A very good triple of operads

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$\mathbb{K}$ is a field of characteristic 0, and $\otimes$ stands for $\otimes_{\mathbb{K}}$. Σ_n denotes the symmetric group, that is the group of permutations of the ordered set $\{1 < 2 < \cdot < n\}$. Given a vector space V, $V^{\otimes n}$ is a representation of Σ_n via

$$\begin{array}{rcl}\phi : \Sigma_n & \to & \mathrm{Aut}_{\mathbb{K}}(V^{\otimes n}) \\ \sigma & \mapsto & (v_1 \otimes \cdots \otimes v_n \mapsto v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)})\end{array}$$

and $\phi(\sigma)$ will always be replaced by σ in our formulas. $\bar{T}V$ will stand for the reduced tensor vector space

$$\bar{T}V := \bigoplus_{n \geqslant 1} V^{\otimes n}$$

Every operad A will be assumed to verify $A(0) = 0$ and $A(1) = \mathbb{K}\mathrm{Id}$. *As*, *Com*, *Leib*, and *Zinb* will stand respectively for the operads encoding associative algebras, commutative algebras, right Leibniz algebras, and left Zinbiel algebras.

1 The triple $(Zinb, Leib, Vect)$

In [Petit Livre Bleu], J-L Loday explores the notion of generalized bialgebra, and gives several examples of such bialgebras. A type of bialgebra is the datum of

- Two operads C and A,
- A compatibility relation $\between$ between each generating operation of A and each generating cooperation of C^c, the cooperad associated to C.

A $(C^c, A, \between)$-bialgebra is then a $\mathbb{K}$-module H, endowed with a C-coalgebra structure, an A-algebra structure, and such that the compatibility relation between any n-ary operation and any m-ary cooperation holds when applied to any n-uple of elements of H.

Examples 1.0.1.

- *Non-unital associative and coassociative Hopf algebras are bialgebras of type* $(As^c, As, \between_{nuHopf})$ *with* $\between_{nuHopf}$ *given by*

$$\delta \circ \mu = \mathrm{Id} \otimes \mathrm{Id} + (12) + (\mathrm{Id} \otimes \mu) \circ (\delta \otimes \mathrm{Id}) + (\mu \otimes \mathrm{Id}) \circ (23) \circ (\delta \otimes \mathrm{Id}) + (\mu \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \delta) + (\mathrm{Id} \otimes \mu) \circ (12) \circ (\mathrm{Id} \otimes \delta) + (\mu \otimes \mu) \circ (23) \circ (\delta \otimes \delta)$$

  where δ *is the generating binary cooperation of* Com^c *and* μ *is the generating binary operation of* As. *Remark that any unital and counital Hopf algebra* $(H, \mu, \Delta, \eta : \mathbb{K} \to H, \varepsilon : H \to \mathbb{K})$ *for which the compatibility relation takes the usual form*

$$\Delta \circ \mu = (\mu \otimes \mu) \circ (23) \circ (\Delta \otimes \Delta)$$

  gives rise to a non-unital one by considering its augmentation ideal $\mathrm{Ker}(\varepsilon)$ *equipped with the reduced coproduct* δ *defined by* $\delta(x) := \Delta(x) - x \otimes 1 - 1 \otimes x$, *for all* x *in* $\mathrm{Ker}(\varepsilon)$.
- *Cocommutative associative Hopf algebras, coassociative commutative Hopf algebras, and cocommutative commutatives Hopf algebras are respectively bialgebras of type* (Com^c, As), (As^c, Com) *and* (Com^c, Com), *when one chooses as compatibility relation the non-unital Hopf relation cited above.*
- *A* $(Zinb^c, As, \between_{nusHopf})$*-bialgebra is a* $\mathbb{K}$*-module* H *equipped with an associative multiplication* $mu : H \otimes H \to H$ *and a Zinbiel coproduct* $\delta : H \to H \otimes H$ *such that the following non-unital semi Hopf relation holds* $\between_{nusHopf}$ *:*

$$\delta \circ \mu = (12) + (\mu \otimes \mathrm{Id}) \circ \otimes (\mathrm{Id} \otimes \delta) + (\mu \otimes \mathrm{Id}) \circ (23) \circ (\delta^{com} \otimes \mathrm{Id}) + (\mathrm{Id} \otimes \mu) \circ (12) \circ (\mathrm{Id} \otimes \delta) + (\mu \otimes \mu) \circ (23) \circ (\delta^{com} \otimes \delta)$$

where δ^{com} is the symmetrized coproduct defined by $\delta^{com} = \delta + (12) \circ \delta$.
This example is due to E. Burgunder [E. Burgunder, A symmetric version of Kontsevich graph complex and Leibniz homology, ArXiv : mathQA/0804.2052] and, to our knowledge, is the only one involving the operad Zinb present in the litterature.

Definition 1.0.2. [Petit Livre Bleu] *Let H be a $(C^c, A, \between)$-bialgebra.*

1. *Its **primitive part**, denoted by* Prim$H$ *is the submodule of elements x of H such that*
$$\delta(x) = 0$$
for any cooperation δ in $C^c(n)$, $n \geqslant 2$.
2. *Let F_rH be defined by $F_rH := \{x \in H, \delta(x) = 0, \delta \in C^c(n), n > r\}$. H is said to be **connected** if*
$$H = \cup_{r\geqslant 1} F_r H$$
3. *The compatibility relation $\between$ is **distributive** if for any m-ary cooperation δ in $C^c(m)$, and any n-ary operation μ in $A(n)$, it takes the form*
$$\delta \circ \mu = \sum_{i\in I} (\mu_1^i \otimes \cdots \otimes \mu_m^i) \circ \omega^i \circ (\delta_1^r \otimes \cdots \otimes \delta_n^i$$
where I is a finite set of indices, and for any i in I
$$\begin{cases} \mu_1^i \in A(k_1), \cdots, \mu_m^i \in A(k_m) \\ \delta_1^i \in C^c(l_1), \cdots, \delta_n^i \in C^c(l_n) \\ k_1 + \cdots + k_m = l_1 + \cdots + l_n = r_i \\ \omega^i \in \mathbb{K}[\Sigma_{r_i}] \end{cases}$$

It is well known that connected $(Com^c, Com, \between_{nuHopf})$-bialgebras are both free and cofree over their primitive part : this is Hopf-Borel's theorem. This result can be seen as a particular case of the following rigidity theorem :

Theorem 1.0.3. [Loday, Petit livre bleu] *Let $(C^c, A, \between)$ be a type of generalized bialgebra and suppose that the following hypothesis hold*

(H0) For any operation μ and any cooperation δ, there is a distributive compatibility relation,

(H1) For any $\mathbb{K}$-module V, the free A-algebra $A(V)$ is naturally equipped with a $(C^c, A, \between)$-bialgebra structure,

(H2iso) The natural C^c-coalgebra map $\varphi : A(V) \to C^c(V)$ lifting the projection on the length one summand $A(V) \to V = A(1) \otimes V$ is an isomorphism.

Then any connected $(C^c, A, \between)$-bialgebra H is both free and cofree over its primitive part i.e.
$$A(\mathrm{Prim}H) \cong H \cong C^c(\mathrm{Prim}H)$$
where the first isomorphism is an isomorphism of A-algebras and the second one is an isomorphism of C^c-coalgebras.

Examples 1.0.4. $(Com^c, Com, \between_{nuHopf})$ *and* $(Zinb^c, As, \between_{nusHopf})$ *both satisfy hypotheses* $(H0)$, $(H1)$ *and* $(H2iso)$.

