NANTES UNIVERSITE
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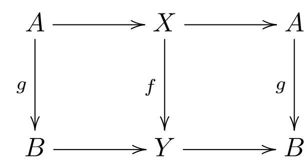
SMALL OBJECT ARGUMENT IN MODEL CATEGORIES

Contents

1 Model structure : definitions and properties

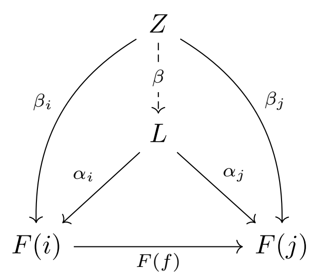
Definition 1.0.1. (1) Suppose $f : X \longrightarrow Y$ and $g : A \longrightarrow B$ are maps in a category $\mathcal{C}$. The map g is a *retract* of the map f if there is a commutative diagram



where the horizontal composites are identities.

(2) A *functorial factorization* is an ordered pair (α, β) of functors $Hom_{\mathcal{C}} \longrightarrow Hom_{\mathcal{C}}$ such that $f = \beta\,(f) \circ \alpha\,(f)$ for all $f \in Hom_{\mathcal{C}}$. In particular, the domain of $\alpha\,(f)$ is th domain of f, the codomain of $\alpha\,(f)$ is the domain of $\beta\,(f)$, and the codomain of $\beta\,(f)$ is the codomain of f.

Definition 1.0.2. (*limits/colimits*)
Let $\mathcal{C}$ be a category, I a small category, and $F : I \longrightarrow \mathcal{C}$ a functor. Then a *limit* of F is $L \in Ob(\mathcal{C})$ equipped with maps $\alpha_i : L \longrightarrow F\,(i)$ where $i \in Ob(I)$ such that for all $f : i \longrightarrow j$, $\alpha_j = F\,(f) \circ \alpha_i$ satisfying the following universal property :
For all object Z equipped with maps $\beta_i : Z \longrightarrow F\,(i)$ where $i \in Ob(I)$ such that for all $f : i \longrightarrow j$, $\beta_j = F\,(f) \circ \beta_i$, there exists an unique morphism $\beta : Z \longrightarrow L$ satisfying $\beta_i = \alpha_i \circ \beta$ for all $i \in Ob(I)$. This provides the following commutative diagram



If the limit exists, it is unique up to isomorphism. Note that the *colimit* is the dual notion of the limit.
A category $\mathcal{C}$ is said to have *all small (co)limits* if, for every small category I and every functor $F : I \longrightarrow \mathcal{C}$, a (co)limit of F exists.

Definition 1.0.3. Suppose $f : A \longrightarrow C$ and $g : B \longrightarrow D$ are maps in a category $\mathcal{C}$. Consider the commutative diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ C & \xrightarrow{g} & D \end{array}$$

Then
- i has the *left lifting property with respect to* p if there is a lift $h : C \longrightarrow B$ such that $hi = f$
- p has the *right lifting property with respect to* i if there is a lift $h : C \longrightarrow B$ such that $ph = g$

Notation : If i has the left lifting property with respect to a class of morphisms I, we denote it by $i \in LLP(I)$.
If p has the right lifting property with respect to a class of morphisms I, we denote it by $p \in RLP(I)$.

Lemma 1.0.1. Let I be a class of maps in a category $\mathcal{C}$. Then
(1) The maps of $LLP(I)$ are closed under retract, composition and pushout.
(2) The maps of $RLP(I)$ are closed under retract, composition and pullback.

Definition 1.0.4. A model category is a category $\mathcal{C}$ equipped with three subcategories
- The subcategory W of weak equivalences denoted by $\xrightarrow{\sim}$
- The subcategory Cof of cofibrations denoted by $\rightarrowtail$
- The subcategory fib of fibrations denoted by $\twoheadrightarrow$

and two functorial factorizations (α, β) and (α, β) satisfying the following properties :

(MC1 or limits/colimits) $\mathcal{C}$ is a category with all small limits and colimits.

(MC2 or 2-out-of-3) If f and g are morphisms of $\mathcal{C}$ such that the following composition is defined

$$A \xrightarrow{f} B \xrightarrow{g} C$$

and two of f, g and $g \circ f$ are weak equivalences, then so is the third.

