ENTROPY OF SEMICLASSICAL MEASURES FOR QUANTIZED CAT-MAPS

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Abstract. In the case of semiclassical measures of the quantized cat-map, we give a simplified proof of Anantharaman-Nonnenmacher's result for semiclassical measures on a riemannian compact manifold of constant negative curvature [2]. We show that for any hyperbolic matrix A in $SL(2,\mathbb{Z})$ and any semiclassical measure μ associated to it, the Kolmogorov-Sinai entropy is bounded from below, i.e.

$$h_{KS}(\mu, A) \geq \frac{\lambda_+}{2},$$

where λ_+ is the positive Lyapunov exponent of A. Thanks to Faure-Nonnenmacher-de Bièvre construction in [8], this bound is optimal.

1. Introduction

In the case of manifolds of negative curvature, the Quantum Unique Ergodicity Conjecture states that *all eigenfunctions of the Laplacian equidistribute on M in the high energy limit* [13]. In [1], Anantharaman proved that for a compact riemannian manifold M of Anosov type, the Kolmogorov-Sinai entropy of any semiclassical measure associated to a sequence of eigenfunctions of Δ is positive. Her result proves in particular that eigenfunctions of the Laplacian cannot concentrate only on a closed geodesic in the large eigenvalue limit. After that, in the case where M is a compact manifold of constant curvature $K=-1$ [2], Anantharaman and Nonnenmacher proved the following explicit bound[1] on the Kolmogorov-Sinai entropy of a semiclassical measure μ:

$$h_{KS}(\mu, g) \geq \frac{d-1}{2}, \tag{1}$$

where g is the geodesic flow on the unit cotangent bundle S^*M and d is the dimension of M. In this second case, the proof was simplified by the use of an entropic uncertainty principle due to Maassen and Uffink [12]. This principle is a consequence of the Riesz-Thorin interpolation theorem and it can be stated as follows [2], [12]:

Theorem 1.1 (Maassen-Uffink). *Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be two Hilbert spaces. Let U be an unitary operator on $\tilde{\mathcal{H}}$. Suppose $(\pi_i)_{i=1}^D$ is a family of operators from $\tilde{\mathcal{H}}$ to $\mathcal{H}$ that satisfies the following property of partition of identity:*

$$\sum_{i=1}^D \pi_i^\dagger \pi_i = Id_{\tilde{\mathcal{H}}}.$$

Then, for any unit vector ψ, we have

$$-\sum_{i=1}^D \|\pi_i\psi\|^2_{\mathcal{H}} \log \|\pi_i\psi\|^2_{\mathcal{H}} - \sum_{i=1}^D \|\pi_i U\psi\|^2_{\mathcal{H}} \log \|\pi_i U\psi\|^2_{\mathcal{H}} \geq -2\log \sup_{i,j} \|\pi_i U \pi_j^\dagger\|_{\mathcal{L}(\mathcal{H})}. \tag{2}$$

In [2], the method was to use this principle for eigenfunctions of the Laplacian and a well-chosen partition of $\mathrm{Id}_{L^2(M)}$ so that the quantity in the left side of (2) can be interpreted as the usual entropy from information theory [14]. One of the main difficulty (that already appeared in [1]) was then to give a sharp estimate on the quantity $\|\pi_i U \pi_j^\dagger\|_{L^2(M)\to L^2(M)}$ for this choice.
One knows that a good way to study phenomena of quantum chaos like these ones is to look at toy models. For instance, one of the simplest toy model is given by the quantized map associated to an hyperbolic matrix A in $SL(2,\mathbb{Z})$ that acts on the torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ [5]. This dynamical

[1] Thanks to [11], proving Quantum Unique Ergodicity would be then equivalent to get rid of 2 in the inequality.

system provides a model of highly chaotic classical behavior (Anosov property for instance) and the usual questions of quantum chaos naturally arise in this setting [5]. Our goal is to give a proof of inequality (1) for this model:

Theorem 1.2. *Let A be an hyperbolic matrix in $SL(2,\mathbb{Z})$ that acts on the torus $\mathbb{T}^2$. Then, for any semiclassical measure μ associated to it, we have*

$$h_{KS}(\mu, A) \geq \frac{\lambda_+}{2},$$

where λ_+ is the positive Lyapunov exponent of A.