We are now going to introduce a new type of bialgebras involving the Zinbiel operad.

Definition 1.0.5. *A $(Zinb^c, Leib, \between_{ZL})$-bialgebra is a $\mathbb{K}$-vector space H endowed with a Zinbiel coproduct $\delta : H \otimes H \otimes H$ and a right Leibniz bracket $\nu : H \otimes H \to H$ such that the following compatibility relation*
$$\delta \circ \nu = \mathrm{Id} \otimes \mathrm{Id} + (\mathrm{Id} \otimes \nu) \circ (\delta \otimes \mathrm{Id}) + (\nu \otimes \mathrm{Id}) \circ (23) \circ (\delta \otimes \mathrm{Id}) - \mathrm{Id} \otimes (\nu \circ \delta) \qquad \between_{ZL}$$
holds.

Proposition 1.0.6. *The free Leibniz algebra $Leib(V)$ is naturally equipped with a $(Zinb^c, Leib, \between_{ZL})$-bialgebra structure.*

Proof. Clearly, $Leib(V) = \bar{T}V = Zinb^c(V)$ as vector spaces which shows that $Leib(V)$ is endowed with a Zinbiel coalgebra structure $\delta : Leib(V) \to Leib(V) \otimes Leib(V)$. So all we have to check is whether relation $()_{ZL}$ is indeed satisfied. Denote by $[a, b]$ the Leibniz bracket $\nu(a \otimes b)$ of two elements a and b of $Leib(V)$. In Sweedler's notation, relation $()_{ZL}$ reads

$$\delta[a, b] = a \otimes b + [a_{(1)}, b] \otimes a_{(2)} + a_{(1)} \otimes [a_{(2)}, b] - a \otimes [b_{(2)}, b_{(1)}] \tag{1}$$

Let's proove that this relation holds by induction on $|b|$, the length and b :

- For $|b| = 1$, i.e. $b \in V$, (1) holds by definition of the cofree half-shuffle zinbiel coproduct δ and because the last term $a \otimes [b_{(2)}, b_{(1)}]$ vanishes since $\delta(b) = 0$.
- For $|b| > 1$. Assume that $()_{ZL}$ holds for any b' of length strictly lower than $|b|$ and for any a. We can suppose that b is of the form

$$b = [b', v] \quad ,: |b'| = |b| - 1, \ v \in V$$

Thus

$$\begin{aligned}\delta[a, b] =& \delta([a, [b', v]]) \\ =& \delta([[a, b'], v]) - \delta([[a, v], b']) \\ =& [[a, b']_{(1)}, v] \otimes [a, b']_{(2)} + [a, b']_{(1)} \otimes [[a, b']_{(2)}, v] + [a, b'] \otimes v - \delta([[a, v], b'])\end{aligned}$$

But the induction hypothesis allows us to express $[a, b']_{(1)} \otimes [a, b']_{(2)} = \delta([a, b'])$ and $\delta[[a, v], b']$ respectively as

$$[a, b']_{(1)} \otimes [a, b']_{(2)} = a \otimes b' + [a_{(1)}, b'] \otimes a_{(2)} + a_{(1)} \otimes [a_{(2)}, b'] - a \otimes [b'_{(2)}, b'_{(1)}]$$

and, using again $()_{ZL}$ to rewrite $\delta[a, v]$,

$$\begin{aligned}\delta([[a, v], b']) =& [a, v] \otimes b' + [[a, v]_{(1)}, b'] \otimes [a, v]_{(2)} + [a, v]_{(1)} \otimes [[a, v]_2, b'] - [a, v] \otimes [b'_{(2)}, b'_{(1)}] \\ =& [a, v] \otimes b' + [[a_{(1)}, v], b'] \otimes a_{,(2)} + [a_{(1)}, b'] \otimes [a_{(2)}, v] + [a, b'] \otimes v \\ & + [a_{(1)}, v] \otimes [a_{(2)}, b'] + a_{(1)} \otimes [[a_{(2)}, v], b'] + a \otimes [v, b'] - [a, v] \otimes [b'_{(2)}, b'_{(1)}]\end{aligned}$$

so that $\delta[a, b]$ takes the form

$$\begin{aligned}\delta[a, b] =& [a, v] \otimes b' + [[a_{(1)}, b'], v] \otimes a_{(2)} + [a_{(1)}, v] \otimes [a_{(2)}, b'] - [a, v] \otimes [b'_{(2)}, b'_{(1)}] \\ & + a \otimes [b', v] + [a_{(1)}, b'] \otimes [a_{(2)}, v] + a_{(1)} \otimes [[a_{(2)}, b'], v] - a \otimes [[b'_{(2)}, b'_{(1)}], v] + [a, b'] \otimes v \\ & - [a, v] \otimes b' - [[a_{(1)}, v], b'] \otimes a_{,(2)} - [a_{(1)}, b'] \otimes [a_{(2)}, v] - [a, b'] \otimes v \\ & - [a_{(1)}, v] \otimes [a_{(2)}, b'] - a_{(1)} \otimes [[a_{(2)}, v], b'] - a \otimes [v, b'] + [a, v] \otimes [b'_{(2)}, b'_{(1)}] \\ =& a \otimes [b', v] + \overbrace{[[a_{(1)}, b'], v] \otimes a_{(2)} - [[a_{(1)}, v], b'] \otimes a_{(2)}}^{A} + \overbrace{a_{(1)} \otimes [[a_{(2)}, b'], v] - a_{(1)} \otimes [[a_{(2)}, v], b']}^{B} \\ & - a \otimes [[b'_{(2)}, b'_{(1)}], v] - a \otimes [v, b']\end{aligned}$$

using the right Leibniz relation satisfied by $[-, -]$ to simplify terms A and B in the above expression, and remembering that $[b', v] = b$ leads to

$$\delta[a, b] = a \otimes b + [a_{(1)}, b] \otimes a_{(2)} + a_{(1)} \otimes [a_{(2)}, b] - a \otimes [[b'_{(2)}, b'_{(1)}], v] - a \otimes [v, b'] \tag{2}$$

Now remark that, applying once again the induction hypothesis gives

$$b_{(1)} \otimes b_{(2)} = \delta(b) = \delta[b', v] = b' \otimes v + [b'_{(1)}, v] \otimes b'_{(2)} + b'_{(1)} \otimes [b'_{(2)}, v]$$

Thus, using the Leibniz relation to get the second equality,

$$[b_{(2)}, b_{(1)}] = [v, b'] + [b'_{(2)}, [b'_{(1)}, v]] + [[b'_{(2)}, v], b'_{(1)}] = [v, b'] + [[b'_{(2)}, b'_{(1)}], v]$$

Which enables us to rewrite equation (2) as

$$\delta[a, b] = a \otimes b + [a_{(1)}, b] \otimes a_{(2)} + a_{(1)} \otimes [a_{(2)}, b] - a \otimes [b_{(2)}, b_{(1)}]$$

i.e. (1) is satisfied by a and b.

This shows that if $()_{ZL}$ holds for any b' such that $|b'| < |b|$ and any a, it holds also for b and any a. Thus $()_{ZL}$ has to hold any elements a and b of $Leib(V)$.