(MC3 or retracts) W, Cof and Fib are closed under retracts i.e if f and g are morphisms of $\mathcal{C}$ such that f is a retract of g and g is a weak equivalence, cofibration or fibration, then so is f.

(MC4 or Lifting) Define the trivial cofibration by $W \cap Cof$ and the trivial fibrations by $W \cap Fib$. Define a commutative square

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \longrightarrow & D \end{array}$$

If $i \in W \cap Cof$ and $p \in Fib$ then there exists a lift h, and if $i \in Cof$ and $p \in W \cap Fib$ there exists a lift h. One says that trivial cofibrations have the left lifting property with respect to fibrations, and cofibrations have left lifting property with respect to trivial fibrations.

(MC5 or Factorization) Any morphism $f : X \longrightarrow Y$ has two functorial factorizations

$$X \rightarrowtail Q_f \overset{\sim}{\twoheadrightarrow} Y$$

denoted by $f = \beta(f) \circ \alpha(f)$ where $\alpha(f)$ is a cofibration and $\beta(f)$ is a trivial fibration,
and

$$X \overset{\sim}{\rightarrowtail} R_f \twoheadrightarrow Y$$

denoted by $f = \delta(f) \circ \gamma(f)$ where $\gamma(f)$ is a trivial cofibration and $\delta(f)$ is a fibration.

In a model category, the cofibrant and fibrant objects play an important role. They are the analogue of projective and injective modules in homological algebra.

Definition 1.0.5. Suppose $\mathcal{C}$ is a category and $X \in Ob(\mathcal{C})$. Then X is said to be :
(1) *cofibrant* if the initial morphism $0 \rightarrowtail X$ is a cofibration.
(2) *fibrant* if the terminal morphism $X \twoheadrightarrow \star$ is a fibration.

Note that by $(MC5)$, for all $X \in Ob(\mathcal{C})$, one has the following factorizations :

$$0 \rightarrowtail L(X) \overset{\sim}{\twoheadrightarrow} X$$

and

$$X \overset{\sim}{\rightarrowtail} R(X) \twoheadrightarrow \star$$

$L(X) \overset{\sim}{\twoheadrightarrow} X$ is said to be a *cofibrant resolution of* X and $X \overset{\sim}{\rightarrowtail} R(X)$ is said to be a *fibrant resolution of* X.

Lemma 1.0.2. (Retract argument)
Suppose we have a factorization $f = p \circ i$ in a category $\mathcal{C}$. Suppose that $f \in LLP(p)$. Then f is a retract of i. Dually, if $f \in RLP(i)$, then f is a retract of p.

Lemma 1.0.3. Suppose $\mathcal{C}$ is a model category. Then a map is a cofibration (a trivial cofibration) iff it has the left lifting property with respect to all trivial fibrations (fibrations).
Dually, a map is a fibration (a trivial fibration) iff it has the right lifting property with respect to all trivial cofibrations (cofibrations).
A map f is a weak equivalence iff $f = p \circ i$ with $i \in W \cap Cof$ and $p \in W \cap Fib$.

Corollary 1.0.1. Suppose $\mathcal{C}$ is a model category. Then
(1) If two of W, Cof and Fib are determined, so is the third.
(2) W, Cof and Fib are closed under composition.
(3) Cof and $Cof \cap W$ are closed under pushout.
(4) Fib and $Fib \cap W$ are closed under pullback.
(5) The isomorphisms are in $W \cap Cof \cap Fib$.

Consequence : In order to get a model category, it suffices to construct :
• Either W and Cof. And then Fib will be all maps satisfying $RLP(Cof \cap W)$.
• Or W and Fib. And then Cof will be all maps satisfying $LLP(Fib \cap W)$.
• Or Cof and Fib. Indeed since Cof and Fib determine respectively trivial fibrations and trivial cofibrations, they determine all the weak equivalences by $(MC2)$.

2 The small object argument

2.1 Some definitions...

Definition 2.1.1. Suppose $\mathcal{C}$ is a category, I a class of maps in $\mathcal{C}$ and $A \in Ob(\mathcal{C})$. Then
(1)A is said to be *sequentially small or* $\mathbb{N}$*-small* if for all functor $B_\star : \mathbb{N} \longrightarrow \mathcal{C}$ the map of sets

$$colim_n Hom_{\mathcal{C}}\left(A, B_n\right) \longrightarrow Hom_{\mathcal{C}}\left(A, colim_n B_\star\right)$$

is an isomorphism.
(2) A is said to be *small relative to* I if the above isomorphism is satisfied for sequences of maps in I.

