A COMBINED FINITE VOLUME–NONCONFORMING FINITE ELEMENT SCHEME FOR COMPRESSIBLE TWO PHASE FLOW IN POROUS MEDIA

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ABSTRACT. We propose and analyze a combined finite volume–nonconforming finite element scheme on general meshes to simulate the two compressible phase flow in porous media. The diffusion term, which can be anisotropic and heterogeneous, is discretized by piecewise linear nonconforming triangular finite elements. The other terms are discretized by means of a cell-centered finite volume scheme on a dual mesh, where the dual volumes are constructed around the sides of the original mesh. The relative permeability of each phase is decentred according the sign of the velocity at the dual interface. This technique also ensures the validity of the discrete maximum principle for the saturation under a non restrictive shape regularity of the space mesh and the positiveness of all transmissibilities. Next, a priori estimates on the pressures and a function of the saturation that denote capillary terms are established. These stabilities results lead to some compactness arguments based on the use of the Kolmogorov compactness theorem, and allow us to derive the convergence of a subsequence of the sequence of approximate solutions to a weak solution of the continuous equations, provided the mesh size tends to zero. The proof is given for the complete system when the density of the each phase depends on the own pressure.

1. INTRODUCTION

The simultaneous flow of immiscible fluids in porous media occurs in a wide variety of applications. A large variety of methods have been proposed for the discretization of degenerate parabolic systems modeling the displacement of immiscible incompressible two-phase flows in porous media. We refer to [1] and [23] for the finite difference method. The finite volume methods have been proved to be well adapted to discretize conservative equations. The cell-centered finite volume scheme has been studied e.g. by [21], [16] and [4]. Recently, the convergence analysis of a finite volume scheme for a degenerate compressible and immiscible flow in porous media has been studied by Bendahmane et al. [3] when the densities of each phase depend on the global pressure, and by B. Saad and M. Saad [26] for the complete system when the density of the each phase depends on the own pressure. In these works, the medium is considered homogeneous, the permeability tensor is proportional to the matrix identity and the mesh is supposed to be admissible in the sense that satisfying the orthogonal property as in [14]. The cell-centered finite volume method with an upwind discretization of the convection term ensures the stability and is extremely robust and have been used in industry because they are cheap, simple to code and robust. However, standard finite volume schemes do not permit to handle anisotropic diffusion on general meshes see e.g. [14].

Various multi-point schemes where the approximation of the flux through an edge involves several scalar unknowns have been proposed, see e.g. Coudière et al. [10], Eymard et al. [15], or Faille [17]. However, such schemes require using more points than the classical 4 points for triangular meshes and 5 points for quadrangular meshes in space dimension two, making the schemes less robust.

On the other hand finite element method allows a very simple discretization of the diffusion term with a full tensor and does not impose any restrictions on the meshes, they were used a lot for the discretization of a degenerate parabolic problems modeling of contaminant transport in porous media. The mixed finite element method by Dawson [11],

the conforming piecewise linear finite element method has been studied e.g. by Barrett and Knabner [2], Chen and Ewing [8], Nochetto et al. [22], and Rulla et al. [25]. However, it is well-known that numerical instabilities may arise in the convection-dominated case.

To avoid these instabilities, the theoretical analysis of the combined finite volume–finite element method has been carried out for the case of a degenerate parabolic problems with a full diffusion tensors. The combined finite volume–conforming finite element method proposed and studied by Debiez et al. [12] or Feistauer et al. [18] for fluid mechanics equations, are indeed quite efficient.

