MR1943884 (2004d:32030) 32Q15 (32L05 53C55 57R57)
Rollin, Yann (4-EDIN-MS)
Surfaces kählériennes de volume fini et équations de Seiberg-Witten. (French. English, French summaries) [Kähler surfaces of finite volume and Seiberg-Witten equations]
Bull. Soc. Math. France **130** (2002), *no. 3,* 409–456.

Let $\overline{M}$ be a complex ruled surface, coming from a rank 2 holomorphic bundle E on a Riemann surface $\overline{\Sigma}$. C. R. LeBrun [Math. Res. Lett. **2** (1995), no. 5, 653–662; MR1359969 (96h:58038)] has proven that the existence of a Kähler metric with constant negative scalar curvature on $\overline{M}$ is equivalent to the stability of E, that is, E comes from a representation of $\pi_1(\Sigma)$ in $\mathrm{PU}(2)$. In this case, the metric must be locally the product of the hyperbolic metric of $\overline{\Sigma}$ and the Fubini-Study metric on the fibres.

In this paper, the author proves a generalisation to the parabolic case: one now looks at $\Sigma = \overline{\Sigma} - \{P_i\}$, where $\{P_i\}$ is a finite set of points on $\overline{\Sigma}$. Let M be the restriction of $\overline{M}$ over Σ. As above, using a finite volume hyperbolic metric on Σ, a representation of $\pi_1(\Sigma)$ into $\mathrm{PU}(2)$ induces a model (finite volume) constant scalar curvature Kähler metric on M. The main theorem (Theorem A) is that if a Kähler metric on M with the same asymptotic behavior as the model has constant nonpositive scalar curvature, then it comes from such a representation.

The existence of the representation is well known to be related to the parabolic stability of the holomorphic bundle E in the sense of V. B. Mehta and C. S. Seshadri [Math. Ann. **248** (1980), no. 3, 205–239; MR0575939 (81i:14010)]. For more links between stability and constant scalar curvature Kähler metrics, see [C. R. LeBrun and M. A. Singer, Invent. Math. **112** (1993), no. 2, 273–313; MR1213104 (94e:53070)] in the case of blowup ruled surfaces, and more generally, [S. K. Donaldson, J. Differential Geom. **59** (2001), no. 3, 479–522; MR1916953 (2003j:32030)].

The proof is based on the idea of LeBrun of using solutions of the Seiberg-Witten equation. The problem here is to produce such a solution in a noncompact, finite volume case. This is achieved (Theorem B) by approximating the finite volume metric by a sequence of compact metrics and studying the convergence of the solutions of the Seiberg-Witten equations (such a method has been used by the reviewer [J. Reine Angew. Math. **490** (1997), 129–154; MR1468928 (98h:53074)]).

Reviewed by *Olivier Biquard*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.