Definition 2.1.2. Let I be a class of maps in a category $\mathcal{C}$.
(1)The maps of $LLP(RLP(I))$ are called the I-cofibrations. We denote I-cof the class of $LLP(RLP(I))$.
(2)The maps of $RLP(LLP(I))$ are called the I-fibrations. We denote I-fib the class of $RLP(LLP(I))$.

Remark : $I \subset I - Cof$ and $I \subset I - fib$

Definition 2.1.3. Let I be a set of maps in a category $\mathcal{C}$ containing all small colimits. A transfinite composition of pushouts of elements of I is called a *relative I-cell complex*. The collection of relative I-cell is denoted by I-cell. $A \in \mathcal{C}$ is an I-cell complex if the map $0 \longrightarrow A$ is a relative I-cell complex.

Let's recall some useful properties of relative I-cell complexes.

Lemma 2.1.1. Suppose I is a class of maps in a category $\mathcal{C}$ containing all small colimits. Then $I-cell \subset I-cof$

Lemma 2.1.2. Suppose $\mathcal{C}$ a category containing all small colimits. Let's consider the sequence

$$G^0 \xrightarrow{i_1} G^1 \xrightarrow{i_2} G^2 \xrightarrow{i_3} \cdots \xrightarrow{i_k} G^k \xrightarrow{i_{k+1}} \cdots \to G^\infty = colim_k G^k$$

in the category $\mathcal{C}$. Let's denote i_∞ the transfinite composition of the maps i_k. If for all $k \geq 0$, $i_k \in LLP(I)$, then $i_\infty \in LLP(I)$.

Lemma 2.1.3. Suppose $\mathcal{C}$ is a category containing all small colimits. Let I be a set of maps of $\mathcal{C}$. Then I-cell is closed under transfinite compositions.

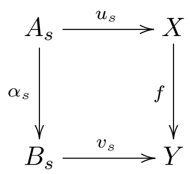
Lemma 2.1.4. Suppose $\mathcal{C}$ is a category containing all small colimits. Let I be a set of maps of $\mathcal{C}$. Then any pushout of coproducts of maps of I is in I-cell.

2.2 Construction

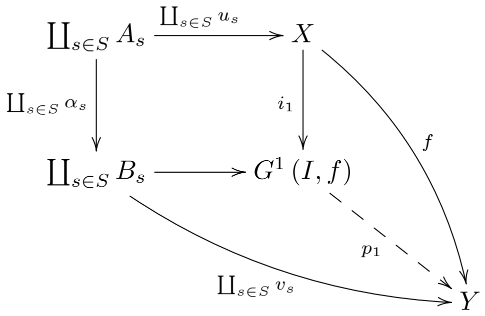
Let I be a set of maps of a category $\mathcal{C}$. Take a map $f : X \longrightarrow Y$ in $\mathcal{C}$. Recursively, we will construct a composition $f = p_\infty \circ i_\infty$ such that $i_\infty \in I - cell$ and $p_\infty \in RLP(I)$. This construction is called *the small object argument.* In addition, this construction is a powerful tool which provides a functorial factorization for any category in which cofibrations and trivial cofibrations are in I-cell.

Theorem 2.2.1. (**The small object argument**)
Suppose $\mathcal{C}$ is a category containing all small colimits, and I is a set of maps in $\mathcal{C}$. Suppose the domains of the maps of I are small relative to I-cell. Then any morphism f in $\mathcal{C}$ has a functorial factorization $f = p_\infty \circ i_\infty$ such that $p_\infty \in RLP(I)$ and $i_\infty \in I - cell$.

