost every ρ in S^*M.

3.2. **Kolmogorov Sinai entropy.** There are several ways to define Kolmogorov-Sinai entropy and we refer the reader to [34] (chapter 4) for the classical definition and the fundamental properties of entropy. This quantity associates to a g^t-invariant measure μ a nonnegative number that characterizes the *complexity of the geodesic flow from the point of view of* μ. A way to define it is to start from a partition $\mathcal{P}=(P_i)_{i=1}^K$ of S^*M. Then, for every ρ in S^*M and for every n in $\mathbb{N}$, there exists a unique sequence $(\alpha_0,\dots,\alpha_{n-1})$ in $\{1,\dots,K\}^n$ such that ρ belongs to $P_{\alpha_0}\cap g^{-1}P_{\alpha_1}\dots\cap g^{-n+1}P_{\alpha_{n-1}}$. We denote this set $B_n(\rho)$. Fix a measure μ in $\mathcal{M}(S^*M,g^t)$. The Shannon-McMillan-Breiman Theorem states [23] that for μ almost ρ in S^*M, the limit

$$\lim_{n+\infty}-\frac{1}{n}\log\mu(B_n(\rho))$$

is well defined. We denote this limit $h_{KS}(\mu_\rho,g,\mathcal{P})$. It defines an element in $L^1(\mu)$ that is g^t-invariant and that measures the exponential decrease of the μ-volume of the "n-balls" $B_n(\rho)$. The Kolmogorov-Sinai entropy is then defined as

$$h_{KS}(\mu,g)=\sup\left\{\int_{S^*M}h_{KS}(\mu_\rho,g,\mathcal{P})d\mu(\rho):\mathcal{P}\text{ is a finite partition of }S^*M\right\}.$$

Recall from the introduction that this quantity is affine for the ergodic decomposition and that it is bounded by the Ruelle-Margulis upper bound. Moreover, Abramov Theorem tells us that $h_{KS}(\mu, g^t) = |t| h_{KS}(\mu, g)$ for every t in $\mathbb{R}$. Finally, we underline that if there exists $C > 0$ and $H_0 \geq 0$ such that for every n in $\mathbb{N}$ and for every $|\alpha| = n$, $\mu(P_{\alpha_0} \cap g^{-1} P_{\alpha_1} \ldots \cap g^{-n+1} P_{\alpha_{n-1}}) \leq Ce^{-nH_0}$, then $h_{KS}(\mu, g) \geq H_0$.

3.3. **Topological pressure.** In corollary 2.2, we made an assumption on the topological pressure of an invariant subset K of S^*M. This quantity can be defined as the Legendre transform of the Kolmogorov-Sinai entropy [24, 34], i.e.

$$\forall f \in \mathcal{C}^0(S^*M),\ P_{top}\left(K, g^t, f\right) := \sup_{\mu \in \mathcal{M}(S^*M, g^t)} \left\{ h_{KS}(\mu, g) + \int_K f d\mu : \mu(K) = 1 \right\}.$$

This definition of topological pressure is known as the variational principle and we used this definition to derive corollary 2.2. There are many other (equivalent) definitions of topological pressure [24, 34]. We just mention one of them here in order to clarify the statements of section 4.

Given $\epsilon > 0$ and $T \geq 0$, a subset F of K is said to be (ϵ, T)-separated if for any $\rho \neq \rho'$ in F, there exists $0 \leq t \leq T$ such that the distance $d(g^t\rho, g^t\rho')$ is $> \epsilon$. Fix now f an element[5] in $\mathcal{C}^0(S^*M)$. Define

$$P_T(K, g^t, f, \epsilon) = \sup \left\{ \sum_{\rho \in F} \exp \left(\int_0^T f \circ g^t(\rho) dt \right) \right\}, \tag{14}$$

where the supremum is taken over all (ϵ, T)-separated subset of K. An equivalent definition of the topological pressure is then

$$P_{top}\left(K, g^t, f\right) = \lim_{\epsilon \to 0} \limsup_{T \to +\infty} \frac{1}{T} \log P_T(K, g^t, f, \epsilon).$$

4. Symbolic coding of the quantum dynamic

We fix C a positive constant (that will be chosen small enough at the end of our proof). Let $(\psi_\hbar)_{0<\hbar\leq\hbar_0}$ be a sequence of normalized vector in $L^2(M)$ such that

$$\mathcal{P}(\hbar, z)\psi_\hbar = z(\hbar)\psi_\hbar,$$

where $z(\hbar)$ belongs to

$$\left[\frac{1}{2} - \hbar, \frac{1}{2} + \hbar\right] + \imath \left[-\frac{C\hbar}{|\log \hbar|}, +\infty\right]. \tag{15}$$

Up to an extraction, we suppose that the distribution $\mu_{\psi_\hbar}$ defined by (10) converges weakly to the g^t-invariant probability measure μ. In the following, we will use the notation $\hbar \to 0$ for the extraction in order to avoid heavy notations and in order to fit semiclassical notations.

Entropic properties of semiclassical measures were already studied in [1, 3]. A common point of their proofs is the introduction of a symbolic coding of the quantum dynamic to study localization properties of semiclassical measures. In these references, the proof relies on a careful study of the "thermodynamical properties" of the symbolic quantum system and on its link with the thermodynamical properties of semiclassical measures. We use also a symbolic presentation of the quantum system (similar to the one in [1]) that we describe in this section.

The main result of this section is proposition 4.3 that will allow us to derive the proof of Theorem 2.1 in section 5. It shows, in a certain sense, that the sequence of eigenmodes defined above cannot concentrate on a subset of small topological pressure (at least for C small enough).

[5] In the following section, we will take $f = \frac{1}{2} \log J^u$.

4.1. **Energy cutoffs.** As the sequence $(\psi_\hbar)_\hbar$ concentrates on S^*M in the semiclassical limit, we can introduce cutoff functions that will allow us to work with observables compactly supported in a small neighborhood of S^*M. First, we define $\tilde{\chi}$ a smooth function on $\mathbb{R}$ which is nonnegative, which is equal to 1 for $|t| \leq 1/2$ and which is equal to 0 for $|t| \geq 1$. Then, we fix $\delta > 0$ and k a positive integer. For every $0 \leq j \leq k-1$, we define

$$\forall \rho = (x,\xi) \in T^*M,\ \chi_{-j}(x,\xi) := \tilde{\chi}\left(4^j\delta^{-1}(\|\xi\|_x^2 - 1)\right).$$

Each of the function χ_{-j} is a smooth function on S^*M which is compactly supported in $\{1-\delta/4^j \leq \|\xi\|_x^2 \leq 1+\delta/4^j\}$. Moreover, by definition, the support of $1-\chi_{-j}$ and χ_{-j-1} are disjoint. Finally, for an eigenfunction $\psi_\hbar$, one has

$$\|\operatorname{Op}_\hbar(\chi_{-j})\psi_\hbar - \psi_\hbar\| = \mathcal{O}(\hbar^\infty),$$

as $\psi_\hbar$ solves (7).

In fact, we will mainly use the cutoff function χ_0 but, at some point of our proof (precisely in the proof of lemma 6.1), we will need to use a finite number k of cutoff functions (the integer k we will take will only depend on δ and on a positive number P_0 with the notations of the following paragraphs).

4.2. **Smooth discretization of M.** In order to give a symbolic analogue of the quantum dynamics induced bu $\mathcal{U}_\hbar^t$, we proceed to a smooth partition of the manifold M. Let $M = M_1 \sqcup \ldots M_K$ be a finite measurabe partition of M of diameter bounded[6] by $\frac{\epsilon}{2}$ and such that the measure μ does not charge the boundary of the partition. By lifting it on T^*M, it can be considered as a partition of T^*M. In [1], Anantharaman explained how to regularize such a partition in a smart way. Without getting into the details of [1] (see paragraph 2.1 and appendix $A.2$ of this reference), we recall that she constructed a family of smooth functions $(P_1^\hbar, \ldots P_K^\hbar)$ on M (that depends on $\hbar$) satisfying in particular the following properties

- for every i, $P_i^\hbar \geq 0$;
- $\forall x \in M,\ \sum_{j=1}^K P_j^\hbar(x) = 1$;
- $\overline{M_i} \subset \operatorname{supp} P_i^\hbar \subset B(\overline{M_i}, \frac{\epsilon}{4})$, where $B(\overline{M_i}, \frac{\epsilon}{4})$ is an $\epsilon/4$-neighborhood of M_i;
- $P_i^\hbar \to 1$ uniformly in every compact subset inside the interior of M_i, as $\hbar$ tends to 0;
- $P_i^\hbar \to 0$ uniformly in every compact subset ouside M_i, as $\hbar$ tends to 0;
- the growth of the derivatives is controlled by powers of $\hbar^{-\overline{\nu}}$ (with $\overline{\nu} < 1/2$) and so the functions are amenable to $\hbar$-pseudodifferential calculus [10, 12] (see also appendix A.1 for a brief reminder);
- There exists a $0 < b < 1/2$ such that

(16) $$\forall 1 \leq i \neq j \leq K,\ \|P_i^\hbar P_j^\hbar \psi_\hbar\| = \mathcal{O}(\hbar^{\frac{b}{2}}) \text{ and } \forall 1 \leq i \leq K,\ \|(P_i^\hbar)^2\psi_\hbar - P_i^\hbar\psi_\hbar\| = \mathcal{O}(\hbar^{\frac{b}{2}}).$$

Property (16) is an important feature in our proof that will allow us to consider the smooth partition as a family of orthogonal projectors – see paragraph 6.4. The parameters $\overline{\nu}$ and b are fixed in the following of the article.

For each of this smooth function $P_i^\hbar$, we can define a multiplication operator on $L^2(M)$

$$\forall u \in L^2(M),\ \pi_i u = P_i^\hbar \times u,$$

which is a bounded operator on $L^2(M)$ (of norm less than 1). One can underline that

$$\sum_{i=1}^K \pi_i = \operatorname{Id}_{L^2(M)}.$$

We also introduce the following operator:

$$\forall \alpha \in \{1,\ldots,K\}^n,\ \Pi_\alpha := \pi_{\alpha_{n-1}}(n-1)\ldots\pi_{\alpha_1}(1)\pi_{\alpha_0},$$

where $A(t) := \mathcal{U}_\hbar^{-t}A\mathcal{U}_\hbar^t$. For a fixed n, Π_α is a pseudodifferential operator in $\Psi_{\overline{\nu}}^{0,0}(M)$ with principal symbol equal to $P_{\alpha_0}^\hbar \times P_{\alpha_1}^\hbar \circ g^1 \times \ldots P_{\alpha_{n-1}}^\hbar \circ g^{n-1}$. We will see in paragraph A.4 that this property extends to short logarithmic times $[\kappa|\log\hbar|]$ when we localize the observables near S^*M. We

[6] We will fix ϵ small enough in a way that depends only on M and on P_0 (see paragraph 4.5).

also underline that the principal symbol of Π_α is the smooth analogue of the refined partition appearing in the classical definition of the Kolmogorov-Sinai entropy.

Finally, we observe that the following property of partition of identity holds:

$$\sum_{|\alpha|=n} \Pi_\alpha = \mathrm{Id}_{L^2(M)}. \tag{17}$$

4.3. **Symbolic coding of the quantum dynamic.** We can now use this smooth discretization of M to introduce a symbolic coding of the quantum dynamic induced by $\mathcal{U}_\hbar^t$. We will then state our main result in terms of this symbolic dynamic – see proposition 4.3.

4.3.1. *Quantum functionals on cylinders.* First, we define

$$\Sigma := \{1, \ldots, K\}^{\mathbb{N}}$$

and denote a cylinder $[\alpha_0, \ldots, \alpha_{n-1}] := \{x \in \Sigma : \forall 0 \leq i \leq n-1, x_i = \alpha_i\}$. We will use $\beta.\alpha$ for the concatenation of two finite words $\beta := (\beta_q, \ldots, \beta_{q+q'})$ and $\alpha := (\alpha_p, \ldots, \alpha_{p+p'})$.

We define the shift on Σ as $\sigma((x_n)_{n\in\mathbb{N}}) = (x_{n+1})_{n\in\mathbb{N}}$ and the following *quantum functional* on the cylinders of Σ

$$\mu_\hbar^\Sigma([\alpha_0, \ldots, \alpha_{n-1}]) = \langle \Pi_\alpha \psi_\hbar, \psi_\hbar \rangle_{L^2(M)}.$$

This object **is not a probability measure**. However, it satisfies the following nice properties

Proposition 4.1. *One has:*

(1) *For every n in $\mathbb{N}$, for every cylinder $[\alpha_0, \ldots, \alpha_{n-1}]$,*

$$\sum_{\alpha_n} \mu_\hbar^\Sigma([\alpha_0, \ldots, \alpha_{n-1}, \alpha_n]) = \mu_\hbar^\Sigma([\alpha_0, \ldots, \alpha_{n-1}]);$$

(2) *For every n in $\mathbb{N}$, for every cylinder $[\alpha_0, \ldots, \alpha_{n-1}]$ and for every k,*

$$\mu_\hbar^\Sigma(\sigma^{-k}[\alpha_0, \ldots, \alpha_{n-1}]) = \mu_\hbar^\Sigma([\alpha_0, \ldots, \alpha_{n-1}]) + o_{n,k}(1);$$

(3) *For every n,*

$$\sum_{|\alpha|=n} \mu_\hbar^\Sigma([\alpha_0, \ldots, \alpha_{n-1}]) = 1.$$

In point 2 of the proposition, we used the notation

$$\mu_\hbar^\Sigma(\sigma^{-k}[\alpha_0, \ldots, \alpha_{n-1}]) = \sum_{\alpha_{-k},\ldots,\alpha_{-1}} \mu_\hbar^\Sigma([\alpha_{-k}, \ldots, \alpha_{-1}, \alpha_0, \ldots, \alpha_{n-1}]).$$

The "quantum functional" $\mu_\hbar^\Sigma$ looks very much like a σ-invariant probability measure. The two mains problems are that it is not positive a priori and that it is not exactly invariant.

Concerning the positivity, one can use Egorov property (51) with $q_1 = \sqrt{2z(\hbar)}V$ and $q_2 = -\sqrt{2z(\hbar)}V$ – see remark A.2. Then, following the proof of proposition 1.3.2 in [1], one finds that the following holds in the semiclassical limit $\hbar \to 0$:

$$\mu_\hbar^\Sigma([\alpha_0, \ldots, \alpha_{n-1}]) \to \mu\left(g^{-n+1} M_{\alpha_{n-1}} \cap \ldots M_{\alpha_0}\right). \tag{18}$$

Thus, in the semiclassical limit, $\mu_\hbar^\Sigma([\alpha_0, \ldots, \alpha_{n-1}])$ defines a nonnegative quantity.

