A mini-course on quasi-categories (partial notes)
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Contents

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Introduction

1 On quasi-categories

See [BV], [Jo1], [Jo2], [Lu1], [Gr], [Ci]

1.1 On simplicial sets

See [JT].

Let Δ be the category of finite non-empty ordinals $[n] = \{0, \dots, n\}$ and order preserving maps. A map $f : [m] \to [n]$ in Δ can be described by the increasing sequence of its values $f = (f(0), f(1), \dots, f(m)) \in [n]^m$. We shall denotes by $d_i^n : [n-1] \to [n]$ the unique injective (order preserving) map with $i \notin Im(d_i^n)$ and by $s_i^n : [n] \to [n-1]$ the unique surjective (order preserving) map such that $s_i^n(i) = s_i^n(i+1)$. For example,

$$d_2^5 = (0,1,3,4,5) \qquad s_2^5 = (0,1,2,2,3,4)$$

For simplicity, we shall often denote the map d_i^n by $d_i : [n-1] \to [n]$ and denote the map s_i^n by $s_i : [n] \to [n-1]$. Notice that the maps $d_0, d_1 : [0] \to [1]$ are defined by $d_0(0) = 1$ and $d_1(0) = 0$. Every injective map $d : [m] \to [n]$ in Δ can be expressed uniquely as a composite $d = d_{i_k} \dots d_{i_1}$ with $0 \le i_1 < \dots < i_k \le n$ and every surjective map $s : [m] \to [n]$ can be expressed uniquely as a composite $s = s_{i_k} \cdots s_{i_1}$ with $0 \le i_1 < \dots < i_k < m$. Every map $f : [m] \to [n]$ in Δ admits a unique decomposition $f = ds : [m] \to [p] \to [n]$ with $s : [m] \to [p]$ a surjective map and $d : [p] \to [n]$ an injective map.

The category Δ has a non-trivial automorphism $\tau : \Delta \to \Delta$ which reverses the order relation on the ordinals $[n] = \{0, \dots, n\}$. By construction, if $f : [m] \to [n]$ then the map $\tau(f) : [m] \to [n]$ is obtained by putting $\tau(f)(i) = n - f(m-i)$ for every $i \in [m]$. Notice that $\tau(d_i^n) = d_{n-i}^n$ and $\tau(s_i^n) = s_{n-1-i}^n$.

By definition, a *simplicial set* X is a presheaf of Δ. We shall adopt the standard convention of denoting the set $X([n])$ by X_n for every $n \ge 0$. The map $d^i := X(d_i) : X_n \to X_{n-1}$ is called a *face operator* and the map $s^i := X(s_i) : X_{n-1} \to X_n$ a *degeneracy operator*. A simplicial set X is often pictured by its diagram of face operators

$$X_0 \overset{d^0}{\underset{d^1}{\leftleftarrows}} X_1 \overset{d^0,\, d^1,\, d^2}{\Lleftarrow} X_2 \qquad \cdots$$

omiting the degeneracy operators:

$$X_0 \xrightarrow{s^0} X_1 \overset{s^0}{\underset{s^1}{\rightrightarrows}} X_2 \overset{s^0,\, s^1,\, s^2}{\Rrightarrow} X_3 \qquad \cdots$$

A *map* of simplicial sets $f : X \to Y$ is a natural transformation. By definition, it is a sequence of maps $f_n : X_n \to Y_n$ and the following squares commute:

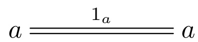
$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ {\scriptstyle d^i}\downarrow & & \downarrow{\scriptstyle d^i} \\ X_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1} \end{array} \qquad \begin{array}{ccc} X_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-1} \\ {\scriptstyle s^i}\downarrow & & \downarrow{\scriptstyle s^i} \\ X_n & \xrightarrow{f_{n-1}} & Y_n \end{array}$$

We shall denote the category of simplicial sets by sSet.

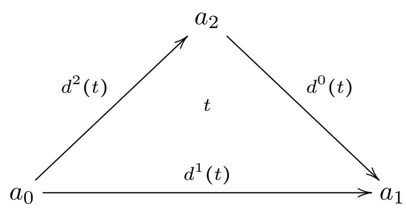
If X is a simplicial set, then an element $x \in X_n$ is said to be a *n-simplex* of X. A 0-simplex $x \in X_0$ is said to be *vertex* and a 1-simplex $f \in X_1$ said to be an *arrow* of X. The *source* of an arrow $f \in X_1$ is the vertex $d^1(f) \in X_0$ and its *target* is the vertex $d^0(f) \in X_0$.

$$d^1(f) \xrightarrow{f} d^0(f)$$

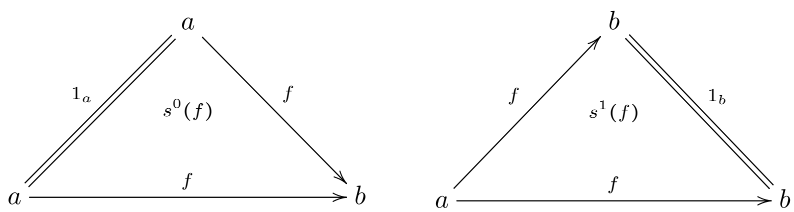
To every vertex $a \in X_0$ is associated a *unit arrow* $1_a := s^0(a) : a \to a$.



A 2-simplex $t \in X_2$ has three faces $d^0(t)$, $d^1(t)$, $d^2(t)$ and three vertices $a_0 = d^1d^2(t) = d^1d^1(t)$, $a_1 = d^0d^2(t) = d^1d^0(t)$ and $a_2 = d^0d^1(t) = d^0d^0(t)$.



A n-simplex $x \in X_n$ is said to be *degenerated* if $n > 0$ and $x = s^i(y)$ for some $y \in X_{n-1}$ and some $0 \le i \le n-1$. For example a unit arrow $1_a := s^0(a) : a \to a$ is degenerated. To every arrow $f : a \to b$ in X_1 is associated two degenerated 2-simplex.



We shall discuss the geometric meaning of degenerated simplices below.

An important example of simplicial set is the *singular complex* $S(X)$ of a topological space. By construction, a n-simplex $x \in S(X)_n$ is a continuous map $x : \Delta^n \to X$, where Δ^n is the convex hull of the set of unit vectors in $\mathbb{R}^{[n]} = \mathbb{R}e_0 \oplus \cdots \oplus \mathbb{R}e_n$,

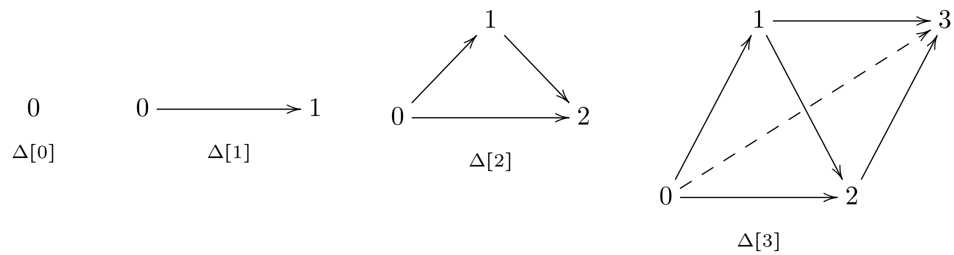
$$\Delta^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 1,\ x_i \ge 0\}.$$

Notice that $\Delta^0 = \{e_0\}$ and that $\Delta^1 = \{(1-t)e_0 + te_1 \mid 0 \le t \le 1\} \simeq [0,1]$. Every map $f : [m] \to [n]$ in Δ can be extended uniquely as a linear map $R(f) : \Delta^m \to \Delta^n$ if we put $R(f)(e_i) = e_{f(i)}$ for every $i \in [m]$. If $x \in S(X)_n$, then $S(X)(f)(x) := xR(f) : \Delta^m \to S(X)$. A vertex of $S(X)$ is a point of X and an arrow $u \in S(X)_1$ is a continuous path $u : [0,1] \to X$; the source of u is the vertex $d^1(u) = uR(d_1)(e_0) = u(0)$ and its target is the vertex $d^0(u) = uR(d_0)(e_0) = u(1)$. If X and Y are topological spaces, then a continuous map $f : X \to Y$ induces a map of simplicial sets $S(f) : S(X) \to S(Y)$. This defines the *singular complex functor*

$$S : \mathsf{Top} \to \mathsf{sSet}$$

where Top is the category of topological spaces and continuous maps.