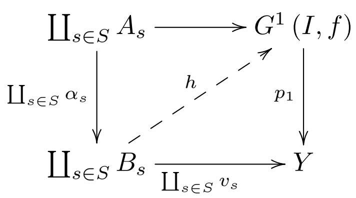
Proof. Let $f : X \longrightarrow Y$ be a map in $\mathcal{C}$. We want to construct a factorization $f = p_\infty \circ i_\infty$ such that $p_\infty \in RLP(I)$ and $i_\infty \in I - cell$.
• Let's start with the construction of the factorization $f = p_\infty \circ i_\infty$. Denote $S(I, f)$ the set of all commutative squares (a commutative square is denoted by s) of the following form



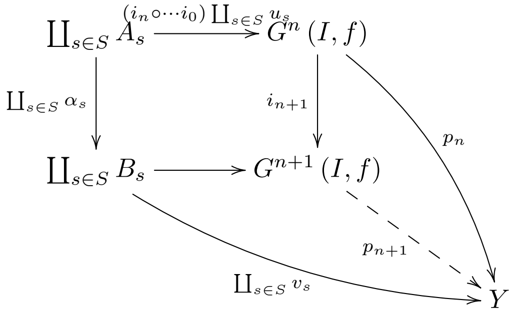
where $\alpha_s \in I$. We will do a recursive construction. Define $G^1(I, f)$ to be the pushout in the diagram below.



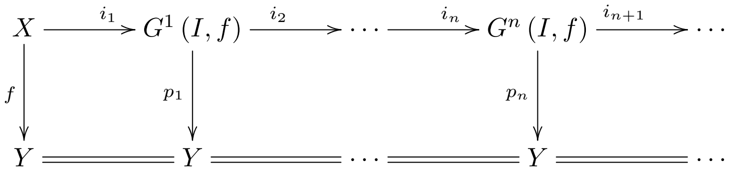
where the map p_1 follows from the universal property of pushout. Note that $i_1 \in I - cell$ by Lemma 2.1.4. In addition, $p_1 \in RLP(I)$ by the induced diagram



The construction is defined recursively for all $n \in \mathbb{N}$ as follows :
$p_0 = f$, $i_0 = id_X$, $G^{n+1}(I,f) = G^1(I,p_n)$
the map i_{n+1} is defined by the following pushout



and the map p_{n+1} follows from the universal property of the previous pushout.
By this construction, one gets the following commutative diagram

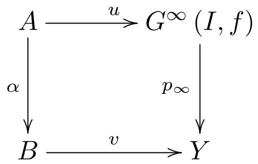


Since the category $\mathcal{C}$ contains all small colimits, one has the following sequence

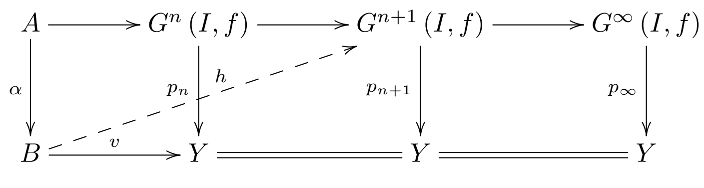
$$X \to G^1(I,f) \to G^2(I,f) \to \cdots \to G^n(I,f) \to \cdots \to G^\infty(I,f) = colim_n G^n(I,f)$$

We denote the transfinite composition of pushout by $i_\infty : X \longrightarrow G^\infty(I,f)$. It follows that $i_\infty \in I - cell$.
By the universal property of the colimit $G^\infty(I,f)$, f factors through the map $p_\infty : G^\infty(I,f) \longrightarrow Y$ such that $f = p_\infty \circ i_\infty$.

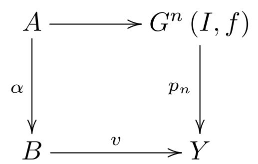
• It remains now to show that $p_\infty \in RLP(I)$. Suppose now we have a commutative square as follows



where α is a map of I. Since A is small relative to I-cell, then $Hom_{\mathcal{C}}(A, G^\infty(I,f)) \cong colim_n Hom_{\mathcal{C}}(A, G^n(I,f))$. Therefore there is $n \in \mathbb{N}$ such that $u : A \longrightarrow G^\infty(I,f)$ factors through $G^n(I,f)$. By construction one gets the following diagram



Since the commutative square



belongs to $S(I,p_n)$, then the map $\coprod_{s\in S} B_s \longrightarrow G^{n+1}(I,f)$ in the pushout defining $G^{n+1}(I,f)$ induces a map $h : B \longrightarrow G^{n+1}(I,f)$. Therefore the map h induces the required lift $r : B \longrightarrow G^\infty(I,f)$ in the diagram (-). So, $p_\infty \in RLP(I)$ ■

2.3 Cofibrantly generated model categories

The main use of the small object argument is the construction of factorizations for a category $\mathcal{C}$ in which the cofibrations and the trivial cofibrations are relative I-cell complexes. Such a category is called *cofibrantly generated model category.*
Recall that in a model category, if I is the class of cofibrations (trivial cofibrations) then $I - cof = I$. And if I is the class of fibrations (trivial fibrations) then $I - fib = I$. Hence the underlying idea in the construction of cofibrantly generated model categories consists to construct two sets of maps I and J such that I-cof is the class of cofibrations and J-cof is the class of trivial cofibrations.