□

As a direct consequence of this proposition, we have the following

Theorem 1.0.7. *The type of bialgebra* $(Zinb^c, Leib, \between_{ZL})$ *satisfies hypotheses* $(H0)$*,* $(H1)$ *and* $(H2iso)$ *of the rigidity theorem 1.0.3. Thus any connected* $(Zinb^c, Leib, \between_{ZL})$*-bialgebra is both free and cofree over its primitive part.*

Proof. It's clear that $\between_{ZL}$ is a distributive compatibility relation so $(H0)$ is fulfilled. The fact that $(H1)$ holds is exactly the content of proposition 1.0.6, and hypothesis $(H2)$ is obviously satisfied since the Zinbiel coproduct defined on $Leib(V) = \bar{T}V$ is by definition the cofree one. □

2 Leibniz dialgebras and the triple $(Zinb, Leib^2, Leib)$

Definition 2.0.8. *A* ***Leibniz dialgebra*** *(or* $Leib^2$*-algebra) is a* $\mathbb{K}$*-module* L *endowed with two linear brackets* $(-,-) : L \otimes L \to L$ *and* $\{-,-\} : L \otimes L \to L$ *such that*

a) $(-,-)$ *is a right Leibniz bracket i.e.*

$$((a,b),c) = ((a,c),b) + (a,(b,c))$$

b) $\{-,-\}$ *is a left Leibniz bracket i.e.*

$$\{a,\{b,c\}\} = \{\{a,b\},c\} + \{b,\{a,c\}\}$$

c) $\{-,-\}$ *is a left derivation of* $(-,-)$ *i.e.*

$$\{a,(b,c)\} = (\{a,b\},c) + (b,\{a,c\})$$

d) $(-,-)$ *is a right derivation for* $\{-,-\}$ *i.e.*

$$(\{a,b\},c) = \{(a,c),b\} + \{a,(b,c)\}$$

for all a*,* b *and* c *in* A*.*

Proposition 2.0.9. *Let* $(L,(-,-),\{-,-\})$ *be a Leibniz dialgebra. Then the linear map* $[-,-] : L \otimes L \to L$ *defined by*

$$[a,b] := (a,b) - \{b,a\}$$

for all a *and* b *in* L *is a right Leibniz bracket.*

Proof. Let a b and c be three elements of L. Then,

$$\begin{aligned}
[[a,b],c] =& ([a,b],c) - \{c,[a,b]\} \\
=& ((a,b),c) - (\{b,a\},c) - \{c,(a,b)\} + \{c,\{b,a\}\} \\
\overset{*}{=}& ((a,c),b) + (a,(b,c)) - \{(b,c),a\} - \{b,(a,c)\} - (\{c,a\},b) - (a,\{c,b\}) + \{\{c,b\},a\} + \{b,\{c,a\}\} \\
=& ([a,c],b) + (a,[b,c]) - \{[b,c],a\} - \{b,[a,c]\} \\
=& [[a,c],b] + [a,[b,c]]
\end{aligned}$$

where the equality $*$ is a direct consequence of relations $a)$, $b)$, $c)$ and $d)$ of definition 2.0.8.

□

Leibniz dialgebras are algebras over an operad we denote by $Leib^2$. This operad is the quotient of $Free(\vdash,\dashv)$, the free operad on two binary generators $\vdash$ and $\dashv$, by the operadic ideal (R) generated the following four relators

$$\begin{aligned}
&(R1) \quad \vdash \circ_1 \vdash - \vdash \circ_1 \vdash (23) - \vdash \circ_2 \vdash, \\
&(R2) \quad \dashv \circ_2 \dashv - \dashv \circ_1 \dashv - \dashv \circ_2 \dashv (12), \\
&(R3) \quad \vdash \circ_1 \dashv - \dashv \circ_1 \vdash (23) - \dashv \circ_2 \vdash, \\
&(R4) \quad \dashv \circ_2 \vdash - \vdash \circ_1 \dashv - \vdash \circ_2 \dashv (12).
\end{aligned}$$

Theorem 2.0.10. *Let V be a $\mathbb{K}$-module and denote by $Leib^2(V)$ the free Leibniz dialgebra generated by V. Then*

$$Leib^2(V) \cong \bar{T}\bar{T}V$$

as a $\mathbb{K}$-module.

Proof. First, notice that the operadic ideal (R) genreated by the four relators $(R1)$, $(R2)$, $(R3)$ and $(R4)$ is also generated by the four following ones

$$\begin{array}{ll}(R1) & \vdash \circ_1 \vdash - \vdash \circ_1 \vdash (23) - \vdash \circ_2 \vdash, \\ (R2) & \dashv \circ_2 \dashv - \dashv \circ_1 \dashv - \dashv \circ_2 \dashv (12), \\ (R3') & \dashv \circ_1 \vdash + \vdash \circ_2 \dashv (123). \\ (R4) & \dashv \circ_2 \vdash - \vdash \circ_1 \dashv - \vdash \circ_2 \dashv (12).\end{array}$$

Choosing as leading terms respectively $\vdash \circ_2 \vdash$, $\dashv \circ_1 \dashv$, $\dashv \circ_1 \vdash$ and $\dashv \circ_2 \vdash$ leads to the four following rewritting rules :

$$\begin{array}{llll}(R1) & \vdash \circ_2 \vdash & \longmapsto & \vdash \circ_1 \vdash - \vdash \circ_1 \vdash (23), \\ (R2) & \dashv \circ_1 \dashv & \longmapsto & \dashv \circ_2 \dashv - \dashv \circ_2 \dashv (12), \\ (R3') & \dashv \circ_1 \vdash & \longmapsto & - \vdash \circ_2 \dashv (123). \\ (R4) & \dashv \circ_2 \vdash & \longmapsto & \vdash \circ_1 \dashv + \vdash \circ_2 \dashv (12).\end{array}$$

One has to check that any critical monomial is confluent. □

Corollary 2.0.11. *$Leib^2$ is a Koszul operad.*

Definition 2.0.12. *A $(Zinb^c, Leib^2, \between_{ZL^2})$-bialgebra is a vector space H endowed with a Zinbiel coproduct $\delta : H \to H \otimes H$ and a $Leib^2$-algebra structure given by brackets $(-,-) : L \otimes L \to L$ and $\{-,-\} : L \otimes L \to L$ such that*

- *$(H, \delta, (-,-))$ is a $(Zinb^c, Leib, \between_{ZL^2})$-bialgebra i.e. δ and $(-,-)$ satisfy relation $\between_{ZL}$ of definition 1.0.5,*
- *$(H, \delta, \{-,-\} \circ (12))$ is also a $(Zinb^c, Leib, \between_{ZL^2})$-bialgebra.*

Proposition 2.0.13. *The free Leibniz dialgebra $Leib^2(V)$ generated by a vector space V is naturally equipped with a Zinbiel coproduct $\delta : Leib^2(V) \to Leib^2(V)^{\otimes 2}$ which turns it into a $(Zinb^c, Leib^2, \between_{ZL^2})$-bialgebra.*

Proof. Set $\delta(v) = 0$ for any v in V and note that δ is then fully determined by induction on the length of monomials since if it exists, it has to satisfy $\between_{ZL^2}$. Moreover, $\between_{ZL^2}$ indeed well-defines a linear map δ on $Leib^2(V)$, because proposition 2.0.10 implies that we have a PBW basis of $Leib^2(V)$ consisting of monomials of the form

$$(\{v_1^1, \cdots, v_{i_1}^1\}, \{v_1^2, \cdots, v_{i_2}^2\}, \cdots, \{v_1^n, \cdots, v_{i_n}^n\})$$

where n ranges over all positive numbers and the v_j^k run over some fixed basis of V, where $(a_1, \cdots, a_n)$ stands for the right-iterated bracket

$$(((\cdots(a_1, a_2), a_3), \cdots), a_n)$$

and $\{v_1, \cdots, v_k\}$ for the iterated left-bracket

$$\{v_1, \{v_2, \{\cdots, \{v_{k-1}, v_k\} \cdots\}\}\}$$

Obviously, the naturality of δ comes from the fact that its definition doesn't depend on the choice of basis of V we are using.