Another example of simplicial set is the *nerve* $N(P)$ of a poset P. By construction, $N(P)_n$ is the set of order preserving maps $[n] \to P$. Equivalently, $N(P)_n$ is the set of (increasing) chains $x_0 \le x_1 \le \cdots \le x_n$ of $n+1$ of elements of P. A n-simplex $x_0 \le x_1 \le \cdots \le x_n$ is non-degenerated if and only if $x_0 < x_1 < \cdots < x_n$. The nerve $N([n])$ of the poset $[n]$ is called the *fundamental simplex* and it is denoted $\Delta[n]$.



Another example of a simplicial set is the nerve $N(\mathcal{C})$ of a category $\mathcal{C}$. Recall that every poset P has the structure of a category if we put $Ob(P) = P$ and

$$Hom(x,y) = \begin{cases} \{(x,y)\} \text{ if } x \leq y \\ \varnothing \text{ otherwise} \end{cases}$$

In particular, the poset $[n]$ is a category with $n+1$ objects $\{0,\dots,n\}$ and with exactly one arrow $i \to j$ for each pair $i \leq j$ in $[n]$. A map of posets $P \to Q$ is the same thing as a functor $P \to Q$. In particular, a map $[m] \to [n]$ in the category Δ is the same thing as a functor $[m] \to [n]$.

If $\mathcal{C}$ is a category, then a n-simplex of the simplicial set $N(\mathcal{C})$ is defined to be a functor $x : [n] \to \mathcal{C}$. If $f : [m] \to [n]$ is a map in Δ, then $N(\mathcal{C})(x) := xf : [m] \to \mathcal{C}$. Notice that category $[n]$ is freely generated by a chain of n arrows

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$$

It follows that a n-simplex of $N(\mathcal{C})$ is a chain $(f_n, f_{n-1}, \dots, f_1)$ of length n of morphisms of $\mathcal{C}$,

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n$$

By definition,

$$d^i(f_n,\dots,f_1) = \begin{cases} (f_n,\dots,f_2) \text{ if } i = 0 \\ (f_n,\dots,f_{i+1}f_i,\dots,f_1) \text{ if } 0 < i < n \\ (f_{n-1},\dots,f_1) \text{ if } i = n \end{cases}$$

$$s^i(f_{n-1},\dots,f_1) = \begin{cases} (f_n,\dots,f_2,f_1,id) \text{ if } i = 0 \\ (f_n,\dots,f_{i+1},id,f_i,\dots,f_1) \text{ if } 0 < i < n \\ (id,f_{n-1},\dots,f_1) \text{ if } i = n \end{cases}$$

where id denotes unit morphisms. A n-simplex $(f_n, f_{n-1}, \dots, f_1)$ is non-degenerated if and only none of the morphisms f_i is a unit.

If $\mathcal{C}$ and $\mathcal{D}$ are categories, then a functor $F : \mathcal{C} \to \mathcal{D}$ induces a map of simplicial sets $N(F) : N(\mathcal{C}) \to N(\mathcal{D})$. This defines a the *nerve functor*

$$N : \mathsf{Cat} \to \mathsf{sSet}$$

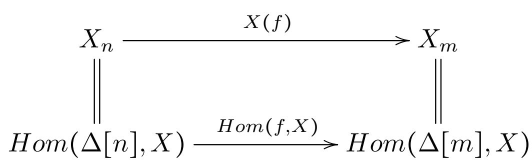
where Cat is the category of small categories and functors.

The functor N is fully faithful. Hence we may use the same notation for a category $\mathcal{C}$ and its nerve $N(\mathcal{C})$.

Recall that every category $\mathcal{C}$ has an *opposite* $\mathcal{C}^{op}$. By definition, the categories $\mathcal{C}$ and $\mathcal{C}^{op}$ have the same objects but $\mathcal{C}^{op}(A,B) := \mathcal{C}(B,A)$ for every pair of objects. Thus, for every arrow $f : B \to A$ in $\mathcal{C}$, there is an arrow $f^{op} : A \to B$ in $\mathcal{C}^{op}$. Moreover, if $g^{op} : B \to C$, then $g^{op}f^{op} := (fg)^{op}$. Similarly, every simplicial set X

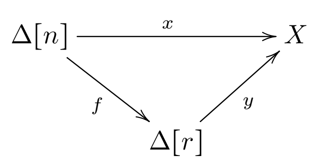
has an *opposite* X^{op} defined by putting $X^{op} = X \circ \tau$, where $\tau : \Delta \to \Delta$ is the non-trivial automorphism of Δ describe above. Thus, for every arrow $f : B \to A$ in $\mathcal{C}$, there is an arrow $f^{op} : A \to B$ in $\mathcal{C}^{op}$. Moreover, if $g^{op} : B \to C$, then $g^{op} f^{op} := (fg)^{op}$. Similarly, every simplicial set X has an *opposite* X^{op} defined by putting $X^{op} = X \circ \tau$, where $\tau : \Delta \to \Delta$ is the non-trivial automorphism of Δ describe above. Every n-simplex $x \in X_n$ has an opposite $x^{op} \in (X^{op})_n$. Moreover $d^i(x^{op}) = d^{n-i}(x)^{op}$ and $s^i(x^{op}) = s^{n-i}(x)^{op}$. It follows from this description that $N(\mathcal{C}^{op}) = N(\mathcal{C})^{op}$.

Recall that the *Yoneda functor* $\mathsf{y} : \Delta \to \mathsf{sSet}$ is defined by putting $\mathsf{y}([n]) = \Delta(-, [n])$ for every $n \geq 0$. We have $\mathsf{y}([n]) = N([n]) = \Delta[n]$ for every $n \geq 0$. Hence the Yoneda functor $\mathsf{y} : \Delta \to \mathsf{sSet}$ coincide with the restriction of the nerve functor $N : \mathsf{Cat} \to \mathsf{sSet}$ to the sub-category $\Delta \subset \mathsf{Cat}$. The Yoneda functor $\mathsf{y} : \Delta \to \mathsf{sSet}$ is fully faithful. Hence we may use the same notation for a map $f : [m] \to [n]$ in Δ and for the map of simplicial sets $\mathsf{y}(f) : \Delta[m] \to \Delta[n]$. By Yoneda lemma, if X is a simplicial set, then the set of maps $\Delta[n] \to X$ is in natural bijection with the set X_n; the bijection takes a map $f : \Delta[n] \to X$ to the n-simplex $f(1_{[n]}) \in X_n$. Notice that $1_{[n]}$ is the n-simplex $(0 < 1 < \cdots < n)$ of the nerve of the poset $[n]$. If X is a simplicial set, we shall identify X_n with the set $Hom(\Delta[n], X)$ by using the same notation for an element $x \in X_n$ and the unique map $x' : \Delta[n] \to X$ such that $x'(1_{[n]}) = x$. In this notation, we have $x(1_{[n]}) = x$ for every $x \in X_n$. If $f : [m] \to [n]$ is a map in Δ, then the simplex $X(f)(x) \in X_m$ is identified with the map $xf := x'f : \Delta[n] \to X$.