Our interest in writing this note was not only to provide a proof in the case of quantized cat-maps but also to give a proof that avoid the major difficulties mentioned above and that appeared in [2]. In particular, we make a quite different choice of partition of identity in the entropic uncertainty principle (precisely (9)) so that the lower bound in (2) can be computed (more) easily. We underline that Faure, Nonnenmacher and de Bièvre proved that in the setting of quantized cat maps, the measure $\frac{1}{2}\delta_0 + \frac{1}{2}\text{Leb}$ is a semiclassical measure. In particular, it says that the bound in the previous theorem is sharp.
This result on the entropy of semiclassical measures of the torus can be compared to previous results on semiclassical measures obtained by Bonechi-de Bièvre [4] and by Faure-Nonnenmacher [9]. A stronger result than theorem 1.2 on the entropy of semiclassical measures is given by Brooks in [7]. So, this note should not be seen as a new result but as a simplification of Anantharaman-Nonnenmacher's proof in the setting of quantized cat-maps and as a new point of view on the methods developed in [2].

2. Quantum mechanics on the torus

In this section, we will briefly recall some facts about quantization of linear symplectic toral automophisms. We follow the approach and notations used by Bouzouina and de Bièvre in [5] and we refer the reader to it for further details and references. In this note, $\rho = (x, \xi)$ will denote a point of the phase space (i.e. $\mathbb{R}^2$ or $\mathbb{T}^2$) and σ the usual symplectic form on $\mathbb{R}^2$, i.e. $\sigma((x,\xi),(x',\xi')) = x'\xi - x\xi'$. In the case of $\mathbb{R}$, the state of a quantum particle is a tempered distribution $\psi \in \mathcal{S}'(\mathbb{R})$. Then, phase space translation operators acting on this space can be defined as

$$U_\hbar(x,\xi) := e^{\frac{\imath}{\hbar}\sigma((x,\xi),(Q,P))},$$

with $(Q\psi)(x) := x\psi(x)$ and $(P\psi)(x) := \frac{\hbar}{\imath}\frac{\partial\psi}{\partial x}(x)$. It can be outlined that it is related to the standard representation on $\mathcal{S}'(\mathbb{R})$ of the Heisenberg group and that it is unitary on $L^2(\mathbb{R})$. Standard facts about this can be found in the book by Folland [10]. In particular, it can be shown that

$$U_\hbar(x,\xi)U_\hbar(x',\xi') = e^{\frac{\imath}{2\hbar}\sigma\left((x,\xi),(x',\xi')\right)}U_\hbar(x+x',\xi+\xi'). \tag{3}$$

To define the quantum states associated to the phase space, it is reasonable to require a kind of invariance under the translation operators $U_\hbar(q,p)$ for $(q,p) \in \mathbb{Z}^2$. It means that the quantum states will have the same periodicity as the phase space. To do this, we let $\kappa = (\kappa_1, \kappa_2)$ be an element of $[0, 2\pi[^2$ and we require that, for all $(q,p) \in \mathbb{Z}^2$, a quantum state ψ should check the following condition:

$$U_\hbar(q,p)\psi = e^{-\imath\kappa_1 q + \imath\kappa_2 p}\psi.$$

It can be remarked that κ different from 0 is allowed as, for $\alpha \in \mathbb{R}$, ψ and $e^{\imath\alpha}\psi$ represent the same quantum state. The states in $\mathcal{S}'(\mathbb{R})$ that satisfy the previous conditions are said to be the quantum states on the 2-torus and their set is denoted $\mathcal{H}_N(\kappa)$ where N satisfies $2\pi\hbar N = 1$. The following lemma can be shown [5]:

Lemma 2.1. *$\mathcal{H}_N(\kappa)$ is not reduced to 0 iff $N \in \mathbb{N}^*$. In this case, $\dim \mathcal{H}_N(\kappa) = N$. Moreover, for all $(q,p) \in \mathbb{Z}^2$, $U_\hbar(\frac{q}{N}, \frac{p}{N})\mathcal{H}_N(\kappa) = \mathcal{H}_N(\kappa)$ and there is a unique Hilbert structure such that $U_\hbar(\frac{q}{N}, \frac{p}{N})$ is unitary for each $(q,p) \in \mathbb{Z}^2$.*

The Hilbert structure on $\mathcal{H}_N(\kappa)$ is not very explicit [5]. However, one can make it more clear using the following map which defines a surjection of $\mathcal{S}(\mathbb{R})$ (Schwartz functions) onto $\mathcal{H}_N(\kappa)$:

$$S(\kappa) := \sum_{(n,m)\in\mathbb{Z}^2} (-1)^{Nnm} e^{\imath(\kappa_1 n - \kappa_2 m)} U_\hbar(n,m).$$

