POLYHEDRAL LAGRANGIAN SURFACES AND MOMENT MAP FLOW

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Abstract. We show that polyhedral surfaces of a symplectic vector space V can be understood as the vanishing locus of a moment map associated to a particular Hamiltonian torus action on a finite dimensional symplectic space. This interpretation allows to introduce an associated modified moment map flow, which is an ODE whose limits are essentially the (parametrized) Lagrangian surfaces. This approach provides an effective method for constructing Lagrangian polyhedral surfaces of $\mathbb{R}^4$. This paper is provided with a computer program which shows the modified flow as a real-time movie, in the case of a 2-dimensional torus.

1. Introduction

1.1. **Motivations.** *Flexibility* issues have been a major center of interest in symplectic geometry. In the opposite direction, rigidity properties, like the *Gromov non-squeezing theorem*, have motivated the introduction of *pseudo-holomorphic curves* techniques and *Lagrangian Floer Homology*, which are cornerstone objects for symplectic geometers and topologists. Similarly to the *sphere eversion theorem*, lots of flexibility questions do not have straightforward answers, particularly in symplectic geometry. Then it was shown by Gromov that solving certain classes of underdetermined partial differential relations can be understood using the point of view of h-principle [7] (see [4] as well). For example, any smoothly immersed surface of $\mathbb{R}^4$ can be approximated by arbitrarily close immersed Lagrangian surfaces, in the C^0-sense, by a theorem of Gromov. Notice that the proof of this theorem uses stability properties of symplectic structures and ordinary differential equations techniques, among many other things. Those properties, specific to smooth manifolds, are not directly available for *piecewise linear* maps and *polyhedral geometry*. Since these essential tools are missing, very little is known about sensible versions of piecewise linear symplectic geometry. As an evidence, we mention several elementary results of symplectic geometry, whose piecewise linear analogues are poorly understood:

(1) In spite of the fact that triangulations are quite common in geometry and topology, it is not known whether a smooth symplectic manifold admits a *symplectic triangulation*. This question studied by

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Distexhe [2] remains opened, even in the basic case of $\mathbb{CP}^n$ for $n \geq 2$. Conversely, it is not known if there are suitable smoothing techniques for piecewise linear symplectic manifolds.

(2) By Darboux theorem, every smooth symplectic structure is locally standard, which shows that smooth symplectic manifold do not have any local invariants. However, Darboux theorem has no analogue for piecewise linear symplectic manifolds. In fact, there might even be local obstructions for the local triviality of piecewise linear symplectic structures up to piecewise linear homeomorphisms!

(3) There are many constructions of Lagrangian manifolds for smooth symplectic manifolds. Lagrangian submanifolds were indeed studied a lot as they are key objects for symplectic geometers and topologists. For example, Lagrangians submanifolds are a natural boundary condition for pseudo-holomorphic curves, so that they appear naturally in Floer homology. The deformation theory of Lagrangians is simple to understand in the smooth setting. Furthermore, the h-principle holds for immersed Lagrangians submanifolds, according to Gromov's theorem. Yet, examples of polyhedral Lagrangian manifolds are scarce. Jauberteau-Rollin-Tapie [8, 11] tackled this question in some particular cases (cf. below for more details).

(4) Suitable piecewise linear analogues of the group of symplectic and Hamiltonian diffeomorphisms are not well understood. A first step was made by Gratza, who introduced some clever (and quite difficult) approximation techniques in [6].

Because piecewise linear or polyhedral geometry is such natural framework, it has attracted a lot of attention in the context of topology, as a tool for studying relations between smooth and topological manifolds. It is our opinions that problems of a symplectic nature should deserve the same treatment, in the piecewise linear setting. Numerical experiments for various geometric flows involving symplectic manifolds, rely on the use of polyhedral geometry for discretization purposes. Thus, obvious questions arise from a numerical perspective as well. In fact, we we realized that so little is known about piecewise linear symplectic geometry as we were in the process of studying certain *mean curvature geometric flows for Lagrangian submanifols* and their possible numerical versions.

Far from the *soft techniques*, like *holonomic approximation*, specific to the h-principle approach, it is worth mentioning that Lagrangian surfaces can be understood as the zeroes of a certain *Donaldson moment map*. This is a rather different point of view where much more geometry comes into the picture. More generally, consider a real symplectic vector space (V, ω_V), a closed surface Σ and the moduli space of smooth maps

$$\mathscr{M} = \{f : \Sigma \to V, f \in C^\infty(\Sigma, V)\}.$$

A map $f \in \mathscr{M}$ called *isotropic*, if the symplectic form vanishes along f. This condition can be written in terms of pullback as $f^*\omega_V = 0$. Hence, the set of isotropic maps is a subspace of the moduli space $\mathscr{I} \subset \mathscr{M}$ given by the equation:

$$\mathscr{I} = \{f \in \mathscr{M}, f^*\omega_V = 0\}.$$

Remark 1.1.1. In the case where V has real dimension 4 and $f : \Sigma \to V$ is an *immersion* (resp. an *embedding*), the image of f is known as an *immersed* (resp. an *embedded*) *Lagrangian surface* of V.

It turns out that the subset $\mathscr{I} \subset \mathscr{M}$ can be understood as the *vanishing locus* of a *moment map*. This fact, coined by Donaldson [3], provided valuable input for constructing polyhedral isotropic surfaces in [8, 11]. The idea is to consider the case of an oriented surface Σ endowed with a volume form ω_Σ. The symplectic forms ω_Σ and ω_V induce a formal symplectic structure Ω on $\mathscr{M}$. The surface Σ admits an action of the group of Hamiltonian diffeomorphisms $\mathrm{Ham}(\Sigma, \omega_\Sigma)$, which induces an action on the moduli space space $\mathscr{M}$ by reparametrization of maps. It turns out that the action of $\mathrm{Ham}(\Sigma, \omega_\Sigma)$ on $(\mathscr{M}, \Omega)$ is Hamiltonian, with moment map

$$\mu^D : \mathscr{M} \to C^\infty_0(\Sigma),$$

where $C^\infty_0(\Sigma)$ is the space of smooth functions with 0 average on Σ, identified to the Lie algebra of $\mathrm{Ham}(\Sigma, \omega_\Sigma)$. In addition, the moment map is explicitly given by the formula

$$\mu^D(f) = \frac{f^*\omega_V}{\omega_\Sigma}.$$

In particular, f is a zero of the moment map if, and only if, the map f is isotropic. A detailed expositions of these claims can be found in [3, 8]. Together with the fixed point principle techniques introduced in [8], the Donaldson moment map geometry leads to construction of isotropic polyhedral surfaces or $\mathbb{R}^{2n}$, for $n \geq 3$. These results where extended to the case where $V = \mathbb{R}^4$ in [11], to show that there exists lots of polyhedral Lagrangian tori of $\mathbb{R}^4$, modelled on the smooth ones:

Theorem 1.1.2 ([11])**.** *Let S be a smoothly immersed 2-dimensional torus of $\mathbb{R}^4$. Then S can be approximated by arbitrarily close immersed polyhedral Lagrangian tori, in the C^0-sense. Furthermore, if S is Lagrangian, it may be approximated by polyhedral Lagrangian immersed tori in the C^1-sense.*

On the one hand, it is expected that Theorem 1.1.2 could be proved by using *soft techniques* only, rather than fixed point techniques. This project is currently being taken care of by Etourneau in [5]. On the other hand, a moment map has an associated *moment map flow*, introduced in [8] whose limits are expected to be Lagrangian surfaces of $\mathbb{R}^4$. Furthermore, this moment map flow has a finite dimensional analogue on the space of polyhedral maps. This work was essentially experimental: the Jauberteau-Rollin-Tapie flow can produce lots of nice effective examples of Lagrangian polyhedral surfaces thanks to a computer. In spite of promising numerical experiments, the flow approach of [8] has some shortcomings:

- Although the moment map flow admits a finite dimensional analogue in the context of polyhedral geometry, the moment map geometry picture does fully transfer to polyhedral geometry. For instance, it is not known what a suitable analogue of the gauge group $\mathrm{Ham}(\Sigma, \omega_\Sigma)$ ought to be in the polyhedral setting.
- In the smooth setting, it is not known whether the moment map flow has *short time existence*. This flow does not appear to be well behaved and the corresponding evolution equation is an ill posed problem, from the point of view of analysis. In particular the flow does not seem to belong to any special framework, like parabolic flows for instance.
- In the case of polyhedral geometry, it is not even clear whether the finite dimensional version of the flow, which is an ordinary differential equation, has long time existence and if it converges. Although this behavior was not observed on computer experiments, we can not not rule out finite time blow-up behavior.

We shall not tackle the above issues directly in this paper. Instead, we are going to introduce a different moment map geometry and a new associated flow, which do not have any of the above pathologies.

1.2. **New flow and moment map interpretation — statement of results.** We consider a surface Σ endowed with an *oriented Euclidean triangulation* $\mathscr{T}$. This structure, which is defined at §3.1, is some kind *abstract Euclidean polyhedron*, with Σ as an underlying topological space. Given a complex Hermitian vector space V of dimension $\dim_{\mathbb{C}} V \geq 2$, we consider the space of *polyhedral maps* $\mathscr{P}$, as the space of continuous function $f : \Sigma \to V$, whose retriction along each face of the triangulation $\mathscr{T}$ is an affine map. A polyhedral map $f \in \mathscr{P}$, such that $f^*\omega_V = 0$ along the complement of the 1-skeleton of $\mathscr{T}$, is called an *isotropic polyhedral map*. Our goal is to provide effective constructions for such *isotropic polyhedral maps*, since they are the natural piecewise linear version of smooth isotropic maps. The usual differential $\mathscr{D}f$ of a polyhedral function $f \in \mathscr{P}$ is generally singular along the 1-skeleton of the triangulation. However, the differential is well defined and locally constant on the interior of each face. Let $\mathscr{F}$ be the space of locally constant V-valued differential 1-forms defined on the complement of the 1-skeleton of Σ. Then we can interpret the differential as a linear map between finite dimensional vector spaces

$$\mathscr{D} : \mathscr{P} \to \mathscr{F}.$$

At §3.9, we define a *canonical Kähler structure* on $\mathscr{F}$ and a *Hamiltonian torus action* on $\mathscr{F}$ at §2.5.5, with moment map μ. Furthemore, the moment map can be interpreted as a *symplectic density* (cf. §3.10). As a consequence, the space of isotropic polyhedral maps $\mathscr{I} \subset \mathscr{T}$ is identified, up to translation, with

$$\mathscr{D}\mathscr{I} = \mathrm{Im}\mathscr{D} \cap \mu^{-1}(0).$$

The *moment map flow* is a popular technique from *geometric invariant theory* (GIT). The idea is to consider the downward gradient flow of the functional

$$\phi : \mathscr{F} \to \mathbb{R}$$

given by $\phi(\zeta) = \frac{1}{2}\|\mu(\zeta)\|^2$. This approach provides an effective method for finding zeroes of the moment map in a given complexified orbits of the gauge group. However, this technique does not work out of the box here, since $\mathrm{Im}\mathscr{D}$ is not invariant under the torus action. In spite of that, we may consider the *modified moment map flow* in our setting, defined by

$$\frac{\partial \zeta_t}{\partial t} = -\Pi(\nabla\phi|_{\zeta_t}), \tag{1.1}$$

where $\Pi : \mathscr{F} \to \mathscr{F}$ is the orthogonal projection onto $\mathrm{Im}\mathscr{D}$. The only difference with the classical moment map flow is the presence of the projection Π, to make sure that $\mathrm{Im}\mathscr{D}$ is invariant under the flow. The outcome of our constructions is summarized in the following theorem:

Theorem A. *Let Σ be a closed surface endowed with an oriented Euclidean triangulation $\mathscr{T}$ and V a complex vector space of dimension* $\dim_{\mathbb{C}} V \geq 2$, *endowed with a Hermitian structure. This data defines a canonical modified moment map flow given by the ordinary differential evolution equation* (2.14), *on the space of exact differentials* $\mathrm{Im}\mathscr{D}$ *of polyhedral maps $\mathscr{P}$ with the following properties:*

1. *Every flow line ζ_t is bounded for future time, so that the maximal time of existence if $+\infty$.*
2. *Every flow line ζ_t converges toward a limit in* $\mathscr{D}\mathscr{I} = \mathrm{Im}\mathscr{D} \cap \mu^{-1}(0)$.
3. *If the limit of a flow line is a regular point of $\mathscr{D}\mathscr{I}$, then it converges exponentially fast.*
4. *The union of flow lines $\mathscr{V} \subset \mathrm{Im}\mathscr{D}$ converging toward a regular limit is an open set of* $\mathrm{Im}\mathscr{D}$.
5. *In the case where Σ is a 2-dimensional torus, the Jauberteau-Rollin-Tapie approximation scheme provides many examples of Euclidean triangulations and regular points of $\mathscr{D}\mathscr{I}$. In particular $\mathscr{V}$ is not empty in such cases.*

1.2.1. *Numerical experiments.* An interesting feature of our moment map construction is that all the objects involved can be be easily computed. For example $\mathrm{Im}\mathscr{D}$ and $\mathscr{F}$ are finite dimensional vector spaces and the Kähler structure of $\mathscr{F}$, the Hamiltonian torus action and the moment map μ are given by simple polynomial expressions. In particular, it is possible to approximate solutions of the modified moment map flow (2.14) by numerical methods. This provides effective constructions of isotropic polyhedral maps $f : \Sigma \to V$ as limits of the flow, relying on the Euler method. This paper comes with a computer program, which shows the modified moment map flow as an evolution equation, in the case where Σ is a 2-dimensional torus endowed with a regular Euclidean triangulation and V is identified to

$\mathbb{C}^2 \simeq \mathbb{R}^4$. The program is written in the Processing language, a dialect of Java and shows the evolution equation as a real-time movie. It is possible to interact with initial conditions and the reader is encouraged to fiddle with the software. For further details, see §**??**.

1.2.2. *The smooth case.* All the moment map constructions described above have been unraveled by first considering the case of smooth differential geometric object. From a formal perspective, the objects are completely analogous, and we shall start with their description at §2. We also discuss a modified moment map flow in the smooth setting and some properties, which subsist in this case.

2. Smooth isotropic maps as zeroes of a moment map

We proceed with the introduction of a new moment map geometry, where smooth isotropic maps can be understood as zeroes of the moment map. Since the spaces are infinite dimensional, all the notions of moduli spaces shall be discussed from a purely *formal perspective*. With some additional effort, it would be possible to define infinite dimensional manifold structures on the moduli spaces of interest, by using Sobolev or Hölder spaces. This work is partially carried out in order to prove short time existence of the modified moment map flow, introduced at §2.7. However the use of Hölder spaces will not be emphasized in the rest of the section.