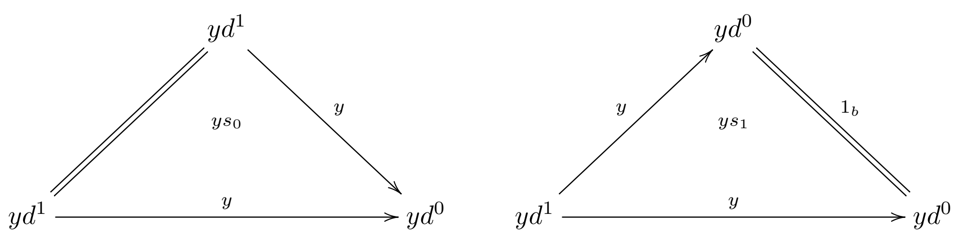


For example, if an arrow $f \in X_1$ is represented by a map $f : \Delta[1] \to X$ then its source $d^1(f) \in X_0$ is represented by the map $fd_1 : \Delta[0] \to X$, where $d_1 : \Delta[0] \to \Delta[1]$ and $d_1(0) = 0$.

A simplex $x : \Delta[n] \to X$ is degenerated if and only there exists a simplex $y : \Delta[r] \to X$ of dimension $r < n$ together with a map $f : \Delta[n] \to \Delta[r]$ such that $x = yf$.



Every simplex $x : \Delta[n] \to X$ admits a unique decomposition $x = yf$, where $y : \Delta[r] \to X$ is a non-degenerated simplex and $f : [r] \to [m]$ is a surjection (Eilenberg-Zilber lemma). For example, if $y : \Delta[1] \to X$ is a non-degenerated arrow then the simplices $ys_0 : \Delta[2] \to X$ and $ys_1 : \Delta[2] \to X$ are degenerated and different, since the surjections $s_0, s_1 : \Delta[2] \to \Delta[1]$ are different. However, the maps ys_0 and ys_1 have the same image in X.



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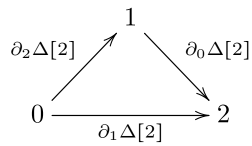
A *sub-simplicial set* of a simplicial set X is a sub-presheaf $S \subseteq X$.

To every non-empty subset $S \subseteq [n]$ corresponds a face $\Delta[S] \subseteq \Delta[n]$ of dimension $Card(S) - 1$. By definition $\Delta[S]_k$ is the set of maps $x : [k] \to [n]$ that factor through the inclusion $S \subseteq [n]$. The simplex $\Delta[n]$ has $n + 1$ faces of codimension 1 denoted $\partial_i \Delta[n]$. By definition, $\partial_i \Delta[n] = \Delta[[n] \setminus i]$.



The *boundary* of $\Delta[n]$ is defined by putting

$$\partial\Delta[n] = \bigcup_{i=0}^{n} \partial_i\Delta[n]$$

The category of simplicial sets sSet has limits and colimits. The limit and the colimit of a diagram of simplicial set $D : I \to \mathsf{sSet}$ are taken pointwise:

$$\big(\varprojlim_{i\in I} D(i)\big)([n]) = \varprojlim_{i\in I} D(i)([n]), \qquad \big(\varinjlim_{i\in I} D(i)\big)([n]) = \varinjlim_{i\in I} D(i)([n])$$

In particular, if $(X_i | i \in I)$ is a family of simplicial sets, then

$$\big(\prod_{i\in I} X_i\big)([n]) = \prod_{i\in I} X_i([n]), \qquad \big(\bigsqcup_{i\in I} X_i\big)([n]) = \bigsqcup_{i\in I} X_i([n])$$

Similary for the construction of amalgamated coproducts (of pushouts) of two maps of simplicial sets $u : C \to A$ and $v : C \to B$ and of the fiber products (pullbacks) of two maps $f : X \to Z$ and $g : Y \to Z$.

$$\begin{array}{ccc} C & \xrightarrow{v} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \mathsf{in}_2} \\ A & \xrightarrow{\mathsf{in}_1} & A \sqcup_C B \end{array} \qquad \begin{array}{ccc} X \times_Z Y & \xrightarrow{\pi_2} & Y \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array}$$

From a simplicial set X and a map $f : \partial\Delta[n] \to X$ we can construct a new simplicial set $\Delta[n] \sqcup_f X$ by taking a pushout

$$\begin{array}{ccc} \partial\Delta[n] & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \mathsf{in}} \\ \Delta[n] & \longrightarrow & \Delta[n] \sqcup_f X \end{array}$$

where i is the inclusion $\partial\Delta[n] \subset \Delta[n]$. The simplicial set $\Delta[n] \sqcup_f X$ is obtain from X by attaching a n-cell along the map $f : \partial\Delta[n] \to X$. Every simplicial set can be constructed from the empty set by attaching cells iteratively. More precisely, a simplicial set X is said to be of dimension $\le n$ if every non-degenerated simplex of X has dimension $\le n$. A simplicial set X is of dimension ≤ 0 if and only if it is a coproduct of $\Delta[0]$; we shall say that it is *discrete*. The n-skeleton $Sk^n(X)$ of a simplicial set X is defined to be the sub-simplicial set of X generated by the non-degenerated simplices of dimension $\le n$. This defines a filtration

$$Sk^0(X) \subseteq Sk^1(X) \subseteq Sk^2(X) \subseteq$$

and $X = \bigcup_n Sk^n(X)$. It turns out that the simplicial set $Sk^{n+1}(X)$ is obtained from the simplicial set $Sk^n(X)$ by attaching a set of $(n+1)$-cells.

The category of simplicial sets sSet is also *cartesian closed*. Recall that this means that the functor $A \times (-) : \mathsf{sSet} \to \mathsf{sSet}$ has a right adjoint $[A, -] : \mathsf{sSet} \to \mathsf{sSet}$ for any simplicial set A. If B is a simplicial set, then

$$[A, B]_n = Map(A \times \Delta[n], B)$$

for every $n \ge 0$. We shall often denote the simplicial set $[A, B]$ by B^A. Notice that a vertex of the simplicial set $A^{\Delta[1]}$ is an arrow in the simplicial set A.

1.2 Realisations

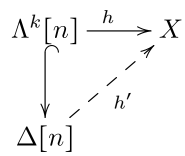
See [JT], [GJ].

The *singular complex functor* $S : \mathsf{Top} \to \mathsf{sSet}$ has a left adjoint $R : \mathsf{sSet} \to \mathsf{Cat}$ which associates to a simplicial set X its *geometric realisation* $R(X)$.

The nerve functor $N : \mathsf{Cat} \to \mathsf{sSet}$. has a left adjoint $\tau_1 : \mathsf{sSet} \to \mathsf{Cat}$ which associates to a simplicial set X its *fundamental category* $\tau_1(X)$. The category $\tau_1(X)$ has an explicit description in terms of generators and relations.