This projector associates to each state in $\mathcal{S}(\mathbb{R})$ a state which is periodic in position and impulsion. Using it, we can define $|\phi,\kappa\rangle := S(\kappa)|\phi\rangle$ and $|\phi',\kappa\rangle := S(\kappa)|\phi'\rangle$ for $|\phi\rangle$ and $|\phi'\rangle$ in $\mathcal{S}(\mathbb{R})$. Then, the following link between scalar products on $L^2(\mathbb{R})$ and $\mathcal{H}_N(\kappa)$ holds:

$$\langle \kappa, \phi|\phi',\kappa\rangle_{\mathcal{H}_N(\kappa)} = \sum_{n,m\in\mathbb{Z}^2} (-1)^{Nnm} e^{\imath(\kappa_1 n - \kappa_2 m)} \langle \phi|U_\hbar(n,m)|\phi'\rangle_{L^2(\mathbb{R})}. \tag{4}$$

We constructed a family of Hilbert spaces associated to the phase space $\mathbb{T}^2$ for every energy level N. In the case of the torus, classical observables are $\mathcal{C}^\infty$ functions on $\mathbb{T}^2$. One can construct a quantization procedure that associates an operator on each $\mathcal{H}_N(\kappa)$ for every f [5]. This can be achieved by considering the anti-Wick quantization. To describe this quantization, we define the coherent state centered at point $(0,0)$ as

$$|0\rangle(x) := \left(\frac{1}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{x^2}{2\hbar}}.$$

We define the translated coherent state at point $\rho\in\mathbb{R}^2$ as $|\rho\rangle := U_\hbar(\rho)|0\rangle$. This defines a state of $L^2(\mathbb{R})$ which is centered in ρ on a ball of radius $\sqrt{\hbar}$ of the phase space $\mathbb{R}^2$. Using these coherent states, one can construct a positive quantization on the torus mimicking the anti-Wick quantization on $\mathbb{R}^2$ [5]. To do this, we project the coherent states on the Hilbert space $\mathcal{H}_N(\kappa)$:

$$|\rho,\kappa\rangle := S(\kappa)|\rho\rangle.$$

This defines states centered in ρ on a ball of radius $\sqrt{\hbar}$ of the phase space $\mathbb{T}^2$. The anti-Wick quantization of an observable f in $\mathcal{C}^\infty(\mathbb{T}^2)$ is defined as follows[2]:

$$\mathrm{Op}_\kappa^{AW}(f) := \int_{\mathbb{T}^2} f(\rho)|\rho,\kappa\rangle\langle\rho,\kappa|\frac{d\rho}{2\pi\hbar}.$$

It satisfies that for a symbol f, $\mathrm{Op}_\kappa^{AW}(f)^* = \mathrm{Op}_\kappa^{AW}(\overline{f})$ and that the quantization is nonnegative. It also satisfies a resolution of identity property [5]

$$\mathrm{Op}_\kappa^{AW}(1) = \mathrm{Id}_{\mathcal{H}_N(\kappa)} = \int_{\mathbb{T}^2} |\rho,\kappa\rangle\langle\rho,\kappa|\frac{d\rho}{2\pi\hbar}.$$

For the cat-map model, the classical evolution on $\mathbb{T}^2$ is given by a discrete time map A (where A is an hyperbolic matrix in $SL(2,\mathbb{Z})$). As in the case of the hamiltonian flow on a manifold, we would like to quantize the dynamic associated to A on the phase space, i.e. define a quantum progator associated to A. This can be done using the metaplectic representation of $SL(2,\mathbb{R})$ [10]. In fact, it defines for each matrix A the unique (up to a phase) operator which satisfies

$$\forall\rho\in\mathbb{R}^2,\ M(A)U_\hbar(\rho)M(A)^{-1} = U_\hbar(A\rho).$$

$M(A)$ is called the quantum propagator associated to A. It is an unitary operator on $L^2(\mathbb{R})$ and it can be shown [5]:

Lemma 2.2. *For each hyperbolic $A\in Sp(2,\mathbb{Z})$ and each $N\in\mathbb{N}^*$, there exists at least one $\kappa\in[0,2\pi[^2$ such that*

$$M(A)\mathcal{H}_N(\kappa) = \mathcal{H}_N(\kappa).$$

$M_\kappa(A)$ denotes then the restriction of $M(A)$ to $\mathcal{H}_N(\kappa)$. It is an unitary operator.