Let us prove that δ is a left-Zinbiel coproduct, i.e. that it satisfies

$$(\delta \otimes \mathrm{Id})\delta = (\mathrm{Id} \otimes \delta^{com}) \circ \delta \tag{3}$$

where, as usual, $\delta^{com} := \delta + (12) \circ \delta$.

This we do one more time by induction on length, using the natural grading of $Leib^2(V)$ given by the PBW basis for which $(\{v_1^1, \cdots, v_{i_1}^1\}, \{v_1^2, \cdots, v_{i_2}^2\}, \cdots, \{v_1^n, \cdots, v_{i_n}^n\})$ has length $i_1 + \cdots + i_n$.

– Clearly, (3) holds on elements of V.
– Assume that (3) holds on any monomial of length strictly lower than n. Any PBW-monomial a of length n is of the form

$$a = (a', w) \quad , |a'| = n - p < n - 1,\ w = \{v_1, \cdots, v_p\},\ v_1, \cdots,\ v_p \in V$$

or of the form

$$b = \{v, b'\} \quad , b' = \{v_1, \cdots, v_{n-1}\},\ v,\ v_1, \cdots,\ v_{n-1} \in V$$

But, since δ is defined so that it satisfies $\between_{ZL^2}$ on (a', w) and on $\{v, b'\}$,

$$\delta(a', w) = a' \otimes w + (a'_{(1)}, w) \otimes a'_{(2)} + a'_{(1)} \otimes (a'_{(2)}, w) - a' \otimes (w_{(2)}, w_{(1)})$$

and

$$\delta\{v, b'\} = b' \otimes v + \{v, b'_{(1)}\} \otimes b'_{(2)} + b'_{(1)} \otimes \{v, b'_{(2)}\}$$

Applying $\delta \otimes \mathrm{Id}$ to both equations and using relation $\between_{ZL^2}$ again gives respectively

$$\begin{aligned}(\delta \otimes \mathrm{Id}) \circ \delta(a', w) =& a'_{(1)} \otimes a'_{(2)} \otimes w + (a'_{(1)}, w)_{(1)} \otimes (a'_{(1)}, w)_{(2)} \otimes a'_{(2)} \\ &+ a'_{(1)_{(1)}} \otimes a'_{(1)_{(2)}} \otimes (a'_{(2)}, w) - a'_{(1)} \otimes a'_{(2)} \otimes (w_{(2)}, w_{(1)}) \\ =& a'_{(1)} \otimes a'_{(2)} \otimes w + a'_{(1)} \otimes w \otimes a'_{(2)} + (a'_{(1)_{(1)}}, w) \otimes a'_{(1)_{(2)}} \otimes a'_{(2)} + a'_{(1)_{(1)}} \otimes (a'_{(1)_{(2)}}, w) \otimes a'_{(2)} \\ &- a'_{(1)} \otimes (w_{(2)}, w_{(1)}) \otimes a'_{(2)} + a'_{(1)_{(1)}} \otimes a'_{(1)_{(2)}} \otimes (a'_{(2)}, w) - a'_{(1)} \otimes a'_{(2)} \otimes (w_{(2)}, w_{(1)})\end{aligned}$$

and

$$\begin{aligned}(\delta \otimes \mathrm{Id}) \circ \delta\{v, b'\} =& b'_{(1)} \otimes b'_{(2)} \otimes v + \{v, b'_{(1)}\}_{(1)} \otimes \{v, b'_{(1)}\}_{(2)}\} \otimes b'_{(2)} + b'_{(1)_{(1)}} \otimes b'_{(1)_{(2)}} \otimes \{v, b'_{(2)}\} \\ =& b'_{(1)} \otimes b'_{(2)} \otimes v + b'_{(1)} \otimes v \otimes b'_{(2)} + \{v, b'_{(1)_{(1)}}\} \otimes b'_{(1)_{(2)}} \otimes b'_{(2)} + b'_{(1)_{(1)}} \otimes \{v, b'_{(1)_{(2)}}\} \otimes b'_{(2)} \\ &+ b'_{(1)_{(1)}} \otimes b'_{(1)_{(2)}} \otimes \{v, b'_{(2)}\}\end{aligned}$$

Now notice that the coZinbiel relation reads

$$c_{(1)_{(1)}} \otimes c_{(1)_{(2)}} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)_{(1)}} \otimes c_{(2)_{(2)}} + c_{(1)} \otimes c_{(2)_{(2)}} \otimes c_{(2)_{(1)}}$$

in Sweedler's notation. Applying it taking $c = a'$ and $c = b'$ in the two preceeding equations, and using $\between_{ZL^2}$ to simplify the terms leads to

$$\begin{aligned}(\delta \otimes \mathrm{Id}) \circ \delta(a', w) =& a'_{(1)} \otimes a'_{(2)} \otimes w + a'_{(1)} \otimes w \otimes a'_{(2)} - a'_{(1)} \otimes (w_{(2)}, w_{(1)}) \otimes a'_{(2)} - a'_{(1)} \otimes a'_{(2)} \otimes (w_{(2)}, w_{(1)}) \\ &+ (a'_{(1)}, w) \otimes a'_{(2)_{(1)}} \otimes a'_{(2)_{(2)}} + (a'_{(1)}, w) \otimes a'_{(2)_{(2)}} \otimes a'_{(2)_{(1)}} \\ &+ a'_{(1)} \otimes (a'_{(2)_{(1)}}, w) \otimes a'_{(2)_{(2)}} + a'_{(1)} \otimes (a'_{(2)_{(2)}}, w) \otimes a'_{(2)_{(1)}} \\ &+ a'_{(1)} \otimes a'_{(2)_{(1)}} \otimes (a'_{(2)_{(2)}}, w) + a'_{(1)} \otimes a'_{(2)_{(2)}} \otimes (a'_{(2)_{(1)}}, w) \\ =& (a'_{(1)}, w) \otimes \delta^{com} a'_{(2)} + a'_{(1)} \otimes \delta^{com}(a'_{(2)}, w) \\ =& (\mathrm{Id} \otimes \delta^{com})((a'_{(1)}, w) \otimes a'_{(2)} + a'_{(1)} \otimes (a'_{(2)}, w)) \\ =& (\mathrm{Id} \otimes \delta^{com}) \circ \delta(a', w) = (\mathrm{Id} \otimes \delta^{com}) \circ \delta(a)\end{aligned}$$

and similarly, since $\delta(v) = 0$, to

$$\begin{aligned}(\delta \otimes \mathrm{Id}) \circ \delta\{v, b'\} =& b'_{(1)} \otimes b'_{(2)} \otimes v + b'_{(1)} \otimes v \otimes b'_{(2)} \\ &+ \{v, b'_{(1)}\} \otimes b'_{(2)_{(1)}} \otimes b'_{(2)_{(2)}} + \{v, b'_{(1)}\} \otimes b'_{(2)_{(2)}} \otimes b'_{(2)_{(1)}} \\ &+ b'_{(1)} \otimes \{v, b'_{(2)_{(1)}}\} \otimes b'_{(2)_{(2)}} + b'_{(1)} \otimes \{v, b'_{(2)_{(2)}}\} \otimes b'_{(2)_{(1)}} \\ &+ b'_{(1)} \otimes b'_{(2)_{(1)}} \otimes \{v, b'_{(2)_{(2)}}\} + b'_{(1)} \otimes b'_{(2)_{(2)}} \otimes \{v, b'_{(2)_{(1)}}\} \\ =& (\mathrm{Id} \otimes \delta^{com})(\{v, b'_{(1)}\} \otimes b'_{(2)} + b'_{(1)} \otimes \{v, b'_{(2)}\}) \\ =& (\mathrm{Id} \otimes \delta^{com}) \circ \delta(\{v, b'\}) = (\mathrm{Id} \otimes \delta^{com}) \circ \delta(b)\end{aligned}$$

which proves that if (3) holds on words of length strictly lower than n, it holds for words of length n. By induction on n, this shows that δ is a coZinbiel coproduct.