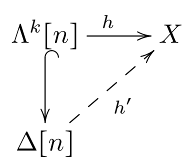
2 On quasi-categories and Kan complexes

The notion of Kan complex was introduced by Daniel Kan []. Recall that a simplicial set X is called a **Kan complex** if every horn $h : \Lambda^k[n] \to X$ with $n > 1$ and $k \in [n]$ admits an extension $h' : \Delta[n] \to X$.



For example, the *singular complex* $S(X)$ of a topological space X is a Kan complex . Recall that $S(X)_n$ is the set of continuous map $\Delta_n \to X$ where $\Delta_n \subseteq \mathbb{R}^n$ is the geometric n-simplex defined by the inequalities $0 \leq x_1 \leq \cdots \leq x_n \leq 1$.

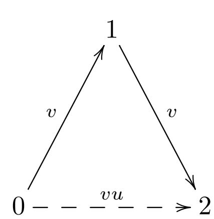
Definition 2.1. *We say that a simplicial set X is a* **quasi-category** *if every horn $h : \Lambda^k[n] \to X$ with $0 < k < n$ admits an extension $h' : \Delta[n] \to X$.*



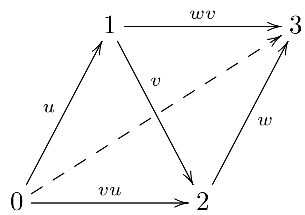
The notion of quasi-category was introduced by Boardman and Vogt [BV], but without a name. It is called an ∞-category by Lurie.

The nerve of a category C is a quasi-category $N(C)$. Every Kan complex is a quasi-category. The opposite X^{op} of a quasi-category (resp. Kan complex) X is a quasi-category (resp. a Kan complex).

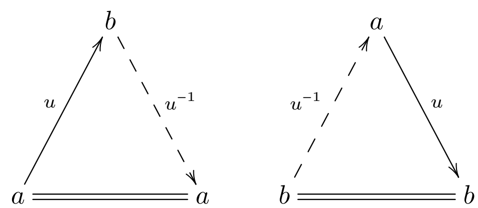
Remark. A simplicial set X is the nerve of a category if and only if every horn $h : \Lambda^k[n] \to X$ with $0 < k < n$ admits a unique extension $h' : \Delta[n] \to X$. For example, in the case $n = 2$ and $k = 1$, the operation $h \mapsto h'$ gives the composition law:



In the case $n = 3$ and $k = 1$ (or $k = 2$), the operation $h \mapsto h'$ gives the associativity law $w(vu) = (wv)u$.



Remark A simplicial set X is the nerve of a groupoid if and only if every map $h : \Lambda^k[n] \to X$ with $n > 1$ admits a unique extension $h' : \Delta[n] \to X$. The operation $h \mapsto h'$ produces left inverses in the case $n = 2$ and $k = 0$, and right inverses in the case $n = 2$ and $k = 2$



The cartesian product $X \times Y$ of two quasi-categories is a quasi-categories. Moreover, if X is a quasi-category, then so is the simplicial set X^A for any simplicial set A.

If X is a simplicial set, then there is a simplicial set of arrows $X(a, b)$ beween any two vertices $a, b \in X$. Recall that a vertex of the simplicial set $X^{\Delta[1]}$ is an arrow in the simplicial set X. From the maps $d^0, d^1 : \Delta[0] \to \Delta[1]$ we obtain two maps $X^{d_0}, X^{d_1} : X^{\Delta[1]} \to X^{\Delta[0]} = X$. The simplicial set $X(a, b)$ is defined to be the fiber of the map $(X^{d_1}, X^{d_0}) : X^{\Delta[1]} \to X \times X$ at the vertex $(a, b) \in X \times X$. In other words, we have a pullback square

$$\begin{array}{ccc} X(a,b) & \longrightarrow & X^{\Delta[1]} \\ \downarrow & & \downarrow{\scriptstyle (X^{d_1}, X^{d_0})} \\ 1 & \xrightarrow{(a,b)} & X \times X \end{array} \tag{1}$$

The simplicial set $X(a, b)$ is a Kan complex when X is a quasi-category. The *homotopy category* $ho(X)$ of a quasi category X is obtained by putting $Ob(ho(X)) := X_0$ and by putting $ho(X)(a, b) := \pi_0 X(a, b)$ for every $a, b \in X_0$. The composition operation

$$\pi_0 X(b, c) \times \pi_0 X(a, b) \to \pi_0 X(a, c)$$

is obtained by filling horns $\Lambda^1[2] \to X$. It turns out that $ho(X) = \tau_1(X)$.

Definition 2.2. *An arrow $f : a \to b$ in a quasi-category X is said to be* invertible*, or to be an* isomorphism*, if its image in the homotopy category $ho(X)$ is invertible.*

Theorem 2.3. [Jo1] *A quasi-category X is a Kan complex iff every arrow in X is invertible.*

In other words, a quasi-category X is a Kan complex iff its homotopy category $ho(X)$ is a groupoid. Kan complexes are to groupoids what quasi-categories are to categories.

Categories	Groupoids
Quasi-categories	Kan complexes

We shall often say that a vertex of a quasi-category X, is an *object* of X and that an arrow $f : a \to b$ is a *morphism.* We may also say that a map between quasi-categories $f : X \to Y$ is a *functor.* We shall say that a functor $f : X \to Y$ is *fully faithful* if the map $X(a, b) \to X(fa, fb)$ induced by f is a homotopy equivalence for every pair of objects $a, b \in X$. We shall say that a functor $f : X \to Y$ is *essentially surjective* if for every object $b \in Y$ there exists an object $x \in X$ together with an isomorphism $f(x) \to y$.

Definition 2.4. *We shall say that a functor between quasi-categories $f : X \to Y$ is a* categorical equivalence*, or an* equivalence of quasi–categories*, if it is fully faithful and essentially surjective.*

For example, a map between Kan complexes $f : X \to Y$ is a homotopy equivalence (as defined by Kan) if and only if it is a categorical equivalence.

If X and Y are quasi-categories, then a *natural transformation* $\alpha : f \to g$ between two functors $f, g : X \to Y$ is defined to be a morphism $f \to g$ in the quasi-category Y^X. A natural transformation $\alpha : f \to g$ is said to be a *natural isomorphism* if the morphism α is invertible in Y^X.

Proposition 2.5. *A natural transformation $\alpha : f \to g$ is invertible if and only if the morphism $\alpha(x) : f(x) \to g(x)$ is invertible in Y for every object $x \in X$.*

Proposition 2.6. *A functor between quasi-categories $f : X \to Y$ is a categorical equivalence if and only if there exists a functor $g : Y \to X$ together with two natural isomorphisms $\alpha : 1_X \to gf$ and $\beta : 1_Y \to fg$,*

2.1 On fibrations

Recall that a map of simplicial sets $f : X \to Y$ is said to be a *Kan fibration* if every commutative square

$$\begin{array}{ccc} \Lambda^k[n] & \xrightarrow{h} & X \\ \big\downarrow & \nearrow & \big\downarrow{\scriptstyle f} \\ \Delta[n] & \xrightarrow{y} & Y \end{array} \tag{2}$$

has a diagonal filler $h' : \Delta[n] \to X$.

Recall also that a map of simplicial sets $f : X \to Y$ is said to be a *trivial fibration* if every commutative square

$$\begin{array}{ccc} \partial\Delta[n] & \xrightarrow{h} & X \\ \big\downarrow & \nearrow & \big\downarrow{\scriptstyle f} \\ \Delta[n] & \xrightarrow{y} & Y \end{array} \tag{3}$$

has a diagonal filler $h' : \Delta[n] \to X$.