This ideas is extended by [24] for the degenerate parabolic problems, to the combination of the mixed-hybrid finite element and finite volume methods, to inhomogeneous and anisotropic diffusion–dispersion tensors, to space dimension three, and finally to meshes only satisfying the shape regularity condition. In order to solve this class of equations, Eymard et al. [24] discretize the diffusion term by means of piecewise linear nonconforming (Crouzeix–Raviart) finite elements over a triangularization of the space domain, or using the stiffness matrix of the hybridization of the lowest order Raviart–Thomas mixed finite element method. The other terms are discretized by means of a finite volume scheme on a dual mesh, with an upwind discretization of the convection term to ensures the stability, where the dual volumes are constructed around the sides of the original triangularization. The intention of this paper is to extend these ideas to a fully nonlinear degenerate parabolic system modeling immiscible gas-water displacement in porous media without simplified assumptions on the state law of the density of each phase, to the combination of the nonconforming finite element and finite volume methods, to inhomogeneous and anisotropic permeability tensors, to space dimension three, and finally to meshes only satisfying the shape regularity condition.

Following [24], let us now introduce the combined scheme that we analyze in this paper. We consider a triangulation of the space domain consisting of simplices (triangles in space dimension two and tetrahedra in space dimension three). We next construct a dual mesh where the dual volumes are associated with the sides (edges or faces). To construct a dual volume, one connects the barycentres of two neighboring simplices through the vertices of their common side.We finally place the unknowns in the barycentres of the sides. The diffusion term, which can be anisotropic and heterogeneous, is discretized by piecewise linear nonconforming triangular finite elements. The other terms are discretized by means of a cell-centered finite volume scheme on a dual mesh, where the dual volumes are constructed around the sides of the original mesh, hence we obtain the combined scheme. To ensures the stability, the relative permeability of each phase is decentred according the sign of the velocity at the dual interface. This technique ensures the validity of the discrete maximum principle for the saturation in the case where all transmissibilities are non-negative.

This paper deals with construction and convergence analysis of a combined finite volume–nonconforming finite element for two compressible and immiscible flow in porous media without simplified assumptions on the state law of the density of each phase. The analysis of this model is based on new energy estimates on the velocities of each phase. Nevertheless, these estimates are degenerate in the sense that they do not permit the control of gradients of pressure of each phase, especially when a phase is not locally present in the domain. The main idea consists to derive from degenerate estimates on pressure of each phase, which not allowed straight bound on pressures, an estimate on global pressure and degenerate capillary term in the whole domain regardless of the presence or the disappearance of the phases. These stabilities results with a priori estimates on the pressures and a function of the saturation that denote capillary terms and some compactness arguments based on the use of the Kolmogorov relative compactness theorem, allow us to derive the convergence of both these approximations to a weak solution of the continuous problem in this paper provided the mesh size tends to zero.

The organization of this paper is as follows. In section 2, we introduce the nonlinear parabolic system modeling the two–compressible and immiscible fluids in a porous media and we state the assumptions on the data and present a weak formulation of the continuous problem. In section 3, we describe the combined finite volume–nonconforming finite element scheme and we present the main theorem of convergence. In section 4, we derive

three preliminary fundamental lemmas. In fact, we present some properties of this scheme and we will see that we can't control the discrete gradient of pressure since the mobility of each phase vanishes in the region where the phase is missing. So we are going to use the feature of global pressure. We show that the control of velocities ensures the control of the global pressure and a dissipative term on saturation in the whole domain regardless of the presence or the disappearance of the phases. Section 5 is devoted to a maximum principle on saturation, a priori estimates on the discrete velocities and existence of discrete solutions. In section 6, we derive estimates on difference of time and space translates for the approximate solutions. In section 7 using the Kolmogorov relative compactness theorem, we prove the convergence of a subsequence of the sequence of approximate solutions to a weak solution of the continuous problem.

2. MATHEMATICAL FORMULATION OF THE CONTINUOUS PROBLEM

Let us state the physical model describing the immiscible displacement of two compressible fluids in porous media. We consider the flow of two immiscible fluids in a porous medium. We focus on the water and gas phase, but the considerations below are also valid for a general wetting phase and a non-wetting phase.