Now notice that we have defined δ so that it satisfies $\between_{ZL^2}$ a priori only on PBW monomials, so it remains to prove that the compatibility relation holds on arbitrary brackets, that is on brackets of PBW monomials. The fact that $(-,-)$ and δ satisfy $\between_{ZL^2}$ can be proved performing exactly the same computation that the one we gave in the proof of proposition 1.0.6, so we only check here that $\{-,-\}$ also satisfies $\between_{ZL^2}$.

Let a and b be PBW monomials.

- If b is of the form $\{v_1,\cdots,v_n\}$, $\between_{ZL^2}$ holds because (a,b) is also a PBW monomial.
- If not, b can be written as $b=(b',w)$, with b' a PBW-monomial, i.e. an iterated bracket $(-,-,\cdots,-)$, with necessarily one less entry than b and on which we can assume that $\between_{ZL^2}$ holds, and where w is of the form $w=\{v_1,\cdots,v_n\}$.

$$\begin{aligned}\delta(a,b)=&\delta(a,(b',w))\\ =&\delta((a,b'),w)-\delta((a,w),b')\\ =&(a,b')\otimes w+((a,b')_{(1)},w)\otimes(a,b')_{(2)}+(a,b')_{(1)}\otimes((a,b')_{(2)},w)-(a,b')\otimes(w_{(2)},w_{(1)})\\ &-(a,w)\otimes b'-((a,w)_{(1)},b')\otimes(a,w)_{(2)}-(a,w)_{(1)}\otimes((a,w)_{(2)},b')+(a,w)\otimes(b'_{(2)},b'_{(1)})\end{aligned}$$

□

We know determine the primitive operad $\mathrm{Prim}_{Zinb^c}(Leib^2)$. The method we us is the one employed by Loday in [Petit Livre Bleu] to show that the primitive operad of the type (As^c, Dup) is Mag.

Proposition 2.0.14. *Let $\mathfrak{g}$ be a right Leibniz algebra with bracket $[-,-]_g$, and define three linear maps $(-,-):\bar{T}\mathfrak{g}\otimes\bar{T}\mathfrak{g}\to\bar{T}\mathfrak{g}$, $\{-,-\}:\bar{T}\mathfrak{g}\otimes\bar{T}\mathfrak{g}\to\bar{T}\mathfrak{g}$ and $\delta:\bar{T}\mathfrak{g}\to\bar{T}\mathfrak{g}\otimes\bar{T}\mathfrak{g}$ by setting :*

- *δ is the cofree conZinbiel coproduct obtained by identifying $\bar{T}\mathfrak{g}$ with $Zinb^c(\mathfrak{g})$, the cofree coZinbiel coalgebra cogenerated by the vector space $\mathfrak{g}$,*
- *$(-,-)$ is the free right Leibniz bracket obtained by identifying $\bar{T}\mathfrak{g}$ with $Leib(\mathfrak{g})$, the free Leibniz algebra generated $\mathfrak{g}$.*
- *$\{a,b\}:=(b,a)-[b,a]$ for all a and b in $\bar{T}\mathfrak{g}$,*

where $[-,-]$ is the unique bracket defined inductively by

- *$[g,h]=[g,h]_{\mathfrak{g}}$ if g and h are in $\mathfrak{g}$,*
- *$[(a,g),b]=([a,b],g)+(a,[g,b])$ for all a, b in $\bar{T}\mathfrak{g}$ and g in $\mathfrak{g}$,*
- *$[g,(b,h)]=(a,(b,h))+(a,(h,b))-(a,[h,b])$ for all b in $\bar{T}\mathfrak{g}$ and g, h in $\mathfrak{g}$.*

Then $(\bar{T}\mathfrak{g},(-,-),\{-,-\},\delta)$ is a $(Zinb^c, Leib^2, \between_{ZL^2})$-bialgebra.

Applying the preceeding proposition to the case $\mathfrak{g}=Leib(V)$, we get the following :

Theorem 2.0.15. *The unique map of Leibniz dialgebras $\psi: Leib^2(V)\to\bar{T}Leib(V)$ lifting the canonical inclusion $V\to\bar{T}Leib(V)$ is an isomorphism of coZinbiel coalgebras.*

Corollary 2.0.16.

$$\mathrm{Prim}_{Zinb^c}(Leib^2)=Leib$$

Definition 2.0.17.

- *An **ideal** of a Leibniz dialgebra $(L,(-,-),\{-,-\})$ is a linear subspace I such that (I,L), (L,I), $\{I,L)\}$ and $\{L,I\}$ are all contained in I.*
- *Let $(\mathfrak{g},[-,-]_{\mathfrak{g}})$ be a Leibniz algebra. The **universal enveloping Leibniz dialgebra** of $\mathfrak{g}$, denoted $U_{LD}(\mathfrak{g})$, is the quotient of the free Leibniz dialgebra generated by $\mathfrak{g}$ by the ideal I generated by elements of the form $(g,h)-\{h,g\}-[g,h]_{\mathfrak{g}}$, i.e.*

$$U_{LD}(\mathfrak{g}):=Leib^2(\mathfrak{g})/<(g,h)-\{h,g\}-[g,h]_{\mathfrak{g}}\ ,\ g,\ h\in\mathfrak{g}>$$

Theorem 2.0.18. *The triple $(Zinb, Leib^2, Leib)$ is a good triple of operads.*

3 Homology of Leibniz dialgebras

Definition 3.0.19. *A **Zinbiel dialgebra** (or $Zinb^2$-algebra) is a vector space Z equipped with two products $\succ: Z\otimes Z\to Z$ and $\prec: Z\otimes Z\to Z$ satisfying the four following relations*

1. *$(a\prec b)\prec c=a\prec(b\prec c+c\prec b)$,*

2. $a \succ (b \succ c) = (a \succ b + b \succ a) \succ c$,

3. $(a \prec b) \succ c = a \prec (b \succ c) + a \succ (c \prec b)$,

4. $a \succ (b \prec c) = (a \succ b) \prec c + (b \prec a) \succ c$,

for all a, b *and* c *in* Z.

Zinbiel dialgebras are encoded by an algebraic operad we denote $Zinb^2$.