Definition 2.3.1. Suppose $\mathcal{C}$ is a model category. $\mathcal{C}$ is said to be *cofibrantly generated* if there are sets I and J of maps such that :
(1)The domains of the maps of I are small relative to I-cell.
(2)The domains of the maps of J are small relative to J-cell.
(3)The class of fibrations is $RLP(J)$.
(4)The class of trivial fibrations is $RLP(I)$.

The elements of I are called the *generating cofibrations* and the elements of J are called the *generating trivial cofibrations*. A cofibrantly generated model category $\mathcal{C}$ is called *finitely generated* if we can choose the sets I and J so that the domains and codomains of I and J are finite relative to I-cell.

We sum up the basic properties of cofibrantly generated model categories, in the following proposition.

Proposition 2.3.1. *Suppose $\mathcal{C}$ is a cofibrantly generated model category, with I as the set of generating cofibrations and J as the set of generating trivial cofibrations. Then :*
(1) The cofibrations form the class I-cof.
(2) Every cofibration is a retract of a relative I-cell complex.
(3) The domains of I are small relative to the cofibrations.
(4) The trivial cofibrations form the class J-cof.
(5) Every trivial cofibration is a retract of a relative J-cell complex.
(6) The domains of J are small relative to the trivial cofibrations.
If $\mathcal{C}$ is finitely generated, then the domains and codomains of I and J are finite relative to the cofibrations.

Proof. See Proposition 2.1.18 in [1]

Thanks to tools given by the small object argument, the following theorem shows how to construct cofibrantly generated model categories.

Theorem 2.3.1. *Suppose $\mathcal{C}$ is a category with all small colimits and limits. Suppose W is a subcategory of $\mathcal{C}$, and I and J are sets of maps of $\mathcal{C}$. Then there is a cofibrantly generated model structure on $\mathcal{C}$ with I as the set of generating cofibrations, J as the set of generating trivial cofibrations, and W as the subcategory of weak equivalences iff the following conditions are satisfied :*
(1) The subcategory W has the 2-out of-3 property and is closed under retracts.
(2) The domains of I are small relative to I-cell.
(3) The domains of J are small relative to J-cell.
(4) $J - cell \subseteq W \cap I - cof$.
(5) $RLP(I) \subseteq W \cap RLP(J)$.
(6) Either $W \cap I - cof \subseteq J - cof$ or $W \cap RLP(J) \subseteq RLP(I)$.

Proof. See Theorem 2.1.19 in [1]

3 Model structure on the category $Ch\,(R)$

We will be using the category of chain complexes of modules over a ring R. We do not assume that our chain complexes are bounded below. In this section, we will describe the standard model structure on $Ch\,(R)$.
Let's first note that the category $Ch\,(R)$ has all small limits and colimits, which are taken degreewise. The initial and terminal object is the chain complex 0, that is 0 in each degree. The category $Ch\,(R)$ is also an abelian category, where short exact sequences are defined degreewise.
Since the small object argument will be useful, we recall the following lemma.

Lemma 3.0.1. Every object in $Ch\,(R)$ is small.

Proof. See Lemma2.3.2 in [1]

Let's define now the **standard model structure** on $Ch\left(R\right)$.

Definition 3.0.1. (1) For all $n \geq 1$, define the chain complex D^n by

$$D_k^n = \begin{cases} 0 \quad if \quad k \neq n, n-1 \\ \mathbb{K}b_{n-1} \quad if \quad k = n-1 \\ \mathbb{K}e_n \quad if \quad k = n \end{cases}$$

where $deg\left(b_{n-1}\right) = n-1$, $deg\left(e_n\right) = n$ and the differential δ is given by $\delta\left(e_n\right) = b_{n-1}$
Define the chain complex S^n by

$$S_k^n = \begin{cases} 0 \quad if \quad k \neq n \\ \mathbb{K}b_n \quad if \quad k = n \end{cases}$$

where $deg\left(b_n\right) = n$.
There is an evident injection $S^{n-1} \longrightarrow D^n$. Define $I = \{i_n : S^{n-1} \longrightarrow D^n\}$ and $J = \{j_n : 0 \longrightarrow D^n\}$

(2) Define a map to be a fibration if it is in $RLP(J)$.
(3) Define a map to be a cofibration if it is in $I - cof$.
(4) Define a map f to be a weak equivalence if the induced map $H_n f$ on homology is an isomorphism for all n.