Let us now explain how one can prove point 2 of the proof. Using the fact that the semiclassical measure μ is g^1 invariant (as $\mathrm{supp}(\mu) \subset \mathcal{N}_V$), one verifies that the limit of the "quantum functional" defines σ-invariant probability measure μ^Σ as follows:

$$\forall [\alpha_0, \ldots, \alpha_{n-1}],\ \mu^\Sigma([\alpha_0, \ldots, \alpha_{n-1}]) := \mu\left(M_{\alpha_0} \cap \ldots g^{-(n-1)} M_{\alpha_{n-1}}\right).$$

Hence, the functional becomes σ-invariant in the semiclassical limit. In particular, it implies point 2 of the proposition, i.e. for fixed n and k,

$$\mu_\hbar^\Sigma(\sigma^{-k}[\alpha_0, \ldots, \alpha_{n-1}]) = \mu_\hbar^\Sigma([\alpha_0, \ldots, \alpha_{n-1}]) + o_{n,k}(1).$$

Remark 4.2. As mentionned in paragraph 2.3, the situation is slightly more complicated than the selfadjoint case treated in [1] where the quantum functional was invariant under σ (or at least invariant modulo a factor of order $k|\log\hbar|^{-1}$ in the case of quasimodes). Here, for a fixed n, there is a priori no reason to obtain a remainder $o_k(1)$ with a really explicit dependence in k. Instead of this, we will prove a subinvariance property (proposition 6.3) that will be sufficient for our proof.

For that purpose, we underline that the quantum functional $\mu_\hbar^\Sigma$ satisfies the following equality

$$\mu_\hbar^\Sigma([\alpha_0,\ldots,\alpha_{n-1}]) = e^{-\frac{2k\,\mathrm{Im}(z(\hbar))}{\hbar}}\sum_{|\beta|=k}\left\langle(\mathcal{U}_\hbar^k)^*\mathcal{U}_\hbar^k\Pi_{\beta.\alpha}\psi_\hbar,\psi_\hbar\right\rangle. \tag{19}$$

Let us verify this fact. First, we write

$$\mu_\hbar^\Sigma([\alpha_0,\ldots,\alpha_{n-1}]) = \langle\mathcal{U}_\hbar^k\Pi_\alpha(k)\mathcal{U}_\hbar^{-k}\psi_\hbar,\mathcal{U}_\hbar^k\mathcal{U}_\hbar^{-k}\psi_\hbar\rangle = e^{-\frac{2k\,\mathrm{Im}(z(\hbar))}{\hbar}}\langle(\mathcal{U}_\hbar^k)^*\mathcal{U}_\hbar^k\Pi_\alpha(k)\psi_\hbar,\psi_\hbar\rangle,$$

where the second equality comes from the eigenmode equation (9). Then, thanks to the partition of identity (17), we obtain

$$\mu_\hbar^\Sigma([\alpha_0,\ldots,\alpha_{n-1}]) = \sum_{|\beta|=k}e^{-\frac{2k\,\mathrm{Im}(z(\hbar))}{\hbar}}\langle(\mathcal{U}_\hbar^k)^*\mathcal{U}_\hbar^k\Pi_\alpha(k)\Pi_\beta\psi_\hbar,\psi_\hbar\rangle,$$

which is the expected equality.

The main Theorem is expressed in terms of the unstable Jacobian J^u. Thus, we define an analogue of this quantity in our discrete setting:

$$J^u(\alpha_0,\alpha_1) := \sup\left(\left\{J^u(\rho):\rho\in M_{\alpha_0}\cap g^{-1}M_{\alpha_1},\|\rho\|\in[1-\delta,1+\delta]\right\}\cup\{\Lambda\}\right),$$

where $0<\Lambda\ll 1$. We also define e^{λ_0} an upper bound on all the $J^u(\alpha_0,\alpha_1)$ that can be choose uniform for $\delta<1/2$ and for any choice of partition. For a given sequence $\alpha:=(\alpha_0,\ldots,\alpha_{n-1})$, we also define

$$J_n^u(\alpha) := J^u(\alpha_0,\alpha_1)\ldots J^u(\alpha_{n-2},\alpha_{n-1}).$$

Finally, we also introduce Σ_n the set of n-cylinders in Σ, i.e.

$$\Sigma_n := \{[\alpha_0,\ldots,\alpha_{n-1}]:\alpha_0,\ldots,\alpha_{n-1}\in\{1,\ldots,K\}\}.$$

4.3.2. *Main proposition.* The proof of the main Theorem relies on the following proposition – see section 5 for details.

Proposition 4.3. *Let P_0 be a positive number. There exist*

$$n_0'(P_0)\in\mathbb{N}\text{ and }\epsilon_0(P_0)>0,$$

such that for every fixed choice of partition of diameter $\leq\epsilon_0(P_0)$, there exists

$$C(P_0)>0\text{ and }0\leq c(P_0)<1$$

such that if

- *for $\hbar$ small enough,*

$$z(\hbar)\in\left[\frac{1}{2}-\hbar,\frac{1}{2}+\hbar\right]+\imath\left[-C(P_0)\frac{\hbar}{|\log\hbar|},+\infty\right];$$

- $n_0\geq n_0'(P_0)$;
- *W_{n_0} is a subset of Σ_{n_0} satisfying*

$$\sum_{[\alpha]\in W_{n_0}}J_{n_0}^u(\alpha)^{\frac{1}{2}}\leq e^{-\frac{n_0P_0}{2}}, \tag{20}$$

then, one has

$$\sum_{[\alpha]\in W_{n_0}}\mu^\Sigma([\alpha])\leq c(P_0).$$

The meaning of this proposition is that eigenmodes cannot concentrate entirely on cylinders satisfying condition (20) which can be interpreted as a condition of "negative topological pressure". In fact, the sum appearing in (20) is a discrete analogue of the quantities appearing in (14). Introducing these thermodynamical quantities generalizes slightly the strategy of [1] where Anantharaman considered discrete versions of topological entropies (and not of topological pressures).

4.4. **Proof of proposition 4.3.** In order to prove this proposition, we will admit several lemmas and show how they allow to derive proposition 4.3. The proof of these intermediary lemmas will be given in section 6. Fix now $P_0 > 0$, a partition of small diameter (less than some $\epsilon_0(P_0) > 0$ that will be precised in paragraph 4.4.3) and two parameters $0 < \overline{\nu}, b < \frac{1}{2}$ that we use to regularize the partition – see paragraph 4.2.

Remark 4.4. Our proof requires the introduction of several small (or large) parameters. In order to avoid any confusion, we will summarize the links between the different parameters in paragraph 4.5.

4.4.1. *Different scales of times.* In order to prove our result, we need to introduce various scales of times. The first one will be a fixed time $n_0 \in \mathbb{N}$ that will play its main part at the *classical level.* We fix n_0 a positive integer and a family W_{n_0} of cylinders of length n_0 satisfying

$$\sum_{[\alpha]\in W_{n_0}} J^u_{n_0}(\alpha) \leq e^{-\frac{P_0 n_0}{2}}. \tag{21}$$

The n_0 will be fixed large enough to apply lemma 4.5.

Fix now $\kappa > 0$. The second important time is a *short logarithmic time*

$$n(\hbar) := [\kappa|\log\hbar|]\,. \tag{22}$$

This time is the one for which the semiclassical approximation will be valid for observables supported in a *small macroscopic neighborhood* of S^*M and for the observables $P_i^{\hbar}$. In particular, all the arguments of paragraphs 4.4.4 and 4.4.5 (and also of the appendix) will be valid for $0 \leq p \leq n(\hbar)$. We underline that we will fix κ small enough in a way that depends on P_0, on the partition, on the size δ of the energy layer we work on and on the parameters $\overline{\nu}$ and b used for the smoothing of the partition. In this article, we will not try to optimize $\kappa > 0$.

Finally, we introduce $k \geq 2$ a large positive integer and a *large logarithmic time* that will be useful at the quantum level

$$N(\hbar) = kn(\hbar). \tag{23}$$

In the following, k and κ will have to be chosen in a way that their product is bounded from below by a positive constant that will depend only on P_0 and on the dimension of M (see paragraph 4.4.3). Precisely, we will suppose that $k\kappa > \frac{8d}{P_0}$.

In the following, we will often omit the dependence of $N(\hbar) = N$ and $n(\hbar) = n$ in $\hbar$ to avoid heavy notations.

4.4.2. *Intermediary lemmas.* We start our proof by providing two intermediary lemmas that we will prove in section 6.

First, we fix τ in $[1/2, 1]$ and we introduce a subfamily of cylinders of length $p \geq n_0$ that spend a lot of time near W_{n_0}:

$$\Sigma_p(W_{n_0}, \tau) := \left\{[\alpha] := [\alpha_0, \dots, \alpha_{p-1}] : \frac{\sharp\left\{j \in [0, p-n_0] : [\alpha_j, \dots, \alpha_{j+n_0-1}] \in W_{n_0}\right\}}{p - n_0 + 1} \geq \tau\right\}.$$

In this paragraph, we will give lower and upper bounds on the following "thermodynamical quantity" associated to the family $\Sigma_p(W_{n_0}, \tau)$:

$$\sum_{[\alpha]\in\Sigma_p(W_{n_0},\tau)} J^u_p(\alpha)^{\frac{1}{2}} = \sum_{[\alpha]\in\Sigma_p(W_{n_0},\tau)} \exp\left(\frac{1}{2}\sum_{j=0}^{p-1}\log J^u(\alpha_j, \alpha_{j+1})\right).$$

Roughly speaking, we will verify that if the eigenmodes concentrate on the cylinders $\Sigma_{n(\hbar)}(W_{n_0},\tau)$, then this thermodynamical quantity must grow faster than it is authorized by the thermodynamical assumption (21) on W_{n_0}. In particular, it will show that eigenmodes cannot concentrate entirely on $\Sigma_{n(\hbar)}(W_{n_0},\tau)$.

The first lemma provides a general upper bound on this "thermodynamical quantity" that relies only on the thermodynamical assumption (21) on W_{n_0}:

Lemma 4.5. *There exists n_0' and p_0 depending only on P_0 such that for every $p\geq p_0$, for every $n_0\geq n_0'$, for every choice of partition, for every W_{n_0} satisfying (21) and for every $\tau\in[1/2,1]$,*

$$\sum_{[\alpha]\in\Sigma_p(W_{n_0},\tau)} J_p^u(\alpha)^{\frac{1}{2}}\leq e^{-\frac{pP_0}{8}+p(1-\tau)(\lambda_0+\log K)}(e^{\lambda_0}K)^{n_0}e^{\frac{n_0P_0}{2}}.$$

We recall that K is the cardinal of the partition of M and e^{λ_0} is an upper bound on all the $J^u(\alpha_0,\alpha_1)$.

This lemma relies only on the *classical properties* of the system. If we apply it for the semi-classical time $p=n=[\kappa|\log\hbar|]$, for $n_0\geq n_0'$ and for $\tau\geq\frac{1}{2}$, we find that, for $\hbar$ small enough, one has

$$(24)\qquad \sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)} J_n^u(\alpha)^{\frac{1}{2}}=\mathcal{O}\left(e^{\left(-\frac{P_0}{8}+(1-\tau)(\lambda_0+\log K)\right)\kappa|\log\hbar|}\right),\ \text{as } \hbar\to 0.$$

In particular, this last equality combined to assumption (15) on the "horizontal" localization of eigenvalues implies that

$$(25)\qquad \sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)} J_n^u(\alpha)^{\frac{1}{2}}e^{-\frac{(n-1)\operatorname{Im}(z(\hbar))}{\hbar}}=\mathcal{O}\left(\hbar^{\kappa(-(1-\tau)(\lambda_0+\log K)+\frac{P_0}{8})}\right),\ \text{as } \hbar\to 0.$$

We now turn to the second lemma which gives a lower bound on the left-hand side of (25) under an assumption on the concentration of the eigenmodes $(\psi_\hbar)_\hbar$. Precisely, we show that if the eigenfunction puts some weight on the cylinders of $\Sigma_n(W_{n_0},\tau)$, then the previous sum is bounded from below by a precise power of $\hbar$:

Lemma 4.6. *Let k, κ and C be as above*[7]*. There exists a constant $c_1>1$ (depending on the choice of partition, on $\overline{\nu}$ and on δ) such that, for any θ in $[0,1]$, if W_{n_0} is a family of n_0-cylinders satisfying (21) and if*

$$(26)\qquad \left\|\sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)^c}\pi_{\alpha_{n-1}}\mathcal{U}_\hbar\ldots\pi_{\alpha_1}\mathcal{U}_\hbar\pi_{\alpha_0}\psi_\hbar\right\|_{L^2(M)}\leq e^{\frac{(n-1)\operatorname{Im}(z(\hbar))}{\hbar}}\frac{e^{-Ck\kappa}}{c_1^k k}\theta,$$

where $\Sigma_n(W_{n_0},\tau)^c$ means the complementary of $\Sigma_n(W_{n_0},\tau)$ in Σ_n; then, one has, for $\hbar>0$ small enough

$$\left(\sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)} J_n^u(\alpha)^{\frac{1}{2}}e^{-\frac{(n-1)\operatorname{Im}(z(\hbar))}{\hbar}}\right)^k$$

$$(27)\qquad \geq e^{-(k-1)\left(\lambda_0-\frac{\operatorname{Im} z(\hbar)}{\hbar}\right)}\left(1-\theta+\mathcal{O}(\hbar^\infty)\right)\hbar^{d/2}e^{-C_0k\kappa\epsilon|\log\hbar|},$$

where C_0 depends only on M.

We recall that ϵ is an upper bound on the diameter of the partition.

The proof of these two lemmas will be given in section 6. Before that, we explain how they allow to derive proposition 4.3 in the subsequent paragraphs.

[7]The constant $C>0$ is the one given by the spectral window (15).