2.1. **Moduli spaces.**

2.1.1. *The target space.* Let V be a complex vector space, of complex dimension m, endowed with a Hermitian metric h_V. That is a definite positive sesquilinear form, anti-$\mathbb{C}$-linear in the first variable. For every $v_1, v_2 \in V$, we can write

$$h_V(v_1, v_2) = g_V(v_1, v_2) + i\omega_V(v_1, v_2),$$

where g_V and ω_V are the real and imaginary parts of h_V. By definition, g_V is a Euclidean inner product and ω_V is a symplectic form. Multiplication by i can be undertood as linear endomorphism of V, also called an *almost complex structure*, denoted $i : V \to V$. By definition h_V, g_V and ω_V are *compatible* with the almost complex structure i, in the sense that $g_V(iv_1, iv_2) = g_V(v_2, v_2)$, $\omega_V(iv_1, iv_2) = \omega_V(v_1, v_2)$ and $g_V(iv_1, v_2) = \omega_V(v_1, v_2)$. for every $v_1, v_2 \in V$.

2.1.2. *The source space.* Let Σ be a smooth closed surface, endowed with a Riemannian metric g_Σ and a *compatible almost complex structure* J_Σ. Similarly to the linear case, an almost complex structure J_Σ on Σ is a smooth family of linear endomorphisms of the tangent bundle of Σ, such that $J_\Sigma^2 = -\mathrm{id}$. The compatibility condition means that J_Σ is g_Σ-orthogonal, in other words $g_\Sigma(J_\Sigma X_1, J_\Sigma X_2) = g_\Sigma(X_1, X_2)$ for all $z \in \Sigma$ and $X_1, X_2 \in T_z\Sigma$. The surface Σ carries a *Kähler form* ω_Σ, defined by $\omega_\Sigma(X_1, X_2) = g_\Sigma(J_\Sigma X_1, X_2)$, for all $z \in \Sigma$ and $X_1, X_2 \in T_z\Sigma$. These structure provide a *Kähler structure* $(\Sigma, J_\Sigma, g_\Sigma, \omega_\Sigma)$ on the surface Σ.

Remark 2.1.3. The tangent bundle $T\Sigma \to \Sigma$ can be thought of as a real vector bundle, or alternatively, as a complex vector bundle, with complex vector bundle structure induced by J_Σ. More precisely, the multiplication by a complex numbers $z \in \mathbb{C}$ on $X \in T_p\Sigma$ is defined by

$$(a + ib) \cdot X = aX + bJ_\Sigma X.$$

for $a, b \in \mathbb{R}$.

2.1.4. *Induced structures on vector bundles.* The bundle $T^*\Sigma \to \Sigma$ is identified to $T\Sigma \to \Sigma$, by using the duality deduced from the Riemannian metric g_Σ. Thus $T^*\Sigma \to \Sigma$ and all the bundles of k-forms $\Lambda^k\Sigma \to \Sigma$ are endowed with an induced Euclidean inner product, also denoted g_Σ. It follows that the complex vector bundles $\Lambda^k\Sigma \otimes_{\mathbb{R}} V \to \Sigma$ have a natural Hermitian structure h induced by g_Σ and h_V and defined as follows: for $\beta_1, \beta_2 \in \Lambda^q_z\Sigma$ and $v_1, v_2 \in V$, we define the Hermitian inner product h by,

$$h(\beta_1 \otimes v_1, \beta_2 \otimes v_2) = g_\Sigma(\beta_1, \beta_2) h_V(v_1, v_2).$$

The Hermitian product h is J_Σ-invariant, by definition, in the sense that for every $\phi, \psi \in \Lambda^q_z\Sigma \otimes V$, we have

$$h(\phi, \psi) = h(\phi \circ J_\Sigma, \psi \circ J_\Sigma).$$

We consider the Riemannian metric g given as the real part of h. Then g is J_Σ-invariant and i-invariant in the sense that

$$g(\phi, \psi) = g(\phi \circ J_\Sigma, \psi \circ J_\Sigma), \quad g(i\phi, i\psi) = g(\phi, \psi).$$

We define a fiberwise almost complex structure on $T^*\Sigma \otimes V \to \Sigma$, by the formula

$$J \cdot F = -F \circ J_\Sigma. \tag{2.1}$$

The metric g is J-invariant so that the formula

$$\omega(F_1, F_2) = g(J \cdot F_1, F_2) \tag{2.2}$$

defines a fiberwise symplectic form on the bundle $T^*\Sigma \otimes V \to \Sigma$.

Remark 2.1.5. By definition, g is the real part of the Hermitian form h. However ω does not agree with the imaginary part of h, which is deduced from g and i, and not J_Σ.

2.1.6. *Moduli spaces of maps and differentials.* The moduli space of maps $\mathscr{M} = C^\infty(\Sigma, V)$ is the obvious ambient space for studying isotropic maps. However the moduli space of V-valued differentials 1-forms

$$\mathscr{F} = \Gamma(T^*\Sigma \otimes_{\mathbb{R}} V)$$

is of interest as well, since it contains all the differentials of functions in $\mathscr{M}$. An element of $\mathscr{F}$ can be understood as a smooth family of fiberwise linear maps from $T\Sigma$ to the Hermitian space V. Thus, we have canonical isomorphisms

$$\mathscr{F} = \Gamma(T^*\Sigma \otimes_{\mathbb{R}} V) = \Omega^1(\Sigma, V)$$

and

$$\mathscr{M} = C^\infty(\Sigma, V) = \Omega^0(\Sigma, V),$$

where $\Omega^q(\Sigma, V)$ is the space of V-valued differential q-forms on Σ. The exterior differential d extends canonically to tensor products

$$\Omega^q(\Sigma, V) \xrightarrow{d} \Omega^{q+1}(\Sigma, V)$$

with the property $d^2 = 0$. The resulting complex defines cohomology spaces known as the V-valued *DeRham cohomology*. The differential $d : \Omega^0(\Sigma, V) \to \Omega^1(\Sigma, V)$ provides, modulo the above identifications, a map denoted

$$\mathscr{M} \xrightarrow{\mathscr{D}} \mathscr{F}.$$

In particular, the image of $\mathscr{D}$ consists of exact differential V-valued 1-forms in $\mathscr{F}$. The kernel of $\mathscr{D}$ is the subspace of locally constant V-valued functions on Σ. The vectors $v \in V$ act on maps $f \in \mathscr{M}$ by translations. The translated map $f + v$ is given by $(f + v)(z) = f(z) + v$. From now on, we shall assume that Σ is connected, so that $\mathscr{D}$ induces an isomorphism

$$\mathscr{M}/V \xrightarrow{\mathscr{D}} \mathrm{Im}\mathscr{D} \subset \mathscr{F}.$$

2.2. **Euclidean structures on moduli spaces and Hodge theory.** The Euclidean metric g on the vector bundle $\Lambda^q\Sigma \otimes V \to \Sigma$ induces a L^2-inner product on $\Omega^q(\Sigma, V)$, given by

$$\mathcal{G}(\phi, \psi) = \int_\Sigma g(\phi, \psi)\, \omega_\Sigma, \quad \text{for } \phi, \psi \in \Omega^q(\Sigma, V). \tag{2.3}$$

The formal adjoint d^* of d is defined by the property

$$\mathcal{G}(\phi, d\psi) = \mathcal{G}(d^*\phi, \psi) \quad \text{for } \psi \in \Omega^q(\Sigma, V) \text{ and } \phi \in \Omega^{q+1}(\Sigma, V)$$

and the *Laplace operator* is defined on $\Omega^q(\Sigma, V)$ by

$$\Delta = dd^* + d^*d.$$

The usual *Hodge theory* for the Laplace operator allows to define a $\mathcal{G}$-orthogonal projection from the space of V-valued 1-forms onto the space of V-valued exact 1-forms denoted

$$\Pi : \mathscr{F} \to \mathrm{Im}\mathscr{D}.$$

The above projection is well behaved with respect to Hölder norms, as proved in Proposition 2.2.1. This fact will turns out to be crucial for proving short time existence of the flow introduced at Equation (2.14). Hölder spaces are well suited for studying elliptic operators, in particular the Laplace operator Δ. Thus, it makes sense to introduce the Hölder analogues $\mathscr{F}^{k,\alpha}$ of $\mathscr{F}$ and $\mathscr{M}^{k,\alpha}$ of $\mathscr{M}$, where k is a non-negative integer and $\alpha \in (0, 1)$ is a Hölder regularity exponent. More precisely

$$\mathscr{M}^{k,\alpha} = C^{k,\alpha}(\Sigma, V)$$

and

$$\mathscr{F}^{k,\alpha} = \{\phi \in C^{k,\alpha}(\Sigma, T^*\Sigma \otimes_{\mathbb{R}} V), \pi \circ \phi = \mathrm{id}_\Sigma\},$$

where

- $C^{k,\alpha}$ denotes the space of C^k-maps, with with Hölder regularity of order α for the k-th derivatives;
- $\pi : T^*\Sigma \otimes V \to \Sigma$ is the canonical projection.

We also denote by $\|\cdot\|_{C^{k,\alpha}}$ the correponding Hölder norm on $C^{k,\alpha}$. The next proposition is a classical fact from Hodge theory. The proof is provided to keep this paper selfcontained.

Proposition 2.2.1. *The orthogonal projection* $\Pi : \mathscr{F}^{k,\alpha} \to \mathscr{F}^{k,\alpha}$ *is a continuous linear map with respect to the* $C^{k,\alpha}$*-norm. In other words, there exists a real constant* $c > 0$ *such that*

$$\|\Pi(F)\|_{C^{k,\alpha}} \leq c\|F\|_{C^{k,\alpha}}, \quad \text{for all } F \in \mathscr{F}^{k,\alpha}.$$

Proof. By classical Hodge theory, every 1-form $F \in \mathscr{F}^{k,\alpha}$ admits an orthogonal decomposition

$$F = F_H + \Delta G,$$

where G is a 1-form orthogonal to harmonic forms and F_H is the harmonic part of F. A $C^{k,\alpha}$-estimate on F provides a control on the $C^{k-2,\alpha}$-norm of ΔF. By definition $\Delta F = \Delta^2 G$, since F_H is harmonic. Hence the $C^{k,\alpha}$-norm of F control the $C^{k-2,\alpha}$ norm of $\Delta^2 G$. The operator Δ is selfajoint, hence ΔG is orthogonal to the kernel of Δ. In particular elliptic Schauder estimates provide a control on the $C^{k,\alpha}$-norm of ΔG. And since G was chosen orthogonal to $\ker \Delta$, we deduce a $C^{k+2,\alpha}$ control on G. In conclusion, there exists a constant $c' > 0$ (independent of F), such that

$$\|G\|_{C^{k+2,\alpha}} \leq c'\|F\|_{C^{k,\alpha}}.$$

The decomposition $F = F_H + d^*dG + dd^*G$ has three terms. The first two terms are orthogonal to $\mathrm{Im}\mathscr{D}$ whereas the last belongs to $\mathrm{Im}\mathscr{D}$. Hence $\Pi(F) = dd^*G$. Finally, the $C^{k+2,\alpha}$-norm of G controls the $C^{k,\alpha}$-norm of dd^*G and the proposition follows. ☐

2.3. **Formal Kähler structures.** The spaces $\Omega^q(\Sigma, V)$, in particular $\mathscr{M}$ and $\mathscr{F}$, are infinite dimensional complex vector spaces. The complex structure is deduced from the complex structure of V. Indeed a V-valued differential form can be multiplied by a complex number in a canonical way.

Furthermore the space $\Omega^q(\Sigma) \otimes V$ is a module over the space of complex valued functions $\Omega^0(\Sigma, \mathbb{C}) = C^\infty(\Sigma, \mathbb{C})$. Indeed, the action is provided by the pointwise complex multiplication:

$$(\xi F)(z) = \xi(z)F(z)$$

for every $\xi \in C^\infty(\Sigma, \mathbb{C})$, $F \in \Omega^q(\Sigma, V)$ and $z \in \Sigma$. However, the space of differential 1-forms $\mathscr{F}$ admits an additional almost complex structure $\mathcal{J}$, defined by

$$(\mathcal{J} \cdot F)(z) \cdot v = -F(z) \circ J_\Sigma \cdot v,$$

for every $z \in \Sigma$ and $v \in T_z\Sigma$.

Recall that the moduli space $\mathscr{F}$ admits a Riemannian metric $\mathcal{G}$ given by (2.3). Since $g(F_1, F_2) = g(F_1 \circ J_\Sigma, F_2 \circ J_\Sigma)$, the metric $\mathcal{G}$ is which compatible with the complex structure $\mathcal{J}$, in the sense that $\mathcal{G}(F_1, F_2) = \mathcal{G}(\mathcal{J}F_1, \mathcal{J}F_2)$. Hence, we obtain a Kähler form $\Omega(\cdot, \cdot) = \mathcal{G}(\mathcal{J}\cdot, \cdot)$ also given by

$$\Omega(F_1, F_2) = \int_\Sigma \omega(F_1, F_2)\omega_\Sigma \quad \text{for all } F_1, F_2 \in \mathscr{F},$$

where $\omega(F_1, F_2) = g(\mathcal{J} \cdot F_1, F_2)$.

Proposition 2.3.1. *The almost complex structure $\mathcal{J}$ is formally integrable. The symplectic form Ω is formally closed. Thus $(\mathscr{F}, \mathcal{G}, \mathcal{J}, \Omega)$ is a formal Kähler structure.*

Proof. The complex structure on $\mathscr{F}$ is deduced from the complex structure J on the vector bundle $\Lambda^1\Sigma \otimes V \to \Sigma$. The fact that J is integrable on the fibers formally implies that $\mathcal{J}$ is integrable. Similarly, we have formally $d\Omega = 0$ since $d\omega = 0$ on each fiber of the vector bundle. □

Remark 2.3.2. Notice that by definition, the map $\mathscr{D} : \mathscr{M} \to \mathscr{F}$ is complex linear with respect to the complex structure i induced by V. Indeed $\mathscr{D}$ commutes with the multiplication by i. In particular, the image or kernel of $\mathscr{D}$ are i-complex subspaces of $\mathscr{F}$. Similarly, the projection map $\Pi : \mathscr{F} \to \mathscr{F}$ is i-complex linear. However Π is not $\mathcal{J}$-complex linear and $\mathrm{Im}\mathscr{D}$ is not stable by $\mathcal{J}$.