Definition 2.7. *We say a map of simplicial sets $f : X \to Y$ is a* left fibration *(resp.* mid fibration*,* right fibration*) if every commutative square (3) with $0 \le k < n$ (resp. $0 < k < n$, $0 < k \le n$) has a diagonal filler.*

These five classes of fibrations are closed under composition and base changes. Recall that the base change of a map $p : X \to B$ along a map $u : A \to B$ is defined to be the map $\pi_1 : A \times_B X \to A$ in a pullback square

$$\begin{array}{ccc} A \times_B X & \xrightarrow{\pi_2} & X \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ A & \xrightarrow{u} & B \end{array} \tag{4}$$

If p is a Kan fibration (resp. trivial fibration, left fibration, mid fibration, right fibration) then so is the map π_1.

A simplicial set X is a Kan complex if and only if the map $X \to 1$ is a Kan fibration. It follows that if $p : X \to B$ is a Kan fibration and B is a Kan complex, then X is a Kan complex.

Similarly, A simplicial set X is a quasi-category if and only if the map $X \to 1$ is a mid fibration. It follows that if $p : X \to B$ is a mid fibration and B is a quasi-category, then X is a quasi-category.

It turns out that a simplicial set X is a Kan complex if and only if the map $X \to 1$ is a left fibration (resp. right fibration). It follows that the fibers of a left fibration (resp. right fibration) $p : X \to B$ are Kan complexes.

2.2 On the coherent nerve functor

See [Cor], [Lu1].

Recall that a *simplicial category* $\mathcal{C}$ is a category enriched over the category of simplicial sets sSet: this means that the set of arrows $\mathcal{C}(A, B)$ between two objects of $\mathcal{C}$ is actually a *simplicial set* rather than an ordinary set, and also that the composition operation

$$\mathcal{C}(B, C) \times \mathcal{C}(A, B) \to \mathcal{C}(A, C)$$

is a map of simplicial sets. We shall denote the category of (small) simplicial categories by SCat.

Let us denote by KCat the category of categories enriched over Kan complexes and by QCat the category of quasi-categories.

We shall describe the *coherent nerve functor* $\tilde{N} : \mathsf{SCat} \to \mathsf{sSet}$ which associate a simplicial set $\tilde{N}(\mathcal{C})$ to any simplicial category $\mathcal{C}$. The simplicial set $\tilde{N}(\mathcal{C})$ is a quasi-category when the simplicial category $\mathcal{C}$ is enriched over Kan complexes.

The simplicial set $\tilde{N}(\mathcal{C})$ is constructed by using a functor $C_* : \Delta \to \mathbf{SCat}$. The objects of the simplicial category $C_\star[n]$ are the elements of $[n]$ and $C_\star[n](i, j) = \varnothing$ unless $i \leq j$, in which case $C_\star[n](i, j)$ is (the nerve of) the poset of subsets $S \subseteq [i, j]$ such that $\{i, j\} \subseteq S$. If $i \leq j \leq k$, the composition operation

$$C_\star[n](j, k) \times C_\star[n](i, j) \to C_\star[n](i, k)$$

is the union $(T, S) \mapsto T \cup S$.

The *coherent nerve* of a simplicial category $\mathcal{C}$ is the simplicial set $\tilde{N}(\mathcal{C})$ defined by putting

$$\tilde{N}(\mathcal{C})_n = \mathsf{SCat}(C_\star[n], X)$$

for every $n \geq 0$. This notion was introduced by Cordier in [Cor]. The simplicial set $\tilde{N}(\mathcal{C})$ is a quasi-category when $\mathcal{C}$ is enriched over Kan complexes [Cor]. The functor $\tilde{N} : \mathsf{SCat} \to \mathsf{sSet}$ has a left adjoint $\tilde{\tau}$ and the pair of adjoint functors

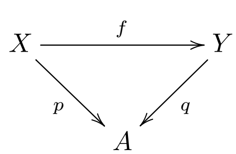
$$\tilde{\tau} : \mathsf{sSet} \longleftrightarrow \mathsf{SCat} : \tilde{N}$$

is a Quillen equivalence of model categories [Lu1].

3 On limits and colimits

3.1 On slices and coslices

We first recall the *slice category* $\mathcal{C}/A$ of a category $\mathcal{C}$ with respect to an object $A \in \mathcal{C}$. Recall that an *object over* A is an object $X \in \mathcal{C}$ equipped with a morphism $p : X \to A$; a morphism $(X, p) \to (Y, q)$ in $\mathcal{C}/A$ is a morphism $f : X \to Y$ such that the following triangle commutes



Dually, there is a coslice category $A\backslash\mathcal{C}$ for any object $A \in \mathcal{C}$.

Similarly, there is a slice simplicial set X/a for any vertex a of a simplicial set X. By construction, a n-simplex $\Delta[n] \to X/a$ is a simplex $x : \Delta[n+1] \to X$ such that $x(n+1) = a$. Dually, there is a coslice simplicial set $a\backslash X$ for any vertex $a \in X$. By construction, a n-simplex $\Delta[n] \to X/a$ is a simplex $x : \Delta[n+1] \to X$ such that $x(0) = a$. The simplicial sets X/a and $a\backslash X$ are quasi-categories when X is a quasi-category.

Recall that a functor $F : \mathcal{E} \to \mathcal{C}$ is said to be a *discrete fibration* if for every object $X \in \mathcal{E}$ and every map $g : Y \to F(X)$ in $\mathcal{C}$, there exists a unique map $f : X' \to X$ in $\mathcal{E}$ such that $F(f) = g$. If A is an object in a category $\mathcal{C}$, then the forgetful functor $\mathcal{C}/A \to \mathcal{C}$ is a discrete fibration.

If a is a vertex of a quasi-category X, then the map $p : X/a \to X$ is defined by putting $px = xd_{n+1}$ for a n-simplex $x : \Delta[n+1] \to X$ of X/a is a right fibration. Dually, the map $p : a\backslash X \to X$ defined by putting $px = xd_0$ for a n-simplex $x : \Delta[n+1] \to X$ in $a\backslash X$ is a left fibration.

3.2 On initial and terminal objects

See [Jo1] [Lu1].
We introduce the notions of initial, terminal and null objects.

Definition 3.1. *If X is a quasi-category, we say that an object $b \in X$ is* terminal *if the simplicial set $X(x,b)$ is contractible for every object $x \in X$. Dually, we say that an object $b \in X$ is initial if the simplicial set $X(b,x)$ is contractible for every object $x \in X$.*

Proposition 3.2. *If X is a quasi-category, then an object $b \in X$ is terminal if and only if the following equivalent conditions hold:*

- *every simplical sphere $x : \partial\Delta[n] \to X$ with $n > 0$ and $x(n) = b$ can be filled;*
- *the projection $X/b \to X$ is a categorical equivalence;*
- *the projection $X/b \to X$ is a trivial fibration.*

The notion of terminal vertex is invariant under categorical equivalence. More precisely, if $u : X \to Y$ is an equivalence of quasi-categories, then an object $a \in X$ is terminal in X iff the object $u(a)$ is terminal in Y. Moreover, the object $1_a \in X/a$ is terminal in X/a for any object $a \in X$.