4.4.3. *Using the intermediary lemmas.* Lemma 4.6 shows that, under an assumption on the concentration of the eigenmodes, one has the lower

$$\sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)} J_n^u(\alpha)^{\frac{1}{2}} e^{-\frac{(n-1)\,\mathrm{Im}(z(\hbar))}{\hbar}} \geq C'\hbar^{\frac{d}{2k}+C_0\kappa\epsilon}, \tag{28}$$

where $C'>0$ can be chosen as an uniform constant in $\hbar$. We can now compare this lower bound to the upper bound (25). Precisely, if we are able to take the different parameters in a way that

$$\frac{d}{2k}+C_0\kappa\epsilon<\kappa\left(-(1-\tau)(\lambda_0+\log K)+\frac{P_0}{8}\right), \tag{29}$$

then the lower bound in (28) is larger than the upper bound in (25) when $\hbar\to 0$. It implies that if (29) is satisfied, then assumption (26) cannot be satisfied and thus, we will have, for $\hbar>0$ small enough,

$$\left\|\sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)^c}\pi_{\alpha_{n-1}}\mathcal{U}_\hbar\ldots\pi_{\alpha_1}\mathcal{U}_\hbar\pi_{\alpha_0}\psi_\hbar\right\|_{L^2(M)} \geq e^{\frac{(n-1)\,\mathrm{Im}(z(\hbar))}{\hbar}}\theta\frac{e^{-Ck\kappa}}{c_1^k k}. \tag{30}$$

In order to obtain relation (29), we first choose ϵ small enough (depending only on P_0). Then, we choose $\tau_0<1$ (depending also on the partition) close enough to 1 to have

$$-(1-\tau_0)(\lambda_0+\log K)+\frac{P_0}{8}-C_0\epsilon\geq\frac{P_0}{16}$$

We underline that this inequality remains true for any $\tau\geq\tau_0$. Finally, we can take any k and κ satisfying $k\kappa>\frac{8d}{P_0}$. So the more κ will be small, the more we will have to take k large.

To summarize our discussion, for this choice of small parameters, relation (29) is satisfied. Hence, an eigenmode cannot put all his weight on the cylinders in $\Sigma_n(W_{n_0},\tau)$, i.e. on a subset of small topological pressure with respect to $\frac{1}{2}\log J^u$ – see inequality (30).

4.4.4. *Using the semiclassical approximation.* In the previous paragraph, we saw that for a partition of small enough diameter and for τ close enough to 1, one has, for $\hbar$ small enough,

$$\left\|\sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)^c}\pi_{\alpha_{n-1}}\ldots\pi_{\alpha_1}(2-n)\pi_{\alpha_0}(1-n)\psi_\hbar\right\|_{L^2(M)}^2\geq\theta^2\frac{e^{-2Ck\kappa}}{c_1^{2k}k^2}. \tag{31}$$

This lower bound holds for any k and κ satisfying $k\kappa>\frac{8d}{P_0}$ with $\kappa>0$ small enough to use the semiclassical arguments from paragraphs 6.2.2 and A.4.

We would like now to relate this lower bound to a lower bound on

$$\sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)^c}\mu_\hbar^\Sigma([\alpha]).$$

The same difficulty already appeared in paragraph 2.4 of [1] and we will briefly recall in paragraph 6.4 how one can prove that, for κ small enough,

$$\left\|\sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)^c}\pi_{\alpha_{n-1}}\ldots\pi_{\alpha_1}(2-n)\pi_{\alpha_0}(1-n)\psi_\hbar\right\|_{L^2(M)}^2$$
$$=\sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)^c}\left\langle\pi_{\alpha_{n-1}}\ldots\pi_{\alpha_1}(2-n)\pi_{\alpha_0}(1-n)\psi_\hbar,\psi_\hbar\right\rangle+\mathcal{O}(\hbar^{\nu_0'}),$$

where ν_0' is a positive constant that depends on the partition $\mathcal{M}$, on the energy layer we work on and on the parameters $\overline{\nu}$ and b used for the smoothing of the partition. Without getting into the details of paragraph 6.4, this can be achieved using the fact that, for κ small enough, $n=[\kappa|\log\hbar|]$ is a short logarithmic time[8] and that, by assumption, the family $(\pi_i)_i$ forms a family of almost orthogonal projectors when it acts on the eigenmodes $\psi_\hbar$ – see property (16).

[8] Thus, the pseudodifferential operators we consider are amenable to semiclassical calculus.

Remark 4.7. In the following, we will make a small abuse of notations as we we will use the exponent ν_0' for all the other remainders due to the semiclassical approximation, meaning that we will always keep the worst remainder term (including the $\mathcal{O}(\hbar^\infty)$ remainders).

Using the fact that $\psi_\hbar$ is an eigenmode, we also find

$$\left\| \sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)^c} \pi_{\alpha_{n-1}}\dots\pi_{\alpha_1}(2-n)\pi_{\alpha_0}(1-n)\psi_\hbar \right\|^2_{L^2(M)}$$
$$= \sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)^c} e^{-\frac{2(n-1)\operatorname{Im}(z(\hbar))}{\hbar}} \left\langle (\mathcal{U}_\hbar^{n-1})^*\mathcal{U}_\hbar^{n-1}\Pi_\alpha\psi_\hbar,\psi_\hbar\right\rangle + \mathcal{O}(\hbar^{\nu_0'}).$$

We have to face here a problem which is due to the "nonselfadjointness" of our problem. Combining remark 6.5 and the fact that there is at most K^n terms in the sum, one gets from lower bound (31)

$$\theta^2\frac{e^{-2Ck\kappa}}{c_1^{2k}k^2}e^{2(n-1)\frac{\operatorname{Im}\ z(\hbar)}{\hbar}} \leq \sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)^c}\mu_\hbar^\Sigma([\alpha]) + \mathcal{O}(K^n\hbar^{\nu_0'}).$$

In particular, for $\kappa>0$ small enough, the remainder in the right-hand side is small as $\hbar$ tends to 0. Finally, thanks to the property of partition of identity (17) and to the lower bound, one also has

$$\sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)}\mu_\hbar^\Sigma([\alpha]) \leq \left(1-\theta^2e^{2(n-1)\frac{\operatorname{Im}\ z(\hbar)}{\hbar}}\frac{e^{-2Ck\kappa}}{c_1^{2k}k^2}\right)+\mathcal{O}(\hbar^{\nu_0'}), \tag{32}$$

where $\nu_0'>0$ is some small constant – see remark 4.7.

4.4.5. *The conclusion: from time $n(\hbar)$ to time n_0.* In inequality (32), n and $\mu_\hbar^\Sigma$ depend both on $\hbar$; hence, one cannot directly take the limit $\hbar\to0$ and derive proposition 4.3. We will start by deriving an estimate on $\mu_\hbar^\Sigma(W_{n_0})$, where we use the notation

$$\forall W\subset\Sigma_p,\ \mu_\hbar^\Sigma(W) := \sum_{[\alpha]\in W}\mu_\hbar^\Sigma([\alpha]).$$

Using proposition 6.3, we write

$$\mu_\hbar^\Sigma(W_{n_0}) \leq \frac{1}{n-n_0}\sum_{k=0}^{n-n_0-1}e^{-2(k-1)\frac{\operatorname{Im}\ z(\hbar)}{\hbar}}\sum_{[\alpha]\in W_{n_0}}\mu_\hbar^\Sigma(\sigma^{-k}[\alpha])+\mathcal{O}_{n_0}(\hbar^{\nu_0'}),$$

where ν_0' has the same properties as above. Then, one can observe that

- the length of the cylinders involved in (33) is of order $\kappa|\log\hbar|$, with κ small enough to have the operators Π_α amenable to semiclassical calculus – lemma A.3;
- $\psi_\hbar$ is localized near S^*M and thus, modulo cutoff functions, we can use the positive quantization procedure $\operatorname{Op}_\hbar^+$ of paragraph A.2;
- there are at most K^n terms in the sums involved in the upper bound (33).

In particular, the functional $\mu_\hbar^\Sigma$ is almost positive at least on these "short" cylinders. Combining this to the compatibility relation of proposition 4.1, we derive that, for $\kappa>0$ small enough,

$$\mu_\hbar^\Sigma(W_{n_0}) \leq e^{-2(n-1)\frac{\operatorname{Im}\ z(\hbar)}{\hbar}}\mu_\hbar^\Sigma\left(\frac{1}{n-n_0}\sum_{k=0}^{n-n_0-1}\mathbf{1}_{\sigma^{-k}W_{n_0}}\right)+\mathcal{O}_{n_0}(\hbar^{\nu_0'}), \tag{33}$$

where we crudely bounded $e^{-2(k-1)\frac{\operatorname{Im}\ z(\hbar)}{\hbar}}$ by $e^{-2(n-1)\frac{\operatorname{Im}\ z(\hbar)}{\hbar}}$. We can now proceed as in [1], i.e. combine the facts that, on $\Sigma_n(W_{n_0},\tau)$, one has

$$\frac{1}{n-n_0}\sum_{k=0}^{n-n_0-1}\mathbf{1}_{\sigma^{-k}W_{n_0}}\leq 1,$$

and that, by definition of the subfamily $\Sigma_n(W_{n_0},\tau)$, one has on $\Sigma_n(W_{n_0},\tau)^c$,

$$\frac{1}{n-n_0}\sum_{k=0}^{n-n_0-1}\mathbf{1}_{\sigma^{-k}W_{n_0}}\leq\tau.$$

Combined one more time to the almost positivity of $\mu_{\hbar}^{\Sigma}$ on "short" cylinders, these properties allow to find that, for κ small enough,

$$\mu_{\hbar}^{\Sigma}(W_{n_0})\leq e^{-2(n-1)\frac{\mathrm{Im}\ z(\hbar)}{\hbar}}\left(\tau\mu_{\hbar}^{\Sigma}\left(\Sigma_n(W_{n_0},\tau)^c\right)+\mu_{\hbar}^{\Sigma}\left(\Sigma_n(W_{n_0},\tau)\right)\right)+\mathcal{O}(\hbar^{\nu_0'}),$$

where $\nu_0'>0$ (with the abuse of notation mentionned in remark 4.7). Thanks to the property of partition of identity, one can verify that

$$\mu_{\hbar}^{\Sigma}(W_{n_0})\leq e^{-2(n-1)\frac{\mathrm{Im}\ z(\hbar)}{\hbar}}\left(\tau+(1-\tau)\mu_{\hbar}^{\Sigma}\left(\Sigma_n(W_{n_0},\tau)\right)\right)+\mathcal{O}(\hbar^{\nu_0'}).$$

At this point of the proof one can use our assumption on the quantum decay rate $\frac{2\,\mathrm{Im}\ z(\hbar)}{\hbar}$. In fact, if we implement property (15) in inequality (32) and if we let $\hbar$ tends to 0, then we derive

$$\mu^{\Sigma}(W_{n_0})\leq e^{2C\kappa}\left(\tau+(1-\tau)\left(1-\frac{\theta^2e^{-2C(k+1)\kappa}}{c_1^{2k}k^2}\right)\right).$$

This inequality holds for any θ in $(0,1)$ and hence,

$$\mu^{\Sigma}(W_{n_0})\leq e^{2C\kappa}\left(\tau+(1-\tau)\left(1-\frac{e^{-2C(k+1)\kappa}}{c_1^{2k}k^2}\right)\right). \tag{34}$$

As all the other parameters were fixed before and as this inequality holds for any $C>0$, one can now take C small enough to have

$$e^{2C\kappa}\left(\tau+(1-\tau)\left(1-\frac{e^{-2C(k+1)\kappa}}{c_1^{2k}k^2}\right)\right)<1.$$

We underline that all the constant involved in the left hand side can be chosen in a way that depends only on P_0 and on the mannifold. It concludes the proof of proposition 4.3.

4.5. **Comments on the choice of the different parameters.** In order to avoid any confusion, we summarize here all the relations between the different parameters.

First, as the eigenfunctions concentrate on S^*M, we fix a small neighborhood of S^*M of size $0<\delta<1/2$ ("the energy layer") and we fix $\overline{\nu}<1/2$ and $0<b<1/2$ that we will use to define the regularized partition.

We also fix a real positive number P_0.

Then, we introduce a partition of small diameter depending only on P_0. Once a partition is fixed, we can introduce symbolic coding of the quantum dynamic. We also fix $\tau<1$ larger than some τ_0 depending on P_0 and on the partition (see paragraph 4.4.2).

Once these parameters are fixed, we fix a parameter $\kappa>0$ small enough depending on P_0, on δ, on the partition $\mathcal{M}$ and on the parameters $\overline{\nu}$ and b used for the smoothing of the partition. It is small enough to make the arguments of paragraphs 4.4.4, 4.4.5 and also of the appendix work. Then, we fix k large enough to have $k\kappa>\frac{8d}{P_0}$.

Finally, once all these parameters are fixed, we fix some C small enough such that the sum in the upper bound of (34) is strictly less than 1.

5. Proof of Theorem 2.1

In the previous section, we gave a symbolic description of the quantum system and of its semi-classical limit. This symbolic coding is a standard procedure in ergodic theory and we will now show how one can relate the results obtained in the symbolic setting to the main Theorem of the introduction. Precisely, we will show how proposition 4.3 implies Theorem 2.1 following an argument from [1].

Let P_0 be a positive number and let $\mathcal{M}:=(M_i)_{i=1}^K$ be a partition of M with small diameter as in proposition 4.3. Let ρ be an element in S^*M and n_0 be positive integer. There exists an unique $[\alpha_\rho]=[\alpha_0,\dots,\alpha_{n_0-1}]$ in Σ_{n_0} such that ρ belongs to $M_{\alpha_0}\cap g^{-1}M_{\alpha_1}\cap\dots g^{-n_0+1}M_{\alpha_{n_0-1}}$. We denote this subset $M_{n_0}(\rho)$. One can remark that if we choose the diameter of the partition ϵ small enough[9] and the size of the energy layer δ small enough, then for every $\rho\in S^*M$, one has

$$\left|\frac{1}{2}\sum_{j=0}^{n_0-1}\log J^u\circ g^j(\rho)-\frac{1}{2}\log J^u_{n_0}(\alpha_\rho)\right|\leq\frac{n_0P_0}{2}.$$

Now, we consider μ in $\mathcal{M}((\psi_\hbar)_{0<\hbar\leq\hbar_0})$ as in the statement of Theorem 2.1 where $C(P_0)$ is given by proposition 4.3. We write the ergodic decomposition of μ [11]

$$\mu=\int_{S^*M}\mu_\rho d\mu(\rho),$$

where every μ_ρ is an ergodic probability measure. According to section 3, one also knows that, for μ almost every $\rho\in S^*M$,

$$\frac{1}{n_0}\sum_{j=0}^{n_0-1}\log J^u\circ g^j(\rho)=\frac{1}{n_0}\log J^u_{n_0}(\rho)\longrightarrow\int_{S^*M}\log J^u d\mu_\rho,\ \text{as } n_0\to+\infty. \tag{35}$$

Introduce now

$$I_{P_0}:=\left\{\rho\in S^*M: h_{KS}(\mu_\rho,g,\mathcal{M})<-\frac{1}{2}\int_{S^*M}\log J^u d\mu_\rho-2P_0\right\}.$$

According to the Shannon-McMillan-Breiman Theorem [23], one knows that for μ almost every ρ in S^*M,

$$h_{KS}(\mu_\rho,g,\mathcal{M})=\lim_{n_0\to+\infty}-\frac{1}{n_0}\log\mu(M_{n_0}(\rho)). \tag{36}$$

We will now prove that $\mu(I_{P_0})\leq c(P_0)$, where $0\leq c(P_0)<1$ is the constant that appears in proposition 4.3.