2.4. **An involution.** The moduli space $\mathscr{F}$ is an infinite dimensional real vector space which admits two complex structures $\mathcal{J}$ and i compatible with the Euclidean metric $\mathcal{G}$. Since $\mathcal{J}$ is defined by acting on the right, whereas i is defined by acting on the left of $F \in \mathscr{F}$, the two actions commute

$$\mathcal{J} \cdot iF = -iF \circ J_\Sigma = i\mathcal{J} \cdot F.$$

In particular

$$\mathscr{R}F = i\mathcal{J} \cdot F = -iF \circ J_\Sigma$$

is a *linear involution*. Since $\mathscr{R}$ commutes with i and $\mathcal{J}$, this is a complex linear endomorphism with respect to both almost complex structures. The involution $\mathscr{R}$ provides an orthogonal decomposition

$$\mathscr{F} = \mathscr{F}^+ \oplus \mathscr{F}^-$$

where $\mathscr{F}^\pm$ are the eigenspaces associated to the eigenvalues ± 1 of $\mathscr{R}$. By definition, we have

$$\mathscr{F}^+ = \{F \in \mathscr{F}, F \circ J_\Sigma = iF\} \quad \text{and}$$
$$\mathscr{F}^- = \{F \in \mathscr{F}, F \circ J_\Sigma = -iF\}.$$

In other words, the elements of $\mathscr{F}^+$ (resp. $\mathscr{F}^-$) are the complex (resp. anti-complex) morphisms from the vector bundle $T\Sigma$ to V. Recall that $\mathscr{F}$ is a module on $C^\infty(\Sigma, \mathbb{C})$, where functions act on V-valued forms by complex multiplication. With these definition $\mathscr{F}^+$ and $\mathscr{F}^-$ are also complex submodules of $\mathscr{F}$.

2.5. **Hamiltonian torus action.** We consider the infinite dimensional complex Lie group

$$\mathbb{T}^{\mathbb{C}} = C^\infty(\Sigma, \mathbb{C}^*).$$

with trivial Lie algebra $\mathfrak{t}^{\mathbb{C}} = C^\infty(\Sigma, \mathbb{C})$. The group $\mathbb{T}^{\mathbb{C}}$ is understood as an infinite dimensional complex torus. We define an action of $\lambda \in \mathbb{T}^{\mathbb{C}}$ on $F = F^+ + F^-$, where $F^\pm$ are the components of F according to the splitting $\mathscr{F} = \mathscr{F}^+ \oplus \mathscr{F}^-$, given by the formula

$$\lambda \cdot F = \bar{\lambda}^{-1} F^+ + \lambda F^-. \tag{2.4}$$

where products in the RHS are given by pointwise complex multiplication and $\bar{\lambda}$ denotes the complex conjugate of λ. The real torus

$$\mathbb{T} = C^\infty(\Sigma, S^1),$$

where $S^1 \subset \mathbb{C}$ is the circle of unit complex numbers, is a subgroup of $\mathbb{T}^{\mathbb{C}}$.

Remark 2.5.1. If $\lambda \in \mathbb{T}$, then $\bar{\lambda}^{-1} = \lambda$, so that the group action is given by

$$\lambda \cdot F = \lambda F, \quad \forall \lambda \in \mathbb{T}.$$

If λ is real valued, we have

$$\lambda \cdot F = \lambda^{-1} F^+ + \lambda F^-, \quad \forall \lambda \in C^\infty(\Sigma, \mathbb{R}^*).$$

2.5.2. *Infinitesimal torus action.* For $\xi \in C^\infty(\Sigma, \mathbb{C}) = \mathfrak{t}^{\mathbb{C}}$ we define an exponential map

$$\exp : C^\infty(\Sigma, \mathbb{C}) \to \mathbb{T}^{\mathbb{C}},$$

by

$$\exp(\xi) = e^{i\xi},$$

so that the space of real valued functions $C^\infty(\Sigma, \mathbb{R})$ is identified to the Lie algebra $\mathfrak{t}$ of $\mathbb{T}$. By definition (2.4), the infinitesimal action of the Lie algebra $\mathfrak{t}^{\mathbb{C}}$ on $\mathscr{F}$ is given by the vector field

$$X_\xi(F) = \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi) \cdot F = i\bar{\xi} F^+ + i\xi F^-. \tag{2.5}$$

In particular, if ξ is a real valued function, we have

$$X_\xi(F) = i\xi F, \quad \forall \xi \in C^\infty(\Sigma, \mathbb{R}), \tag{2.6}$$

whereas if ξ is purely imaginary, we have

$$X_\xi(F) = -i\xi \mathscr{R} F, \quad \forall \xi \in C^\infty(\Sigma, i\mathbb{R}). \tag{2.7}$$

By (2.6) and (2.7), the infinitesimal action of $\xi \in C^\infty(\Sigma, \mathbb{R})$ satisfies

$$\mathcal{J} \cdot X_\xi(F) = \mathcal{J} \cdot i\xi F = \xi \mathscr{R} F = -i(i\xi)\mathscr{R} F = X_{i\xi}(F),$$

and we deduce the folowing result:

Proposition 2.5.3. *The action of $\mathbb{T}^{\mathbb{C}}$ on $\mathscr{F}$ is holomorphic on $(\mathscr{F}, \mathcal{J})$ and is the complexification of the action of $\mathbb{T}$.*

In addition, the action of $\mathbb{T}$ on $\mathscr{F}$ isometric. Indeed for every $\lambda \in \mathbb{T}$ and $F \in \mathscr{F}$,

$$\|\lambda \cdot F\|_g^2 = \|\bar{\lambda}^{-1} F^+\|_g^2 + \|\lambda F^-\|_g^2 = \|F^+\|_g^2 + \|F^-\|_g^2 = \|F\|_g^2,$$

so that $\|\lambda \cdot F\|_{\mathcal{G}}^2 = \|F\|_{\mathcal{G}}^2$. Thus we have the following result:

Proposition 2.5.4. *The action of $\mathbb{T}$ on $\mathscr{F}$ is $\mathcal{G}$-isometric and preserves the Kähler structure $(\mathscr{F}, \mathcal{J}, \mathcal{G}, \Omega)$.*

2.5.5. *Moment map.* We are going to show that the map

$$\begin{aligned} \mu : \mathscr{F} &\longrightarrow \mathfrak{t} = C^\infty(\Sigma, \mathbb{R}) \\ F &\longmapsto \tfrac{1}{2} g(F, \mathscr{R} F) \end{aligned} \tag{2.8}$$

is in fact a moment map for the action of $\mathbb{T}$ on $\mathscr{F}$. In addition, this map can be interpreted as the symplectic density:

Lemma 2.5.6. *The following formulas hold*

$$\mu(F) = \frac{1}{2}\omega(iF, F) = \frac{1}{2}(\|F^+\|_g^2 - \|F^-\|_g^2) = \frac{F^*\omega_V}{\omega_\Sigma}, \tag{2.9}$$

for every $F \in \mathscr{F}$, where $(F^*\omega_V)(X_1, X_2) = \omega_V(F(X_1), F(X_2))$, for every $p \in \Sigma$ and $X_1, X_2 \in T_p\Sigma$.

Proof. Using the fact that $g(\mathscr{R}F, F) = g(i\mathcal{J}F, F) = g(\mathcal{J}iF, F) = \omega(iF, F)$ we deduce the first identity.

Using the decomposition $F = F^+ + F^-$, we write $g(\mathscr{R}F, F) = g(F^+ - F^-, F^+ + F^-)$. Using the fact the F^+ and F^- are pointwise g-orthogonal, we deduce that $g(\mathscr{R}F, F) = \|F^+\|_g^2 - \|F^-\|_g^2$, which proves the second identity.

Let (ϕ, ψ) be a local oriented g_Σ-orthonormal frame of $T\Sigma$. In particular $\omega_\Sigma(\phi, \psi) = 1$ and $J_\Sigma \phi = \psi$. Hence $(\mathcal{J} \cdot F)(\phi) = -F(J_\Sigma \phi) = -F(\psi)$. Similarly $(\mathcal{J} \cdot F)(\psi) = F(\phi)$. Hence $\mathscr{R}F(\phi) = -iF(\psi)$ and $\mathscr{R}F(\psi) = iF(\phi)$. By definition

$$\begin{aligned} \mu(F) &= \frac{1}{2} g(\mathscr{R}F, F) \\ &= \frac{1}{2} g_V(\mathscr{R}F(\phi), F(\phi)) + \frac{1}{2} g_V(\mathscr{R}F(\psi), F(\psi)) \\ &= \frac{1}{2}(-g_V(iF(\psi), F(\phi)) + g_V(iF(\phi), F(\psi))) \\ &= \omega_V(F(\phi), F(\psi)) \\ &= \frac{(F^*\omega_V)(\phi, \psi)}{\omega_\Sigma(\phi, \psi)}, \end{aligned}$$

which proves the last identity of the lemma. □

As a trivial consequence we deduce the corollary below:

Corollary 2.5.7. *Let F be an element of $\mathscr{F}$. The following properties are equivalent:*

(1) *The morphism* $F : T\Sigma \to V$ *is isotropic in the sense that* $F^*\omega_V = 0$ *identically on* Σ.
(2) $\mu(F) = 0 \in \mathfrak{t}$.
(3) $\|F^+\|_g = \|F^-\|_g$ *identically on* Σ.

Remark 2.5.8. According to the identity (2.9), we have the commutative diagram

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{\mathscr{D}} & \mathscr{F} \\ & \searrow^{\mu^D} & \downarrow \mu \\ & & \mathfrak{t} \end{array}$$

Hence, the space of isotropic maps $\mathscr{I} \subset \mathscr{M}$, more accurately, its image $\mathscr{D}(\mathscr{I}) \subset \mathscr{F}$, can be understood as the intersection between the image of $\mathscr{D}$ and the zero set of the map μ:

$$\mathscr{D}(\mathscr{I}) = \mathrm{Im}\mathscr{D} \cap \mu^{-1}(0). \tag{2.10}$$

Remark 2.5.8 becomes handy, as μ is a moment map for the Hamiltonian action of $\mathbb{T}$ on $\mathscr{F}$. This is the main property of the construction:

Theorem 2.5.9. *The action of* $\mathbb{T}$ *on* $\mathscr{F}$ *is Hamiltonian with moment map* μ. *More precisely,* μ *is* $\mathbb{T}$*-invariant and*

$$\mathcal{G}(D\mu|_F \cdot \dot{F}, \xi) = \Omega(X_\xi(F), \dot{F})$$

for all $F, \dot{F} \in \mathscr{F}$ *and* $\xi \in \mathfrak{t}$. *In other words,* X_ξ *is a Hamiltonian vector field on* $\mathscr{F}$ *and the map* $F \mapsto \mathcal{G}(\mu(F), \xi)$ *is a Hamiltonian function of* X_ξ.

Proof. Let $\lambda \in \mathbb{T}$ and $F \in \mathscr{F}$. Then $2\mu(\lambda \cdot F) = g(\mathscr{R}(\lambda F), \lambda F)$. Since $\mathscr{R}$ is a module morphism, we have $\mathscr{R}\lambda \cdot F = \lambda \cdot \mathscr{R}F$. Hence $2\mu(\lambda \cdot F) = g(\lambda \cdot \mathscr{R}F, \lambda F) = 2|\lambda|^2\mu(F) = 2\mu(F)$. In conclusion μ is $\mathbb{T}$-invariant.

Lemma 2.5.10.

$$D\mu|_F \cdot \dot{F} = g(\mathscr{R}F, \dot{F}).$$

Proof. This is an obvious calculation: $2D\mu|_F \cdot F = g(\mathscr{R}F, \dot{F}) + g(F, \mathscr{R}\dot{F})$ by bilinearity. The lemma follows from the fact that $\mathscr{R}$ is pointwise g-selfadjoint. ☐

Since $\xi \in \mathfrak{t}$, we have $X_\xi(F) = i\xi F$ and it follows that

$$\begin{aligned} \Omega(X_\xi(F), \dot{F}) &= \Omega(i\xi F, \dot{F}) \\ &= \mathcal{G}(\mathcal{J} i\xi F, \dot{F}) \\ &= \mathcal{G}(\xi\mathscr{R}F, \dot{F}) \\ &= \int_\Sigma g(\xi\mathscr{R}F, \dot{F})\omega_\Sigma \\ &= \int_\Sigma \xi(D\mu|_F \cdot \dot{F})\omega_\Sigma \end{aligned}$$

The last term is equal to $\mathcal{G}(D\mu|_F \cdot F, \xi)$, which proves the theorem. ☐

2.6. **Isotropic maps and geometric invariant theory.** We are now in a typical situation of moment map geometry, with a grain of salt. The moduli space of differentials $\mathscr{F}$ is endowed with a Kähler structure and a Hamiltonian action of the real gauge group $\mathbb{T}$, by Theorem 2.5.9. In the case where the Lie group $\mathbb{T}$ is finite dimensional, which is not the case here, *geometric invariant theory* together with the *Kempf-Ness theorem*, show that each complexified orbit of the gauge group action contains a zero of the moment map if, and only if, the orbit is polystable in the GIT sense. Furthermore, zeroes of the moment map are unique up to the real group action. The fact that $\mathbb{T}$ is infinite dimensional is not such a big issue here. We shall see that most of these statements go through for obvious reasons, in our very special setting.

However, our original goal was to construct isotropic maps $f \in \mathscr{M}$, rather than differentials. Hence the problem splits into two sub-problems according to Remark 2.5.8:

(1) Find the zeroes of μ in $\mathscr{F}$.
(2) Find the intersection points of $\mu^{-1}(0)$ and $\mathrm{Im}\mathscr{D}$.

The answer to Question (1) is explicitely understood in our case. Question (2) is less usual in the context of GIT. The space $\mathrm{Im}\mathscr{D}$ is not invariant under the action of $\mathbb{T}$ and the orthogonal projection $\Pi : \mathscr{F} \to \mathrm{Im}\mathscr{D}$ does not seem to behave nicely with respect to the moment map and GIT constructions. To tackle this problem, we turn toward an additional feature of the GIT theory: there exists a natural *moment map flow* on the moduli space $\mathscr{F}$, obtained as the downward gradient flow $-\nabla\phi$ of the functional $\phi : \mathscr{F} \to \mathbb{R}$ given by

$$\phi = \frac{1}{2}\|\mu\|^2_{\mathcal{G}}.$$

Such a flow preserves complexified orbits. In the finite dimensional setting, in every given stable orbit, the flow is converging toward a zero of the moment map. In our situation, zeroes of the moment map in $\mathscr{F}$ are explicit and this interpretation is of little interest. Yet, the restriction of ϕ to $\mathrm{Im}\mathscr{D}$ allows to define a modified moment map flow along the subspace $\mathrm{Im}\mathscr{D} \subset \mathscr{F}$. In the polyhedral setting, this flow will provide effective approach for constructing isotropic polyhedral maps.