Proposition 3.0.20. $Zinb^2$ *is the Koszul dual of* $Leib^2$, *i.e.*

$$(Leib^2)^! = Zinb^2$$

The knowledge of the Koszul dual of $Leib^2$ enables us to determine the operadic homology theory of Leibniz dialgebras :

Definition 3.0.21. *Let* $(L, (-,-), \{-,-\})$ *be a Leibniz dialgebra. The* ***Leibniz dialgebra chain complex*** *of* L, *denoted by* $CLD_*(L)$, *is the graded vector space defined in degree* n *by*

$$CLD_n(L) := \mathbb{K}[\{0,1\}^{n-1}] \otimes L^{\otimes n}$$

equipped with the differential $d^{LD} : CLD_*(L) \to CLD_{*-1}(L)$ *defined in degree* n *by*

$$d^{LD}\big((\varepsilon_2, \cdots, \varepsilon_n) < x_1|\cdots|x_n > \big) := \sum_{1 \leqslant i < j \leqslant n} (-1)^j (\varepsilon_2, \cdots, \widehat{\varepsilon_j}, \cdots, \varepsilon_n) < x_1|\cdots|x_{i-1}|(x_i, x_j)_{\varepsilon_j}|\cdots|\widehat{x_j}|\cdots|x_n >$$

for all ε_2, ..., ε_n *in* $\{0,1\}$, *and* x_1, ..., x_n *in* L, *where the bracket* $(x,y)_\varepsilon$ *is defined by*

$$(x,y)_\varepsilon := \begin{cases} (x,y) & \text{if } \varepsilon = 1, \\ \{y,x\} & \text{if } \varepsilon = 0. \end{cases}$$

and where the notation $(\varepsilon_2, \cdots, \varepsilon_n) < x_1|\cdots|x_n >$ *stands for the length* n *elementary tensor* $(\varepsilon_2, \cdots, \varepsilon_n) \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_n$ *and the symbol* $\widehat{x_j}$ *means as usual that* x_j *has been omitted.*

Proposition 3.0.22. *The graded vector space* $CLD_*(L)$ *equipped with the degree* -1 *map* d^{LD} *defined above is indeed a chain complex, i.e.*

$$d^{LD} \circ d^{LD} = 0$$

Moreover, its homology, that we denote $HLD_*(L) := H_*(CLD_*(L), d^{LD})$, *is the operadic homology of the* L *prescribed by the theory of operads, meaning that it can obtained as the homology of the (graded) cofree* $Zinb^2$*-coalgebra cogenerated by* $L[1]$ *endowed with the unique coderivation extending the canonical twisting cochain* $\kappa : (Zinb^2)^c(L[1]) \to L[1]$.

The canonical morphism of operads $Leib \to Leib^2$ induces a morphism of operads $(Leib^2)^! \to Leib^!$ and thus, for any Leibniz dialgebra $(L, (-,-), \{-,-\})$, a morphism of chain complexes

$$CL_*(L_{Leib}) \to CLD_*(L)$$

where L_{Leib} denotes the underlying Leibniz algebra obtained from L by only remembering the Leibniz bracket $[-,-] := (-,-) - \{-,-\} \circ (12)$.

Taking $L = U_{LD}(\mathfrak{g})$ for some Leibniz algebra $\mathfrak{g}$ and precomposing this morphism with the one induced at the level of chin complexes by the inclusion of Leibniz algebras $\mathfrak{g} \hookrightarrow (U_{LD}(\mathfrak{g}))_{Leib}$, we get a morphisms of chain complexes

$$\phi : CL_*(\mathfrak{g}) \to CLD_*(U_{LD}(\mathfrak{g})) \tag{4}$$

Proposition 3.0.23. *The morphism of chain complexes* $\phi : CL_*(\mathfrak{g}) \to CLD_*(U_{LD}(\mathfrak{g}))$ *defined above is given in degree* n *by the following explicit formula :*

$$\phi(< g_1|\cdots|g_n >) = \sum_{\varepsilon \in \{0,1\}^{n-1}} (-1)^{c(\varepsilon)} \varepsilon < g_1|\cdots|g_n >$$

where the integer $c(\varepsilon)$ *is the number of* 0*'s in the multi-index* ε.

Notice that $(Leib(V), [-,-])$, the free Leibniz algebra generated by some vector space V, can be seen as a Leibniz dialgebra by setting $(-,-) := [-,-]$ and $\{-,-\} := [-,-] \circ (12)$.

Proposition 3.0.24. *The universal enveloping dialgebra of a vector space V seen as abelian Leibniz algebra is $Leib(V)$ endowed with the dialgebra structure defined above.*

Proposition 3.0.25. *For any vector space V seen as an abelian Leibniz algebra, the map ϕ of (4) induces an isomorphism in homology i.e.*

$$H_n(\phi) : HL_n(V) = V^{\otimes n} \xrightarrow{\cong} HLD_n(U_{LD}(V)) = HLD_n(Leib(V))$$

is an isomorphism for all $n \geqslant 0$

Proof. Let n be a positive integer and recall that, as a vector space, $Leib(V) = \bar{T}V$ and $V \subset Leib(V)$, and that the $Leib^2$-brackets are given by $(-,-)_1 = (-,-)_0 = [-,-]$, the free Leibniz bracket on $Leib(V)$.

The fact that $HLD_n(Leib(V))$ is "smaller" than $\mathbb{K}[\{0,1\}^{n-1}] \otimes V^{\otimes n}$ is a consequence of the following

Lemma 3.0.26. *Any n-cycle in $CLD_n(Leib(V))$ is homologeous to a n-cycle in $\mathbb{K}[\{0,1\}^{n-1}] \otimes V^{\otimes n} \subset CLD_n(Leib(V))$*

Before proving the lemma, let us show how to use it to establish the proposition. Fix an integer n.

First notice that $H_n(\phi)$ is clearly injective because ϕ takes its values in $\mathbb{K}[\{0,1\}^{n-1}] \otimes V^{\otimes n}$ which is not hit by d^{LD} since any tensor in the image of an element under d^{LD} has to contain a factor of length greater than 2.

Let us show that $H_n(\phi)$ is surjective, i.e. that any cycle ω in $CLD_n(Leib(V))$ is, modulo some boundary, in the image of ϕ. Thanks to lemma 3.0.26, we can restrict to the case when ω is a cycle of the form

$$\omega = \sum_{\varepsilon} \varepsilon < v_1^{\varepsilon} | \cdots | v_n^{\varepsilon} >$$

where the v_i^{ε}'s are in V.

Proof of lemma 3.0.26. Let $\omega = \sum_{\varepsilon} \varepsilon < x_1^{\varepsilon} | \cdots , x_n^{\varepsilon} >$ be an arbitrary element in $CLD_n(Leib(V))$, where the ε's are multi-indices in $\{0,1\}^{n-1}$ (possibly redundant) and the x_i's are elements of $Leib(V) = \bar{T}V$. Since for any $\varepsilon = (\varepsilon_2 \cdots , \varepsilon_n)$, any y in $Leib(V)$ and $v \in V$

$$\begin{aligned} d^{LD}\big((\varepsilon_2, \cdots , \varepsilon_n, 1) < x_1 | \cdots | x_{n-1} | y | v > \big) =& (-1)^n \varepsilon < x_1 | \cdots | x_{n-1} | [y,v] > + \sum_{\varepsilon'} \varepsilon' < z_1^{\varepsilon'} | \cdots z_{n-1}^{\varepsilon'} | y > \\ &+ \sum_{\varepsilon''} \varepsilon' < z_1^{\varepsilon''} | \cdots z_{n-1}^{\varepsilon''} | v > \end{aligned}$$

where the $z_i^{\varepsilon'}$'s and the $z_i^{\varepsilon''}$'s are elements of $Leib(V)$, we can assume that x_n^{ε} is in V for all ε (because applying the preceeding equality recursively to lower the degree of the last factor shows that ω is at least homologeous to a sum of tensors having this property).