One knows that, for every $\eta>0$, there exists $\mathcal{R}$ such that $\mu(\mathcal{R})\leq\eta$ and such that the previous limits (36) and (35) hold uniformly for $\rho\in I_{P_0}-\mathcal{R}$. Denote $I^\eta_{P_0}=I_{P_0}-\mathcal{R}$. According to the definition of $I^\eta_{P_0}$, one knows that there exists $n_0'(P_0,\eta)$ such that for every $n_0\geq n_0'(P_0,\eta)$ and for every $\rho\in I^\eta_{P_0}$, one has

$$-\log\mu(M_{n_0}(\rho))+\frac{1}{2}\sum_{j=0}^{n_0-1}\log J^u\circ g^j(\rho)\leq -P_0n_0.$$

This implies that, for every ρ in $I^\eta_{P_0}$, one has

$$J^u_{n_0}(\alpha_\rho)^{\frac{1}{2}}\leq e^{-\frac{n_0P_0}{2}}\mu(M_{n_0}(\rho)).$$

Consider now F a finite subset of $I^\eta_{P_0}$ such that $\rho\neq\rho'$ in F implies that $M_{n_0}(\rho)\neq M_{n_0}(\rho')$ and such that

$$I^\eta_{P_0}\subset\bigsqcup_{\rho\in F}M_{n_0}(\rho).$$

One has that $\sum_{\rho\in F}J^u_{n_0}(\alpha_\rho)^{\frac{1}{2}}\leq e^{-\frac{n_0P_0}{2}}$. According to proposition 4.3, one knows that

$$\mu(I^\eta_{P_0})\leq\mu\left(\bigsqcup_{\rho\in F}M_{n_0}(\rho)\right)\leq c(P_0).$$

This inequality holds for every $\eta>0$ small enough and we obtain that $\mu(I_{P_0})\leq c(P_0)$. Finally, we have

$$\mu\left(\left\{\rho\in S^*M: h_{KS}(\mu_\rho,g,\mathcal{M})\geq-\frac{1}{2}\int_{S^*M}\log J^u d\mu_\rho-2P_0\right\}\right)\geq 1-c(P_0)>0.$$

[9] Underline that $(M_{\alpha_0}\cap g^{-1}M_{\alpha_1})_{\alpha_0,\alpha_1}$ defines then a partition of small diameter of S^*M.

This property holds for any $P_0>0$ and it concludes the proof of Theorem 2.1 by taking $c_0(P_0)=1-c(P_0/2)$.

6. Proof of intermediary lemmas

In this section, we give the proof of intermediary results that were at the heart of our proof of proposition 4.3.

6.1. **Proof of lemma 4.5.** The proof of this lemma relies only on the classical properties of our problem (and not on its quantum structure). As mentionned above, we fix n_0 a positive integer and a family W_{n_0} of cylinders of length n_0 such that

$$\sum_{[\alpha]\in W_{n_0}} J^u_{n_0}(\alpha)^{\frac{1}{2}}\leq e^{-\frac{P_0n_0}{2}}.$$

Our only assumption on n_0 is that it is large enough so that for p large enough, one has

$$\begin{pmatrix} p \\ \left[\frac{p}{n_0}\right]\end{pmatrix}\leq e^{p\frac{P_0}{16}}. \tag{37}$$

We underline that n_0 is large in a way that depends only on P_0.

The proof follows a similar strategy as its analogue in [1], §2.3, except that we consider slightly different dynamical quantities related to dynamical pressures and not to entropies.

Decomposition of cylinders in $\Sigma_p(W_{n_0},\tau)$. Let $[\alpha]=[\alpha_0,\ldots,\alpha_{p-1}]$ be an element in $\Sigma_p(W_{n_0},\tau)$. We will first show that α can be decomposed into the concatenation of well-chosen cylinders. In order to describe this decomposition, we introduce an increasing sequence of stopping times. First, one sets

$$t_0:=\inf\left\{0\leq j\leq p-n_0:[\alpha_j,\ldots,\alpha_{j+n_0-1}]\in W_{n_0}\right\}.$$

As $\tau\geq 1/2$, one knows from the definition of $\Sigma_p(W_{n_0},\tau)$ that t_0 is well defined. Then, as long as the induction is well defined, we define the following increasing sequence of integers:

$$t_1:=\inf\left\{t_0+n_0\leq j\leq p-n_0:[\alpha_j,\ldots,\alpha_{j+n_0-1}]\in W_{n_0}\right\},\ldots$$

$$t_{l'+1}:=\inf\left\{t_{l'}+n_0\leq j\leq p-n_0:[\alpha_j,\ldots,\alpha_{j+n_0-1}]\in W_{n_0}\right\}.$$

We associate to this sequence a sequence of intervals of length n_0

$$I_0=[t_0,t_0+n_0-1],\ldots,I_l=[t_l,t_l+n_0-1].$$

From the definition of our sequence, one knows that, for $0\leq j\leq\max\{t_l+n_0-1,p-n_0\}$ outside $\sqcup_{j'}I_{j'}$, $[\alpha_j,\ldots,\alpha_{j+n_0-1}]$ does not belong to W_{n_0}. From the definition of $\Sigma_p(W_{n_0},\tau)$, there are at most $(1-\tau)(p-n_0)$ such j. In particular, it implies that one must have $ln_0\geq(p-n_0)-(1-\tau)(p-n_0)$, i.e. $l\geq\frac{\tau p}{n_0}-\tau$. Moreover, as the length of cylinders is equal to p, one has $l\leq\frac{p}{n_0}$. Using the fact that $\tau\geq 1/2$, we finally have that $\frac{p}{2n_0}-1\leq\frac{\tau p}{n_0}-\tau\leq l\leq\frac{p}{n_0}$.

We have showed that $[\alpha]$ in $\Sigma_p(W_{n_0},\tau)$ can be written as $[b_0;a_0;\ldots;b_{l-1};a_{l-1};b_l]$ where

- every subcylinder a_j belongs to W_{n_0};
- every subcylinder b_j (if not empty) contains letters α_k such that $[\alpha_k,\ldots,\alpha_{k+n_0-1}]$ (when it makes sense, i.e. $k\leq p-n_0$) does not belong to W_{n_0}.

Again, from the definition of $\Sigma_p(W_{n_0},\tau)$, one knows that $\sum_{j=0}^{l}|b_j|\leq(1-\tau)(p-n_0)+n_0\leq(1-\tau)p+n_0$.

Upper bound on $\sum_{[\alpha]\in\Sigma_p(W_{n_0},\tau)} J^u_p(\alpha)^{\frac{1}{2}}$. From the previous paragraph, a cylinder $[\alpha_0,\dots,\alpha_{p-1}]$ in $\Sigma_p(W_{n_0},\tau)$ is determined by the following data:

(1) the subcylinders $(a_j)_{0\leq j\leq l-1}$ where $\frac{p}{2n_0}-1\leq l\leq p/n_0$;
(2) the subcylinders $(b_j)_{0\leq j\leq l}$ where $l\leq p/n_0$.

Regarding this decomposition, we will now give an upper bound on

$$\sum_{[\alpha]\in\Sigma_p(W_{n_0},\tau)} J^u_p(\alpha)^{\frac{1}{2}}=\sum_{[\alpha]\in\Sigma_p(W_{n_0},\tau)} J^u(\alpha_0,\alpha_1)^{\frac{1}{2}}\dots J^u(\alpha_{n-2},\alpha_{n-1})^{\frac{1}{2}}.$$

Choosing a family of positions for the subcylinders (a_j) is equivalent to choose a family of endpoints; so there are at most $\begin{pmatrix} p \\ \left[\frac{p}{n_0}\right]\end{pmatrix}^2$ choices of family of position for the subcylinders a_j.

For a given position of these subcylinders, the sum runs (for every a_j) over the cylinders in W_{n_0}, for which one has

$$\sum_{\alpha\in W_{n_0}} J^u_{n_0}(\alpha)^{\frac{1}{2}}\leq e^{-\frac{n_0P_0}{2}}.$$

We explained above that there were at least $\frac{p}{2n_0}-1$ such subcylinders in any cylinder of $\Sigma_p(W_{n_0},\tau)$. Thus the contribution of these subcylinders can be bounded by $(e^{-\frac{n_0P_0}{2}})^{\frac{p}{2n_0}-1}$.

For a fixed choice of positions for the subcylinders $(b_j)_{0\leq j\leq l}$, the number of possibility for the values of $(b_j)_{0\leq j\leq l}$ is bounded by $K^{(1-\tau)p+n_0}$, where K is the cardinal of the partition. In this case, we bound $J^u(\alpha_i,\alpha_{i+1})^{\frac{1}{2}}$ by e^{λ_0}. Thus the contribution of these subcylinders can be bounded by $(e^{\lambda_0}K)^{(1-\tau)p+n_0}$.

Finally, we find that

$$\sum_{[\alpha]\in\Sigma_p(W_{n_0},\tau)} J^u_p(\alpha)^{\frac{1}{2}}\leq \begin{pmatrix} p \\ \left[\frac{p}{n_0}\right]\end{pmatrix}^2 (e^{\lambda_0}K)^{(1-\tau)p+n_0}(e^{-\frac{n_0P_0}{2}})^{\frac{p}{2n_0}-1}.$$

This last equality concludes the proof of the lemma thanks to the assumption (37) on n_0.

6.2. **Proof of lemma 4.6.** In order to prove lemma 4.6, we introduce a family of cylinders of length $N=kn$ related to $\Sigma_n(W_{n_0},\tau)$, where k is a fixed integer.

Precisely, we define $\Sigma_n(W_{n_0},\tau)^k$ as the family of cylinders of the form $[\gamma]:=[\gamma^0;\dots;\gamma^{k-1}]$ where every γ^j is an element of $\Sigma_n(W_{n_0},\tau)$. We use the notation $(\Sigma_n(W_{n_0},\tau)^k)^c$ for its complementary in Σ_N. We will prove lemma 4.6 using the following lemma:

Lemma 6.1 (submultiplicativity property)**.** *Let k, κ and C be as in paragraph 4.5. There exists a constant $c_1>0$ (depending only on the choice of partition, on $\overline{\nu}$ and on $\delta>0$) such that, for any θ in $[0,1]$, if*

$$\left\|\sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)^c}\pi_{\alpha_{n-1}}\mathcal{U}_\hbar\dots\pi_{\alpha_1}\mathcal{U}_\hbar\pi_{\alpha_0}\psi_\hbar\right\|_{L^2(M)}\leq\frac{e^{-Ck\kappa}}{c_1^k k}\theta e^{\frac{(n-1)\operatorname{Im}(z(\hbar))}{\hbar}},\tag{38}$$

where $\Sigma_n(W_{n_0},\tau)^c$ means the complementary of $\Sigma_n(W_{n_0},\tau)$ in Σ_n; then, one has

$$\left\|\sum_{[\gamma]\in(\Sigma_n(W_{n_0},\tau)^k)^c}\pi_{\gamma_{N-1}}\mathcal{U}_\hbar\dots\pi_{\gamma_1}\mathcal{U}_\hbar\pi_{\gamma_0}\psi_\hbar\right\|\leq\theta e^{\frac{(N-1)\operatorname{Im}(z(\hbar))}{\hbar}}+\mathcal{O}(\hbar^\infty).$$

This lemma is a crucial step as it allows to connect relations on cylinders of short logarithmic length $n(\hbar)$ to relations on cylinders of large logarithmic length $kn(\hbar)$ (where k is an arbitrary integer). We postpone the proof of this lemma to the end of this paragraph and first show how we can derive lemma 4.6 from it.

6.2.1. *Combining lemma 6.1 to hyperbolic dispersive estimates from* [1, 31]. In order to prove lemma 4.6, one can write that

$$\sum_{[\gamma]\in(\Sigma_n(W_{n_0},\tau)^k)^c}\langle\pi_{\gamma_{N-1}}\mathcal{U}_\hbar\ldots\pi_{\gamma_1}\mathcal{U}_\hbar\pi_{\gamma_0}\psi_\hbar,\psi_\hbar\rangle+\sum_{[\gamma]\in\Sigma_n(W_{n_0},\tau)^k}\langle\pi_{\gamma_{N-1}}\mathcal{U}_\hbar\ldots\pi_{\gamma_1}\mathcal{U}_\hbar\pi_{\gamma_0}\psi_\hbar,\psi_\hbar\rangle=e^{\frac{\imath(N-1)\overline{z(\hbar)}}{\hbar}}.$$

If (38) is satisfied, then lemma 6.1 implies that

$$\left|\sum_{[\gamma]\in(\Sigma_n(W_{n_0},\tau)^k)^c}\langle\pi_{\gamma_{N-1}}\mathcal{U}_\hbar\ldots\pi_{\gamma_1}\mathcal{U}_\hbar\pi_{\gamma_0}\psi_\hbar,\psi_\hbar\rangle\right|\leq\theta e^{\frac{(N-1)\,\mathrm{Im}(z(\hbar))}{\hbar}}+\mathcal{O}(\hbar^\infty).$$

This allows to derive that

$$e^{-\frac{(kn-1)\,\mathrm{Im}(z(\hbar))}{\hbar}}\left|\sum_{[\gamma]\in\Sigma_n(W_{n_0},\tau)^k}\langle\pi_{\gamma_{N-1}}\mathcal{U}_\hbar\ldots\pi_{\gamma_1}\mathcal{U}_\hbar\pi_{\gamma_0}\psi_\hbar,\psi_\hbar\rangle\right|\geq 1-\theta+\mathcal{O}(\hbar^\infty). \tag{39}$$

At this point, one can use hyperbolic[10] estimates on quantum cylinders that were first used in [1] and then in several other articles to derive quantitative properties of semiclassical measures [1, 3, 26]. Recall that these estimates tell us that for every $\mathcal{K}>0$, there exists $\hbar_{\mathcal{K}}>0$ such that for every $\hbar\leq\hbar_{\mathcal{K}}$, for every $0\leq N\leq\mathcal{K}|\log\hbar|$ and for every $[\gamma]$ in Σ_N,

$$\|\Pi_\gamma\,\mathrm{Op}_\hbar(\chi_0)\|\leq 2(2\pi\hbar)^{-\frac{d}{2}}J_N^u(\gamma)^{\frac{1}{2}}(1+\mathcal{O}(\epsilon))^N, \tag{40}$$

where ϵ is an upper bound on the diameter of our partition, $\mathcal{O}(\hbar)$ depends only on V and on M and the constant in $\mathcal{O}(\epsilon)$ is uniform in γ and $\hbar$.