2.6.1. *Kempf-Ness theory for $\mathscr{F}$.* Let F be an element of the moduli space $\mathscr{F}$. For simplicity we shall assume that F is *non-singular*, in the sense that the linear map F_z does not vanish at any point $z \in \Sigma$. This condition is invariant under the action of $\mathbb{T}^{\mathbb{C}}$. A differential 1-form is a smooth section of the real bundle $T^*\Sigma \otimes_{\mathbb{R}} V \to \Sigma$. This bundle has real rank at least 8 if $\dim_{\mathbb{C}} V \geq 2$, so that the condition of being non singular is generic, by transversality. In particular, the differential $F = df$ of an immersion $f \in \mathscr{M}$, is non-singular.

For F non-singular, we say that F is *unstable* if F^+_z or F^-_z vanishes at some point $z \in \Sigma$. Otherwise, we say that F is *stable*. Notice that the stability condition is also invariant under the action of $\mathbb{T}^{\mathbb{C}}$. The next proposition provides a rather general (and easy) answer to Question (1).

Proposition 2.6.2. *Let F be a non-singular element of $\mathscr{F}$. Then F is stable if, and only if, there exists a zero of the moment map μ in its $\mathbb{T}^{\mathbb{C}}$-orbit. If F is stable, zeroes of the moment map in its complexified orbit are unique up to the action of $\mathbb{T}$.*

Proof. If F is a non-singular zero of the moment map, then it is stable. Indeed, if F^+ vanishes at some point $z \in \Sigma$, the condition $2\mu(F) = \|F^+\|^2 - \|F^-\|^2 = 0$ forces the vanishing of F^- at z (and vice versa), which is impossible since F is non-singular, by assumption. Hence, F is stable.

Assume that F is non-singular and stable. By definition, $\|F^\pm\|$ are positive smooth functions on Σ. As a consequence, there exists a unique positive smooth real function $\lambda : \Sigma \to \mathbb{R}^*$, defined by the equation

$$\lambda^{-1}\|F^+\| - \lambda\|F^-\| = 0.$$

By definition of λ, we have $\mu(\lambda \cdot F) = 0$. The uniqueness statement follows by a similar argument. □

2.6.3. *A dead end for Kempf-Ness.* Proposition 2.6.2 provides an abundance of isotropic differential 1-forms $F \in \mathscr{F}$. But we are more interested in the space of maps $f \in \mathscr{M}$ such that μ vanishes at $\mathscr{D}(f)$, that is isotropic maps. Unfortunately, $\mathrm{Im}\mathscr{D}$ is not preserved by the $\mathbb{T}^{\mathbb{C}}$-action, not even by the $\mathbb{T}$-action as shown in Lemma 2.6.4. Hence it is unclear how to seek zeroes of the moment map that belong to $\mathrm{Im}\mathscr{D}$ with the usual Kempf-Ness approach.

Lemma 2.6.4. *Consider the family of constant functions $\lambda_t = e^{it\xi} \in \mathbb{T}$, for $\xi \in C^\infty(\Sigma, \mathbb{R})$. Let $F = \mathscr{D}f$ for some function $f \in \mathscr{M}$ and $F_t = \lambda_t \cdot F$. Then*

$$\left.\frac{\partial}{\partial t}\right|_{t=0} dF_t = id\xi \wedge F.$$

In particular, there exists a function ξ such that this quantity does not vanish unless f is constant. If f is not constant there exists some family $\lambda_t \in \mathbb{T}$, such that $\lambda_t \cdot F$ does not belong to $\mathrm{Im}\mathscr{D}$ for every t sufficiently small.

Proof. We compute the variation of $d(\lambda_t \cdot F)$ at $t = 0$. Since $\lambda_t \in \mathbb{T}$ the action $F_t = \lambda_t \cdot F$ is given by complex multiplication. Hence $dF_t = d\lambda_t \wedge F + \lambda_t \wedge dF = d\lambda_t \wedge F$. Taking the derivative with respect to t give the first identity of the lemma.

If F does not vanish identically, there exists a function ξ such that $d\xi \wedge F$ does not vanish identically. However F does not vanish identically unless f is constant. In this case, the choice of a function ξ as above shows that F_t is not closed for every sufficiently small t, and in particular, does not belong to $\mathrm{Im}\mathscr{D}$. □

Corollary 2.6.5. *The space $\mathrm{Im}\mathscr{D}$ is not stable under the $\mathbb{T}$-action.*

2.7. **Modified moment map flow.** The stage has been set to implement a modified version of the moment map flow, a traditional technique to find zeroes of a moment map in a given complexified orbit. According to Corollary 2.5.7, a map $f \in \mathscr{M}$ is isotropic if and only if $\mathscr{D}f \in \mu^{-1}(0)$.

2.7.1. *A functional on the moduli space.* The idea is to use the downward gradient flow of the functional

$$\begin{aligned} \phi : \mathscr{F} &\longrightarrow \mathbb{R} \\ F &\longmapsto \phi(F) = \tfrac{1}{2}\|\mu(F)\|^2_{\mathcal{G}} \end{aligned} \tag{2.11}$$

restricted to Im$\mathscr{D}$. We start with various formulas about the differential and the gradient of the functional ϕ on $\mathscr{F}$.

Proposition 2.7.2. *The gradient of the functional $\phi : \mathscr{F} \to \mathbb{R}$ is given by the formula*

$$\nabla\phi = -\mathcal{J}Z, \tag{2.12}$$

where Z is the vector field on $\mathscr{F}$ defined by

$$Z(F) = -X_{\mu(F)}(F) = -i\mu(F)F.$$

Moreover, we have the formulas

$$\nabla\phi|_F = \mu(F)\mathscr{R}F \quad \text{and} \quad D\phi|_F \cdot F = 2\|\mu(F)\|^2_{\mathcal{G}}. \tag{2.13}$$

Proof. By definition of $\phi(F) = \frac{1}{2}\|\mu(F)\|^2_{\mathcal{G}}$ as a square norm, we have

$$D\phi|_F \cdot \dot{F} = \mathcal{G}\left(\mu(F), D\mu|_F \cdot \dot{F}\right).$$

By Lemma 2.5.10, we have $D\mu|_F \cdot \dot{F} = g(\mathscr{R}F, \dot{F})$ so that

$$D\phi|_F \cdot \dot{F} = \mathcal{G}\left(\mu(F), g(\mathscr{R}F, \dot{F}))\right).$$

Now

$$\begin{aligned} \mathcal{G}(\mu(F), g(\mathscr{R}F, \dot{F})) &= \int_\Sigma \mu(F) g(\mathscr{R}F, \dot{F})\omega_\Sigma \\ &= \int_\Sigma g(\mu(F)\mathscr{R}F, \dot{F})\omega_\Sigma \\ &= \mathcal{G}(\mu(F)\mathscr{R}F, \dot{F}) \end{aligned}$$

and we deduce that $\nabla\phi|_F = \mu(F)\mathscr{R}F$. We put $Z = \mathcal{J}\nabla\phi$ so that $\mathcal{J}Z = -\nabla\phi$. Hence $Z(F) = \mu(F)\mathcal{J}\mathscr{R}F = \mu(F)\mathcal{J}i\mathcal{J}F = -i\mu(F)F = -X_{\mu(F)}(F)$. In particular, for $\dot{F} = F$, we have $D\phi|F \cdot F = \mathcal{G}(\nabla\phi|_F, F)$. Since

$$\mathcal{G}(\nabla\phi|_F, F) = \int_\Sigma \mu(F) g(\mathscr{R}F, F)\omega_\Sigma,$$

and $g(\mathscr{R}F, F) = 2\mu(F)$, by definition of the moment map, we have

$$\mathcal{G}(\nabla\phi|_F, F) = 2\int_\Sigma |\mu(F)|^2\omega_\Sigma.$$

In conclusion $D\phi|_F \cdot F = 2\|\mu(F)\|^2_{\mathcal{G}}$. □

Remark 2.7.3. According to Lemma 2.6.4, $\mathrm{Im}\mathscr{D}$ is not stable under the $\mathbb{T}^{\mathbb{C}}$-action (even the $\mathbb{T}$-action). Hence, the gradient $\nabla\phi$ is generally not tangent to $\mathrm{Im}\mathscr{D}$ so that the usual gradient flow approach cannot be used directly to find zeros of the moment map in $\mathrm{Im}\mathscr{D}$.

2.7.4. *Definition of a new flow.* We consider the restriction of the functional ϕ to the subspace $\mathrm{Im}\mathscr{D} \subset \mathscr{F}$:

$$\phi : \mathrm{Im}\mathscr{D} \longrightarrow \mathbb{R}.$$

The above functional has a gradient vector field $\nabla'\phi$ closely related to the gradient $\nabla\phi$ on $\mathscr{F}$. Indeed, for all points $F \in \mathrm{Im}\mathscr{D}$, we have

$$\nabla'\phi|_F = \Pi(\nabla\phi|_F) = -\Pi(\mathcal{J}Z(F)).$$

We define the *modified moment map flow*

$$\frac{\partial F}{\partial t} = \Pi(\mathcal{J}Z) \tag{2.14}$$

along $\mathrm{Im}\mathscr{D}$. A smooth solution of the flow is a smooth family of maps $F_t \in \mathscr{F}$, for t in some interval I, such that $\frac{\partial}{\partial t}F_t = \Pi(\mathcal{J}Z(F_t))$ along I, with some initial condition $F_{t_0} \in \mathrm{Im}\mathscr{D}$. Notice that by definition of the flow, the initial condition implies that $F_t \in \mathrm{Im}\mathscr{D}$ for every $t \in I$.

The modified moment map flow is an evolution equation where the RHS is a pseudodifferential operator of order 0. If $F_0 = df$ is smooth, short time existence is insured by the Cauchy-Lipschitz theorem as we shall see in Theorem 2.8.3. However it may lose regularity on the long run and we shall not investigate this problem in the current paper.

2.8. **Basic properties of the modified moment map flow.** An interesting property of this flow is that the L^2-norm of F must decrease along flow lines:

Theorem 2.8.1. *Let $F_t \in \mathrm{Im}\mathscr{D}$ be a smooth solution of the modified moment map flow on $\mathrm{Im}\mathscr{D}$, for t in some interval of $\mathbb{R}$. Then $\|F_t\|^2_{\mathcal{G}}$ is a decreasing function of t, which satisfies the evolution equation*

$$\frac{\partial}{\partial t}\|F_t\|^2_{\mathcal{G}} = -2\|\mu(F_t)\|^2_{\mathcal{G}}. \tag{2.15}$$

Proof. We have

$$\frac{\partial}{\partial t}\|F_t\|^2_{\mathcal{G}} = 2\mathcal{G}\left(\frac{\partial F_t}{\partial t}, F_t\right) = \mathcal{G}\left(\Pi(\mathcal{J}Z(F_t)), F_t\right)$$

Since $F_t \in \mathrm{Im}\mathscr{D}$, the RHS is equal to $\mathcal{G}(\mathcal{J}Z(F_t), F_t) = -\mathcal{G}(\nabla\phi|_{F_t}, F_t)$. By (2.13), we have

$$\frac{\partial}{\partial t}\|F_t\|^2_{\mathcal{G}} = -2\|\mu(F_t)\|^2_{\mathcal{G}}.$$

In particular $\|F_t\|^2_{\mathcal{G}}$ can only decrease along a flow line. □

Another important easy result is that the stationary points of the modified moment map flow are exactly the zeroes of the moment map.

Proposition 2.8.2. *The critical points of the modified moment map flow on* $\mathrm{Im}\mathscr{D}$ *are precisely the points* $F \in \mathrm{Im}\mathscr{D}$. *such that* $\mu(F) = 0$.

Proof. If F is a point of $\mathrm{Im}\mathscr{D}$ such that $\mu(F) = 0$, we have by definition $Z(F) = 0$. Conversely, if F is a stationary point of the modified flow, we have

$$0 = \Pi(\nabla\phi|_F) = \Pi(\mu(F)\mathscr{R}F).$$

We deduce that $\mathcal{G}(\Pi(\mu(F)\mathscr{R}F), F) = 0 = \mathcal{G}(\mu(F)\mathscr{R}F, F)$, since $F \in \mathrm{Im}\mathscr{D}$. But

$$\begin{aligned}\mathcal{G}(\mu(F)\mathscr{R}F, F) &= \int_\Sigma g(\mu(F)\mathscr{R}F, F)\omega_\Sigma \\ &= \int_\Sigma \mu(F)g(\mathscr{R}F, F)\omega_\Sigma \\ &= \int_\Sigma 2\mu(F)^2\omega_\Sigma \\ &= 2\|\mu(F)\|^2_{\mathcal{G}}\end{aligned}$$

and we deduce that $\mu(F) = 0$. □

The modified moment map flow on $\mathrm{Im}\mathscr{D}$ is an evolution equation which involves a non linear pseudo-differential operator of order 0 in its RHS. The flow is unlikely to have nice regularizing features like heat flows. However, it is easy to show short time existence of solutions for regular smooth initial conditions.

Theorem 2.8.3. *Let* $F \in \mathscr{F}^{k,\alpha}$ *be an exact* 1*-form. Then, there exists* $\varepsilon > 0$ *and a unique solution of the modified moment map flow* $F_t \in \mathscr{F}^{k,\alpha}$*, for* $t \in (-\varepsilon, \varepsilon)$ *with* F_t *exact and* $F_0 = F$.

Proof. The quadratic map $F \mapsto \mu(F)$ is locally Lipschitz with respect to the $C^{k,\alpha}$-norm. Thus, $F \mapsto X_\mu(F)(F) = Z(F)$ is also locally Lipschitz As recalled in Proposition 2.2.1

$$\Pi : \mathscr{F}^{k,\alpha} \to \mathscr{F}^{k,\alpha}$$

is a pseudodifferential linear operator of order zero, which is continuous for the $C^{k,\alpha}$-norm. Therefore the maps $F \mapsto \Pi\mathcal{J}Z(F)$ is locally Lipschitz on $\mathscr{F}^{k,\alpha}$. Since we are working on a Banach space, the Cauchy-Lipschitz theorem applies and we have the local existence and uniqueness of solutions of the flow for short time. □

2.9. **Towards flow stability.** We would like to understand the behavior of the modified moment map flow near a critical point. Let $f : \Sigma \to V$ be an isotropic immersion and $F = \mathscr{D}f \in \mathscr{F}$ the corresponding differential. By [8], the map

$$\mu^D : \mathscr{M} \to C^\infty_0(\Sigma, \mathbb{R})$$

is a submersion near f. Since $\mu^D = \mu \circ \mathscr{D}$, We deduce that $\mu : \mathscr{F} \to C^\infty_0(\Sigma, \mathbb{R})$ is also a submersion at $F = \mathscr{D}f$. Some additional work would be required to show that $\mu^{-1}(0)$ is, in a suitable sense, a regular submanifold of $\mathscr{F}$ near F.