The mathematical model is given by the mass balance equation and Darcy's law for both phases $\alpha = l, g$. Let $T > 0$ be the final time fixed, and let be Ω a bounded open subset of $\mathbb{R}^d$ ($d \geq 1$). We set $Q_T = (0, T) \times \Omega$, $\Sigma_T = (0, T) \times \partial\Omega$. The mass conservation of each phase is given in Q_T

(2.1)
$$\phi(x)\partial_t(\rho_\alpha(p_\alpha)s_\alpha)(t, x) + \operatorname{div}(\rho_\alpha(p_\alpha)\mathbf{V}_\alpha)(t, x) + \rho_\alpha(p_\alpha)s_\alpha f_P(t, x) = \rho_\alpha(p_\alpha)s_\alpha^I f_I(t, x),$$

where ϕ, ρ_α and s_α are respectively the porosity of the medium, the density of the α phase and the saturation of the α phase. Here the functions f_I and f_P are respectively the injection and production terms. Note that in equation (2.1) the injection term is multiplied by a known saturation s_α^I corresponding to the known injected fluid, whereas the production term is multiplied by the unknown saturation s_α corresponding to the produced fluid.

The velocity of each fluid $\mathbf{V}_\alpha$ is given by the Darcy law:

(2.2)
$$\mathbf{V}_\alpha = -\mathbf{\Lambda}\frac{k_{r_\alpha}(s_\alpha)}{\mu_\alpha}\big(\nabla p_\alpha - \rho_\alpha(p_\alpha)\mathbf{g}\big), \qquad \alpha = l, g.$$

where $\mathbf{\Lambda}$ is the permeability tensor of the porous medium, k_{r_α} the relative permeability of the α phase, μ_α the constant α-phase's viscosity, p_α the α-phase's pressure and $\mathbf{g}$ is the gravity term. Assuming that the phases occupy the whole pore space, the phase saturations satisfy

(2.3)
$$s_l + s_g = 1.$$

The curvature of the contact surface between the two fluids links the jump of pressure of the two phases to the saturation by the capillary pressure law in order to close the system (2.1)-(2.3)

(2.4)
$$p_c(s_l(t, x)) = p_g(t, x) - p_l(t, x).$$

With the arbitrary choice of (2.4) (the jump of pressure is a function of s_l), the application $s_l \mapsto p_c(s_l)$ is non-increasing, ($\frac{\mathrm{d}p_c}{\mathrm{d}s_l}(s_l) < 0$, for all $s_l \in [0, 1]$), and usually $p_c(s_l = 1) = 0$ when the wetting fluid is at its maximum saturation.

2.1. **Assumptions and main result.** The model is treated without simplified assumptions on the density of each phase, we consider that the density of each phase depends on its corresponding pressure. The main point is to handle a priori estimates on the approximate solution. The studied system represents two kinds of degeneracy: the degeneracy for evolution terms $\partial_t(\rho_\alpha s_\alpha)$ and the degeneracy for dissipative terms $\operatorname{div}(\rho_\alpha M_\alpha \nabla p_\alpha)$ when the saturation vanishes. We will see in the section 5 that we can't control the discrete gradient of pressure since the mobility of each phase vanishes in the region where the phase is missing. So, we are going to use the feature of global pressure to obtain uniform estimates on the discrete gradient of the global pressure and the discrete gradient of the capillary term $\mathcal{B}$ (defined on (2.7)) to treat the degeneracy of this system.

Let us summarize some useful notations in the sequel. We recall the conception of the global pressure as describe in [7]

$$M(s_l)\nabla p = M_l(s_l)\nabla p_l + M_g(s_g)\nabla p_g,$$

with the α-phase's mobility M_α and the total mobility are defined by

$$M_\alpha(s_\alpha) = k_{r_\alpha}(s_\alpha)/\mu_\alpha, \quad M(s_l) = M_l(s_l) + M_g(s_g).$$

This global pressure p can be written as

(2.5) $$p = p_g + \tilde{p}(s_l) = p_l + \bar{p}(s_l),$$

or the artificial pressures are denoted by $\bar{p}$ and $\tilde{p}$ defined by:

(2.6) $$\tilde{p}(s_l) = -\int_0^{s_l} \frac{M_l(z)}{M(z)} p_c'(z)\mathrm{d}z \text{ and } \overline{p}(s_l) = \int_0^{s_l} \frac{M_g(z)}{M(z)} p_c'(z)\mathrm{d}z.$$

We also define the capillary terms by

$$\gamma(s_l) = -\frac{M_l(s_l)M_g(s_g)}{M(s_l)}\frac{\mathrm{d}p_c}{\mathrm{d}s_l}(s_l) \geq 0,$$

and let us finally define the function $\mathcal{B}$ from $[0,1]$ to $\mathbb{R}$ by:

$$\mathcal{B}(s_l) = \int_0^{s_l} \gamma(z)\mathrm{d}z = -\int_0^{s_l} \frac{M_l(z)M_g(z)}{M(z)}\frac{\mathrm{d}p_c}{\mathrm{d}s_l}(z)\mathrm{d}z$$

(2.7) $$= -\int_0^{s_g} M_l(z)\frac{\mathrm{d}\bar{p}}{\mathrm{d}s_l}(z)\mathrm{d}z = \int_0^{s_l} M_g(z)\frac{\mathrm{d}\tilde{p}}{\mathrm{d}s_l}(z)\mathrm{d}z.$$

Using these notations, we derive the fundamental relationship between the velocities and the global pressure and the capillary term:

(2.8) $$M_l(s_l)\nabla p_l = M_l(s_l)\nabla p + \nabla\mathcal{B}(s_l), \quad M_g(s_g)\nabla p_g = M_g(s_l)\nabla p - \nabla\mathcal{B}(s_l).$$

As mentioned above that the mobilities vanish and consequently the control of the gradient of the pressure of each phase is not possible. A main point of the paper is to give sense for the term ∇p_α, $\alpha = l, g$. This term is a distribution and it is not enough to give a sense of the velocity. Our approach is based on the control of the velocity of each phase. Thus the gradient of the global pressure and the function $\mathcal{B}$ are bounded which give a rigorous justification for the degenerate problem.

We complete the description of the model (2.1) by introducing boundary conditions and initial conditions. To the system (2.1)–(2.4) we add the following mixed boundary conditions. We consider the boundary $\partial\Omega = \Gamma_l\cup\Gamma_{imp}$, where Γ_l denotes the water injection boundary and Γ_{imp} the impervious one.

(2.9) $$\begin{cases} p_l(t,x) = p_g(t,x) = 0 \text{ on } (0,T)\times\Gamma_l, \\ \rho_l\mathbf{V}_l\cdot\mathbf{n} = \rho_g\mathbf{V}_g\cdot\mathbf{n} = 0 \text{ on } (0,T)\times\Gamma_{imp}, \end{cases}$$

where $\mathbf{n}$ is the outward normal to Γ_{imp}.

The initial conditions are defined on pressures

(2.10) $$p_\alpha(t = 0) = p_\alpha^0 \text{ for } \alpha = l, g \text{ in } \Omega.$$

Next we introduce some physically relevant assumptions on the coefficients of the system.

(A1) There is two positive constants ϕ_0 and ϕ_1 such that $\phi_0 \leq \phi(x) \leq \phi_1$ almost everywhere $x \in \Omega$.

(A2) $\mathbf{\Lambda}_{ij} \in L^\infty(\Omega)$, $|\mathbf{\Lambda}_{ij}| \leq \frac{C_\mathbf{\Lambda}}{d}$ a.e. in Ω, $1 \leq i,j \leq d$, $C_\mathbf{\Lambda} > 0$, $\mathbf{\Lambda}$ is a symmetric and there exist a constant $c_\mathbf{\Lambda} > 0$ such that

$$\langle\mathbf{\Lambda}(x)\xi,\xi\rangle \geq c_\mathbf{\Lambda}|\xi|^2, \forall\xi \in \mathbb{R}^d.$$

(A3) The functions M_l and M_g belongs to $\mathcal{C}^0([0,1],\mathbb{R}^+)$, $M_\alpha(s_\alpha = 0) = 0$. In addition, there is a positive constant $m_0 > 0$ such that for all $s_l \in [0,1]$,

$$M_l(s_l) + M_g(s_g) \geq m_0.$$

(A4) $(f_P, f_I) \in (L^2(Q_T))^2$, $f_P(t,x)$, $f_I(t,x) \geq 0$ almost everywhere $(t,x) \in Q_T$.