Remark 6.2. We underline that the proof of (40) in [1] (Theorem 1.3.3) was given in a selfadjoint setting ($V\equiv 0$). The generalization of Anantharaman's result to the nonselfadjoint setting $V\geq 0$ was performed by Schenck in [31]. In this reference, the author generalized the strategy of [1] by taking into account the nonselfadjoint contribution in the WKB Ansatz. In particular, as $V\geq 0$, one can verify that the exponential decrease due to the damping term can be crudely bounded by 1 in the WKB expansion. We underline that the proof of Schenck was given for the same propagator $\mathcal{U}_\hbar^t$ as ours. Yet, he considered observables that do not depend on $\hbar>0$, not like our family $(P_i^\hbar)_{i=1}^K$. Still, in order to get (40), one can combine the inputs of Schenck to the proof of Anantharaman in [1] which allows such observables.

Applying this hyperbolic dispersive estimate, one has, for $\hbar$ small enough,

$$\left|\sum_{[\gamma]\in\Sigma_n(W_{n_0},\tau)^k}\langle\pi_{\gamma_{N-1}}\mathcal{U}_\hbar\ldots\pi_{\gamma_1}\mathcal{U}_\hbar\pi_{\gamma_0}\psi_\hbar,\psi_\hbar\rangle\right|\leq 2(2\pi\hbar)^{-\frac{d}{2}}(1+\mathcal{O}(\epsilon))^N\sum_{[\gamma]\in\Sigma_n(W_{n_0},\tau)^k}J_N^u(\gamma)^{\frac{1}{2}}+\mathcal{O}(\hbar^\infty).$$

Then, thanks to the multiplicative structure of J_N^u, one has

$$\sum_{[\gamma]\in\Sigma_n(W_{n_0},\tau)^k}J_N^u(\gamma)^{\frac{1}{2}}\leq e^{(k-1)\lambda_0}\left(\sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)}J_n^u(\alpha)^{\frac{1}{2}}\right)^k,$$

where the term $e^{(k-1)\lambda_0}$ comes from the fact that we bounded $J^u(\gamma_{jn-1},\gamma_{jn})$ by e^{λ_0} for every $1\leq j\leq k-1$.

Combining these last two bounds to (39), one obtains, for $\hbar$ small enough,

$$\left(\sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)}J_n^u(\alpha)^{\frac{1}{2}}e^{-\frac{(n-1)\,\mathrm{Im}\,z(\hbar)}{\hbar}}\right)^k$$
$$\geq e^{-(k-1)\left(\lambda_0-\frac{\mathrm{Im}\,z(\hbar)}{\hbar}\right)}\left(1-\theta+\mathcal{O}(\hbar^\infty)\right)\hbar^{d/2}e^{-C_0k\kappa\epsilon|\log\hbar|},$$

[10] The Anosov assumption is only used for this point of our proof.

where C_0 is a positive constant that depends only on M and λ_0 is an upper bound on all the $\log J^u(\alpha_0,\alpha_1)$. This concludes the proof of lemma 4.6.

6.2.2. *Proof of lemma 6.1.* Our proof of lemma 4.6 relied on lemma 6.1 that we will prove now. We will show that the strategy from [1] (proof of lemma 2.2.3) can be adapted in a nonselfadjoint setting thanks to our assumption on the localization property (15) of $z(\hbar)$.

First simplification of the sum. We introduce the following notation for the purpose of our proof:

$$\forall[\gamma]\in\Sigma_n,\ \tilde{\Pi}_\gamma=\mathcal{U}_\hbar\pi_{\gamma_{n-1}}\mathcal{U}_\hbar\dots\pi_{\gamma_1}\mathcal{U}_\hbar\pi_{\gamma_0}.$$

Let $[\gamma]$ be an element in $(\Sigma_n(W_{n_0},\tau)^k)^c$. It can be decomposed into the concatenation of n-cylinders $[\gamma]:=[\gamma^0,\dots,\gamma^{k-1}]$ where at least one of the cylinders $[\gamma^j]$ does not belong to $\Sigma_n(W_{n_0},\tau)$. Thus, we can write the decomposition

$$(\Sigma_n(W_{n_0},\tau)^k)^c=\bigsqcup_{j=0}^{k-1}B_j, \tag{41}$$

where B_j is the subfamily of cylinders $[\gamma^0,\dots,\gamma^j,\dots,\gamma^{k-1}]$ satisfying

$$\forall\ i>j,\ [\gamma^i]\in\Sigma_n(W_{n_0},\tau),\ [\gamma^j]\in\Sigma_n(W_{n_0},\tau)^c,\ \forall i<j,\ [\gamma^i]\in\Sigma_n.$$

We now use lemma A.5 to introduce cutoffs in each term of the sum. Precisely, we write

$$\sum_{[\gamma]\in(\Sigma_n(W_{n_0},\tau)^k)^c}\tilde{\Pi}_{\gamma^{k-1}}\dots\tilde{\Pi}_{\gamma^j}\dots\tilde{\Pi}_{\gamma^0}\psi_\hbar$$

(42)
$$=\sum_{[\gamma]\in(\Sigma_n(W_{n_0},\tau)^k)^c}\left(\tilde{\Pi}_{\gamma^{k-1}}\operatorname{Op}_\hbar(\chi_0)\right)\dots\left(\tilde{\Pi}_{\gamma^j}\operatorname{Op}_\hbar(\chi_{-k+j+1})\right)\dots\left(\tilde{\Pi}_{\gamma^0}\operatorname{Op}_\hbar(\chi_{-k+1})\right)\psi_\hbar+\mathcal{O}(\hbar^\infty),$$

Then, using this equality, the eigenmode equation (7) and the property (17) of partition of identity, we find that

$$\left\|\sum_{[\gamma]\in(\Sigma_n(W_{n_0},\tau)^k)^c}\pi_{\gamma_{N-1}}\mathcal{U}_\hbar\dots\pi_{\gamma_1}\mathcal{U}_\hbar\pi_{\gamma_0}\psi_\hbar\right\|\leq\|\mathcal{U}_\hbar^{-1}\|\mathbf{R}'\sum_{j=0}^{k-1}\mathbf{R}_{k-j-1}e^{\frac{jn\operatorname{Im}(z(\hbar))}{\hbar}}+\mathcal{O}(\hbar^\infty), \tag{43}$$

where

$$\mathbf{R}':=\left\|\sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)^c}\mathcal{U}_\hbar\pi_{\alpha_{n-1}}\mathcal{U}_\hbar\dots\pi_{\alpha_1}\mathcal{U}_\hbar\pi_{\alpha_0}\psi_\hbar\right\|_{L^2(M)},$$

$\mathbf{R}_{k-1}=1$ and, for $1\leq j\leq k-1$,

$$\mathbf{R}_{k-j-1}:=\prod_{l=j+1}^{k-1}\left\|\sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)}\mathcal{U}_\hbar\pi_{\alpha_{n-1}}\mathcal{U}_\hbar\dots\pi_{\alpha_1}\mathcal{U}_\hbar\pi_{\alpha_0}\operatorname{Op}_\hbar(\chi_{-(k-l)+1})\right\|.$$

Bound on $\mathbf{R}_{k-j-1}$. Fix now a family W in Σ_n and any $1\leq j\leq k-1$. Then, one has

$$\left\|\sum_{[\alpha]\in W}\pi_{\alpha_{n-1}}\dots\mathcal{U}_\hbar\pi_{\alpha_0}\mathcal{U}_\hbar\operatorname{Op}_\hbar(\chi_{-(k-j)+1})\right\|\leq\left\|\sum_{[\alpha]\in W}\pi_{\alpha_{n-1}}\dots\mathcal{U}_\hbar\pi_{\alpha_0}\mathcal{U}_\hbar\operatorname{Op}_\hbar(\chi_{-(k-l)+1})\mathcal{U}_\hbar^{-n}\right\|\|\mathcal{U}_\hbar^n\|.$$

One knows that $\|\mathcal{U}_\hbar^n\| \leq 1$. Moreover, one can verify (lemma A.4 in the appendix) that, for $\kappa > 0$ small enough and $\hbar > 0$ small enough,

$$\left\| \sum_{[\alpha]\in W} \pi_{\alpha_{n-1}} \ldots \mathcal{U}_\hbar \pi_{\alpha_0} \mathcal{U}_\hbar \operatorname{Op}_\hbar(\chi_{-(k-j)+1}) \mathcal{U}_\hbar^{-n} \right\| \leq c_1,$$

where $c_1 \geq 1$ depends only on the choice of the partition (and not on W), on the regularization parameter $\overline{\nu}$ and on the size of the energy layer δ. Finally, one has, for any $0 \leq j \leq k-1$,

$$\mathbf{R}_{k-j-1} \leq c_1^k.$$

The constant $c_1 \geq 1$ appearing here is the one we use in the assumptions of lemma 6.1

The conclusion. We now implement these upper bounds in inequality (43) and we find that, for $\hbar > 0$ small enough,

$$\left\| \sum_{[\gamma]\in(\Sigma_n(W_{n_0},\tau)^k)^c} \pi_{\gamma_{N-1}} \mathcal{U}_\hbar \ldots \pi_{\gamma_1} \mathcal{U}_\hbar \pi_{\gamma_0} \psi_\hbar \right\|$$
$$\leq c_1^k \sum_{j=0}^{k-1} e^{\frac{jn \operatorname{Im}(z(\hbar))}{\hbar}} \left\| \sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)^c} \pi_{\alpha_{n-1}} \mathcal{U}_\hbar \ldots \pi_{\alpha_1} \mathcal{U}_\hbar \pi_{\alpha_0} \psi_\hbar \right\|_{L^2(M)} + \mathcal{O}(\hbar^\infty).$$

Then, thanks to the assumption (38) on the concentration of $\psi_\hbar$, we find

$$\left\| \sum_{[\gamma]\in(\Sigma_n(W_{n_0},\tau)^k)^c} \pi_{\gamma_{N-1}} \mathcal{U}_\hbar \ldots \pi_{\gamma_1} \mathcal{U}_\hbar \pi_{\gamma_0} \psi_\hbar \right\| \leq \frac{\theta e^{-Ck\kappa}}{k} \sum_{j=0}^{k-1} e^{\frac{((j+1)n-1) \operatorname{Im}(z(\hbar))}{\hbar}} + \mathcal{O}(\hbar^\infty).$$

Recall now that we made the assumption that

$$\frac{\operatorname{Im}(z(\hbar))}{\hbar} \geq -C|\log \hbar|^{-1}.$$

At this point of the proof, we crucially use this hypothesis in order to bound $e^{\frac{((j+1)n-1) \operatorname{Im}(z(\hbar))}{\hbar}}$ by $e^{\frac{(kn-1) \operatorname{Im}(z(\hbar))}{\hbar}} e^{Ck\kappa}$ for every $0 \leq j \leq k-1$. It implies

$$\left\| \sum_{[\gamma]\in(\Sigma_n(W_{n_0},\tau)^k)^c} \pi_{\gamma_{N-1}} \mathcal{U}_\hbar \ldots \pi_{\gamma_1} \mathcal{U}_\hbar \pi_{\gamma_0} \psi_\hbar \right\| \leq \theta e^{\frac{(kn-1) \operatorname{Im}(z(\hbar))}{\hbar}} + \mathcal{O}(\hbar^\infty),$$

that concludes the proof of lemma 6.1.

6.3. **Subinvariance of the quantum functional $\mu_\hbar^\Sigma$.** In paragraph 4.3, we constructed a quantum functional $\mu_\hbar^\Sigma$ on a set Σ. The set Σ was defined from a regularized partition that we suppose to be fixed in this section. As mentioned in paragraph 4.3, this functional is not invariant under the shift σ and it only satisfies

$$\forall [\alpha_0, \ldots, \alpha_{n_1-1}],\ \forall p \geq 0,\ \mu_\hbar^\Sigma(\sigma^{-p}[\alpha_0, \ldots, \alpha_{n_1-1}]) = \mu_\hbar^\Sigma([\alpha_0, \ldots, \alpha_{n_1-1}]) + o_{n_1,p}(1).$$

This remainder term in this property is not explicit enough to use it directly. Yet, we recall from (19) that one has the following exact equality

$$\mu_\hbar^\Sigma([\alpha_0, \ldots, \alpha_{n_1-1}]) = e^{-\frac{2p \operatorname{Im}(z(\hbar))}{\hbar}} \sum_{|\beta|=p} \left\langle (\mathcal{U}_\hbar^p)^* \mathcal{U}_\hbar^p \Pi_{\beta.\alpha} \psi_\hbar, \psi_\hbar \right\rangle. \tag{44}$$

Starting from this observation, we will prove the following subinvariance property:

Proposition 6.3. *Let $(P_i^\hbar)_{i=1,\ldots K}$ be a fixed partition satisfying the assumptions from paragraph 4.2. There exist*[11] *$\kappa_0 > 0$ and ν_0 such that for every $0 \leq p \leq \kappa_0 |\log \hbar|$, one has*

$$\forall [\alpha_0, \ldots, \alpha_{n_1-1}],\ \mu_\hbar^\Sigma([\alpha_0, \ldots, \alpha_{n_1-1}]) \leq e^{-\frac{2p \operatorname{Im}(z(\hbar))}{\hbar}} \left(\mu_\hbar^\Sigma(\sigma^{-p}[\alpha_0, \ldots, \alpha_{n_1-1}]) + K^p \mathcal{O}_{n_1}(\hbar^{\nu_0}) \right).$$

[11]Even if we will not mention it at every step of the proof, both κ_0 and ν_0 depend on the partition, on the energy layer we work on and on the parameter $\overline{\nu}$ used for the smoothing of the partition.

Before giving the proof of this proposition, let us mention that once the partition is fixed (hence K is fixed), there exists a $\kappa_1>0$ such that for every $\kappa\leq\kappa_1$, the remainder $K^p\mathcal{O}_{n_1}(\hbar^{\nu_0})$ in the proposition is of order $\mathcal{O}_{n_1}(\hbar^{\nu'(\kappa)})$ for some positive $\nu'(\kappa)$.