All these definitions allow to introduce the notion of semiclassical measures for the quantized cat-maps [5]:

[2] One can show that it is related to the Weyl quantization $\|\mathrm{Op}_\kappa^w(f) - \mathrm{Op}_\kappa^{AW}(f)\|_{\mathcal{L}(\mathcal{H}_N(\kappa))} = O_f(N^{-1})$

Definition 2.3. Let A be an hyperbolic matrix in $SL(2,\mathbb{Z})$. We call semiclassical measure of $(\mathbb{T}^2, A)$ any accumulation point of a sequence of measures of the form

$$\forall f \in \mathcal{C}^{\infty}(\mathbb{T}^2,\mathbb{C}),\ \mu^N(f) := \langle \psi^N | \mathrm{Op}_{\kappa}^{AW}(f) | \psi^N \rangle_{\mathcal{H}_N(\kappa)} = \int_{\mathbb{T}^2} f(\rho) N \left| \langle \psi^N | \rho, \kappa \rangle_{\mathcal{H}_N(\kappa)} \right|^2 d\rho,$$

where $(\psi^N)_N$ is a sequence of eigenvectors of $M_\kappa(A)$ in $\mathcal{H}_N(\kappa)$.

The set of semiclassical measure defines a nonempty set of probability measures on the torus $\mathbb{T}^2$. They are A-invariant measures using the following Egorov property:

Theorem 2.4 (Egorov property). *Let A be an hyperbolic matrix in $SL(2,\mathbb{Z})$. For every a in $\mathcal{C}^{\infty}(\mathbb{T}^2)$, one has*

$$\forall t \in \mathbb{R},\ M_\kappa(A)^{-t} Op_{\kappa}^{AW}(f) M_\kappa(A)^t = Op_{\kappa}^{AW}(f \circ A^t) + O_f(e^{-2\lambda_+ t} N),$$

where the constant involved in the remainder depends only on f and λ_+ is the positive Lyapunov exponent of A, i.e. $\lambda_+ := \sup\{\log|\gamma| : \gamma \in Sp(A)\}$.

In other words, this proposition says that the quantum propagation is related to the classical evolution for times under the Ehrenfest time:

$$m_E(N) := \left[\frac{1-\epsilon}{2\lambda_+} \log N \right],$$

where ϵ is some small positive number. In particular, one has that the measures μ^N defined by 2.3 are almost A-invariant until the Ehrenfest time:

Corollary 2.5. *Let $(\psi^N)_N$ be a sequence of eigenvectors of $M_\kappa(A)$ in $\mathcal{H}_N(\kappa)$ and μ^N the associated sequence of measures. Then, for every positive ϵ, one has*

$$(5) \qquad \forall f \in \mathcal{C}^{\infty}(\mathbb{T}^2,\mathbb{C}),\ \forall |t| \leq m_E(N),\ \mu^N(f \circ A^t) = \mu^N(f) + o_{f,\epsilon}(1),$$

where the constant in remainder depends only on f and ϵ.

3. Proof of theorem 1.2

We consider a semiclassical measure μ associated to a sequence of eigenvectors ψ_{N_k} of $M_\kappa(A)$ in $\mathcal{H}_{N_k}(\kappa)$, i.e. $\mu(f) = \lim_{k\to+\infty} \langle \psi_{N_k} | \mathrm{Op}_{\kappa}^{AW}(f) | \psi_{N_k} \rangle_{\mathcal{H}_{N_k}(\kappa)}$. To simplify notations , we will not mention k in the following of this note. We start our proof by fixing a finite measurable partition $\mathcal{Q}$ of small diameter δ ($< 1/100$ for instance)[3] whose boundary is not charged by μ [2]. We denote $\eta(x) := -x\log x$ (with the convention $0\log 0 = 0$). We recall that the Kolmogorov-Sinai entropy of the measure μ for the partition $\mathcal{Q}$ can be defined as [14]

$$h_{KS}(\mu, A, \mathcal{Q}) := \lim_{m\to+\infty} \frac{1}{2m} \sum_{|\alpha|=2m} \eta\left(\mu\left(A^m Q_{\alpha_{-m}} \cdots \cap A^{-(m-1)} Q_{\alpha_{m-1}} \right) \right),$$

where α_j varies in $\{1, \cdots, K\}$ (K is the cardinal of $\mathcal{Q}$).