However, the subspace $\mu^{-1}(0) \subset \mathscr{F}$ is generally not regular in no reasonable sense. However, we know that $d\mu|_F \cdot \dot{F} = g(\mathscr{R}F, \dot{F})$ (after Lemma 2.5.6), and we define its *formal tangent space* as

$$T_F\mu^{-1}(0) = \{\dot{F} \in \mathscr{F}, g(\mathscr{R}F, \dot{F}) = 0\}.$$

Then we have the following result:

Lemma 2.9.1. *At every $F \in \mu^{-1}(0) \subset \mathscr{F}$, the Hessian of the functional ϕ is degenerate along the formal tangent space at F and definite positive along any vector space transverse to of $T_F\mu^{-1}(0)$.*

Proof. By Proposition 2.7.2, we have

$$W(F) = \nabla\phi|_F = \mu(F)\mathscr{R}F = \frac{1}{2}g(\mathscr{R}F, F)\mathscr{R}F.$$

Hence

$$DW|_F \cdot \dot{F} = g(\mathscr{R}F, \dot{F})\mathscr{R}F + \frac{1}{2}g(\mathscr{R}F, F)\mathscr{R}\dot{F}.$$

By assumption, $\mu(F) = 0$, so that

$$DW|_F \cdot \dot{F} = g(\mathscr{R}F, \dot{F})\mathscr{R}F.$$

Thus

$$g(DW|_F \cdot \dot{F}, \dot{F}) = (g(\mathscr{R}F, \dot{F}))^2.$$

In conclusion

$$\mathcal{G}(DW|_F\dot{F}, \dot{F}) > 0,$$

unless $\dot{F} \in T_F\mathscr{I}$. Since

$$DW|_F = \mathrm{Hess}|_F(\phi),$$

the proposition follows. □

Remark 2.9.2. In finite dimension, one can consider downward gradient flow of a *Morse-Bott* function $f : M \to \mathbb{R}$, where M is a closed smooth manifold. Let $C \subset M$ be a connected component of the set of critical points of f. By definition of a Morse-Bott furnction, the Hessian of f is non degenerate in directions transverse to C. In this setting there is an adapted version of the classical Morse lemma. If the Hessian is definite positive in directions transverse to C the Morse lemma shows that C is stable under the downward gradient flow of f.

Lemma 2.9.1 together with the above remark suggests that the modified moment map flow along $\mathrm{Im}\mathscr{D}$ should have some stability properties along $\mu^{-1}(0)$. However, nothing is clear since we do not have a version of the Morse lemma in this infinite dimensional setting.

3. Polyhedral surfaces and symplectic geometry

This section is devoted to recasting the smooth geometry presented in §2 in the context of *polyhedral* and *piecewise linear* geometry. In a nuttshell, we are going to replace smooth surfaces and smooth maps $f : \Sigma \to V$ with *Euclidean triangulations*, *polyhedral surfaces* and *piecewise linear maps*.

As in the smooth setting, the target space is a complex vector space V, with complex dimension $n \geq 2$, endowed with a Hermitian structure h_V. The corresponding metric and symplectic form of V are denoted g_V and ω_V. Smooth surfaces shall be replaced by some kind of abstract polyhedra, called *Euclidean triangulations*, modelled on the *Euclidean affine plane* $\mathbb{E}$. Its underlying Euclidean vector space is denoted $\vec{\mathbb{E}}$ and carries a Euclidean metric $g_{\mathbb{E}}$. It is convenient to endow $\mathbb{E}$ with a prescribed orientation. Thus, we can make sense of a rotation $J_{\mathbb{E}}$ of angle $+\frac{\pi}{2}$, as an isometry of $\vec{E}$. The rotation $J_{\mathbb{E}}$, satisfies $J_{\mathbb{E}}^2 = -\mathrm{id}$, which defines an almost complex structure. In turn, we can define the Kähler form given by $\omega_{\mathbb{E}}(X,Y) = g_{\mathbb{E}}(J_{\mathbb{E}}X,Y)$ for every $X, Y \in \vec{\mathbb{E}}$.

3.1. **Euclidean triangulations.**

From now on and in all §3, Σ is a closed and connected *topological* surface, endowed with a *Euclidean triangulation*. It turns out that a Euclidean triangulation induces a smooth structure on Σ with a flat Riemannian metric, with conical singularities at a finite number of vertices. Furthermore, if the Euclidean triangulation is oriented, there is an induced compatible smooth almost complex structure on J_Σ on Σ, which provides a Kähler structure with conical singularities $(\Sigma, J_\Sigma, g_\Sigma, \omega_\Sigma)$. By definition, a Euclidean triangulation consists of the following data:

- The surface Σ is endowed with a triangulation $\mathscr{T}$, that consist of a finite number of vertices $\mathbf{v}_j$, edges $\mathbf{e}_k$ and faces $\mathbf{f}_k$, glued together as simplicial complex, with a total space homeomorphic to Σ.
- Each face $\mathbf{f}_k$ of $\mathscr{T}$ is equipped with a topological embedding
$$\Psi_k : \mathbf{f}_k \to \mathbb{E},$$
where $\mathbb{E}$ is the standard 2-dimensional affine Euclidean space, with the additional requirement that the image $\hat{\mathbf{f}}_k = \Psi_k(\mathbf{f}_k)$ is a Euclidean 2-simplex of $\mathbb{E}$.
- If two faces $\mathbf{f}_k$ and $\mathbf{f}_l$ have an edge in common, the transition map $\Psi_k \circ \Psi_l^{-1}$ is an isometry between the corresponding edges of $\hat{\mathbf{f}}_k$ and $\hat{\mathbf{f}}_l$ in $\mathbb{E}$.

A Euclidean 2-simplex of $\mathbb{E}$ carries an orientation induced by the orientation of $\mathbb{E}$ and so does the boundary edges, by the usual convention. If for every pair of distinct faces $\mathbf{f}_k$ and $\mathbf{f}_l$ of $\mathscr{T}$ with a common edge $\mathbf{e}$, the transition map $\Psi_k \circ \Psi_l^{-1}$ of the corresponding edges $\Psi_k(\mathbf{e})$ and $\Psi_l(\mathbf{e})$ is orientation reversing, we say that *the Euclidean triangulation is oriented.* In fact, this paper is only concerned with oriented Euclidean triangulations.

3.2. **Smooth induced structure.** A surface Σ endowed with a Euclidean triangulation admits a particular atlas, deduced form the maps Ψ_k. First, it is clear that the open set

$$\Sigma' = \Sigma \setminus \bigcup_j \{\mathbf{v}_j\},$$

which is the complement of the 0-skeleton of $\mathscr{T}$ in Σ, has such a canonical atlas, constructed as follows:

- For any interior point z of a face $\mathbf{f}_k$, the map Ψ_k provides a local chart for an open neighborhood of z.
- If z is an interior point of an edge $\mathbf{e}$ of $\mathscr{T}$, there are exactly two distinct faces $\mathbf{f}_k$ and $\mathbf{f}_l$ with $\mathbf{e}$ as a common edge. Up to composition by an isometry of $\mathbb{E}$, we may assume that the images $\hat{\mathbf{f}}_k$ and $\hat{\mathbf{f}}_l$ of $\mathbb{E}$ have a common edge as well. Thus, the two maps Ψ_k and Ψ_l define a map between the joined faces $\Psi_{kl} : \mathbf{f}_k \cup \mathbf{f}_l \to \hat{\mathbf{f}}_k \cup \hat{\mathbf{f}}_l$. This defines a local chart for an open neighborhood of z into $\mathbb{E}$.

The above recipe defines a smooth atlas for Σ'. If the Euclidean triangulation is oriented, we can use orientation preserving isometries and this atlas is compatible with the orientation of $\mathbb{E}$. In particular Σ' admits a tangent bundle. It is easy to see that the Euclidean metric $g_{\mathbb{E}}$ descend to as a natural smooth flat Riemannian metric g_Σ on Σ', since all the transition maps of the atlas are isometries of $\mathbb{E}$. If the Euclidean triangulation is oriented, the almost complex structure also induces an almost complex structure J_Σ on Σ'.

In a neighborhood of a vertex $\mathbf{v}_j$, the surface Σ is identified with a star of Euclidean triangles. It follows that we have local normal coordinates (r, θ) at $\mathbf{v}_j$, where $r \geq 0$ is the Euclidean distance from $\mathbf{v}_j$ and θ is the Euclidean angle around the vertex. The total angle around $\mathbf{v}_j$ is the sum $\frac{2\pi}{\gamma}$ the angles of the Euclidean triangles at the vertex. This is precisely a conical singularity of angle $\frac{2\pi}{\gamma}$ so that $\theta \in \mathbb{R}/\frac{2\pi}{\gamma}\mathbb{Z}$. We define local coordinates $\hat{\Psi}_j : U \to \mathbb{C}$ on a neighborhood U of $\mathbf{v}_j$, given by

$$\hat{\Psi}_j(r, \theta) = r^\gamma e^{i\gamma\theta}$$

In the oriented case, such local coordinates are holomorphic, in the sense that $id\hat{\Psi}_j = d\hat{\Psi}_j \circ J_\Sigma$ along Σ'. This provides a smooth atlas for the topological surface Σ, such that J_Σ extends smoothly at the vertices.

In conclusion, an oriented Euclidean triangulation of Σ provides a smooth structure on Σ, a smooth almost complex structure J_Σ on Σ and a flat Kähler metric g_Σ with conical singularities at the vertices of $\mathscr{T}$.

3.3. **Parallel transport.** Given two points $z_0, z_1 \in \Sigma'$, we choose a smooth path $\gamma : [0, 1] \to \Sigma'$ with $z_0 = \gamma(0)$ and $z_1 = \gamma(1)$. The Levi-Civita connection associated to g_Σ and its parallel transport provide a holonomy map $\Phi_\gamma : T_{z_0}\Sigma \to T_{z_1}\Sigma$, which is a linear isometry. Since g_Σ is flat, the holonomy depends only on the homotopy class of γ in Σ'. Is z_0 and z_1 both belong to $\mathbf{f}_k \cap \Sigma'$, we can choose a path γ contained in $\mathbf{f}_k \cap \Sigma'$. Since this space is contractible, the holonomy map does not depend on the choice of path γ and we have a canonical isomorphism

$$\Phi_{z_0,z_1} : T_{z_0}\Sigma \to T_{z_1}\Sigma.$$

The equivalence class of such tangent spaces, up to parallel transport, will be denoted $T\mathbf{f}_k$:

$$T\mathbf{f}_k = (\bigsqcup_{z\in \mathbf{f}_k\cap\Sigma'} T_z\Sigma)/\sim \tag{3.1}$$

where $X_0 \in T_{z_0}\Sigma$ and $X_1 \in T_{z_1}\Sigma$ are equivalent if, and only if, $\Phi_{z_0,z_1}(X_0) = X_1$. Since the holonomy map respects all the structures, we have induced metric g_Σ and almots complex structure J_Σ on $T\mathbf{f}_k$.

3.4. **The 2-torus case.** The quotient

$$\Sigma = \mathbb{E}/\Gamma$$

where Γ is a lattice of $\vec{\mathbb{E}}$ acting by translation on $\mathbb{E}$, is diffeomorphic to a torus of real dimension 2. The Kähler structure of $\mathbb{E}$ descends to the quotient and we obtain a flat Kähler structure $(\omega_\Sigma, J_\Sigma, g_\Sigma, \omega_\Sigma)$.

We pick any Γ-invariant triangulation $\mathscr{T}_\mathbb{E}$ of $\mathbb{E}$, such that every face of $\mathscr{T}_\mathbb{E}$ is a Euclidean 2-simplex of $\mathbb{E}$. We assume that every face $\hat{\mathbf{f}}_j$ of $\mathscr{T}_E$ is *small enough*, in the sense that the restriction of the canonical projection $\hat{\mathbf{f}}_j \to \Sigma$ is an embedding with image $\mathbf{f}_j$. In particular, the inverse of this restrction provides a map $\Psi_j : \mathbf{f}_j \to \hat{\mathbf{f}}_j \subset \mathbb{E}$, which give an oriented Euclidean triangulation structure on Σ. The quotient Kähler structure coincides obviously with the Kähler structure induced by the Euclidean triangulation. In this case, the metric g_Σ does not have any conical singularities, and it is globally smooth. Furthermore, the holonomy map do not depend on anything. In fact, the differential of the projection map provides canonical isomorphisms

$$T_z\Sigma \to \vec{\mathbb{E}}$$

which is the holonomy.

3.4.1. *Equilateral triangulations.* In the experimental part of this paper, we shall use a particular example of torus with Euclidean triangulation. With some additional programming effort, if would be possible to implement the full-blown framework on a computer, for surfaces any genus and any type of Euclidean triangulation. However, we felt that restricting to a torus $\Sigma = \mathbb{E}/\Gamma$ with the type of lattice given below, would be a sufficiently convincing testing ground. We define

$$\Gamma = span\left\{1, e^{i\frac{\pi}{3}}\right\} \subset \mathbb{C} \simeq \mathbb{E}.$$

The corresponding fundamental domain is a rhombus composed of two equilateral triangles. Now, Γ is a sublattice of

$$\Lambda_N = N^{-1}\Gamma,$$

where N is some positive integer. Then Λ_N can be understood as the set of vertices of a natural triangulation $\mathscr{T}_N$ of $\mathbb{E}$ by equilateral triangle, with sides of length N^{-1}. The triangulation $\mathscr{T}_N$ is represented in Figure 1 for $N = 10$. The picture is restricted to a fundamental domain of of Γ and the vertices of Γ are highlighted.

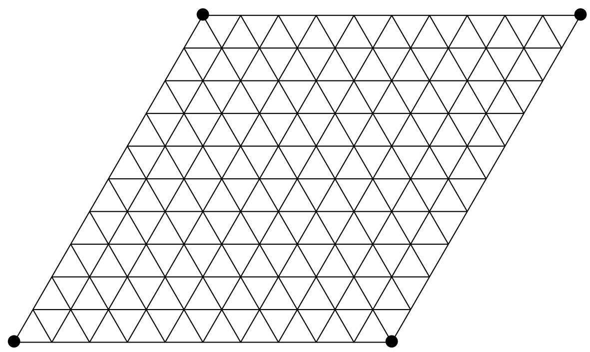

FIGURE 1. Equilateral Euclidean triangulation of a torus

3.5. **Polyhedral and piecewise linear maps.** We go back to the case where Σ is a surface endowed with an oriented Euclidean triangulation. Recall that there are preferred charts $\Psi_k : \mathbf{f}_k \to \mathbb{E}$, by definition of an Euclidean triangulation.