Proof of proposition 6.3. The proof of proposition 6.3 is a direct consequence of equality (44) and of the following lemma:

Lemma 6.4. *Let $(P_i^{\hbar})_{i=1,\dots K}$ be a fixed partition satisfying the assumptions from paragraph 4.2. There exists $\kappa_0>0$ and ν_0 such that for every $0\leq p\leq\kappa_0|\log\hbar|$, one has*

$$\forall[\alpha]\in\Sigma_{n_1},\ \forall[\beta]\in\Sigma_p,\ \langle(\mathcal{U}_{\hbar}^p)^*\mathcal{U}_{\hbar}^p\Pi_{\beta.\alpha}\psi_{\hbar},\psi_{\hbar}\rangle\leq\left(\langle\Pi_{\beta.\alpha}\psi_{\hbar},\psi_{\hbar}\rangle+\mathcal{O}_{n_1}(\hbar^{\nu_0})\right).$$

Remark 6.5. Our proof allows to take $[\alpha]$ empty, i.e. we have

$$\forall[\beta]\in\Sigma_p,\ \langle(\mathcal{U}_{\hbar}^p)^*\mathcal{U}_{\hbar}^p\Pi_{\beta}\psi_{\hbar},\psi_{\hbar}\rangle\leq\left(\langle\Pi_{\beta}\psi_{\hbar},\psi_{\hbar}\rangle+\mathcal{O}_{n_1}(\hbar^{\nu_0})\right).$$

Proof. In order to prove lemma 6.4, we first use lemma A.3 on long products of pseudodifferential operators from the appendix and the fact that $\mathrm{Op}_{\hbar}(\chi_0)\psi_{\hbar}=\psi_{\hbar}+\mathcal{O}(\hbar^{\infty})$. It allows us to write, for $0\leq p\leq\kappa_0|\log\hbar|$,

$$\Pi_{\beta.\alpha}\psi_{\hbar}=\mathrm{Op}_{\hbar}\left(P_{\beta_0}\dots P_{\beta_{p-1}}\circ g^{p-1}P_{\alpha_p}\circ g^p\dots P_{\alpha_{n_1+p-1}}\circ g^{n_1+p-1}\chi_0\right)\psi_{\hbar}+\mathcal{O}_{n_1}(\hbar^{\nu_0}),$$

where $\nu_0>0$ and κ_0 are given by the lemmas in the appendix.

We recall from the Egorov property (see the appendix) that, for every fixed $t\geq0$, the operator $(\mathcal{U}_{\hbar}^t)^*\mathcal{U}_{\hbar}^t$ is a pseudodifferential operator with principal symbol equal to $e^{-2\int_0^t V\circ g^s ds}$. As we are localized in a small neighborhood of S^*M, this property remains true up to short logarithmic times after multiplication by an appropriate cutoff function. Thanks to lemma A.3, $P_{\beta_0}\dots P_{\beta_{p-1}}\circ g^{p-1}P_{\alpha_p}\circ g^p\dots P_{\alpha_{n_1+p-1}}\circ g^{n_1+p-1}\chi_0$ belongs to some good class of symbols $S^{-\infty,0}_{\overline{\nu}_0}(T^*M)$. Moreover, when restricted to a small neighborhood of S^*M, it is also the case for $e^{-2\int_0^t V\circ g^s ds}$. Thus, the operators are amenable to pseudodifferential calculus and one has, for $\kappa_0>0$ small enough and for $0\leq p\leq[\kappa_0|\log\hbar|]$,

$$(\mathcal{U}_{\hbar}^p)^*\mathcal{U}_{\hbar}^p\Pi_{\beta.\alpha}\psi_{\hbar}$$
$$=\mathrm{Op}_{\hbar}\left(e^{-2\int_0^p V\circ g^s ds}P_{\beta_0}\dots P_{\beta_{p-1}}\circ g^{p-1}P_{\alpha_p}\circ g^p\dots P_{\alpha_{n_1+p-1}}\circ g^{n_1+p-1}\chi_0\right)\psi_{\hbar}+\mathcal{O}_{n_1}(\hbar^{\nu_0}),$$

where $\nu_0>0$ is still a positive constant. As the symbol of the previous operator is compactly supported in T^*M, it can also be quantized using $\mathrm{Op}_{\hbar}^+$ (see paragraph A.2). It implies that

$$(\mathcal{U}_{\hbar}^p)^*\mathcal{U}_{\hbar}^p\Pi_{\beta.\alpha}\psi_{\hbar}$$
$$=\mathrm{Op}_{\hbar}^+\left(e^{-2\int_0^p V\circ g^s ds}P_{\beta_0}\dots P_{\beta_{p-1}}\circ g^{p-1}P_{\alpha_p}\circ g^p\dots P_{\alpha_{n_1+p-1}}\circ g^{n_1+p-1}\chi_0\right)\psi_{\hbar}+\mathcal{O}_{n_1}(\hbar^{\nu_0}),$$

where $\nu_0>0$ – with the abuse of notations mentionned in remark 4.7. Thanks to the positivity of $\mathrm{Op}_{\hbar}^+$, one finds then

$$\langle(\mathcal{U}_{\hbar}^p)^*\mathcal{U}_{\hbar}^p\Pi_{\beta.\alpha}\psi_{\hbar},\psi_{\hbar}\rangle$$
$$\leq\langle\mathrm{Op}_{\hbar}^+\left(P_{\beta_0}\dots P_{\beta_{p-1}}\circ g^{p-1}P_{\alpha_p}\circ g^p\dots P_{\alpha_{n_1+p-1}}\circ g^{n_1+p-1}\chi_0\right)\psi_{\hbar},\psi_{\hbar}\rangle+\mathcal{O}_{n_1}(\hbar^{\nu_0}).$$

Using finally the same arguments backward, one finds

$$\langle(\mathcal{U}_{\hbar}^p)^*\mathcal{U}_{\hbar}^p\Pi_{\beta.\alpha}\psi_{\hbar},\psi_{\hbar}\rangle\leq\langle\Pi_{\beta.\alpha}\psi_{\hbar},\psi_{\hbar}\rangle+\mathcal{O}_{n_1}(\hbar^{\nu_0}),$$

where $\nu_0>0$ still satisfies the same properties as above.

□

6.4. **Using the fact that $(P_j^\hbar)_{j=1,\dots,K}$ are almost orthogonal.** In this last paragraph, we will prove that, for κ small enough and for $n=[\kappa|\log\hbar|]$,

$$\left\|\sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)^c}\pi_{\alpha_{n-1}}\dots\pi_{\alpha_1}(2-n)\pi_{\alpha_0}(1-n)\psi_\hbar\right\|^2_{L^2(M)}$$
$$=\sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)^c}\left\langle\pi_{\alpha_{n-1}}\dots\pi_{\alpha_1}(2-n)\pi_{\alpha_0}(1-n)\psi_\hbar,\psi_\hbar\right\rangle+\mathcal{O}(\hbar^{\nu_0'}),$$

where ν_0' is a positive constant that depends on the choice of the partition, on the size of the energy layer and on the parameter $\overline{\nu}$ and b used for the smoothing of the partition. The proof of this property relies on the fact that, for κ small enough, cylinders of length $n=[\kappa|\log\hbar|]$ are amenable to semiclassical rules and that $(P_j^\hbar)_{j=1,\dots,K}$ acts almost like a family of orthogonal projectors on $\psi_\hbar$ – property (16).

The proof of this equality was already given in [1] in the case $V\equiv 0$. If proceeding carefully, the arguments can be adapted in our setting and for the sake of completeness, we give a proof of this equality below.

For simplicity of notations, we introduce $\mathbf{P}_\alpha^\hbar=P_{\alpha_{n-1}}^\hbar\dots P_{\alpha_1}^\hbar\circ g^{2-n}P_{\alpha_0}^\hbar\circ g^{1-n}$. As in lemma A.3, one can prove that, for $0\leq\kappa\leq\kappa_0$,

$$\left\|\sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)^c}\pi_{\alpha_{n-1}}\dots\pi_{\alpha_1}(2-n)\pi_{\alpha_0}(1-n)\psi_\hbar\right\|^2_{L^2(M)}$$
$$=\sum_{[\alpha],[\alpha']\in\Sigma_n(W_{n_0},\tau)^c}\left\langle\mathrm{Op}_\hbar\left(\mathbf{P}_\alpha^\hbar\chi_0\right)\psi_\hbar,\mathrm{Op}_\hbar\left(\mathbf{P}_{\alpha'}^\hbar\chi_0\right)\psi_\hbar\right\rangle+(\sharp\Sigma_n)^2\mathcal{O}(\hbar^{\nu_0})+\mathcal{O}(\hbar^\infty),$$

where we used the fact that $\mathrm{Op}_\hbar(\chi_0)\psi_\hbar=\psi_\hbar+\mathcal{O}(\hbar^\infty)$ and where $\nu_0>0$. We recall that $\sharp\Sigma_n=K^n$. Thus, as before, for $\kappa>0$ small enough, the remainder of the right-hand side is small as $\hbar\to 0$. The parameter ν_0 also depends on the choice of the partition, on the choice of the energy layer and on the regularization parameter $\overline{\nu}$ used for the smoothing of the partition. We will omit to mention this dependence in the following of the proof and we will also allow to take $\nu_0>0$ to be smaller from line to line in order to have the semiclassical arguments below work – see remark 4.7.

We underline that $\mathbf{P}_\alpha^\hbar\chi_0$ belongs to a class of symbols of type $S^{-\infty,0}_{\overline{\nu}_0}(T^*M)$ (where $0\leq\overline{\nu}_0<1/2$). Thanks to composition rules for pseudodifferential operators, we derive that

$$\left\|\sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)^c}\pi_{\alpha_{n-1}}\dots\pi_{\alpha_1}(2-n)\pi_{\alpha_0}(1-n)\psi_\hbar\right\|^2_{L^2(M)}$$
$$=\sum_{[\alpha],[\alpha']\in\Sigma_n(W_{n_0},\tau)^c}\left\langle\mathrm{Op}_\hbar\left(\mathbf{P}_{\alpha'}^\hbar\mathbf{P}_\alpha^\hbar\chi_0^2\right)\psi_\hbar,\psi_\hbar\right\rangle+(\sharp\Sigma_n)^2\mathcal{O}(\hbar^{\nu_0}).\tag{45}$$

We will now distinguish two kind of terms: $\alpha=\alpha'$ and $\alpha\neq\alpha'$. We will use the fact that $(P_j^\hbar)_j$ acts as a family of orthogonal projectors in order to show that the terms with $\alpha\neq\alpha'$ are small in the semiclassical limit and that we can replace $(\mathbf{P}_\alpha^\hbar)^2$ by $\mathbf{P}_\alpha^\hbar$ when $\alpha=\alpha'$ (up to a small error term).

First case $\alpha=\alpha'$. We use the composition formula and the long time Egorov property to derive that

$$\left\|\mathrm{Op}_\hbar\left((\mathbf{P}_\alpha^\hbar)^2\chi_0^2\right)-\mathrm{Op}_\hbar\left(\frac{(\mathbf{P}_\alpha^\hbar)^2}{(P_{\alpha_0}^\hbar\circ g^{1-n})^2}\chi_0^2\right)\pi_{\alpha_0}^2(1-n)\right\|_{L^2\to L^2}=\mathcal{O}(\hbar^{\nu_0}),$$

where $\nu_0>0$ and the remainder can be chosen uniform in α. Then, one can use the specific properties of our partition – precisely, the fact that it behaves like orthogonal projectors when it acts on $\psi_\hbar$ (see property (16)). It allows to prove that

$$\left\|\left(\pi_{\alpha_0}(1-n)-\pi_{\alpha_0}^2(1-n)\right)\psi_\hbar\right\|=\mathcal{O}(\hbar^{\frac{b}{2}-2\kappa\|V\|_\infty}),$$

as $-2\hbar\|V\|_\infty \leq \text{Im } z \leq 0$. Thus, for $\kappa > 0$ small enough, the remainder is a small power of $\hbar$. Using again the composition formula and the long time Egorov property, we find that

$$\left\|\text{Op}_\hbar\left((\mathbf{P}^\hbar_\alpha)^2\chi_0^2\right)\psi_\hbar - Op_\hbar\left(\frac{(\mathbf{P}^\hbar_\alpha)^2}{P^\hbar_{\alpha_0}\circ g^{1-n}}\chi_0^2\right)\psi_\hbar\right\| = \mathcal{O}(\hbar^{\nu_0}) + \mathcal{O}(\hbar^{\frac{b}{2}-2\kappa\|V\|_\infty}),$$

where the constant in the remainder is still uniform in α. Proceeding by induction, we get the following approximation:

$$\left\|\text{Op}_\hbar\left((\mathbf{P}^\hbar_\alpha)^2\chi_0^2\right)\psi_\hbar - Op_\hbar\left(\mathbf{P}^\hbar_\alpha\chi_0^2\right)\psi_\hbar\right\| = |\log\hbar|\left(\mathcal{O}(\hbar^{\nu_0}) + \mathcal{O}(\hbar^{\frac{b}{2}-2\kappa\|V\|_\infty})\right).$$

Finally, thanks to lemma A.3, we obtain

$$\left\|\text{Op}_\hbar\left((\mathbf{P}^\hbar_\alpha)^2\chi_0^2\right)\psi_\hbar - \pi_{\alpha_{n-1}}\dots\pi_{\alpha_1}(2-n)\pi_{\alpha_0}(1-n)\psi_\hbar\right\| = |\log\hbar|\left(\mathcal{O}(\hbar^{\nu_0}) + \mathcal{O}(\hbar^{\frac{b}{2}-2\kappa\|V\|_\infty})\right),$$

with an uniform constant in α in the remainder.