Theorem 3.0.1. *$Ch\left(R\right)$ is a cofibrantly generated model category with I as its generating set of cofibrations, J as its generating set of trivial cofibrations, and homology isomorphisms as its weak equivalences. The fibrations are the surjections.*

Note that this theorem claims that the category $Ch\left(R\right)$ (of unbounded chain complex over R) is given the structure of model category such that :
• The weak equivalences are quasi-isomorphisms.
• The fibrations are degreewise surjection maps.
• The cofibrations are dimensionwise split inclusion with cofibrant cokernel.

The proof of the Theorem 3.0.1 will consist to verify the hypotheses of the Theorem 2.3.1. Let's go...

Note that the hypotheses (2) and (3) of Theorem 2.3.1 are satisfied. This is a consequence of Lemma 3.0.1. In fact, the complexes 0, S^n and D^n are small. Particularly, they are small relative to I-cell and J-cell.

3.1 Characterization of weak equivalences

We simply verify that the hypothese (1) of Theorem 2.3.1 is satisfied. By definition, a chain map f is a weak equivalence if the induced map $H_n f$ on homology is an isomorphism. Since homology is functorial, the weak equivalences are closed under retracts. The axiom 2-out of-3 is satisfied.
In fact, since isomorphisms satisfy the axiom 2-out-of-3, so do the quasi-isomorphisms. Moreover, since the retract of an isomorphism is an isomorphism, so the retract of a quasi-isomorphism is a quasi-isomorphism.

3.2 Characterization of fibrations and trivial fibrations

Proposition 3.2.1. *(1) A map $f : D^n \longrightarrow X$ is entirely determined by an element of X_n*
(2) A map $p : X \longrightarrow Y$ in $Ch\left(R\right)$ is a fibration iff $p_n : X_n \longrightarrow Y_n$ is sujective for all n.

Proof. For the first part, by linearity $f : D^n \longrightarrow X$ is determined by $f_{n-1}\left(b_{n-1}\right)$ and $f_n\left(e_n\right)$. Moreover, f commutes with the differential. Thus, one gets : $f_{n-1}\left(b_{n-1}\right) = f_{n-1}\left(de_n\right) = df_n\left(e_n\right)$
So, f is uniquely determined by $f_n\left(e_n\right)$.

For the second part, let p be a fibration such that $p \in RLP\left(J\right)$. A commutative square

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & \circlearrowleft & \downarrow p \\ D^n & \longrightarrow & Y \end{array}$$

is determined by an element $y_n \in Y_n$. A lift in this square is equivalent to an element $x_n \in X_n$ such that $p_n\left(x_n\right) = y_n$. Therefore, $p \in RLP\left(J\right)$ iff each $p_n : X_n \longrightarrow Y_n$ is sujective. ■

Proposition 3.2.2. *(1) A map $f : S^n \longrightarrow X$ is equivalent to an n-cycle in X.*
(2) A map $p : X \longrightarrow Y$ in $Ch(R)$ is a trivial fibration iff $f \in RLP(I)$.

Proof. For the first part, $f : S^n \longrightarrow X$ is determined by $f(e_n)$. Moreover, f commutes with the differential and then : $df_n(e_n) = f_{n-1}(de_n) = 0$. So, f is determined by an n-cycle in X.

For the second part, let's consider the map $p : X \longrightarrow Y$. The set of commutative squares

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{f} & X \\ \downarrow & \circlearrowleft & \downarrow p \\ D^n & \xrightarrow{g} & Y \end{array}$$

is one-to-one correspondence with $\{(y,x) \in Y_n \oplus Z_{n-1}X : p(x) = d(y)\}$. A lift in a such square is an element $z \in X_n$ such that $d(z) = x$ and $p(z) = y$. In fact, the upper triangle is commutative since f is equivalent to an element x of degree $n-1$ and the composition map $S^{n-1} \longrightarrow D^n \longrightarrow X$ is equal to $d(z)$ in degree $n-1$ by commutation with the differentials. The lower triangle is also commutative.