3.1. Using the entropic uncertainty principle.

Our quantization is defined for smooth observables on the torus. So we start by defining a smooth partition $(P_i)_{i=1}^K$ of observables in $\mathcal{C}^{\infty}(\mathbb{T}^2, [0,1])$ (of small support of diameter less than 2δ) that satisfies the following property of partition of $\mathbb{T}^2$:

$$(6) \qquad \forall \rho \in \mathbb{T}^2,\ \sum_{i=1}^K P_i^2(\rho) = 1.$$

Mimicking the definition of Kolmogorov-Sinai entropy, we define the quantum entropy of ψ_N with respect to $\mathcal{P}$:

$$(7) \qquad h_{2m}(\psi_N, \mathcal{P}) := -\sum_{|\alpha|=2m} \mu^N(\mathbf{P}_\alpha^2) \log \mu^N(\mathbf{P}_\alpha^2),$$

[3] The parameter δ is small and fixed for all the note: it has no vocation to tend to 0.

where $\mathbf{P}_\alpha := \prod_{j=-m}^{m-1} P_{\alpha_j} \circ A^j$ for $\alpha := (\alpha_{-m}, \cdots, \alpha_{m-1})$. One can verify that for fixed m, we have

$$h_{2m}(\mu, \mathcal{P}) := -\sum_{|\alpha|=2m} \mu(\mathbf{P}_\alpha^2) \log \mu(\mathbf{P}_\alpha^2) = \lim_{N\to\infty} h_{2m}(\psi_N, \mathcal{P}). \tag{8}$$

So, for a fixed m, the quantum entropy we have just defined tends as $N \to \infty$ to the usual entropy of μ at time $2m$ (with the notable difference that we consider smooth partitions). Our crucial observation to apply the entropic uncertainty principle is that we have the following partition of identity for $\mathcal{H}_N(\kappa)$:

$$\sum_{|\alpha|=2m} \int_{\mathbb{T}^2} \mathbf{P}_\alpha^2(\rho) |\rho, \kappa\rangle\langle\rho, \kappa| N d\rho = \mathrm{Id}_{\mathcal{H}_N(\kappa)}. \tag{9}$$

The entropic uncertainty principle can be applied for $\mathcal{H} = L^2(\mathbb{T}^2)$ and $\tilde{\mathcal{H}} = \mathcal{H}_N(\kappa)$. For ρ in $\mathbb{T}^2$ and ψ in $\mathcal{H}_N(\kappa)$, we define $\pi_\alpha\psi(\rho) := \sqrt{N} P_\alpha(\rho)\langle\rho, \kappa|\psi\rangle$. This defines a linear application from $\mathcal{H}_N(\kappa)$ to $L^2(\mathbb{T}^2, \mathbb{C})$ and its adjoint is given by $\pi_\alpha^\dagger f := \int_{\mathbb{T}^2} \sqrt{N} \mathbf{P}_\alpha(\rho) f(\rho) |\rho, \kappa\rangle d\rho$, for f in $L^2(\mathbb{T}^2, \mathbb{C})$. It defines a quantum partition of identity as it satisfies the relation $\sum_{|\alpha|=2m} \pi_\alpha^\dagger \pi_\alpha = \mathrm{Id}_{\mathcal{H}_N(\kappa)}$. Applying (2) for this partition and $U = M_\kappa(A)^n$, we bound $\|\pi_\alpha M_\kappa(A)^n \pi_\beta^\dagger\|_{\mathcal{L}(L^2(\mathbb{T}^2,\mathbb{C}))}$ and finally find as a corollary of the Entropic Uncertainty Principle:

Corollary 3.1. *Using the previous notations, one has*

$$\forall n \in \mathbb{N},\ \forall m \in \mathbb{N},\ h_{2m}(\psi_N, \mathcal{P}) \geq -\log \sup_{\rho,\rho' \in \mathbb{T}^2} \{N |\langle\kappa, \rho'| M_\kappa(A)^n |\rho, \kappa\rangle|\} - \log \sup_{|\alpha|=2m} \{Leb(\mathbf{P}_\alpha^2)\}.$$

We underline that the last corollary holds for any integers n and m and our strategy will consist now into optimizing n and m to prove the main theorem.