Second case $\alpha \neq \alpha'$. In this case, there exists j such that $\alpha_j \neq \alpha'_j$. As in the first case, we use the composition formula and the Egorov property to write

$$\left\|\text{Op}_\hbar\left(\mathbf{P}^\hbar_\alpha\mathbf{P}^\hbar_{\alpha'}\chi_0^2\right) - \text{Op}_\hbar\left(\frac{\mathbf{P}^\hbar_\alpha\mathbf{P}^\hbar_{\alpha'}}{P^\hbar_{\alpha_j}\circ g^{1-n}\times P^\hbar_{\alpha'_j}\circ g^{1-n}}\chi_0^2\right)(\pi_{\alpha_j}\pi_{\alpha'_j})(1-n)\right\|_{L^2\to L^2} = \mathcal{O}(\hbar^{\nu_0}),$$

with the same properties as above for the remainder. Recall again that our partition behaves like orthogonal projectors when it acts on $\psi_\hbar$ (see paragraph 4.2). Hence, one finds

$$\left\|(\pi_{\alpha_j}\pi_{\alpha'_j})(1-n)\psi_\hbar\right\| = \mathcal{O}(\hbar^{\frac{b}{2}-2\kappa\|V\|_\infty}).$$

Thanks to the Calderón-Vaillancourt Theorem, the operator $\text{Op}_\hbar\left(\frac{\mathbf{P}^\hbar_\alpha\mathbf{P}^\hbar_{\alpha'}}{P^\hbar_{\alpha_j}\circ g^{1-n}\times P^\hbar_{\alpha'_j}\circ g^{1-n}}\chi_0^2\right)$ has a norm bounded by a constant uniform in α and in $\hbar$. Finally, we obtain

$$\left\|\text{Op}_\hbar\left(\mathbf{P}^\hbar_\alpha\mathbf{P}^\hbar_{\alpha'}\chi_0^2\right)\psi_\hbar\right\| = \mathcal{O}(\hbar^{\nu_0}) + \mathcal{O}(\hbar^{\frac{b}{2}-2\kappa\|V\|_\infty}).$$

Combining the two cases with (45). To conclude the proof of this paragraph, we combine equality (45) with the two cases treated above. We find that, for κ small enough, there exists a constant $\nu_0 > 0$ (depending also on the partition $\mathcal{M}$, the size of the energy layer and the smoothing parameters $\overline{\nu}$ and b) and such that, for $n = [\kappa|\log\hbar|]$,

$$\left\|\sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)^c}\pi_{\alpha_{n-1}}\dots\pi_{\alpha_1}(2-n)\pi_{\alpha_0}(1-n)\psi_\hbar\right\|^2_{L^2(M)}$$

$$= \sum_{[\alpha]\in\Sigma_n(W_{n_0},\tau)^c}\left\langle\pi_{\alpha_{n-1}}\dots\pi_{\alpha_1}(2-n)\pi_{\alpha_0}(1-n)\psi_\hbar,\psi_\hbar\right\rangle + (\sharp\Sigma_n)^2\mathcal{O}(\hbar^{\nu_0}).$$

As $\sharp\Sigma_n$ is equal to K^n as $\nu_0 > 0$ can be chosen uniformly for κ small enough, one can find κ small enough to have a remainder which goes to 0 as a positive power of $\hbar$ (which was the expected property).

Appendix A. Pseudodifferential calculus on a manifold

In this appendix, we review some basic facts on semiclassical analysis that can be found for instance in [10, 12]. We also give several lemmas that we use at different steps of the paper.

A.1. **General facts.** Recall that we define on $\mathbb{R}^{2d}$ the following class of symbols:

$$S^{m,k}(\mathbb{R}^{2d}) := \left\{ (a_\hbar(x,\xi))_{\hbar\in(0,1]} \in C^\infty(\mathbb{R}^{2d}) : |\partial_x^\alpha \partial_\xi^\beta a_\hbar| \leq C_{\alpha,\beta} \hbar^{-k} \langle \xi \rangle^{m-|\beta|} \right\}.$$

Let M be a smooth Riemannian d-manifold without boundary. Consider a smooth atlas (f_l, V_l) of M, where each f_l is a smooth diffeomorphism from $V_l \subset M$ to a bounded open set $W_l \subset \mathbb{R}^d$. To each f_l correspond a pull back $f_l^* : C^\infty(W_l) \to C^\infty(V_l)$ and a canonical map $\tilde{f}_l$ from T^*V_l to T^*W_l:

$$\tilde{f}_l : (x,\xi) \mapsto \left(f_l(x), (Df_l(x)^{-1})^T \xi \right).$$

Consider now a smooth locally finite partition of identity (ϕ_l) adapted to the previous atlas (f_l, V_l). That means $\sum_l \phi_l = 1$ and $\phi_l \in C^\infty(V_l)$. Then, any observable a in $C^\infty(T^*M)$ can be decomposed as follows: $a = \sum_l a_l$, where $a_l = a\phi_l$. Each a_l belongs to $C^\infty(T^*V_l)$ and can be pushed to a function $\tilde{a}_l = (\tilde{f}_l^{-1})^* a_l \in C^\infty(T^*W_l)$. As in [10, 12], define the class of symbols of order m and index k

$$S^{m,k}(T^*M) := \left\{ (a_\hbar(x,\xi))_{\hbar\in(0,1]} \in C^\infty(T^*M) : |\partial_x^\alpha \partial_\xi^\beta a_\hbar| \leq C_{\alpha,\beta} \hbar^{-k} \langle \xi \rangle^{m-|\beta|} \right\}. \tag{46}$$

Then, for $a \in S^{m,k}(T^*M)$ and for each l, one can associate to the symbol $\tilde{a}_l \in S^{m,k}(\mathbb{R}^{2d})$ the standard Weyl quantization

$$\mathrm{Op}_\hbar^w(\tilde{a}_l)u(x) := \frac{1}{(2\pi\hbar)^d} \int_{R^{2d}} e^{\frac{i}{\hbar}\langle x-y,\xi\rangle} \tilde{a}_l\left(\frac{x+y}{2}, \xi; \hbar\right) u(y) dy d\xi,$$

where $u \in \mathcal{S}(\mathbb{R}^d)$, the Schwartz class. Consider now a smooth cutoff $\psi_l \in C_c^\infty(V_l)$ such that $\psi_l = 1$ close to the support of ϕ_l. A quantization of $a \in S^{m,k}$ is then defined in the following way:

$$\mathrm{Op}_\hbar(a)(u) := \sum_l \psi_l \times \left(f_l^* \mathrm{Op}_\hbar^w(\tilde{a}_l)(f_l^{-1})^* \right)(\psi_l \times u), \tag{47}$$

where $u \in C^\infty(M)$. This quantization procedure $\mathrm{Op}_\hbar$ sends (modulo $\mathcal{O}(\hbar^\infty)$) $S^{m,k}(T^*M)$ onto the space of pseudodifferential operators of order m and of index k, denoted $\Psi^{m,k}(M)$ [10, 12]. It can be shown that the dependence in the cutoffs ϕ_l and ψ_l only appears at order 1 in $\hbar$ (using for instance Theorem 18.1.17 in [15]) and the principal symbol map $\sigma_0 : \Psi^{m,k}(M) \to S^{m,k}/S^{m,k-1}(T^*M)$ is then intrinsically defined. Most of the rules (for example the composition of operators, the Egorov and Calderón-Vaillancourt Theorems) that holds in the case of $\mathbb{R}^{2d}$ still holds in the case of $\Psi^{m,k}(M)$. Because our study concerns behavior of quantum evolution for logarithmic times in $\hbar$, a larger class of symbols should be introduced as in [10, 12], for $0 \leq \overline{\nu} < 1/2$,

$$S_{\overline{\nu}}^{m,k}(T^*M) := \left\{ (a_\hbar(x,\xi))_{\hbar\in(0,1]} \in C^\infty(T^*M) : |\partial_x^\alpha \partial_\xi^\beta a_\hbar| \leq C_{\alpha,\beta} \hbar^{-k-\overline{\nu}|\alpha+\beta|} \langle \xi \rangle^{m-|\beta|} \right\}. \tag{48}$$

Results of [10, 12] can be applied to this new class of symbols. For example, a symbol of $S_{\overline{\nu}}^{0,0}(T^*M)$ gives a bounded operator on $L^2(M)$ (with norm uniformly bounded with respect to $\hbar$).

A.2. **Positive quantization.** Even if the Weyl procedure is a natural choice to quantize an observable a on $\mathbb{R}^{2d}$, it is sometimes preferrable to use a quantization procedure $\mathrm{Op}_\hbar$ that satisfies the property : $\mathrm{Op}_\hbar(a) \geq 0$ if $a \geq 0$. This can be achieved thanks to the anti-Wick procedure $\mathrm{Op}_\hbar^{AW}$, see [13]. For a in $S_{\overline{\nu}}^{0,0}(\mathbb{R}^{2d})$, that coincides with a function on $\mathbb{R}^d$ outside a compact subset of $T^*\mathbb{R}^d = \mathbb{R}^{2d}$, one has

$$\| \mathrm{Op}_\hbar^w(a) - \mathrm{Op}_\hbar^{AW}(a) \|_{L^2} \leq C \sum_{|\alpha|\leq D} \hbar^{\frac{|\alpha|+1}{2}} \|\partial^\alpha da\|, \tag{49}$$

where C and D are some positive constants that depend only on the dimension d. To get a positive procedure of quantization on a manifold, one can replace the Weyl quantization by the anti-Wick one in definition (47). We will denote $\mathrm{Op}_\hbar^+(a)$ this new choice of quantization, well defined for every element in $S_{\overline{\nu}}^{0,0}(T^*M)$ of the form $c_0(x) + c(x,\xi)$ where c_0 belongs to $S_{\overline{\nu}}^{0,0}(T^*M)$ and c belongs to $\mathcal{C}_o^\infty(T^*M) \cap S_{\overline{\nu}}^{0,0}(T^*M)$.

This positivity assumption was used at several steps of the paper when we argued that the functional $\mu_\hbar^\Sigma$ was "almost positive" (see for instance paragraphs 4.4.4 and 4.4.5) or when we proved that it was "subinvariant" (paragraph 6.3).

A.3. **Egorov property for long times.** In this paragraph, we recall an Egorov property for times of order $\kappa_0|\log\hbar|$, where κ_0 is a small enough constant that we will not try to optimize. Consider q_1 and q_2 two symbols belonging to $S^{0,0}(T^*M)$ (for the sake of simplicity, we also assume that these symbols depend smoothly on $\hbar\in(0,1]$). In this article, we will use the symbols q_i equal to $\sqrt{2z(\hbar)}V$ or $-\sqrt{2\bar{z}(\hbar)}V$ – see remark A.2.

A.3.1. *The case of fixed times.* We consider a smooth function b on T^*M which is compactly supported in a neighborhood of S^*M, say $\text{supp}(b)\subset\{(x,\xi):\|\xi\|^2\in[1/2,3/2]\}$ and which belongs to $S^{-\infty,0}(T^*M)$. The following operator is a pseudodifferential operator, for every $t\in\mathbb{R}$,

$$A(t,b)=\left(e^{-\frac{\imath t}{\hbar}\left(-\frac{\hbar^2\Delta}{2}-\imath\hbar\operatorname{Op}_\hbar(q_1)\right)}\right)^*\operatorname{Op}_\hbar(a)e^{-\frac{\imath t}{\hbar}\left(-\frac{\hbar^2\Delta}{2}-\imath\hbar\operatorname{Op}_\hbar(q_2)\right)}.$$

We briefly recall how such a fact can be proved by a direct adaptation of the arguments used in the selfadjoint case [10, 12, 6, 28]. Take $q=\overline{q_1}+q_2$, and introduce, for $t,s\in\mathbb{R}$, the symbol

$$A_t(s):=a\circ g^{t-s}\exp\left(-\int_0^{t-s}q\circ g^\tau d\tau\right).$$

To alleviate our notations, we call

$$\mathcal{U}_\hbar^s(q_i):=e^{-\frac{\imath s}{\hbar}\left(-\frac{\hbar^2\Delta}{2}-\imath\hbar\operatorname{Op}_\hbar(q_i)\right)},\quad i=1,2\,,$$

so that the operator $A(t,b)=(\mathcal{U}_\hbar^t(q_1))^*\operatorname{Op}_\hbar(a)\,\mathcal{U}_\hbar^t(q_2)$. Fixing t, we then introduce the auxiliary operators

$$R(\hbar,s)=(\mathcal{U}_\hbar^s(q_1))^*\operatorname{Op}_\hbar(A_t(s))\,\mathcal{U}_\hbar^s(q_2).$$

Like in the classical proof of the Egorov Theorem (i.e. in the selfadjoint case), one can compute the derivative of $R(\hbar,s)$:

$$\frac{d}{ds}\left(R(\hbar,s)\right)=(\mathcal{U}_\hbar^s(q_1))^*\left(\frac{\imath}{\hbar}\left[-\frac{\hbar^2\Delta}{2},\operatorname{Op}_\hbar(A_t(s))\right]-\operatorname{Op}_\hbar(q_1)^*\operatorname{Op}_\hbar(A_t(s))-\operatorname{Op}_\hbar(A_t(s))\operatorname{Op}_\hbar(q_2)\right)\mathcal{U}_\hbar^s(q_2)$$
$$-(\mathcal{U}_\hbar^s(q_1))^*\big(\operatorname{Op}_\hbar\left(\{p_0,A_t(s)\}\right)-\operatorname{Op}_\hbar(A_t(s)(\overline{q_1}+q_2))\big)\mathcal{U}_\hbar^s(q_2).$$

We integrate this equality between 0 and t [6]:

$$\left(\mathcal{U}_\hbar^t(q_1)\right)^*\operatorname{Op}_\hbar(a)\,\mathcal{U}_\hbar^t(q_2)=\operatorname{Op}_\hbar\left(a\circ g^te^{-\int_0^tq\circ g^\tau d\tau}\right)+\int_0^t(\mathcal{U}_\hbar^s(q_1))^*\tilde{R}(\hbar,s)\mathcal{U}_\hbar^s(q_2)ds,$$

where $\tilde{R}(\hbar,s)$ is a pseudodifferential operator in $\Psi^{-\infty,-1}(M)$ thanks to pseudodifferential rules. Proceeding by induction and using pseudodifferential calculus perfomed locally on each chart [10, 12] (respectively Chapter 7 and 4) and the fact that $\mathcal{U}_\hbar^s(q_2)$ is a bounded operator (with a norm depending[12] on q_2 and s), one in fact finds that $\left(\mathcal{U}_\hbar^t(q_1)\right)^*\operatorname{Op}_\hbar(a)\mathcal{U}_\hbar^t(q_2)$ is a pseudodifferential operator in $\Psi^{-\infty,0}(M)$,

$$\left(\mathcal{U}_\hbar^t(q_1)\right)^*\operatorname{Op}_\hbar(a)\mathcal{U}_\hbar^t(q_2)=\operatorname{Op}_\hbar(\tilde{a}(t))+\mathcal{O}(\hbar^\infty),\tag{50}$$

where $\tilde{a}(t)\sim\sum_{j\geq0}\hbar^ja_j(t)$,

$$a_0(t)=A_t(0)=a\circ g^t\exp\left(-\int_0^t(\overline{q_1}+q_2)\circ g^\tau d\tau\right),$$

and all the higher order terms $(a_j(t))_{j\geq1}$ in the asymptotic expansion depend on a, t, q_1, q_2 and the choice of coordinates on the manifold. Moreover, for a fixed $t\in\mathbb{R}$, one can verify that every term $a_j(t)$ is supported in $g^{-t}\text{supp}(b)$. Each $a_j(t)$ can be written as $b_j(t)\exp\left(-\int_0^t(\overline{q_1}+q_2)\circ g^\tau d\tau\right)$, where $b_j(t)\in S^{-\infty,0}(T^*M)$. The Calderón-Vaillancourt Theorem [12, Chap.5] tells us that there exist constants $C_{a,t}$ and $C'_{a,t}$ (depending on b, q_1, q_2, t and M) such that

$$\left\|\left(\mathcal{U}_\hbar^t(q_1)\right)^*\operatorname{Op}_\hbar(a)\,\mathcal{U}_\hbar^t(q_2)\right\|_{L^2(M)\to L^2(M)}\leq C_{a,t}\|a_0(t)\|_\infty,$$

[12] It is in fact bounded by a constant of order $e^{|s|\|q_2\|_\infty}$.

and also

$$\left\| \left(\mathcal{U}_\hbar^t(q_1)\right)^* \operatorname{Op}_\hbar(a)\mathcal{U}_\hbar^t(q_2) - \operatorname{Op}_\hbar(a_0(t))\right\|_{L^2(M)\to L^2(M)} \leq C'_{a,t}\hbar. \tag{51}$$

A.3.2. *The case of logarithmic times.* All the above discussion was done for a fixed $t\in\mathbb{R}$. In this article, we needed to apply Egorov property for long range of times of order $\kappa_0|\log\hbar|$ [6, 3]. This can be achieved as all the arguments above can be adapted if we use more general classes of symbols, i.e. $S^{-\infty,0}_{\overline{\nu}}(T^*M)$ where $\overline{\nu}<1/2$ is a fixed constant[13].