(A5) The density ρ_α is $\mathcal{C}^1(\mathbb{R})$, increasing and there exist two positive constants $\rho_m > 0$ and $\rho_M > 0$ such that $0 < \rho_m \leq \rho_\alpha(p_g) \leq \rho_M$.

$(A6)$ The capillary pressure fonction $p_c \in \mathcal{C}^1([0,1];\mathbb{R}^+)$, decreasing and there exists $\underline{p_c} > 0$ such that $0 < \underline{p_c} \leq |\frac{\mathrm{d}p_c}{\mathrm{d}s_l}|$.

$(A7)$ The function $\gamma \in C^1([0,1];\mathbb{R}^+)$ satisfies $\gamma(s_l) > 0$ for $0 < s_l < 1$ and $\gamma(s_l = 1) = \gamma(s_l = 0) = 0$. We assume that $\mathcal{B}^{-1}$ (the inverse of $\mathcal{B}(s_l) = \int_0^{s_l} \gamma(z)\mathrm{d}z$) is a Hölder[1] function of order θ, with $0 < \theta \leq 1$, on $[0, \mathcal{B}(1)]$.

The assumptions (A1)–(A7) are classical for porous media. Note that, due to the boundedness of the capillary pressure function, the functions $\tilde{p}$ and $\bar{p}$ defined in (2.6) are bounded on $[0, 1]$.

We now give the definition of a weak solution of the problem (2.1)–(2.4).

Definition 1. (*Weak solutions*). *Under assumptions (A1)-(A7) and suppose $(p_l^0,\ p_g^0)$ belong to $(L^2(\Omega))^2$ and $0 \leq s_\alpha^0(x) \leq 1$ almost everywhere in Ω, then the pair (p_l, p_g) is a weak solution of problem (2.1) satisfying :*

$$(2.11) \qquad p_\alpha \in L^2(Q_T),\ 0 \leq s_\alpha(t,x) \leq 1\ a.e\ in\ Q_T,\ (\alpha = l, g),$$

$$(2.12) \qquad p \in L^2(0,T;H^1(\Omega)),\ \mathcal{B}(s_l) \in L^2(0,T;H^1_{\Gamma_l},(\Omega)),$$

$$(2.13) \qquad M_\alpha(s_\alpha)\nabla p_\alpha \in (L^2(Q_T))^d,$$

such that for all $\varphi,\ \psi \in \mathbb{C}^1([0,T];H^1_{\Gamma_l}(\Omega))$ with $\varphi(T,\cdot) = \psi(T,\cdot) = 0$,

$$
\begin{aligned}
&- \int_{Q_T} \phi\rho_l(p_l)s_l\partial_t\varphi\mathrm{d}x\mathrm{d}t - \int_\Omega \phi(x)\rho_l(p_l^0(x))s_l^0(x)\varphi(0,x)\mathrm{d}x \\
(2.14) \qquad &+ \int_{Q_T} \rho_l(p_l)M_l(s_l)\mathbf{\Lambda}\nabla p_l \cdot \nabla\varphi\mathrm{d}x\mathrm{d}t - \int_{Q_T} \mathbf{\Lambda}M_l(s_l)\rho_l^2(p_l)\mathbf{g} \cdot \nabla\varphi\mathrm{d}x\mathrm{d}t \\
&+ \int_{Q_T} \rho_l(p_l)s_l f_P\varphi\mathrm{d}x\mathrm{d}t = \int_{Q_T} \rho_l(p_l)s_l^I f_I\varphi\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{Q_T} \phi\rho_g(p_g)s_g\partial_t\psi\mathrm{d}x\mathrm{d}t - \int_\Omega \phi(x)\rho_g(p_g^0(x))s_g^0(x)\psi(0,x)\,dx \\
(2.15) \qquad &+ \int_{Q_T} M_g(s_g)\rho_g(p_g)\mathbf{\Lambda}\nabla p_g \cdot \nabla\psi\mathrm{d}x\mathrm{d}t - \int_{Q_T} \mathbf{\Lambda}M_g(s_g)\rho_g^2(p_g)\mathbf{g} \cdot \nabla\psi\mathrm{d}x\mathrm{d}t \\
&+ \int_{Q_T} \rho_g(p_g)s_g f_P\psi\mathrm{d}x\mathrm{d}t = \int_{Q_T} \rho_g(p_g)s_g^I f_I\psi\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