A continuous map $\phi : \Sigma \to \mathbb{R}$ is called *polyhedral* (resp. *piecewise linear*) with respect to a Euclidean triangulation, if the restriction to every face of the triangulation

$$\phi \circ \Psi_k^{-1} : \hat{\mathbf{f}}_k \to \mathbb{R}$$

is an *affine* (resp. a *piecewise linear*) map from $\hat{\mathbf{f}}_k \subset \mathbb{E}$ to $\mathbb{R}$. Similarly, we say that a continuous map $\phi : \Sigma \to V$, into a vector space V, is polyhedral (resp. piecewise linear) if all its coordinate functions are also affine (resp. piecewise linear), in some basis of V. On the one hand, a polyhedral map is obviously a special case piecewise linear map. On the other hand, it is always possible to refine a Euclidean triangulation so that a piecewise linear map can be considered as a polyhedral map with respect to the refined triangulation. We will use the notation

$$\mathscr{P} = \mathscr{P}(\mathscr{T}, V),$$

for the space of V-valued polyhedral maps on Σ, with respect to a Euclidean triangulation $\mathscr{T}$ of Σ.

The image of Σ by a piecewise linear map $\Sigma \to V$ is called a *polyhedral surface* of V. Notice that a polyhedral surface can be very degenerate. At some stage, we will be interested in topologically embedded or immersed polyhedral surfaces. However, polyhedral surfaces may not be immersed, although this is the case generically.

3.6. **Triangular meshes.** We define *the moduli space of triangular meshes* as the space of V-valued functions on the set of vertices of a Euclidean triangulation $\mathscr{T}$ of Σ

$$\mathscr{M} = C^0(\mathscr{T}, V).$$

By definition, a *triangular mesh* $\tau \in \mathscr{M}$ associates V-coordinates to each vertex of $\mathscr{T}$. In particular, for every $f \in \mathscr{P}$ we can define a triangular mesh by restriction

$$\tau(\mathbf{v}) = f(\mathbf{v}).$$

This defines a canonical map

$$ev : \mathscr{P} \to \mathscr{M}$$

given by the evaluation at vertices. It turns out that this map is an isomorphism, almost by construction.

Lemma 3.6.1. *The canonical map $ev : \mathscr{P} \to \mathscr{M}$ is an isomorphism.*

Proof. The injectivity as well a the surjectivity follow from the fact that there exists a unique affine map from a 2-simplex of $\mathbb{E}$ to V with prescribed values at the vertices of the simplex. □

Since the map $ev : \mathscr{P} \to \mathscr{M}$ is invertible, we denote by $f_\tau \in \mathscr{P}$ the unique polyhedral map such that $ev(f_\tau) = \tau$.

3.7. **Differential triangular meshes.** Recall from §3.2 that a surface Σ endowed with an oriented Euclidean triangulation has an induced smooth structure, an almost complex structure and a flat Riemannian metric with conical singularities at the vertices of the triangulation. Using the Levi-Civita connection associated to the metric and its parallel transport, we can make sense of locally constant tensors, at least on Σ'.

In particular, consider the space of V-valued differential 1-forms defined on

$$\Sigma'' = \Sigma \setminus \left(\bigcup_l \{\mathbf{e}_l\} \cup \bigcup_j \{\mathbf{v}_j\} \right),$$

which the complement of the 1-skeleton of $\mathscr{T}$. In other words, Σ'' is the union of the interior domains of the faces of the triangulation. We say that a differential form β smoothly defined over Σ'' is *locally constant* if, and only if, for every face $\mathbf{f}_k$, the push forward $(\Psi_k)_*\beta$ is constant, as a differential form on the interior of $\hat{\mathbf{f}}_k \subset \mathbb{E}$. Then we introduce the space locally constant V-valued differential 1-forms defined along Σ'':

$$\mathscr{F} = \{\zeta \in \Gamma(T^*\Sigma'' \otimes V), \zeta \text{ is locally constant}\}$$

A 1-form of $\mathscr{F}$ is characterized by its value at single interior point of each face, since it is locally constant. Equivalently, there exists a canonical projection map

$$\mathscr{F} \to \prod_{\mathbf{f}} T^*\mathbf{f} \otimes V,$$

where $T\mathbf{f}$ is the tangent space to a face defined at (3.1) and $T^*\mathbf{f}$ is the dual space. By definition, the above map is an isomorphism. Thus an element $\zeta \in \mathscr{F}$ can be understood either as a locally constant 1-form on Σ'', or as a collection of forms on a finite number of vector spaces $T\mathbf{f}$. For the second point of view we use the notation $\zeta(\mathbf{f})$ for each component of ζ in $T^*\mathbf{f} \otimes V$.

3.8. **Differential of a polyhedral map.** The usual differential df, of a polyhedral map $f \in \mathscr{P}$ is well defined along Σ'' as a smooth section of $T^*\Sigma \otimes V \to \Sigma''$. Furthermore, df coincides with the linear part of f since f is locally affine along each face. In particular, df is locally constant, so that we have a natural differentiation map

$$\mathscr{D} : \mathscr{P} \simeq \mathscr{M} \to \mathscr{F}.$$

Thus, the differential $\mathscr{D}f$ can be understood as a locally constant V-valued differential 1-form on Σ'', or as a finite collection of V-valued forms on the tangent spaces $T\mathbf{f}$, depending on what is the most convenient point of view. The space $\mathscr{M}$ is acted on by V, acting by translation on itself and the differential descends to the quotient

$$\mathscr{D} : \mathscr{M}/\mathrm{V} \to \mathscr{F}$$

If Σ is connected, the above map is injective.

3.9. **Fiberwise structures for differential meshes.** Although the metric g_Σ has conical singularites, most constructions of the smooth setting go through without any problems. We already mentioned that an oriented Euclidean triangulation of Σ induces a smooth structure with a smooth almost complex structure J_Σ. The induced metric g_Σ is flat with conical singularities at the vertices of the triangulation. Furthermore, the metric is compatible with J_Σ and has an associated Kähler form ω_Σ with conical singularities at the vertices.

As in the smooth case, we deduce a fiberwise metric g from the Riemannian metric g_Σ and g_V on the bundle $T^*\Sigma \otimes_{\mathbb{R}} V \to \Sigma'$. More generally, we have fiberwise inner products, denoted g as well, on all all bundles of differential forms over Σ'. The only noticeable difference with the smooth case is that the metric g has conical singularities at the vertices of the triangulation. Using parallel transport, we can identify all the tangent spaces of a face $\mathbf{f}_k$. Since parallel transport leave all the structures g_Σ, J_Σ, ω_Σ and g invariant, they can be understood as structures defined on $T\mathbf{f}_k$ and the corresponding tensors $T^*\mathbf{f}_k \otimes V$, for instance.

As in the smooth case, we can introduce a Euclidean metric on the space of differential forms, obtained by integration on Σ. In particular for $\zeta_1, \zeta_2 \in \mathscr{F}$, we define

$$\mathcal{G}(\zeta_1, \zeta_2) = \int_\Sigma g(\zeta_1, \zeta_2)\omega_\Sigma,$$

where the integrand is understood as a locally constant function defined on Σ''. Using the alternate interpretation, ζ_i can be understood as a collection of forms $\zeta_i(\mathbf{f}_k) \in T^*\mathbf{f}_k \otimes V$. Then we have the obvious identity

$$\mathcal{G}(\zeta_1, \zeta_2) = \sum_{\mathbf{f}} g(\zeta_1(\mathbf{f}), \zeta_2(\mathbf{f}))\mathrm{Area}(\mathbf{f}),$$

where

$$\mathrm{Area}(\mathbf{f}) = \int_{\mathbf{f}} \omega_\Sigma$$

is the Euclidean area of the face $\mathbf{f}$, with respect to g_Σ.

As in the smooth case, the almost complex structure $i : V \to V$ acts by multiplication on V-valued forms, whereas J_Σ acts by composition on the right. We use the same almost complex structure J on V-valued 1-forms defined by

$$J \cdot \zeta = -\zeta \circ J_\Sigma$$

and the fiberwise symplectic form on $T^*\Sigma \otimes V \to \Sigma$

$$\omega(\cdot, \cdot) = g(J\cdot, \cdot).$$

As in the smooth case, there is a fiberwise involution $\mathscr{R} = iJ$ acting on the bundle $T^*\Sigma \otimes V \to \Sigma$. This provides a splitting into ± 1 eigenspaces.

The almost complex structure J provides an almost complex structure $\mathcal{J}$ on $\mathscr{F}$ and we have a Kähler form $\Omega(\cdot, \cdot) = \mathcal{G}(\mathcal{J}\cdot, \cdot)$, given also by

$$\Omega(\zeta_1, \zeta_2) = \int_\Sigma \omega(\zeta_1, \zeta_2)\omega_\Sigma$$

This defines a Kähler structure on $\mathscr{F}$, given by the almost complex structure $\mathcal{J}$, the metric $\mathcal{G}$ and Kähler form Ω. As in the smooth case, the involution $\mathscr{R} = i\mathcal{J}$ provides a splitting of $\mathscr{F}$ into complex eigenspaces

$$\mathscr{F} = \mathscr{F}^+ \oplus \mathscr{F}^-.$$

We introduce the $\mathcal{G}$-orthogonal projection

$$\Pi : \mathscr{F} \to \mathscr{F}$$

onto $\mathrm{Im}\mathscr{D}$.

3.10. **Symplectic density for differential meshes.** We denote by $C^2(\mathscr{T}, \mathbb{R})$ the space of real-valued functions on the set of faces of the triangulation $\mathscr{T}$. Alternatively, $C^2(\mathscr{T}, \mathbb{R})$ can be understood as the space of locally constant real-valued functions on Σ''. We introduce an analogue of the moment map in the smooth setting given by the map

$$\mu : \mathscr{F} \to C^2(\mathscr{T}, \mathbb{R})$$

defined by

$$\mu(\zeta) = \frac{1}{2} g(\mathscr{R}\zeta, \zeta),$$

for every $\zeta \in \mathscr{F}$.

We can make sense of the pull-back $\zeta^*\omega_V$ defined by

$$(\zeta^*\omega_V)(X, Y) = \omega_V(\zeta \cdot X, \zeta \cdot Y)$$

for every $X, Y \in T_z\Sigma$, where $z \in \Sigma''$, which defines a locally constant 2-form $\zeta^*\omega_V$ on Σ''. Alternatively this locally constant 2-form can be understood as a collection of 2-forms on the tangent spaces $T\mathbf{f}$. As in the smooth case, the map μ is a symplectic density, in the sense of the following lemma:

Lemma 3.10.1. *Fore every $\zeta \in \mathscr{F}$, we have the identity*

$$\mu(\zeta) = \frac{\zeta^*\omega_V}{\omega_\Sigma}.$$

Remark 3.10.2. Again, the above quotient can be understood as a locally constant function on Σ'', or as a function on the space of faces of the triangulation.

Proof. Along Σ'', an element $\zeta \in \mathscr{F}$ is a locally constant V-valued 1-form. In particular, ζ is smooth on Σ'' so that Lemma 2.5.6 applies locally, along Σ'' and the result follows. □

The map μ admits a natural interpretation in the context of polyhedral geometry. Indeed, let $\tau \in \mathscr{M}$ be a triangular mesh, and $f_\tau \in \mathscr{P}$ be the associated piecewise linear function (cf. §3.6). The differential $\zeta = \mathscr{D} f_\tau$ of f_τ is well defined along Σ''. In particular $f_\tau^*\omega_V$ is well defined along Σ'', in the usual sense of differential calculus. By definition, we have

$$\zeta^*\omega_V = f_\tau^*\omega_V.$$

In particular

$$\mu \circ \mathscr{D}(\tau) = \frac{f_\tau^*\omega_V}{\omega_\Sigma}. \tag{3.2}$$

This motivates the following definition:

Definition 3.10.3. *A triangular polyhedral map $f \in \mathscr{P}$ is called isotropic if $f^*\omega_V$ vanishes along Σ'. Similarly, a triangular mesh $\tau \in \mathscr{M}$ is called isotropic if its corresponding polyhedral map $f_\tau \in \mathscr{P}$ is isotropic in the previous sense.*

In particular, by Equation (3.2), we have

Lemma 3.10.4. *A triangular mesh $\tau \in \mathscr{M}$ is isotropic if, and only if*

$$\mu \circ \mathscr{D}(\tau) = 0.$$

By Equation (3.2), we deduce the following result:

Lemma 3.10.5. *For every $\tau \in \mathscr{M}$, the density function $\sigma_\tau = \mu \circ \mathscr{D}(\tau)$ is $\mathcal{G}$-orthogonal to constant functions, in the sense that*

$$\mathcal{G}(\sigma_\tau, \mathbf{1}) = \int_\Sigma \sigma_\tau \, \omega_\Sigma = 0,$$

where $\mathbf{1} \in C^2(\mathscr{T}, \mathbb{R})$ is the constant function equal to 1 on every face of the triangulation.

Proof. By definition, we have

$$\int_\Sigma \sigma_\tau \omega_\Sigma = \int_\Sigma f_\tau^*\omega_V.$$

The symplectic form ω_V is an exact 2-form on V. There exists a Liouville 1-form λ_V on V, such that $\omega_V = d\lambda_V$. Thus $\int_{\mathbf{f}} f_\tau^*\omega_V = \int_{\partial \mathbf{f}} f^*\lambda_V$ by Stockes theorem over Σ, where $\partial\mathbf{f}$ consists of a union of the three boundary edges of $\mathbf{f}$. Cancellations of faces with a common boundary edges show that the total integral over Σ vanishes. □

The next corollary shows that the restriction map $\mu : \mathrm{Im}\mathscr{D} \to C^2(\mathscr{T},\mathbb{R})$ can never be a submersion.

Corollary 3.10.6. *The map μ admits a restriction*

$$\mu : \mathrm{Im}\mathscr{D} \to C^2_0(\mathscr{T},\mathbb{R}),$$

where $C^2_0(\mathscr{T},\mathbb{R})$ is the space of constant functions on faces of the triangulation, which are $\mathcal{G}$-orthogonal to constants.