3.2. **Estimate of the quantum correlation function.** In [8], using relation (4) for the scalar product on $\mathcal{H}_N(\kappa)$, Faure, Nonnenmacher and de Bièvre were able to give an estimate of the quantum correlation function $|\langle\kappa, \rho'| M_\kappa(A)^t |\rho, \kappa\rangle|$ (see also [3]). In fact, using (4), we know that

$$\left|\langle\kappa, \rho'| M_\kappa(A)^t |\rho, \kappa\rangle_{\mathcal{H}_N(\kappa)}\right| \leq \sum_{r\in\mathbb{Z}^2} \left| \left\langle 0 \left| M(A)^{-\frac{t}{2}} U_\hbar \left(r + A^{-\frac{t}{2}}\rho' + A^{\frac{t}{2}}\rho\right) M(A)^{\frac{t}{2}} \right| 0 \right\rangle_{L^2(\mathbb{R})} \right|.$$

From [10] (chapter 4), the metaplectic representation of a matrix D_γ of the form $\begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix}$ is given, for every f in $\mathcal{S}(\mathbb{R})$, by $M(D_\gamma) f(x) = \frac{1}{\sqrt{\gamma}} f(\gamma^{-1}x)$. Combining these two observations, they proved that as long as $|t| \leq \frac{1-\epsilon}{\lambda_+} \log N$, the only term that contributes to the previous sum is the term $(0,0)$, precisely:

Lemma 3.2. *There exists a constant $C > 0$ such that, for $t \leq 2m_E(N)$, one has:*

$$\forall \rho \in \mathbb{T}^2,\ \forall \rho' \in \mathbb{T}^2,\ \left|\langle\kappa, \rho'| M_\kappa(A)^t |\rho, \kappa\rangle_{\mathcal{H}_N(\kappa)}\right| \leq C e^{-\frac{\lambda_+ t}{2}}.$$

As mentioned above, the strategy to prove this lemma essentially consists in estimating the scalar products of two Gaussian states (one is spread in the unstable direction and the other one in the stable direction). Then, taking $n = 2m_E(N)$ in corollary 3.1, we find that there exists a constant C such that the quantum entropy at time $2m$ is bounded from below as follows:

$$\forall m \geq 1,\ h_{2m}(\psi_N, \mathcal{P}) \geq -\log N + \lambda_+ m_E(N) - \log \sup_{|\alpha|=2m} \{\mathrm{Leb}(\mathbf{P}_\alpha^2)\} + C \tag{10}$$

This estimate is our main simplification compared with [2]. We underline that lemma 3.2 plays a crucial role in our proof[4] as it replaces all the discussion of section 3 in [2].

[4]The other term on the lower bound will be estimate thanks to the computation of the entropy of the Lebesgue measure.

3.3. **Subadditivity of the quantum entropy.** Now, we have to find a time m for which previous inequality is optimal. It will depend on N and the last difficulty is that if $m(N)$ grows too fast with N, $h_{2m(N)}(\psi_N,\mathcal{P})$ has no particular reason to tend to $h_{KS}(\mu,A)$ in the semiclassical limit. We have to be careful and we first verify that classical arguments from ergodic theory (subadditivity of the entropy) can be adapted for the quantum entropy as long as $m\leq \log N/(2\lambda_+)$. In particular, we prove that the sequence $\frac{1}{2m_0}h_{2m_0}(\psi_N,\mathcal{P})$ is 'almost' decreasing until the Ehrenfest time (see paragraph 3.5):

Lemma 3.3. *We fix an integer* $m_0\leq m_E(N)$. *We have then*

$$\frac{1}{2m_E(N)}h_{2m_E(N)}(\psi_N,\mathcal{P})\leq \frac{1}{2m_0}h_{2m_0}(\psi_N,\mathcal{P})+R(m_0,N),$$

where $R(m_0,N)$ *is a remainder that satisfies* $\forall m_0\in\mathbb{N},\ \lim_{N\to\infty}R(m_0,N)=0$.

Combining this lemma with the entropic estimation (10), we have, for every fixed $m_0>0$,

$$\frac{1}{2m_0}h_{2m_0}(\psi_N,\mathcal{P})+\tilde{R}(m_0,N)\geq -(1+\epsilon)\frac{\lambda_+}{2}-\frac{1}{2m_E(N)}\log \sup_{|\alpha|=2m_E(N)}\{\mathrm{Leb}(\mathbf{P}_\alpha^2)\}. \tag{11}$$

where $\tilde{R}(m_0,N)$ is a remainder that satisfies $\forall m_0\in\mathbb{N},\ \lim_{N\to\infty}\tilde{R}(m_0,N)=0$.