In particular, one can show that, for $a\in S^{-\infty,0}(T^*M)$ supported near S^*M as above and κ_1 small enough (depending on the support of a, on $\overline{\nu}$, on q_1 and on q_2), the operator $A(t,b)$ is a pseudodifferential operator in $\Psi^{-\infty,0}_{\overline{\nu}}(M)$ for all $|t|\leq\kappa_1|\log\hbar|$. Precisely, its symbol has an asymptotic expansion of the same form as in the case of fixed times, except that for every $j\geq 0$ the symbol $b_j(t)$ belongs to $S^{-\infty,k_j}_{\overline{\nu}}(T^*M)$ for every $|t|\leq\kappa_1|\log\hbar|$, where $j-k_j$ is an increasing sequence of real numbers converging to infinity as $j\to+\infty$.

We also mention that all the seminorms of the symbols $b_j(t)$ can be bounded uniformly for $|t|\leq\kappa_1|\log\hbar|$. Finally, using pseudodifferential calculus (performed locally on every chart), one can verify that the following uniform estimates hold:

Proposition A.1. *There exist constants $\kappa_1>0$ and $\nu_0>0$ (depending only on q_1, q_2, $\overline{\nu}$ and M) such that for every smooth function a compactly supported in $\{(x,\xi):\|\xi\|^2\in[1/2,3/2]\}$, there exists a constant $C_a>0$ such that for every $|t|\leq\kappa_1|\log\hbar|$, one has*

$$\left\| \left(\mathcal{U}_\hbar^t(q_1)\right)^* \operatorname{Op}_\hbar(a)\mathcal{U}_\hbar^t(q_2)\right\|_{L^2(M)\to L^2(M)} \leq C_a\|a_0(t)\|_\infty,$$

and

$$\left\| \left(\mathcal{U}_\hbar^t(q_1)\right)^* \operatorname{Op}_\hbar(a)\mathcal{U}_\hbar^t(q_2) - \operatorname{Op}_\hbar(a_0(t))\right\|_{L^2(M)\to L^2(M)} \leq C_a\hbar^{\nu_0}.$$

Remark A.2. We will mostly use evolutions involving the propagator $\mathcal{U}_\hbar^t$ of (8). Then, the expression $(\mathcal{U}_\hbar^t)^*\operatorname{Op}_\hbar(a)\mathcal{U}_\hbar^t$ has the form of (50), with $q_1=q_2=\sqrt{2z(\hbar)}V$. As a result, in this case the principal symbol is $a_0(t)=a\circ g^t\, e^{-2\int_0^t V\circ g^\tau d\tau}$.

Another operator will be used: $(\mathcal{U}_\hbar^t)^{-1}\operatorname{Op}_\hbar(a)\mathcal{U}_\hbar^t$ also has the form (50), now with $q_1=-\sqrt{2\overline{z}}V$, $q_2=\sqrt{2z(\hbar)}V$. In this case, the principal symbol $a_0(t)=a\circ g^t$.

A.4. **Product of pseudodifferential operators.** In the last two paragraphs of this appendix, we fix a smooth partition satisfying the assumptions of paragraph 4.2. In particular, all the functions $P_j^\hbar$ belong to a class of symbol $S^{0,0}_{\overline{\nu}}(T^*M)$ with $0<\overline{\nu}<1/2$. Then, one can verify that the following lemma holds:

Lemma A.3. *Let χ_* be one of the cutoff function supported in a small neighborhood of S^*M that were defined in paragraph 4.1. There exists $\kappa_0>0$ depending only on δ (the size of the energy layer), on $\overline{\nu}$ (the parameter for the regularization of the partition) and on the choice of the partition such that*

$$\forall 0\leq m\leq\kappa_0|\log\hbar|,\ \pi_{\alpha_{m-1}}(m-1)\dots\pi_{\alpha_1}(1)\pi_{\alpha_0}\operatorname{Op}_\hbar(\chi_*)$$

is a pseudodifferential operator in $\Psi^{0,-\infty}_{\overline{\nu}_0}(M)$ (where $0<\overline{\nu}_0<1/2$) with principal symbol equal to

$$P^\hbar_{\alpha_{m-1}}\circ g^{m-1}\dots\times P^\hbar_{\alpha_1}\circ g^1\times P^\hbar_{\alpha_0}\chi_*.$$

The proof[14] of this lemma relies on the fact that for κ_0 small enough, the operators we consider are amenable to semiclassical calculus: composition rules, Egorov property (paragraph A.3).

Using this lemma, one also has the following property that we used in the proof of lemma 4.6:

[13] In order to avoid too many indices, we take the same $\overline{\nu}$ as in the definition of $\Theta_{\Lambda,\hbar,\overline{\nu}}$.

[14] For instance, similar properties on product of pseudodifferential operators were proved in [26] (section 7) in a selfadjoint context. They can be adapted in a nonselfadjoint setting and the situation is even simpler here as we do not try to optimize the parameter $\kappa_0>0$.

Lemma A.4. *Let χ_* be one of the cutoff function supported in a small neighborhood of S^*M that were defined in paragraph 4.1. There exist $\kappa_0 > 0$ and $c_1 \geq 1$ depending only on δ (the size of the energy layer), on $\overline{\nu}$ (the parameter for the regularization of the partition) and on the choice of the partition such that for $\hbar > 0$ small enough, for every $0 \leq n \leq \kappa_0 |\log \hbar|$ and for every subset $W \subset \Sigma_n$,*

$$\left\| \sum_{\gamma \in W} \mathcal{U}_\hbar \pi_{\gamma_{n-1}} \ldots \mathcal{U}_\hbar \pi_{\gamma_0} \mathcal{U}_\hbar \operatorname{Op}_\hbar(\chi_*) \mathcal{U}_\hbar^{-n} \right\|_{L^2(M) \to L^2(M)} \leq c_1.$$

Proof. Let γ be an element in Σ_n. As in lemma A.3, one can verify that, for κ_0 small enough (uniform in γ) and for any $0 \leq n \leq \kappa_0 |\log \hbar|$,

$$\mathcal{U}_\hbar \pi_{\gamma_{n-1}} \ldots \mathcal{U}_\hbar \pi_{\gamma_0} \operatorname{Op}_\hbar(\chi_*) \mathcal{U}_\hbar^{-n} = \pi_{\gamma_{n-1}}(-1) \ldots \pi_{\gamma_1}(1-n) \left(\pi_{\gamma_0} \operatorname{Op}_\hbar(\chi_*) \right)(-n)$$

is a pseudodifferential operator in $\Psi^{0,-\infty}_{\overline{\nu}'}(M)$ (for some $\overline{\nu} < \overline{\nu}' < 1/2$) with principal symbol equal to

$$P^\hbar_{\gamma_{n-1}} \circ g^{-1} \ldots \times P^\hbar_{\gamma_1} \circ g^{1-n} \times P^\hbar_{\gamma_0} \circ g^{-n} \chi_* \circ g^{-n}.$$

From the definition of χ_*, one has $\chi_* \circ g^t = \chi_*$ for every t in $\mathbb{R}$. Moreover, there exists $\nu_0 > 0$ such that, for every $0 \leq n \leq \kappa_0 |\log \hbar|$, one has

$$\left\| \mathcal{U}_\hbar \pi_{\gamma_{n-1}} \ldots \mathcal{U}_\hbar \pi_{\gamma_0} \operatorname{Op}_\hbar(\chi_*) \mathcal{U}_\hbar^{-n} - \operatorname{Op}_\hbar \left(P^\hbar_{\gamma_{n-1}} \circ g^{-1} \ldots \times P^\hbar_{\gamma_1} \circ g^{1-n} \times P^\hbar_{\gamma_0} \circ g^{-n} \chi_* \right) \right\| = \mathcal{O}(\hbar^{\nu_0}),$$

where the constant in the remainder can be chosen uniform[15] for $\gamma \in \Sigma_n$ and $0 \leq n \leq \kappa_0 |\log \hbar|$.

One knows that $\sharp W$ is at most equal to K^n. Hence, one can verify that, for $\kappa_0 > 0$ small enough, there exists $\nu_0' > 0$ (both independent of W) such that, for every $0 \leq n \leq \kappa_0 |\log \hbar|$,

$$\sum_{\gamma \in W} \mathcal{U}_\hbar \pi_{\gamma_{n-1}} \ldots \mathcal{U}_\hbar \pi_{\gamma_0} \operatorname{Op}_\hbar(\chi_*) \mathcal{U}_\hbar^{-n}$$

$$= \operatorname{Op}_\hbar \left(\sum_{\gamma \in W} P^\hbar_{\gamma_{n-1}} \circ g^{-1} \ldots \times P^\hbar_{\gamma_1} \circ g^{1-n} \times P^\hbar_{\gamma_0} \circ g^{-n} \chi_* \right) + \mathcal{O}_{L^2(M) \to L^2(M)}(\hbar^{\nu_0'}).$$

As $(P_i^\hbar)_{i=1,\ldots,K}$ is a family of nonnegative functions satisfying a property of partition of identity(see paragraph 4.2), one knows that

$$\left\| \sum_{\gamma \in W} P^\hbar_{\gamma_{n-1}} \circ g^{-1} \ldots \times P^\hbar_{\gamma_1} \circ g^{1-n} \times P^\hbar_{\gamma_0} \circ g^{-n} \chi_* \right\|_\infty \leq 1.$$

By similar arguments, one can verify that $\mathbf{P}_W = \sum_{\gamma \in W} P^\hbar_{\gamma_{n-1}} \circ g^{-1} \ldots \times P^\hbar_{\gamma_1} \circ g^{1-n} \times P^\hbar_{\gamma_0} \circ g^{-n} \chi_*$ belongs to some class $S^{-\infty,0}_{\overline{\nu}'}(T^*M)$ for some $0 < \overline{\nu}' < 1/2$ – with seminorms that can be chosen uniform in W and $0 \leq n \leq \kappa_0 |\log \hbar|$. Thus, one can apply Calderón Vaillancourt Theorem – Chapter 5 in [12] for instance. It tells us that there exist constant $C' > 0$ and $C'' > 0$ (depending only on M and on the choice of coordinate charts) such that

$$\| \operatorname{Op}_\hbar(\mathbf{P}_W) \|_{L^2(M) \to L^2(M)} \leq C' \sum_{|\alpha| \leq C'' d} \hbar^{\frac{|\alpha|}{2}} \| \partial^\alpha \mathbf{P}_W \|_\infty.$$

Thus, combined to the rest of the proof, it implies the existence of κ_0 and $c_1 \geq 1$ (depending on the choice of the partition $(M_i)_{i=1}^K$, on $\overline{\nu}$, on the size of the size energy layer but not on $\hbar$) such that, for $\hbar > 0$ small enough, one has

$$\forall 0 \leq n \leq \kappa_0 |\log \hbar|, \ \forall W \subset \Sigma_n, \ \left\| \sum_{\gamma \in W} \mathcal{U}_\hbar \pi_{\gamma_{n-1}} \ldots \mathcal{U}_\hbar \pi_{\gamma_0} \mathcal{U}_\hbar \operatorname{Op}_\hbar(\chi_*) \mathcal{U}_\hbar^{-n} \right\|_{L^2(M) \to L^2(M)} \leq c_1.$$

□

[15] It depends on the choice of the partition $(M_i)_{i=1}^K$, on the regularization parameter $\overline{\nu}$ and on the size of the energy layer.

A.5. **Inserting cutoffs functions.** Recall that we used the notation, for $\gamma=[\gamma_0,\gamma_1,\dots,\gamma_{p-1}]$ in Σ_p,

$$\tilde{\Pi}_\gamma=\mathcal{U}_\hbar\pi_{\gamma_{p-1}}\dots\mathcal{U}_\hbar\pi_{\gamma_1}\mathcal{U}_\hbar\pi_{\gamma_0}.$$

As we needed it in paragraph 6.2.2, we briefly explain here that we can insert a finite number k of cutoff functions in a cylinder of length kn, where n is a short logarithmic time for which the operators are amenable to pseudodifferential calculus. Precisely, it is a consequence of the following lemma which can be obtained as a combination of lemma A.3 on long product of pseudodifferential operators and of the property of the cutoff functions.

Lemma A.5. *There exists $\kappa>0$ small enough (depending on the smooth partition and on the size of the energy layer) such that for every $0\leq j\leq k-2$, for every $0\leq p\leq[|\kappa|\log\hbar|]$ and for every $[\gamma]\in\Sigma_p$,*

$$(\mathrm{Id}-\mathrm{Op}_\hbar(\chi_{-j}))\tilde{\Pi}_\gamma\,\mathrm{Op}_\hbar(\chi_{-j-1})=\mathcal{O}(\hbar^\infty).$$

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