3.11. **Hamiltonian torus action.** We define the complex gauge group $\mathbb{T}^{\mathbb{C}}$ as the space of non vanishing complex valued functions on the set of faces

$$\mathbb{T}^{\mathbb{C}} = C^2(\mathscr{T},\mathbb{C}^*)$$

is endowed with a canonical multiplicative law, that provides an Abelian Lie group structure on $\mathbb{T}^{\mathbb{C}}$. In fact $\mathbb{T}^{\mathbb{C}}$ is identified to a complex torus, with dimension equal to the number of faces of the triangulation. We consider the real subgroup $\mathbb{T} \subset \mathbb{T}^{\mathbb{C}}$ given by the set of functions

$$\mathbb{T} = C^2(\mathscr{T}, S^1)$$

We define an action of $\mathbb{T}^{\mathbb{C}}$ on $\mathscr{F}$ by

$$\lambda \cdot \zeta = \bar{\lambda}^{-1}\zeta^+ + \lambda\zeta^-,$$

where $\zeta^{\pm}$ are the component of ζ with respect to the splitting $\mathscr{F} = \mathscr{F}^+ \oplus \mathscr{F}^-$.

The complex torus has Lie algebra

$$\mathfrak{t}^{\mathbb{C}} = C^2(\mathscr{T},\mathbb{C}).$$

We consider the exponential map

$$\exp : \mathfrak{t}^{\mathbb{C}} \to \mathbb{T}^{\mathbb{C}}$$

defined by

$$\exp \xi = e^{i\xi},$$

so that the real torus $\mathbb{T}$ has Lie algebra identified to

$$\mathfrak{t} = C^2(\mathscr{T},\mathbb{R}).$$

In particular, the map μ can be regarded as a map

$$\mu : \mathscr{F} \to \mathfrak{t}.$$

For $\xi \in \mathfrak{t}^{\mathbb{C}}$, we denote by X_ξ the vector field on $\mathscr{F}$, defined by

$$X_\xi(\zeta) = \frac{\partial}{\partial t} \exp(\xi) \cdot \zeta \Big|_{t=0}.$$

The vector field X_ξ represents the infinitesimal action of $\xi \in \mathfrak{t}^{\mathbb{C}}$ on $\mathscr{F}$. By definition, we have

$$X_\xi(\zeta) = i\bar{\xi}\zeta^+ + i\xi\zeta^-.$$

As in the smooth case, the construction has the following nice properties:

Lemma 3.11.1. *The $\mathbb{T}^{\mathbb{C}}$-action on $\mathscr{F}$ is holomorphic with respect to $\mathcal{J}$. The $\mathbb{T}$-action preserves the metric $\mathcal{G}$ and the symplectic form Ω as well.*

By construction, the torus action is Hamiltonian and everything was designed so that the following result holds, as in the smooth case:

Proposition 3.11.2. *The map $\mu : \mathscr{F} \to \mathfrak{t}$ is a moment map for the $\mathbb{T}$-action on $\mathscr{F}$. More precisely, μ is $\mathbb{T}$-invariant and*

$$\mathcal{G}(D\mu|_\zeta \cdot \dot{\zeta}, \xi) = \Omega(X_\xi(\zeta), \dot{\zeta}).$$

Proof. The proof is formally identical to the smooth setting. □

3.12. **Finite dimensional flow.** Recall that by Lemma 3.10.4, a triangular meshes $\tau \in \mathscr{M}$ is isotropic if, and only if, it is solution of the equation

$$\mu \circ \mathscr{D}(\tau) = 0.$$

Thus, the space of isotropic meshes $\mathscr{I}$ is identified, up to translations, to

$$\mathscr{D}\mathscr{I} = \mathrm{Im}\mathscr{D} \cap \mu^{-1}(0).$$

Since the subspace $\mathrm{Im}\mathscr{D} \subset \mathscr{F}$ is not invariant under the $\mathbb{T}^{\mathbb{C}}$-action, the usual moment map flow does not leave $\mathrm{Im}\mathscr{D}$ invariant and it is not suited for our purpose.

3.12.1. *Modified moment maps flow.* We are looking for a flow preserving $\mathrm{Im}\mathscr{D}$ that converges toward zeroes of the moment map. As in the smooth case we define a non negative functional

$$\phi : \mathscr{F} \to \mathbb{R}$$

such that $\phi^{-1}(0) = \mu^{-1}(0)$ by

$$\phi(\zeta) = \frac{1}{2}\|\mu(\zeta)\|^2_{\mathcal{G}}.$$

We denote by $\nabla\phi$ the gradient of $\phi : \mathscr{F} \to \mathbb{R}$ with respect to the metric $\mathcal{G}$. Then we define the flow along $\mathrm{Im}\mathscr{D}$ by

$$\frac{\partial \zeta}{\partial t} = -\Pi\left(\nabla\phi|_\zeta\right), \tag{3.3}$$

where Π is the $\mathcal{G}$-orthogonal projection onto $\mathrm{Im}\mathscr{D}$. This flow preserves $\mathrm{Im}\mathscr{D}$, by definition. In fact, this flow restricted to $\mathrm{Im}\mathscr{D}$ is just the downward gradient flow of the functional ϕ restricted to $\mathrm{Im}\mathscr{D}$. In addition, we have

$$\Pi(\nabla\phi) = -\Pi\mathcal{J}Z$$

where Z is the vector field on $\mathscr{F}$ given by

$$Z(\zeta) = -X_{\mu(\zeta)}(\zeta) = -i\mu(\zeta)\zeta.$$

so that the modified moment map flow is also given by

$$\frac{\partial \zeta}{\partial t} = \Pi\mathcal{J}Z.$$

As in the smooth case, the same formal properties follow. For instance we have:

Proposition 3.12.2. *The fixed points of the modified moment map flow on $\mathrm{Im}\mathscr{D}$ are precisely the zeroes of the moment map.*

Proposition 3.12.3. *A solution ζ_t of the modified moment map flow, for t in some interval, satisfies the differential equation:*

$$\frac{\partial}{\partial t}\|\zeta_t\|^2_{\mathcal{G}} = -2\|\mu(\zeta_t)\|^2_{\mathcal{G}}.$$

The last proposition has much stronger consequences in the finite dimensional setting than in the analogous smooth setting. In particular, the flow cannot blow up as stated in the following corollay:

Corollary 3.12.4. *Let $\zeta_t \in \mathrm{Im}\mathscr{D}$ be a solution of modified moment map flow, defined for t in some interval $I = [t_0, T)$, where T is the maximal time of existence. Then ζ_t is trapped in a compact set of $\mathrm{Im}\mathscr{D}$ for all $t \in I$ and $T = +\infty$. Furthermore, the limiting orbit of ζ_t*

$$\bigcap_{t'\geq t_0} \overline{\{\zeta_t, t \geq t'\}}$$

is compact and consists of zeroes of the moment map in $\mathrm{Im}\mathscr{D}$.

Proof. The function $\|\zeta_t\|^2_{\mathcal{G}}$ can only decrease along flow lines by Proposition 3.12.3. Hence, every flow line is bounded in $\mathrm{Im}\mathscr{D}$ for $t \in [t_0, T)$. By the Cauchy-Lipschitz theorem, the maximality of T implies $T = +\infty$. The boundedness of the flow lines also implies that the limiting orbit is a compact set of $\mathrm{Im}\mathscr{D}$.

The fact that the limiting orbit consists of zeroes of the moment map is slightly less trivial and relies on the fact that we have a gradient flow: let ζ_∞ be an element of the limiting orbit of the flow ζ_t. We would like to show that $\mu(\zeta_\infty) = 0$. Assume that it is not the case. Let $t_k \in [t_0, +\infty)$ be an increasing sequence such that $t_0 = t_0$, $t_{k+1} \geq t_k + 1$ and $\lim \zeta_{t_k} = \zeta_\infty$. For k sufficiently large, all the terms ζ_{t_k} are contained in an arbitrarily small open neighborhood U of ζ_∞. By the Cauchy-Lipschitz theorem, there exists $0 < \delta < 1$ such that the flow lines are contained in U for every $t \in [t_k - \delta, t_k + \delta]$. If U is chosen sufficiently small, we may assume that $\|\mu(\zeta)\|^2_{\mathcal{G}} \geq \frac{1}{2}\|\mu(\zeta_\infty)\|^2_{\mathcal{G}} = c > 0$ for every $\zeta \in U$. In particular

$$\begin{aligned}
\|\zeta_{t_k+\delta}\|^2 - \|\zeta_{t_0}\|^2 &= \int_{t_0}^{t_k+\delta} \frac{\partial}{\partial t}\|\zeta_t\|^2 dt \\
&= -2\int_{t_0}^{t_k+\delta} \|\mu(\zeta_t)\|^2 dt \\
&\leq -4\delta k c
\end{aligned}$$

This shows that $\|\zeta_{t_k+\delta}\|^2$ is negative for k sufficiently large, which is a contradiction. □

3.13. **Stability near the regular locus.** The moduli space of isotropic meshes $\mathscr{I} \subset \mathscr{M}$ is identified, up to translations, with

$$\mathscr{D}(\mathscr{I}) = \mathrm{Im}\mathscr{D} \cap \mu^{-1}(0),$$

according to Lemma 3.10.4. The moduli space may contain various singularities. For example, the origin $0 \in \mathrm{Im}\mathscr{D}$ is most singular since the differential of μ at $\zeta = 0$ vanishes. However, regular points of the moduli space $\mathscr{D}\mathscr{I}$ are expected to be generic under mild circumstances, although this claim would have to be proved.

Recall that, by Corollary 3.10.6, the moment map μ admits a restriction $\mu : \mathrm{Im}\mathscr{D} \to C^2_0(\mathscr{T}, \mathbb{R})$.

Definition 3.13.1. *A point $\zeta \in \mathscr{D}\mathscr{I}$ is called* regular*, if $\mu : \mathrm{Im}\mathscr{D} \to C^2_0(\mathscr{T}, \mathbb{R})$ is a submersion at ζ. In particular $\mathscr{D}\mathscr{I}$ is a submanifold of $\mathrm{Im}\mathscr{D}$ in a neighborhood of ζ, with codimension $\dim C^2_0(\mathscr{T}, \mathbb{R}) = \#\mathfrak{C}_2(\mathscr{T}) - 1$ and tangent space*

$$T_\zeta \mathscr{D}\mathscr{I} = \left\{ \dot{\zeta} \in \mathrm{Im}\mathscr{D}, g(\mathscr{R}\zeta, \dot{\zeta}) = 0 \right\}.$$

Proposition 3.13.2. *The Hessian of $\phi : \mathrm{Im}\mathscr{D} \to \mathbb{R}$ is positive transversely to the regular locus of $\mathscr{D}\mathscr{I}$.*

Proof. The formal computation of the Hessian of ϕ is the same as in the smooth case. Let ζ_0 be a regular point of $\mathscr{D}\mathscr{I}$. We obtain

$$\mathcal{G}(\mathrm{Hess}(\phi)|_{\zeta_0} \cdot \dot{\zeta}, \dot{\zeta}) = \int_\Sigma g(\mathscr{R}\zeta_0, \dot{\zeta})^2 \omega_\Sigma.$$

This quantity is positive unless $g(\mathscr{R}\zeta_0, \dot{\zeta}) = 0$, which means that $\dot{\zeta}$ belongs to $T_{\zeta_0}\mathscr{D}\mathscr{I}$. □

In particular, the functional ϕ is Morse-Bott near regular points:

Corollary 3.13.3. *Critical points of the function $\phi : \mathrm{Im}\mathscr{D} \to \mathbb{R}$ coincide with its minimum $\phi = 0$ and $\mathscr{D}\mathscr{I}$. Furthermore ϕ satisfies the Morse-Bott condition along the regular locus of $\mathscr{D}\mathscr{I}$.*

The Morse lemma, well known for Morse function, can be adapted to the case of Morse-Bott function: let ζ_0 be a regular point of $\mathscr{D}\mathscr{I}$. Then there exist local coordinates $x_1, \cdots, x_m$ on $\mathrm{Im}\mathscr{D}$, vanishing at ζ_0, such that $\mathscr{D}\mathscr{I}$ is given locally by the equations $x_1 = \cdots = x_k = 0$ and

$$\phi = x_1^2 + \cdots + x_k^2.$$

The differential of ϕ is explicit in these coordinates, and downward flow of the corresponding gradient can be studied almost explicitly, using classical techniques of ODE and Morse theory. In particular, we obtain the following stability result:

Corollary 3.13.4. *Let ζ_0 be a regular point of $\mathscr{D}\mathscr{I}$. There exists an open neighborhood V of ζ_0 in $\mathrm{Im}\mathscr{D}$ with the following properties:*

1. *Let $\phi_t : V \to \mathrm{Im}\mathscr{D}$ be the modified moment map flow for $t \in \mathbb{R}^+$. Then the neighborhood V in invariant under the flow in the sense that $\phi_t(V) \subset V$ for every $t \in \mathbb{R}^+$.*
2. *For every $\zeta \in V$, the flow $t \mapsto \phi_t(\zeta)$ converges exponentially fast toward a point of $\mathscr{D}\mathscr{I}$ as t goes to $+\infty$.*

This provides strong constraints on limiting orbits:

Corollary 3.13.5. *Let $\zeta_t \in \mathscr{F}$ be a solution of the modified moment map flow defined for $t \in [0, +\infty)$. If the limiting orbit of ζ_t contains a regular point ζ_∞ of $\mathscr{D}\mathscr{I}$, then ζ_t converges exponentially fast toward ζ_∞. Furthermore, the map $V \times [0, +\infty] \to \mathscr{D}\mathscr{I}$ which maps any point ζ_0 to $\zeta_t = \phi_t(\zeta_0)$ under the flow is continuous. Here, it is understood that the image of $(\zeta_0, +\infty)$ is the limit ζ_∞ of the flow.*

Proof. Let ζ_∞ be a regular point $\mathscr{I}$ contained in the limiting orbit of ζ_t. We choose a neighborood V of ζ_∞ as in Ccorollary 3.13.4. There exists a time t' such that $\zeta_{t'} \in V$. By definition of V, ζ_t converge exponentially fast toward a point of $\mathscr{I}$. This point must be ζ_∞ since it belongs to the limiting orbit. $\square$

These results can be reformulated more globally as follows:

Theorem 3.13.6. *Let $\mathscr{V}$ be the set of differential meshes of* $\mathrm{Im}\mathscr{D}$ *that converge under the flow toward a limit which is a regular point of $\mathscr{D}\mathscr{I}$. Then, set $\mathscr{V}$ has the following properties*

(1) *The set $\mathscr{V}$ is invariant under the modified moment map flow ϕ_t, for $t \geq 0$.*
(2) *The set $\mathscr{V}$ is an open set of* $\mathrm{Im}\mathscr{D}$.
(3) *The set $\mathscr{V}$ continuously retracts onto the regular locus of $\mathscr{D}\mathscr{I}$.*

Proof. The claim (1) is obvious, by definition of $\mathscr{V}$.