3.4. **The conclusion.** To conclude, it remains to bound the quantity $\sup_{|\alpha|=2m_E(N)}\{\mathrm{Leb}(\mathbf{P}_\alpha^2)\}$. To do this, we underline that, for each α of length $2m$,

$$\forall x\in\mathrm{supp}(\mathbf{P}_\alpha^2),\ \mathrm{Leb}(\mathbf{P}_\alpha^2)\leq \mathrm{Leb}(\mathrm{supp}(\mathbf{P}_\alpha^2))\leq \mathrm{Leb}(B(x,2\delta,2m)),$$

where $B(x,2\delta_0,2m):=\{y\in\mathbb{T}^2:\forall j\in[-m,m-1],\ d(A^jx,A^jy)<2\delta_0\}$, where d is the metric induced on $\mathbb{T}^{2d}$ by the Euclidean norm on $\mathbb{R}^{2d}$. By induction and using the invariance of the metric d, we know that for every x in $\mathbb{T}^{2d}$ and for every k in $\mathbb{Z}$, $A^{-k}B(A^kx,2\delta_0)=x+A^{-k}B(0,2\delta_0)$. Then, using the invariance by translation of the Lebesgue measure, we know that for every x in $\mathbb{T}^{2d}$, $\mathrm{Leb}(B(x,2\delta_0,2m))=\mathrm{Leb}(B(0,2\delta_0,2m))$. Combining [6] and theorem 8.15 from [14], we know that $\mathrm{Leb}(B(0,2\delta_0,2m))\leq C_{\delta_0}e^{-2m(\Lambda_+-\epsilon)}$. We use this last inequality and we make N tends to infinity in (11). It gives, for every positive m_0,

$$\frac{1}{2m_0}h_{2m_0}(\mu,\mathcal{P})\geq \frac{\lambda_+}{2}(1-2\epsilon).$$

This last inequality holds for all (small enough) smoothing $\mathcal{P}$ of the partition $\mathcal{Q}$. The lower bound does not depend on the derivatives of $\mathcal{P}$ so we we can replace the smooth partition $\mathcal{P}$ by the true partition $\mathcal{Q}$ in the definition of $h_{2m_0}(\mu,\mathcal{P})$. We let m_0 tends to infinity and then ϵ to 0 in order to find

$$h_{KS}(\mu,A)\geq h_{KS}(\mu,A,\mathcal{Q})\geq\frac{\lambda_+}{2}.\square$$

3.5. **Proof of lemma 3.3.** To complete the proof of the previous paragraph, it remains to prove lemma 3.3. To prove this lemma, we use classical properties of the entropy of a partition [14] (chapter 4) that we briefly prove here (see theorem 4.3 and 4.9 in [14] for details). We fix three integers p, n and m. To simplify our notations, we define the p-translated entropy as follows:

$$h_{2m}^p(\psi_N,\mathcal{P}):=\sum_{|\alpha|=2m}\eta\left(\mu^N(\mathbf{P}_\alpha^2\circ A^p)\right).$$

Mimicking the usual proof for the subadditivity of the entropy of a partition [14] (chapter 4), we write

$$\begin{aligned} h_{2(n+m)}^p(\psi_N,\mathcal{P}) &= -\textstyle\sum_{|\alpha|=2(n+m)}\mu^N\left(\prod_{j=-m-n}^{n+m-1}P_{\alpha_j}^2\circ A^{j+p}\right)\log\mu^N\left(\prod_{j=-m+n}^{m+n-1}P_{\alpha_j}^2\circ A^{j+p}\right)\\ &+ \textstyle\sum_{|\alpha|=2(n+m)}\eta\left(\frac{\mu^N\left(\prod_{j=-m-n}^{m+n-1}P_{\alpha_j}^2\circ A^{j+p}\right)}{\mu^N\left(\prod_{j=-m+n}^{m+n-1}P_{\alpha_j}^2\circ A^{j+p}\right)}\right)\mu^N\left(\prod_{j=-m+n}^{m+n-1}P_{\alpha_j}^2\circ A^{j+p}\right). \end{aligned}$$