Suppose now that $p \in RLP(I)$. Let $y \in Z_nY$ be a cycle. Then the pair $(y,0)$ defines a square as above. So there exists $z \in X_n$ such that $d(z) = 0$ and $p(z) = y$. Hence $Z_np : Z_nX \longrightarrow Z_nY$ is surjective. It follows that H_np is surjective. One can deduce that p is surjective. Indeed, let $y \in Y_n$. Then $d(y) \in Z_{n-1}Y$. . So there is a class $x \in Z_{n-1}X$ such that $p(x) = d(y)$ by the sujectivity of $Z_{n-1}p$. The pair (y,x) determines a square as above. Since $p \in RLP(I)$, there exists a lift in the square and then there exists an element $z \in X_n$ such that $p(z) = y$. Hence p is surjective. By Proposition 3.2.1, p is a fibration.
It remains to prove that H_np is injective. Take $z \in Z_nX$ such that $p(x) = d(y)$ for some $y \in Y_{n+1}$. Then (y,x) determines a commutative square as above. Since $p \in RLP(I)$, then there exists a lift in this square, and then an element $z \in X_n$ such that $d(z) = x$. Then, only the zero class is mapped to the zero class. So H_np is injective and then H_np is an isomorphism. Therefore p is a trivial fibration.

Now suppose that $p : X \longrightarrow Y$ is a trivial fibration. We have to show that $p \in RLP(I)$ ie given a pair $(y,x) \in Y_n \oplus Z_{n-1}X$ such that $p(x) = d(y)$ there exists $z \in X_n$ such that $d(z) = x$ and $p(z) = y$. Since p is surjective, one gets the following exact sequence

$$0 \longrightarrow Ker(p) \longrightarrow X \longrightarrow Y \longrightarrow 0$$

Since p is a quasi-isomorphism, $H_\star(Ker(p)) = 0$ by the long exact homology.
Let's choose $w \in X_n$ such that $p(w) = y$. So $dp(w) = p(dw) = dy = p(x)$. Then $dw - x \in ker(p)_{n-1}$ since $p(dw - x) = 0$. Moreover, $d(dw - x) = d^2w - dx = 0$. Then $dw - x \in Z_{n-1}Ker(p)$. Since $H_\star(Ker(p)) = 0$ there is $v \in Ker(p)_n$ such that $dv = dw - x$. Set $z = w - v$. Then $p(z) = p(w) - p(v) = y - 0 = y$ and $d(z) = dw - dv = x$. This proves that $p \in RLP(I)$ ■

From this Proposition, it follows that the hypotheses (5) and (6) of Theorem 2.3.1 are satisfied.

3.3 Characterization of cofibrations and trivial cofibrations

It remains to verify the hypothese (4) of Theorem 2.3.1. For this purpose, we will characterize cofibrations and trivial cofibrations.

Definition 3.3.1. Two chain maps $f, g : X \longrightarrow Y$ are said to be chain homotopic if there is a collection of R-modules $D_n : X_n \longrightarrow Y_n$ such that $dD_n + D_{n-1}d = f_n - g_n$

Lemma 3.3.1. Suppose R is a ring. If A is a cofibrant chain complex, then A_n is a projective R-module for all n. Conversely, any bounded below complex of projective R-modules is cofibrant.

Proof. See Lemma 2.3.6 in [1]

Remark : Not every complex of projective R-modules is cofibrant.

Lemma 3.3.2. Suppose R is a ring, C is a cofibrant chain complex, and $H_\star(K) = 0$. Then every map from C to K is chain homotopic to 0.

Proof. See Lemma 2.3.8 in [1]

The previous results are useful to characterize cofibrations and trivial cofibrations.