Let ζ_0 be a point of $\mathscr{V}$. We want to show that $\mathscr{V}$ is a neighborhood of ζ_0, which will prove the claim (2). Let $\zeta_t = \phi_t(z_0) \in \mathscr{V}$ to $t \in [0, +\infty)$ be the solution of the modified moment map flow. By definition, ζ_t converges exponentially fast toward a limit ζ_∞, which is a regular point of $\mathscr{D}\mathscr{I}$. We choose an open neighborhood V of ζ_∞ in $\mathrm{Im}\mathscr{D}$ given by Corollary 3.13.5. There exists $T > 0$ sufficiently large, such that $\zeta_T \in V$ since $\lim \zeta_t = \zeta_\infty$. The flow $\phi_T : \mathrm{Im}\mathscr{D} \to \mathrm{Im}\mathscr{D}$ is a continous map. In particular $V' = \phi_T^{-1}(V)$ is an open set of $\mathrm{Im}\mathscr{D}$. Furthermore $\zeta_T \in V$ so that $\zeta_0 \in V'$. For every $z'_0 \in V'$, we have $z'_T \in V$. By Corollary 3.13.5, the flow starting at z'_T converges exponentially fast toward a regular point z'_∞ in the regular locus of $\mathscr{D}\mathscr{I}$, which implies that $V' \subset \mathscr{V}$. We conclude that $\mathscr{V}$ is a neighborhood of ζ_0.

The claim (3) is proved in a similar way. We consider the map $\mathscr{V} \times [0, +\infty] \to \mathscr{D}\mathscr{I}$ which associates $\zeta_t = \phi_t(\zeta_0)$ to any pair (ζ_0, t). The continuity follows from Corollary 3.13.5. $\square$

Remark 3.13.7. We expect the set $\mathscr{V} \subset \mathrm{Im}\mathscr{D}$ of Theorem 3.13.6 to be *generic* in a suitable sense, although we do not have a proof of this fact. However, we were able to prove that $\mathscr{V}$ is not empty (cf. §3.14) in certain situation based on the constructions of [8].

3.14. **Approximation of smooth isotropic surfaces and stability.** We consider a smooth isotropic immersion

$$\ell : \Sigma \to V$$

where Σ is a 2 dimensional torus. A scheme of approximations of ℓ by triangular isotropic meshes was investigated by Jauberteau, Rollin and Tapie. Their result can be reformulated as follows:

Theorem 3.14.1 ([8] and [11])**.** *There exists a sequence of Eulidean triangulation $\mathscr{T}_N$ of the torus Σ and isotropic meshes $\tau_n \in \mathscr{M}(\mathscr{T}_N)$ defined for N sufficiently large, such that the polyhedral maps f_{τ_N} are isotropic polyhedral topological immersions, which converge in the C^1-sense toward ℓ. Furthermore, the differential meshes $\zeta_N = \mathscr{D}_N\tau_N$ are regular points of the set of isotropic differential meshes in $\mathscr{D}_N\mathscr{I}_N$ in* $\mathrm{Im}\mathscr{D}_N$*.*

Proof. We provide some comment to help the reader extract the above theorem from [8, 11]. The first part of the theorem is straightforward from [8, Theorem A and Theorem 7.4.2] for the C^0-convergence. The C^1-convergence, as well as the fact that the maps are topological immersions, is obtained in [11].

Only the regularity statement may need some clarification. For this, we have to contemplate the proofs of [8]. The first point is that the triangulations $\mathscr{T}_N$ are obtained from certain quadrangulation $\mathscr{Q}_N$. The first step of their construction is to obtain isotropic quadrangular meshes $\tau'_N \in \mathscr{M}(\mathscr{Q}_N)$ as approximation ℓ perturbed using the fixed point principle. The proof rests on the fact that a certain version of the symplectic density for quadrangular meshes

$$\hat{\mu}^D_N : \mathscr{M}(\mathscr{Q}_N) \to C^2_0(\mathscr{Q}_N, \mathbb{R})$$

is a submersion at τ'_N. The triangular mesh τ_N of the theorem are obtained by refining the quadrangular mesh into a triangulation. The idea is to replace every isotropic quadrilateral by an isotropic pyramid. This subproblem is easy to solve since it is only a linear problem. In this process, we obtain a triangular τ_N, which is by construction a regular zero of the symplectic density

$$\mu^D_N = \mu \circ \mathscr{D}_N : \mathscr{M}(\mathscr{T}_N) \to C^2_0(\mathscr{T}_N, \mathbb{R}).$$

Since $\mu \circ \mathscr{D}_N = \mu^D_N$, we deduce that μ^D_n is a submersion at $\zeta_N = \mathscr{D}_N\tau_N$, which shows that ζ_N is regular. □

Corollary 3.14.2. *For every torus Σ, smooth isotropic immersion $\ell : \Sigma \to V$ and N sufficiently large, the Euclidean triangulation $\mathscr{T}_N$ constructed in Theorem 3.14.1 have non empty spaces of orbits $\mathscr{V}_N \subset$* $\mathrm{Im}\mathscr{D}_N$*.*

3.15. **Convergence of the modified moment map flow.** We have a good understanding of the behavior of orbits that converge toward regular points of $\mathrm{Im}\mathscr{D}$. However, the Duistermaat theorem asserts that, under sensible assumptions, every orbit of a moment map flow converge. We are going to adapt the proof of the Dusitermaat theorem presented in [9] to our modified setting and show that the flow line of the modified moment map flow always converge, whether or not the limit is regular:

Theorem 3.15.1. *For every* $\zeta_0 \in \mathrm{Im}\mathscr{D}$*, the limiting orbit of the flow* $\zeta_t = \varphi_t(\zeta_0)$ *is reduced to a point* $\zeta_\infty \in \mathscr{D}\mathscr{I}$*. In other words* $\lim \zeta_t = \zeta_\infty \in \mathrm{Im}\mathscr{D} \cap \mu^{-1}(0)$*. Furthemore, the map*

$$[0, +\infty] \times \mathrm{Im}\mathscr{D} \to \mathrm{Im}\mathscr{D}$$

given by $(t, \zeta_0) \mapsto \zeta_t = \varphi_t(\zeta_0)$*, with the convention that* $(+\infty, \zeta_0) \mapsto \lim \varphi_t(\zeta_0) = \zeta_\infty$*, is a continuous retraction of* $\mathrm{Im}\mathscr{D}$ *onto* $\mathscr{D}\mathscr{I}$*.*

We use a result due to Lojasiewicz (cf. [9, Lemma 2.2] or Proposition 1 p. 67 of [10], or Proposition 6.8 in [1]).

Lemma 3.15.2 (Lojasiewicz gradient inequality)**.** *Let* ϕ *be a real analytic function on an open set* $W \subset \mathbb{R}^k$ *and* $x \in W$*, a critival point of* ϕ*, such that* $\phi(x) = 0$*. Then, there exists a neighborhood* U *of* x*, some constants* $c > 0$ *and* $\alpha \in (0, 1)$*, such that*

$$\|\nabla \phi(y)\| \geq c|\phi(y)|^\alpha$$

for every $y \in W$*, where* $\|\cdot\|$ *is an Euclidean norm on* $\mathbb{R}^k$ *and* ∇ *is the gradient with respect to the Euclidean inner product.*

Proof of Theorem 3.15.1. The function $\phi : \mathrm{Im}\mathscr{D} \to \mathbb{R}$ is polynomial of order 4. In particular, it is analytic and Lojasiewicz gradient inequality applies. Let $\zeta_0 \in \mathrm{Im}\mathscr{D}$ with orbit $\zeta_t = \varphi_t(\zeta_0)$ and $\mathscr{O}$ its limiting orbit. We know that $\mathscr{O} \subset \phi^{-1}(0)$. In particular

$$\lim_{t \to +\infty} \phi(\zeta_t) = 0.$$

By Proposition 3.12.3, the norm decreases along the flow ζ_t. Put $K = \|\zeta_0\|$. Then the orbit ζ_t for $t \geq 0$ and its limiting orbit $\mathscr{O}$ are contained in the interior of the closed ball $\overline{B}(0, 2K) \subset \mathrm{Im}\mathscr{D}$ of radius $2K$ centered at 0.

For each $\zeta \in \overline{B}(0, 2K)$, there exists an open neighborhood U_ζ of ζ in $\mathrm{Im}\mathscr{D}$ and constants $c_\zeta > 0$ and $\alpha_\zeta \in (0, 1)$, which satisfy Lemma 3.15.2 for the function $\phi : \mathrm{Im}\mathscr{D} \to \mathbb{R}$. In other words, for all $\zeta \in U_\zeta$

$$\|\nabla \phi(\zeta)\| \geq c_\zeta \phi(\zeta)^{\alpha_\zeta}$$

By compactness of $\overline{B}(0, 2K)$, we can extract a finite cover $U_1, \cdots, U_k$ of the ball $\overline{B}(0, 2K)$ by open sets of the family U_ζ. We put

$$U = \bigcup U_i, \quad \alpha = \max \alpha_i \quad \text{and} \quad c = \min c_i,$$

where α_i and c_i are the constants corresponding to the open sets U_i. Then, for every $\zeta \in U$ such that $\phi(\zeta) \leq 1$, we have $\phi(\zeta)^{\alpha_i} \geq \phi(\zeta)^\alpha$. Thus, for every $\zeta \in U$ such that $\phi(\zeta) \leq 1$, we have

$$\|\nabla \phi(\zeta)\| \geq c\phi(\zeta)^\alpha.$$

Now, for every $\zeta'_0 \in B(0, 2K)$, we have $\zeta'_t \in \overline{B}(0, 2K) \subset U$ for every $t \geq 0$, since the norm decreases along the flow. As $\lim \phi(\zeta'_t) = 0$, we have $\phi(\zeta'_t) \leq 1$ for every t sufficiently large, so that

$$\|\nabla \phi(\zeta'_t)\| \geq c\phi(\zeta'_t)^\alpha. \tag{3.4}$$

We compute

$$-\frac{\partial}{\partial t}\phi(\zeta'_t)^{1-\alpha} = -(1-\alpha)\phi(\zeta'_t)^{-\alpha}\|\nabla\phi(\zeta'_t)\|^2.$$

For t sufficiently large, we deduce by (3.4) the differential inequality

$$-\frac{\partial}{\partial t}\phi(\zeta'_t)^{1-\alpha} \geq (1-\alpha)c\|\nabla\phi(\zeta'_t)\|.$$

Hence, for $t_1 > t_0$ sufficiently large, we obtain by integrating the above inequality

$$\phi(\zeta'_{t_0})^{1-\alpha} - \phi(\zeta'_{t_0})^{1-\alpha} = -\int_{t_0}^{t_1}\frac{\partial}{\partial t}\phi(\zeta'_t)^{1-\alpha}dt \geq (1-\alpha)c\int_{t_0}^{t_1}\|\nabla\phi(\zeta'_t)\|dt.$$

Put $C = \frac{1}{(1-\alpha)c}$ so that we have

$$\begin{aligned} d(\zeta_{t_0}, \zeta'_{t_1}) &\leq \int_{t_0}^{t_1}\left\|\frac{\partial}{\partial t}\zeta'_t\right\|dt \\ \Rightarrow d(\zeta'_{t_0}, \zeta'_{t_1}) &\leq \int_{t_0}^{t_1}\|\nabla\phi(\zeta'_t)\|dt \\ \Rightarrow d(\zeta'_{t_0}, \zeta'_{t_1}) &\leq C\left(\phi(\zeta'_{t_0})^{1-\alpha} - \phi(\zeta'_{t_1})^{1-\alpha}\right). \end{aligned} \tag{3.5}$$

Since $\lim\phi(\zeta'_t)^{1-\alpha} = 0$, the function $t \mapsto \phi(\zeta'_t)^{1-\alpha}$ satisfies the Cauchy criterion, and so does $t \mapsto \zeta'_t$ by Inequality (3.5). In particular ζ'_t converges as t goes to infinity. This shows that the flow $\zeta'_t = \varphi_t(\zeta'_0)$ converges as t goes to infinity for every $\zeta'_0 \in \overline{B}(0, 2K)$. In fact, we could have chosen any constant $K > 0$ in our argument so that the flow converges for any initial condition $\zeta'_0 \in \mathrm{Im}\mathscr{D}$. Thus the flow extends as a map $[0, +\infty] \times \mathrm{Im}\mathscr{D} \to \mathrm{Im}\mathscr{D}$ by setting $\varphi_\infty(\zeta_0) = \zeta_\infty$, where ζ_∞ is the limit of the flow.

Since the flow is a continuous map, it is sufficient to prove that the limit ζ_∞ of ζ_t depends continously on ζ_0 to prove that the retraction map is continuous. To do this, first observe that passing to the limit in Inequality (3.5) provides a control between ζ'_t and its limit ζ'_∞. More precisely, for all $\zeta'_0 \in B(0, 2K)$ and $T > 0$ sufficiently large, such that $\phi(\zeta'_t) \leq 1$, we have

$$d(\zeta'_t, \zeta'_\infty) \leq C\phi(\zeta'_t)^{1-\alpha} \tag{3.6}$$

for every $t \geq T$.

In conclusion, starting from $\zeta_0 \in \mathrm{Im}\mathscr{D}$ with $\|\zeta_0\| = K$ as before, we pick $T > 0$ sufficiently large so that $\phi(\zeta_T) \leq \frac{1}{2}$ and $C\phi(\zeta_T)^{1-\alpha} \leq \varepsilon$. Since φ_T is continuous, there exists a neigborhood $U' \subset U \cap B(0, 2K)$ of ζ_0, such that for every $\zeta \in \varphi_T(U')$, we have

$$d(\zeta, \zeta_T) \leq \varepsilon, \quad \phi(\zeta) \leq 1$$

and

$$C\phi(\zeta)^{1-\alpha} \leq 2\varepsilon.$$

We deduce that for every $\zeta'_0 \in U'$, with corresponding flow ζ'_t

$$d(\zeta_\infty, \zeta'_\infty) \leq d(\zeta_\infty, \zeta_T) + d(\zeta_T, \zeta'_T) + d(\zeta'_T, \zeta'_\infty) \leq \varepsilon + \varepsilon + 2\varepsilon = 4\varepsilon.$$

We conclude that the retraction map is continuous. □

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