Now suppose that ω is homologeous to a cycle of the form

$$\omega' := \sum_{\varepsilon} \varepsilon < x_1^{\varepsilon} | \cdots | x_k^{\varepsilon} | v_{k+1}^{\varepsilon} | \cdots | v_n^{\varepsilon} >$$

where $k < n$ and the v_i^{ε}'s have length 1. Then

$$\begin{aligned} 0 = d^{LD}\omega = d^{LD}\omega' =& \sum_{\varepsilon} \sum_{i<j\leqslant k} (-1)^j \varepsilon^j < x_1^{\varepsilon} | \cdots | [x_i^{\varepsilon}, x_j^{\varepsilon}] | \cdots | \widehat{x_j^{\varepsilon}} | \cdots | x_k^{\varepsilon} | v_{k+1}^{\varepsilon} | \cdots | v_n^{\varepsilon} > \\ &+ \sum_{\varepsilon} \sum_{i\leqslant k<j} (-1)^j \varepsilon^j < x_1^{\varepsilon} | \cdots | [x_i^{\varepsilon}, v_j^{\varepsilon}] | \cdots | x_k^{\varepsilon} | v_{k+1}^{\varepsilon} | \cdots | \widehat{v_j^{\varepsilon}} | \cdots | v_n^{\varepsilon} > \\ &+ \sum_{\varepsilon} \sum_{k<i<j} (-1)^j \varepsilon^j < x_1^{\varepsilon} | \cdots | x_k^{\varepsilon} | v_{k+1}^{\varepsilon} | \cdots | [v_i^{\varepsilon}, v_j^{\varepsilon}] | \cdots | \widehat{v_j^{\varepsilon}} | \cdots | v_n^{\varepsilon} > \end{aligned}$$

where $\varepsilon^j := (\varepsilon_2, \cdots , \widehat{\varepsilon_j}, \cdots , \varepsilon_n)$ if $\varepsilon = (\varepsilon_2, \cdots , \varepsilon_n)$. But looking at the elements located at the k-th place and at the $i-th$ place of each tensor appearing in the right-hand side of the preceeding equation, we can see that for length reasons, it implies

$$(S)\left\{\begin{array}{l}\displaystyle\sum_{\substack{\varepsilon\\|x_k^\varepsilon|>1}}\sum_{k<j}(-1)^j\varepsilon^j < x_1^\varepsilon|\cdots|[x_k^\varepsilon, v_j^\varepsilon]|\cdots|\widehat{v_j^\varepsilon}|\cdots|v_n^\varepsilon >= 0\\[2ex]\displaystyle\sum_{\substack{\varepsilon\\|x_k^\varepsilon|>1}}\sum_{k<i<j}(-1)^j\varepsilon^j < x_1^\varepsilon|\cdots|x_k^\varepsilon|\cdots|[v_i^\varepsilon, v_j^\varepsilon]|\cdots|\widehat{v_j^\varepsilon}|\cdots|v_n^\varepsilon >= 0\end{array}\right.$$

Now write any x_k^ε of length greater than 1 as a bracket of the form $x_k^\varepsilon = [y_k^\varepsilon, v_k^\varepsilon]$, with v_k^ε in V and y_k^ε in $Leib(V)$. As the Leibniz bracket $[-,-]$ is the free one, we can replace every x_k^ε by $y_k^\varepsilon|v_k^\varepsilon$ and any ε^j by $(\varepsilon_2,\cdots,\varepsilon_k,1,\cdots,\widehat{\varepsilon_j},\cdots,\varepsilon_n)$ in the second equation of (S) so that

$$\sum_{\substack{\varepsilon\\|x_k^\varepsilon|>1}}\sum_{k<i<j}(-1)^j(\varepsilon_2,\cdots,\varepsilon_k,1,\cdots,\widehat{\varepsilon_j},\cdots,\varepsilon_n) < x_1^\varepsilon|\cdots|y_k^\varepsilon|v_k^\varepsilon|\cdots|[v_i^\varepsilon, v_j^\varepsilon]|\cdots|\widehat{v_j^\varepsilon}|\cdots|v_n^\varepsilon >= 0 \tag{5}$$

Similarly, the definition of the free bracket enables us to replace each $[x_k^\varepsilon, v_j^\varepsilon] = [[y_k^\varepsilon, v_k^\varepsilon], v_j^\varepsilon]$ by $y_k^\varepsilon|v_k^\varepsilon|v_j^\varepsilon$ and any ε^j by $(\varepsilon_2,\cdots,\varepsilon_k,1,\cdots,\widehat{\varepsilon_j},\cdots,\varepsilon_n)$ in the first equation of (S) to get

$$\sum_{\substack{\varepsilon\\|x_k^\varepsilon|>1}}\sum_{k<j}(-1)^j(\varepsilon_2,\cdots,\varepsilon_k,1,\cdots,\widehat{\varepsilon_j},\cdots,\varepsilon_n) < x_1^\varepsilon|\cdots|y_k^\varepsilon|v_k^\varepsilon|v_j^\varepsilon|\cdots|\widehat{v_j^\varepsilon}|\cdots|v_n^\varepsilon >= 0 \tag{6}$$

which implies both

$$\sum_{\substack{\varepsilon\\|x_k^\varepsilon|>1}}\sum_{k<j}(-1)^j(\varepsilon_2,\cdots,\varepsilon_k,1,\cdots,\widehat{\varepsilon_j},\cdots,\varepsilon_n) < x_1^\varepsilon|\cdots|y_k^\varepsilon|[v_k^\varepsilon, v_j^\varepsilon]|\cdots|\widehat{v_j^\varepsilon}|\cdots|v_n^\varepsilon >= 0 \tag{7}$$

and

$$\sum_{\substack{\varepsilon\\|x_k^\varepsilon|>1}}\sum_{k<j}(-1)^j(\varepsilon_2,\cdots,\varepsilon_k,1,\cdots,\widehat{\varepsilon_j},\cdots,\varepsilon_n) < x_1^\varepsilon|\cdots|[y_k^\varepsilon, v_j^\varepsilon]|v_k^\varepsilon|\cdots|\widehat{v_j^\varepsilon}|\cdots|v_n^\varepsilon >= 0 \tag{8}$$

by applying respectively $\mathrm{Id}^{\otimes k}\otimes[-,-]\otimes\mathrm{Id}^{\otimes n-k-1}$ and $\mathrm{Id}^{\otimes k-1}\otimes[-,-]\otimes\mathrm{Id}^{\otimes n-k}\circ(k\ k+1)$ to (6).