3. COMBINED FINITE VOLUME–NONCONFORMING FINITE ELEMENT SCHEME

We will describe the space and time discretizations, define the approximation spaces, and introduce the combined finite volume–nonconforming finite element scheme in this section.

3.1. Space and time discretizations.

In order to discretize the problem (2.1), we perform a triangulation $\mathcal{T}_h$ of the domain Ω, consisting of closed simplices such that $\overline{\Omega} = \cup_{K \in \mathcal{T}_h} K$ and such that if $K, \mathrm{L} \in \mathcal{T}_h$, $K \neq \mathrm{L}$, then $K \cap \mathrm{L}$ is either an empty set or a common face, edge, or vertex of K and L. We denote by $\mathcal{E}_h$ the set of all sides, by $\mathcal{E}_h^{\mathrm{int}}$ the set of all interior sides, by $\mathcal{E}_h^{\mathrm{ext}}$ the set of all exterior sides, and by $\mathcal{E}_K$ the set of all the sides of an element $K \in \mathcal{T}_h$. We define $h := max\{\mathrm{diam}(K), K \in \mathcal{T}_h\}$ and make the following shape regularity assumption on the family of triangulations $\{\mathcal{T}_h\}_h$:
There exists a positive constant $\kappa_\mathcal{T}$ such that

$$(3.1) \qquad \min_{K \in \mathcal{T}_h} \frac{|K|}{\mathrm{diam}(K)^d} \geq \kappa_T, \quad \forall h > 0.$$

Assumption (3.1) is equivalent to the more common requirement of the existence of a constant $\theta_\mathcal{T} > 0$ such that

$$(3.2) \qquad \max_{K \in \mathcal{T}_h} \frac{\mathrm{diam}(K)}{\mathfrak{D}_K} \geq \kappa_T, \quad \forall h > 0,$$

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[1]This means that there exists a positive constant c such that for all $a, b \in [0, \mathcal{B}(1)]$, one has $|\mathcal{B}^{-1}(a) - \mathcal{B}^{-1}(b)| \leq c|a - b|^\theta$.

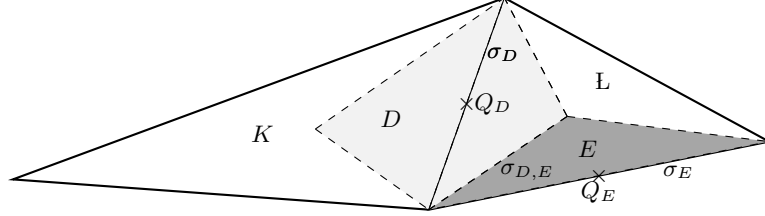where $\mathfrak{D}_K$ is the diameter of the largest ball inscribed in the simplex K.