The full simplicial subset spanned by the terminal (resp. initial) objects of a quasi-category is a contractible Kan complex when non-empty.

3.3 On join and cones

See [Jo1] [Lu1].
In this section we study the notions of limit and colimit in a quasi-category. We define the notions of cartesian product, of fiber product, of coproduct and of pushout. The notion of limit in a quasi-category subsume the notion of homotopy limits. For example, the loop space of a pointed object is a pullback and its suspension a pushout. We consider various notions of complete and cocomplete quasi-categories. Many results of this section are taken from [Jo2] and [Jo3].

If X is a quasi-category and A is a simplicial set, we say that a map $d : A \to X$ is a *diagram* indexed by A in X

The *join* of the simplices $\Delta[m]$ and $\Delta[n]$ is defined by putting $\Delta[m] \star \Delta[n] = \Delta[m+1+n]$. In general, the join of two simplicial sets X and Y can be defined by the formula

$$(X \star Y)_n = X_n \sqcup Y_n \sqcup \bigsqcup_{i+1+j=n} X_i \times Y_j$$

By construction, $X \sqcup Y \subseteq X \star Y$ and there is a n-simplex $\Delta[i] \star \Delta[i] \to X \star Y$ for each pair of simplices $\Delta[i] \to X$ and $\Delta[j] \to Y$ with $i+1+j = n$. For example, $(X \star Y)_1 = X_1 \sqcup Y_1 \sqcup X_0 \times Y_0$. The join operation $(X,Y) \mapsto X \star Y$ gives the category sSet the structure of a monoidal category $(\mathsf{sSet}, \star)$ with the empty simplicial set as the unit object. Notice that $A \star B \neq B \star A$ is general. The simplicial set $1 \star A = \Delta[0] \star A$ is a *cone* with base $A = \varnothing \star A$ and apex $1 = 1 \star \varnothing$. Dually, we shall say that the simplicial set $A \star 1 = A \star \Delta[0]$ is a *cocone* with base $A = A \star \varnothing$ and apex $1 = \varnothing \star 1$.

If X is a quasi-category, we shall say that a map of simplicial sets $d : A \to X$ is a *diagram indexed* by A in X.

Recall that for any vertex b of a simplicial set X there is a slice simplicial set X/b. By construction, a n-simplex $\Delta[n] \to X/b$ is a simplex $x : \Delta[n+1] \to X$ such that $x(n+1) = b$. More generally, for any map of simplicial sets $d : A \to X$, there is a slice simplicial set X/d; by construction, a n-simplex of X/d is a map $f : \Delta[n] \star A \to X$ such that $f|A = d$ (recall that $A \subset \Delta[n] \star A$). In particular, a vertex of X/d is a map $f : 1 \star A \to X$ such that $f|A = d$; we shall say that f is a *cone with base* d in X. We shall say that X/d is the simplicial set of cones with base d in X. The simplicial set X/d is a quasi-category if X is a quasi-category.

3.4 On limit cones

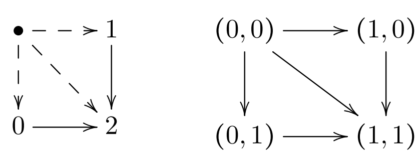
If X is a quasi-category, we shall say that a cone $c : 1 \star A \to X$ with base $d = c \mid A : A \to X$ is a *limit cone* if c is a terminal object of the quasi-category X/d; in which case, the vertex $c(1) \in X$ is said to be the (*homotopy*) *limit* of d and we write

$$c(1) = \varprojlim_{a \in A} d(a) = \varprojlim_{A} d.$$

Remark If $d : A \to X$ is a diagram in a quasi-category X, then the full simplicial subset of X/d spanned by the limit cones with base d is a contractible Kan complex when non-empty. It follows that the limit of the diagram $d : A \to X$ is homotopy unique when it exists.

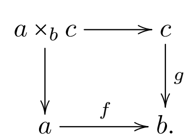
For example, a family of objects $(a_i \mid i \in I)$ in a quasi-category X is the same thing as a map of simplicial sets $d : I \cdot 1 \to X$ where $I \cdot 1 = \bigsqcup_I 1 = \bigsqcup_I \Delta[0]$. We shall write that $d : I \to X$. The product $a = \varprojlim_I d$ is equipped with a family of morphisms $\pi_i : a \to a_i$ (the *projections*) one for each $(i \in I)$; for every object $b \in X$, the maps $X(b, \pi_i) : X(b, a) \to X(b, a_i)$ are defined up to homotopy and the resulting map $X(b, a) \to \prod_{i \in I} X(b, a_i)$ is a homotopy equivalence.

The cone $\{\bullet\} \star \Lambda^2[2]$ is isomorphic to the square $\Delta[1] \times \Delta[1]$,



We say that a square $S : \Delta[1] \times \Delta[1] \to X$ is *cartesian*, or a *pullback*, if it is a limit cone.

A map $\Lambda^2[2] \to X$ is the same thing as a couple (f, g) of morphisms $f : a \to b$ and $g : c \to b$ having a commun target (in this case b). The limit of the diagram $(f, g) : \Lambda^2[2] \to X$ is called the *fiber product* f and g and it is often denoted $a \times_b c$.



The diagonal morphism $a \times_b c \to b$ is actually the cartesian product of f and g as objects of X/b.

A quasi-category X is said to have *fiber products* if every diagram $\Lambda^2[2] \to X$ has a limit. A quasi-category X has fiber product if and only if the quasi-category X/b has binary fiber products for every object $b \in X$.

We say that a quasi-category X has *finite limits* if every finite diagram $A \to X$ has a limit.

Proposition 3.3. *A quasi-category has finite limits if and only if it has a terminal object and fiber products.*

We say that a (large) quasi-category X is *complete* if every (small) diagram $A \to X$ has a limit. There is a dual notion of a *cocomplete* quasi-category.

We shall say that a functor between quasi-categories $f : X \to Y$ *preserves limits* if it takes every limit cone in X to a limit cone.

A right adjoint preserves all limits that exists.

The canonical functor $X \to ho(X)$ preserves all products that exist.

Let us say that a commutative square in the category of sets Set

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle v} \\ B & \xrightarrow{u} & D \end{array} \tag{5}$$

is *pseudo-cartesian* if the induced map $A \to B \times_D C$ is surjective. More generally we shall say that a square (5) in category $\mathcal{C}$ is *pseudo-cartesian* if its image by the functor $\mathcal{C}(K, -) : \mathcal{C} \to \mathsf{Set}$ is pseudo-cartesian for every object $K \in \mathcal{C}$.

Lemma 3.4. *If X is a quasi-category, then the canonical functor $X \to ho(X)$ takes cartesian squares to pseudo-cartesian squares.*

3.5 On fiber sequences

We shall say that a vertex $0 \in X$ in a quasi-category X is *null* if it is both initial and terminal. We shall say that a quasi-category X is *pointed* if it has a nul object $0 \in X$. If X is a pointed quasi-category then the projection $(X^{d_0}, X^{d_1}) : X^{\Delta[1]} \to X \times X$ admits a (homotopy unique) section which associates to a pair of objects $x, y \in X$ a *null morphism* $0 : x \to y$ obtained by composing the morphisms $x \to 0 \to y$.