Using the concavity of the function η and the property of partition of identity (6), we can write the following inequality:

$$h^p_{2(n+m)}(\psi_N,\mathcal{P}) \leq \sum_{|\alpha|=2m} \eta\left(\mu^N\left(\prod_{j=-m+n}^{m+n-1} P^2_{\alpha_j}\circ A^{j+p}\right)\right) + \sum_{|\alpha|=2n} \eta\left(\mu^N\left(\prod_{j=-m-n}^{-m+n-1} P^2_{\alpha_j}\circ A^{j+p}\right)\right).$$

Under a more compact form, it can be reformulated as follows:

Lemma 3.4. *Using previous notations, one has*

$$\forall p\in\mathbb{N},\ \forall n\geq 0,\ \forall m\geq 0,\ h^p_{2(n+m)}(\psi_N,\mathcal{P}) \leq h^{n+p}_{2m}(\psi_N,\mathcal{P}) + h^{-m+p}_{2n}(\psi_N,\mathcal{P}). \tag{12}$$

We fix now two integers $m_0 < m$ and write the Euclidean division $m = qm_0 + r$ where $0 \leq r < m_0$. We use inequality (12) to derive

$$h_{2m}(\psi_N,\mathcal{P}) \leq h^r_{2qm_0}(\psi_N,\mathcal{P}) + h^{-qm_0}_{2r}(\psi_N,\mathcal{P}).$$

We apply one more time inequality (12) to find

$$h_{2m}(\psi_N,\mathcal{P}) \leq h^{r+m_0}_{2(q-1)m_0}(\psi_N,\mathcal{P}) + h^{-(q-1)m_0+r}_{2m_0}(\psi_N,\mathcal{P}) + h^{-qm_0}_{2r}(\psi_N,\mathcal{P}).$$

By induction, we finally have the following corollary:

Corollary 3.5. *Using previous notations, one has*

$$h_{2m}(\psi_N,\mathcal{P}) \leq h^{-qm_0}_{2r}(\psi_N,\mathcal{P}) + \sum_{j=1}^{q} h^{-(q+1-2j)m_0+r}_{2m_0}(\psi_N,\mathcal{P}). \tag{13}$$

Proof of lemma 3.3. This last inequality is true for any integers (m, m_0, r) satisfying $m = qm_0 + r$. We can now give the proof of lemma 3.3. To do this, we fix a positive integer m_0 and consider (q, r) in $\mathbb{N}\times\mathbb{N}$ satisfying $qm_0 + r = m_E(N)$ where $0 \leq r < m_0$. Recall that according to Egorov property (proposition 2.5), one has, for every a in $\mathcal{C}^\infty(\mathbb{T}^2)$,

$$\forall\ |t| \leq m_E(N),\ \mu^N\left(a\circ A^t\right) = \mu^N(a) + o_a(1),\ \text{as } N\to+\infty.$$

We underline that the remainder tends to 0 uniformly for t in the allowed interval. We now apply this property to $\mathbf{P}^2_\alpha$ where $|\alpha| = 2m_0$. Using the continuity of η, we find that

$$\forall\ |t| \leq m_E(N),\ \eta\left(\mu^N\left(\mathbf{P}^2_\alpha\circ A^t\right)\right) = \eta\left(\mu^N(\mathbf{P}^2_\alpha)\right) + o_\alpha(1),\ \text{as } N\to+\infty.$$

As m_0 is fixed, we can deduce from the definition of $h^p_{2m_0}(\psi_N,\mathcal{P})$ that

$$\forall\ |p| \leq m_E(N),\ h^p_{2m_0}(\psi_N,\mathcal{P}) = h_{2m_0}(\psi_N,\mathcal{P}) + o_{m_0}(1),\ \text{as } N\to+\infty.$$

We can apply this result in inequality (13). In this case, one has that $p = -(q+1-2j)m_0+r$ belongs to $[-m_E(N), m_E(N)]$. As $|qm_0| \leq m_E(N)$, we can also write $h^{-qm_0}_{2r}(\psi_N,\mathcal{P}) = h_{2r}(\psi_N,\mathcal{P}) + o_r(1)$ as N tends to infinity. Finally, we find that

$$h_{2m_E(N)}(\psi_N,\mathcal{P}) \leq h_{2r}(\psi_N,\mathcal{P}) + qh_{2m_0}(\psi_N,\mathcal{P}) + (q+1)R'(m_0,N),$$

where $R'(m_0,N)$ is a remainder that satisfies $\forall m_0\in\mathbb{N},\ \lim_{N\to\infty} R'(m_0,N) = 0$. The conclusion of the lemma follows from this last statement.□

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