Now define the $n+1$-chain α by

$$\alpha := \sum_{\substack{\varepsilon,\\|x_k^\varepsilon|>1}}(\varepsilon_2,\cdots,\varepsilon_k,1,\cdots,\varepsilon_n) < x_1^\varepsilon|\cdots|y_k^\varepsilon|v_k^\varepsilon|\cdots|v_n^\varepsilon >$$

So that

$$d^{LD}(\alpha) = \sum_{\substack{\varepsilon \\ |x_k^\varepsilon|>1}} \sum_{k<i<j} (-1)^{j+1} (\varepsilon_2, \cdots, \varepsilon_k, 1, \cdots, \widehat{\varepsilon_j}, \cdots, \varepsilon_n) < x_1^\varepsilon | \cdots | y_k^\varepsilon | v_k^\varepsilon | \cdots | [v_i^\varepsilon, v_j^\varepsilon] | \cdots | \widehat{v_j^\varepsilon} | \cdots | v_n^\varepsilon >$$
$$+ \sum_{\substack{\varepsilon \\ |x_k^\varepsilon|>1}} \sum_{k<j} (-1)^{j+1} (\varepsilon_2, \cdots, \varepsilon_k, 1, \cdots, \widehat{\varepsilon_j}, \cdots, \varepsilon_n) < x_1^\varepsilon | \cdots | y_k^\varepsilon | [v_k^\varepsilon, v_j^\varepsilon] | \cdots | \widehat{v_j^\varepsilon} | \cdots | v_n^\varepsilon >$$
$$+ \sum_{\substack{\varepsilon \\ |x_k^\varepsilon|>1}} \sum_{k<j} (-1)^{j+1} (\varepsilon_2, \cdots, \varepsilon_k, 1, \cdots, \widehat{\varepsilon_j}, \cdots, \varepsilon_n) < x_1^\varepsilon | \cdots | [y_k^\varepsilon, v_j^\varepsilon] | v_k^\varepsilon | \cdots | \widehat{v_j^\varepsilon} | \cdots | v_n^\varepsilon >$$
$$+ \sum_{\substack{\varepsilon \\ |x_k^\varepsilon|>1}} \sum_{i<k<j} (-1)^{j+1} (\varepsilon_2, \cdots, \varepsilon_k, 1, \cdots, \widehat{\varepsilon_j}, \cdots, \varepsilon_n) < x_1^\varepsilon | \cdots | [x_i^\varepsilon, v_j^\varepsilon] | \cdots | y_k^\varepsilon | v_k^\varepsilon | \cdots | \widehat{v_j^\varepsilon} | \cdots | v_n^\varepsilon >$$
$$+ (-1)^{k+1} \sum_{\substack{\varepsilon \\ |x_k^\varepsilon|>1}} \varepsilon < x_1^\varepsilon | \cdots | x_k^\varepsilon | v_{k+1}^\varepsilon | \cdots | v_n^\varepsilon >$$
$$+ (-1)^{k+1} \sum_{\substack{\varepsilon \\ |x_k^\varepsilon|>1}} \sum_{i<k} \varepsilon < x_1^\varepsilon | \cdots | [x_i^\varepsilon, v_k^\varepsilon] \cdots | y_k^\varepsilon | v_{k+1}^\varepsilon | \cdots | v_n^\varepsilon >$$
$$+ (-1)^{k} \sum_{\substack{\varepsilon \\ |x_k^\varepsilon|>1}} \sum_{i<k} (\varepsilon_2, \cdots, \varepsilon_{k-1}, 1, \cdots, \varepsilon_n) < x_1^\varepsilon | \cdots | [x_i^\varepsilon, y_k^\varepsilon] | \cdots | x_{k-1}^\varepsilon | v_k^\varepsilon | \cdots | v_n^\varepsilon >$$
$$+ \sum_{\substack{\varepsilon \\ |x_k^\varepsilon|>1}} \sum_{i<j<k} (-1)^{j} (\varepsilon_2, \cdots, \widehat{\varepsilon_j}, \cdots, \varepsilon_k, 1, \cdots, \varepsilon_n) < x_1^\varepsilon | \cdots | [x_i^\varepsilon, x_j^\varepsilon] | \cdots | \widehat{x_j^\varepsilon} | \cdots | y_k^\varepsilon | v_k^\varepsilon | \cdots | v_n^\varepsilon >$$

Equations (5), (7) and (8) imply that the first three sums in this expression have to vanish, which leads to

$$\sum_{\substack{\varepsilon \\ |x_k^\varepsilon|>1}} \varepsilon < x_1^\varepsilon | \cdots | x_k^\varepsilon | v_{k+1}^\varepsilon | \cdots | v_n^\varepsilon > = (-1)^k d^{LD}(\alpha)$$
$$+ \sum_{\substack{\varepsilon \\ |x_k^\varepsilon|>1}} \sum_{i<k<j} (-1)^{j+k} (\varepsilon_2, \cdots, \varepsilon_k, 1, \cdots, \widehat{\varepsilon_j}, \cdots, \varepsilon_n) < x_1^\varepsilon | \cdots | [x_i^\varepsilon, v_j^\varepsilon] \cdots | y_k^\varepsilon | v_k^\varepsilon | \cdots | \widehat{v_j^\varepsilon} | \cdots | v_n^\varepsilon >$$
$$+ \sum_{\substack{\varepsilon \\ |x_k^\varepsilon|>1}} \sum_{i<k} \varepsilon < x_1^\varepsilon | \cdots | [x_i^\varepsilon, v_k^\varepsilon] \cdots | y_k^\varepsilon | v_{k+1}^\varepsilon | \cdots | v_n^\varepsilon >$$
$$- \sum_{\substack{\varepsilon \\ |x_k^\varepsilon|>1}} \sum_{i<k} (\varepsilon_2, \cdots, \varepsilon_{k-1}, 1, \cdots, \varepsilon_n) < x_1^\varepsilon | \cdots | [x_i^\varepsilon, y_k^\varepsilon] | \cdots | x_{k-1}^\varepsilon | v_k^\varepsilon | \cdots | v_n^\varepsilon >$$
$$- \sum_{\substack{\varepsilon \\ |x_k^\varepsilon|>1}} \sum_{i<j<k} (-1)^{j+k} (\varepsilon_2, \cdots, \widehat{\varepsilon_j}, \cdots, \varepsilon_k, 1, \cdots, \varepsilon_n) < x_1^\varepsilon | \cdots | [x_i^\varepsilon, x_j^\varepsilon] | \cdots | \widehat{x_j^\varepsilon} | \cdots | y_k^\varepsilon | v_k^\varepsilon | \cdots | v_n^\varepsilon >$$

This proves that $\sum_{\substack{\varepsilon \\ |x_k|>1}} \varepsilon < x_1^\varepsilon | \cdots | x_k^\varepsilon | v_{k+1}^\varepsilon | \cdots | v_n^\varepsilon >$ is homologeous to a chain of the form

$$\sum_{\varepsilon'} \varepsilon' < x_1^{\varepsilon'} | \cdots | x_k^{\varepsilon'} | v_{k+1}^{\varepsilon'} | \cdots | v_n^{\varepsilon'} >$$

such that

$$\max_{\varepsilon'} |x_k^{\varepsilon'}| < \max_\varepsilon |x_k^\varepsilon| =: N_k(\omega')$$

and thus so is ω'. By decreasing induction on $N_k(\omega')$, this proves that ω' is homologeous to a cycle of the form

$$\sum_{\varepsilon''} \varepsilon'' < x_1^{\varepsilon''} | \cdots | x_{k-1}^{\varepsilon''} | v_k^{\varepsilon''} | \cdots v_n^{\varepsilon''} > \quad , v_i^{\varepsilon''} \in V$$

and thus so is ω.

By decreasing induction on the integer k, this shows that ω is indeed homologeous to a cycle in elements of V, i.e. in $\mathbb{K}[\{0,1\}^{n-1}] \otimes V^{\otimes} \subset CLD_n(Leib(V))$, which concludes the proof of the lemma.

□

□

Theorem 3.0.27. *For any Leibniz algebra* $\mathfrak{g}$,

$$HLD_*(U_{LD}(\mathfrak{g})) \cong HL_*(\mathfrak{g})$$

Proof. The general case can be reduced to the abelian one by the following standard spectral sequence argument :

□