Let X be a pointed quasi-category. A *null sequence* $a \to b \to c$ in X can be defined to be a commutative square $[S] : \Delta[1] \times \Delta[1] \to X$ with boundary $\partial[S] = ([S](d_1 \times id), [s](d_0 \times id), [s](id \times d_1), [s](id \times d_0)) = (0, v, u, 0)$,

$$\begin{array}{ccc} a & \xrightarrow{u} & b \\ \downarrow & [S] & \downarrow{\scriptstyle v} \\ 0 & \longrightarrow & c. \end{array} \tag{6}$$

A null sequence $a \to b \to c$ is a *fiber sequence* is the square $[s]$ is a pullback. For example, the *loop space* $\Omega(x)$ of an object $x \in X$ is defined by a fiber sequence $\Omega(x) \to 0 \to x$, that is, by a pullback square

$$\begin{array}{ccc} \Omega(x) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & x. \end{array}$$

Every fiber sequence $a \to b \to c$ has a canonical extension $\Omega(c) \to a \to b \to c$

$$\begin{array}{ccc} \Omega(c) & \longrightarrow & 0 \\ {\scriptstyle \partial}\downarrow & [S'] & \downarrow \\ a & \xrightarrow{u} & b \\ \downarrow & [S] & \downarrow{\scriptstyle v} \\ 0 & \longrightarrow & c. \end{array} \tag{7}$$

By iterating, we obtain a long fiber sequence

$$\cdots \longrightarrow \Omega^2(a) \xrightarrow{\Omega^2(u)} \Omega^2(b) \xrightarrow{\Omega^2(v)} \Omega^2(c) \xrightarrow{\Omega(\partial)} \Omega(a) \xrightarrow{\Omega(u)} \Omega(b) \xrightarrow{\Omega(v)} \Omega(c) \xrightarrow{\partial} a \xrightarrow{u} b \xrightarrow{v} c$$

$$\begin{array}{ccccccccc}
\Omega^3(c) & \longrightarrow & 0 & & & & & & \\
{\scriptstyle \Omega^2(\partial)}\downarrow & & \downarrow & & & & & & \\
\Omega^2(a) & \xrightarrow{\Omega(u)} & \Omega^2(b) & \longrightarrow & 0 & & & & \\
 & & {\scriptstyle \Omega(v)}\downarrow & & \downarrow & & & & \\
 & & \Omega^2(c) & \xrightarrow{\Omega(\partial)} & \Omega(a) & \longrightarrow & 0 & & \\
 & & \downarrow & & {\scriptstyle \Omega(u)}\downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \Omega(b) & \xrightarrow{\Omega(v)} & \Omega(c) & \longrightarrow & 0 \\
 & & & & \downarrow & & {\scriptstyle \partial}\downarrow & & \downarrow \\
 & & & & 0 & \longrightarrow & a & \xrightarrow{u} & b \\
 & & & & & & \downarrow & & \downarrow{\scriptstyle v} \\
 & & & & & & 0 & \longrightarrow & c
\end{array} \tag{8}$$

Dually, a null sequence $[S] : a \to b \to c$ is said to be a *cofiber sequence* if the square

$$\begin{array}{ccc}
a & \xrightarrow{u} & b \\
\downarrow & [S] & \downarrow{\scriptstyle v} \\
0 & \longrightarrow & c.
\end{array} \tag{9}$$

is a pushout, in which case the morphism $v : b \to c$ is said to be the *cofiber* of the morphism $u : a \to b$. For example, the *suspension* $\Sigma(x)$ of an object $x \in X$ is defined to be the cofiber of the morphism $x \to 0$,

$$\begin{array}{ccc}
x & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \Sigma(x)
\end{array} \tag{10}$$

In a pointed category with finite limits X, every cofiber sequence $a \to b \to c$ has a canonical extension $a \to b \to c \to \Sigma(a)$.

$$\begin{array}{ccccc}
a & \xrightarrow{u} & bv & \longrightarrow & 0 \\
\downarrow & [S] & \downarrow & [S'] & \downarrow \\
0 & \longrightarrow & c & \xrightarrow{\partial} & \Sigma(a)
\end{array} \tag{11}$$

By iterating, we obtain a long cofiber sequence

$$a \xrightarrow{u} b \xrightarrow{v} c \xrightarrow{\partial} \Sigma(a) \xrightarrow{\Sigma(u)} \Sigma(b) \xrightarrow{\Sigma(v)} \Sigma(c) \xrightarrow{\Sigma(\partial)} \Sigma^2(a) \xrightarrow{\Sigma^2(u)} \Sigma^2(b) \xrightarrow{\Sigma^2(v)} \Sigma^2(c) \longrightarrow \cdots$$

3.6 On adjoint functors

See [RV1].

Recall that *an adjunction* $\theta : F \dashv G$ between two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ is a family of bijections

$$\theta = \theta_{AB} : Hom(F(A), B) \simeq Hom(A, G(B))$$

natural in $A \in \mathcal{C}$ and $B \in \mathcal{D}$. We can then define a natural transformation $\eta : Id_{\mathcal{C}} \to GF$ by putting $\eta_A := \theta(1_{F(A)} : A \to G(F(A))$ for every object $A \in \mathcal{C}$ and a natural transformation $\epsilon : FG \to Id_{\mathcal{D}}$ by putting $\epsilon_B := \theta^{-1}(1_{G(B)} : FG(B) \to B$ for every object $B \in \mathcal{D}$. The natural transformation $\eta : Id_{\mathcal{C}} \to GF$ is called the *unit of the adjunction* and the natural transformation $\epsilon : FG \to Id_{\mathcal{D}}$ called the *counit of the adjunction.* The adjunction $\theta : F \dashv G$ is determined by its unit and by its counit.

A functor $F : \mathcal{C} \to \mathcal{D}$ has a right adjoint if and only if the category F/B defined by the pullback square

$$\begin{array}{ccc} F/B & \longrightarrow & \mathcal{D}/B \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array}$$

has terminal object $(G(B), \epsilon_B)$ for every object $B \in \mathcal{D}$. Dually, a functor $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint if and only if the category $A\backslash G$ defined by the pullback square

$$\begin{array}{ccc} A\backslash G & \longrightarrow & A\backslash \mathcal{C} \\ \downarrow & & \downarrow \\ \mathcal{D} & \xrightarrow{G} & \mathcal{C} \end{array}$$

has an initial object $(F(A), \eta_A)$ for every object $A \in \mathcal{C}$.

We shall say that functor between quasi-categories $f : X \to Y$ has a *right adjoint* if the quasi-category f/b defined by the pullback square

$$\begin{array}{ccc} f/b & \longrightarrow & Y/b \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

has terminal object $(g(b), \epsilon_b)$ for every object $b \in \mathcal{D}$, where $\epsilon_b : f(g(b)) \to b$. We can then construct a functor $g : Y \to X$ together with a natural transformation $\epsilon : fg \to id_X$.

Dually, a functor $g : Y \to X$ has a left adjoint if and only if the quasi-category $a\backslash g$ defined by the pullback square

$$\begin{array}{ccc} a\backslash g & \longrightarrow & a\backslash X \\ \downarrow & & \downarrow \\ Y & \xrightarrow{g} & X \end{array}$$

has an initial object $(f(a), \eta_a)$ for every object $a \in X$, where $\eta_a : a \to gf(a)$. We can then construct a functor $g : Y \to X$ together with a natural transformation $\epsilon : fg \to id_X$.

4 On stable quasi-categories

See [Lu2].

4.1 On bicartesian squares

A pointed quasi-category $\mathcal{E}$ is said to be **stable** if it has pullbacks, pushouts and cartesian squares coincide with cocartesian squares.

Obviously, the opposite of a stable quasi-category is stable.