Proposition 3.3.1. *Suppose R is a ring. Then a map $i : A \longrightarrow B$ in $Ch(R)$ is a cofibration iff i is a dimensionwise split inclusion with cofibrant cokernel.*

Proof. See Proposition 2.3.9 [1]

Proposition 3.3.2. *Suppose R is a ring. Then a map $i : A \longrightarrow B$ is in J-cof iff i is an injection whose cokernel is projective as a chain complex. In particular, every map in J-cof is a trivial cofibration.*

Proof. We will prove the two parts.
• For the first part, suppose $i : A \longrightarrow B$ is an injection with projective cokernel $j : B \longrightarrow C$. Consider a commutative square as follows

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array}$$

where $p \in RLP\,(J)$ i.e p is surjective. Since the cokernel C of i is projective, then C is a direct summand in B and there is a retraction $r : B \longrightarrow A$. The map $pfr - g : B \longrightarrow Y$ satisfies $(pfr - g)\,i = 0$. In fact $(pfr - g)\,i = pf(ri) - gi = pf - gi = 0$. By the universal property of cokernel, the map $pfr - g$ factors through a map $s : C \longrightarrow Y$. Since C is projective, there is a map $t : C \longrightarrow X$ such that $p \circ t = s$. On gets

$$p\,(fr - tj) = pfr - sj = pfr - (pfr - g) = g$$

and

$$(fr - tj)i = f(ri) - t(ji) = f$$

Then $fr - tj$ is the desired lift in the above square. Therefore $i \in J - cof$.
Conversely, suppose $i : A \longrightarrow B$ is in $J - cof$ i.e i has the left lifting property with respect to surjections. In particular, there exists a lift in the following commutative square

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow \\ B & \longrightarrow & 0 \end{array}$$

Hence i is injective. Denote $j : B \longrightarrow C$ the cokernel of i. We have to show that C is projective. Suppose $p : X \longrightarrow Y$ is a surjection and $f : C \longrightarrow Y$ is a map. One gets the following commutative square

$$\begin{array}{ccc} A & \xrightarrow{0} & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow[f\circ j]{} & Y \end{array}$$

Indeed, since $j \circ i = 0$ the square is commutative. Moreover since $p \in RLP(J)$ and $i \in J - cof$ there exists a lift h. Since $h \circ i = 0$ h factors through a map $g : C \longrightarrow X$ such that $h = g \circ j$. In addition $pg = f$ since j is surjective and $fj = ph = pgj$. Hence g is the desired lift. Therefore C is projective as desired.

• For the second part, suppose $i : A \longrightarrow B$ is in $J - cof$. So i is an injection whose cokernel C is projective as chain complex. Particularly C is dimensionwise projective and then i is dimensionwise split injection. In addition, since C is projective, C has a lift with respect to surjections particularly trivial surjections. Since trivial surjections are in $RLP(I)$, the initial morphism $0 \longrightarrow C$ is in $LLP(RLP(I)) = I - cof$. This means that C is cofibrant. The map i is a dimensionwise split inclusion with cofibrant cokernel. So $i : A \longrightarrow B$ is a cofibration by Proposition 3.3.1. It remains to prove that i is a weak equivalence i.e a quasi-isomorphism.
One has the following short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

In order to prove that i is a quasi-isomorphism (i.e i induces an isomorphism in homology), it suffices to prove that C is acyclic. Let C be projective. Let's consider the complex defined by $P_n = C_n \oplus C_{n+1}$ where $d(x, y) = (dx, x - dy)$ and the surjection $\pi : P \longrightarrow C$ is given by $(x, y) \mapsto x$. Since C is projective, there is map $\psi : C \longrightarrow P$ such that $\pi \circ \psi = Id_C$. Then the first component of ψ is Id_C. The second component is a collection of maps $D_n : C_n \longrightarrow C_{n+1}$ such that ψ commutes with the differentials. But $d\psi = (d, Id - dD)$ and $\psi d = (d, Dd)$. And then $Id = Dd + dD$. In particular, if x is a cycle, then $dD(x) = x$ so x is also a boundary. Hence C is acyclic. ■

From this Proposition, it follows that the hypothese (4) of Theorem 2.3.1 is satisfied.

Since all the hypotheses of Theorem 2.3.1 are satisfied, we have just proved the Theorem 3.0.1.

Remark : From Theorem 3.0.1, it follows that every trivial cofibration is in J-cof, and so is an injection with projective cokernel. In particular, X is projective iff it is cofibrant and acyclic. Note that every chain complex is fibrant in this standard model structure on $Ch(R)$.

References

[1] M. HOVEY, *Model Category*, Second edition, Mathematical Surveys and Monographs, vol.**63**, American Mathematical Society, 1999.

[2] S. YALIN, *Cours sur les catégories de Modèles*, Nantes Université, 2023.