Example 4.1. *The quasi-category of spectra* Sp *is the basic example of a stable quasi-category.*

Definition 4.2. *We shall say that a commutative square in a stable category* $\mathcal{E}$

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow \\ c & \longrightarrow & d \end{array}$$

is **bicartesian** *if the square is cartesian (hence also cocartesian).*

Lemma 4.3. (3-for-2 for bicartesian squares) *Consider the following commutative diagram in a stable quasi-category.*

$$\begin{array}{ccccc} a_0 & \longrightarrow & a_1 & \longrightarrow & a_2 \\ \downarrow & [A] & \downarrow & [B] & \downarrow \\ b_0 & \longrightarrow & b_1 & \longrightarrow & b_2 \end{array} \qquad \begin{array}{ccc} a_0 & \longrightarrow & a_2 \\ \downarrow & [A{+}B] & \downarrow \\ b_0 & \longrightarrow & b_2 \end{array}$$

If two of the three squares $[A]$, $[B]$ *and* $[A+B]$ *are bicartesian, then so is the third.*

The suspension $\Sigma(a)$ of an object a in a stable category $\mathcal{E}$ is defined by a pushout square

$$\begin{array}{ccc} a & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \Sigma(a) \end{array}$$

and the loop $\Omega(a)$ by a pullback square

$$\begin{array}{ccc} \Omega(a) & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & a \end{array}$$

We have $\Sigma\Omega(a) = a$ and $\Omega\Sigma(a) = a$, since the squares are bicartesian. We shall put $a[n] = \Sigma^n(a)$ and $a[-n] = \Omega^n(a)$ for every $n \geq 0$.

A null sequence $[S] : a \to b \to c$ in a stable quasi-category $\mathcal{E}$ is said to be **exact** if the corresponding square

$$\begin{array}{ccc} a & \xrightarrow{u} & b \\ \downarrow & [S] & \downarrow v \\ 0 & \longrightarrow & c. \end{array} \tag{12}$$

is bicartesian. A null sequence is exact if and only if it is a fiber sequence if and only if it is a cofiber sequence.

Every exact sequence $a \to b \to c$ in a stable quasi-category $\mathcal{E}$ can be extended naturally as a two sided long exact sequence

$$\cdots \longrightarrow \Omega(a) \xrightarrow{\Omega(u)} \Omega(b) \xrightarrow{\Omega(v)} \Omega(c) \xrightarrow{\partial} a \xrightarrow{u} b \xrightarrow{v} c \xrightarrow{\partial} \Sigma(a) \xrightarrow{\Sigma(u)} \Sigma(b) \xrightarrow{\Sigma(v)} \Sigma(c) \xrightarrow{\Sigma(\partial)} \cdots$$

If $[a,b]$ denotes the (pointed) hom space between two objects a and b in a stable category $\mathcal{E}$, then

$$\Omega[a,b] = [a,\Omega(b)] = [\Sigma(a),b]$$

The sequence of spaces $[a,b[n]]$ for $n \geq 0$ has the structure of a spectrum, since

$$\Omega[a,b[n+1]] = [a,\Omega b[n+1]] = [a,\Omega b[n]]$$

for every $n \geq 0$. It follows that the space $[a,b]$ has the structure of an infinite loop space. Thus, $Ext^0(a,b) := \pi_0[a,b]$ is an abelian group. We shall put $Ext^n(a,b) := \pi_0[a,b[n]]$ for every $n \in \mathbb{Z}$. Notice that for every $n \geq 0$,

$$\pi_n[a,b] = \pi_0\Omega^n[a,b] = \pi_0[a,\Omega^n b] = \pi_0[a,b[-n]] = Ext^{-n}(a,b).$$

For every object $x \in \mathcal{E}$, the functor

$$Ext^0(x,-) : \mathcal{E} \to \textsf{Abelian groups}$$

takes exact sequences to exact sequences of abelian groups. Dually, the contravariant functor

$$Ext^0(-,x) : \mathcal{E} \to \textsf{Abelian groups}$$

takes exact sequences to exact sequences of abelian groups.

An exact sequence $a \to b \to c$ in a stable quasi-category $\mathcal{E}$ gives rise to long exact sequences of abelian groups for every object $x \in \mathcal{E}$.

$$\cdots \longrightarrow Ext^{-1}(x,c) \xrightarrow{\partial} Ext^0(x,a) \longrightarrow Ext^0(x,b) \longrightarrow Ext^0(x,c) \xrightarrow{\partial} Ext^1(x,a) \longrightarrow \cdots$$

$$\cdots \longrightarrow Ext^{-1}(a,x) \xrightarrow{\partial} Ext^0(c,x) \longrightarrow Ext^0(b,x) \longrightarrow Ext^0(a,x) \xrightarrow{\partial} Ext^1(c,x) \longrightarrow \cdots$$

4.2 On t-structures

See [Lu2], Definition 1.2.1.4. and Remark 1.2.1.8.

Definition 4.4. *Let $\mathcal{E}$ be a triangulated quasi-category. We say that a pair $(\mathcal{E}^{\geq 0}, \mathcal{E}_{\leq -1})$ of full sub-quasi-categories of $\mathcal{E}$ is a* truncation structure *if the following conditions hold:*

1. *$\mathcal{E}(X,Y) = 0$ for every $X \in \mathcal{E}^{\geq 0}$ and $Y \in \mathcal{E}^{\leq -1}$.*
2. *the inclusion functor $\mathcal{E}^{\geq 0} \subseteq \mathcal{E}$ has a right adjoint $\tau^{\geq 0} : \mathcal{E} \to \mathcal{E}^{\geq 0}$ and the the inclusion functor $\mathcal{E}_{\leq -1} \subseteq \mathcal{E}$ has a left adjoint $\tau_{\leq -1} : \mathcal{E} \to \mathcal{E}_{\leq -1}$.*
3. *The null sequence $\tau^{\geq 0}(X) \to X \to \tau_{\leq -1}(X)$ is exact for every $X \in \mathcal{E}$.*

5 On universes

To be completed.

5.1 On universal left fibrations

See [Ci].

5.2 On Grothendieck fibrations

See [Ngu].

Definition 5.1. *Let $p : E \to B$ be a mid fibration between simplicial sets. We say that an arrow $f : a \to b$ in $\mathcal{E}$ is* cartesian *with respect to p if the map $E/f \to B/p(f) \times_{B/p(b)} E/b$ obtained from the commutative square*

$$\begin{array}{ccc} E/f & \longrightarrow & E/b \\ \downarrow & & \downarrow \\ B/p(f) & \longrightarrow & B/p(b) \end{array}$$

is a trivial fibration.

An arrow $f \in E$ is cartesian if and only if every commutative square

$$\begin{array}{ccc} \Lambda^n[n] & \xrightarrow{x} & E \\ \downarrow & & \downarrow p \\ \Delta[n] & \longrightarrow & B \end{array}$$

with $n > 1$ and $x(n-1, n) = f$ has a diagonal filler.

Definition 5.2. *We say that a map of simplicial sets $p : E \to B$ is a* Grothendieck fibration *if it is a mid fibration and for every vertex $b \in E$ and every arrow $g \in B$ with target $p(b)$ there exists a cartesian arrow $f \in E$ with target b such that $p(f) = g$.*

6 On sheaves of quasi-categories

To be completed.

6.1 On sheaves of Kan complexes

6.2 On sheaves of quasi-categories

6.3 On sheaves of stable quasi-categories

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