FIGURE 1. Triangles $K, \mathrm{Ł} \in \mathcal{T}_h$ and dual volumes $D, E \in \mathcal{D}_h$ associated with edges $\sigma_D, \sigma_E \in \mathcal{E}_h$

We also use a dual partition $\mathcal{D}_h$ of Ω such that $\overline{\Omega} = \cup_{D \in \mathcal{D}_h} D$. There is one dual element D associated with each side $\sigma_D \in \mathcal{E}_h$. We construct it by connecting the barycentres of every $K \in \mathcal{T}_h$ that contains σ_D through the vertices of σ_D. For $\sigma_D \in \mathcal{E}_h^{\mathrm{ext}}$, the contour of D is completed by the side σ_D itself. We refer to Fig. 1 for the two-dimensional case. We denote by Q_D the barycentre of the side σ_D. As for the primal mesh, we set $\mathcal{F}_h$, $\mathcal{F}_h^{\mathrm{int}}$, $\mathcal{F}_h^{\mathrm{ext}}$ and $\mathcal{F}_D$ for the dual mesh sides. We denote by $\mathcal{D}_h^{\mathrm{int}}$ the set of all interior and by $\mathcal{D}_h^{\mathrm{ext}}$ the set of all boundary dual volumes. We finally denote by $\mathcal{N}(D)$ the set of all adjacent volumes to the volume D,

$$\mathcal{N}(D) := \{E \in \mathcal{D}_h; \exists \sigma \in \mathcal{F}_h^{\mathrm{int}} \text{ such that } \sigma = \partial D \cap \partial E\}$$

and remark that

$$(3.3) \qquad\qquad |K \cap D| = \frac{|K|}{d+1},$$

for each $K \in \mathcal{T}_h$ and $D \in \mathcal{D}_h$ such that $\sigma_D \in \mathcal{E}_K$. For $E \in \mathcal{N}(D)$, we also set $d_{D,E} := |Q_E - Q_D|$, $\sigma_{D,E} := \partial D \cap \partial E$ and $K_{D,E}$ the element of $\mathcal{T}_h$ such that $\sigma_{D,E} \subset K_{D,E}$.

The problem under consideration is time-dependent, hence we also need to discretize the time interval $(0, T)$. The time discretization of $(0, T)$ is given by an integer value N and by a strictly increasing sequence of real values $(t^n)_{n \in [0,N]}$ with $t^0 = 0$ and $t^N = T$. Without restriction, we consider a uniform step time $\delta t = t^n - t^{n-1}$, for $n \in [1, N]$.

We define the following finite-dimensional spaces:

$$X_h := \{\varphi_h \in L^2(\Omega); \varphi_h|_K \text{ is linear } \forall K \in \mathcal{T}_h,$$
$$\varphi_h \text{ is continuous at the points } Q_D, D \in \mathcal{D}_h^{\mathrm{int}}\},$$
$$X_h^0 := \{\varphi_h \in X_h; \varphi_h(Q_D) = 0 \quad \forall D \in \mathcal{D}_h^{\mathrm{ext}}\}.$$

The basis of X_h is spanned by the shape functions φ_D, $D \in \mathcal{D}_h$, such that $\varphi_{\mathcal{D}}(Q_E) = \delta_{DE}$, $E \in \mathcal{D}_h$, δ being the Kronecker delta. We recall that the approximations in these spaces are nonconforming since $X_h \not\subset H^1(\Omega)$. We equip X_h with the seminorm

$$\|u_h\|_{X_h}^2 := \sum_{K \in \mathcal{T}_h} \int_K |\nabla u_h|^2 \, \mathrm{d}x,$$

which becomes a norm on X_h^0.

For a given value u_D, $D \in \mathcal{D}_h$ (resp. $u_D^n, D \in \mathcal{D}_h, n \in [0, N]$), we define a constant piecewise function as : $u(x) = u_D$ for $x \in D$ (resp. $u(t, x) = u_D^n$ for $x \in D$, $t \in]t^{n-1}, t^n]$). Next, we define the discret differential operateur :

$$\delta_{D|E}(u) = u_E - u_D, \text{ and } \delta_{D|E}^n(u) = u_E^n - u_D^n.$$

Definition 2. *Let the values u_D^n, $D \in \mathcal{D}_h$, $n \in \{0, 1, \cdots, N\}$. As the approximate solutions of the problem by means of the combined finite volume–nonconforming finite element scheme, we understand:*

(1) A function $u_{\delta t,h}$ such that

$$u_{\delta t,h}(x,0) = u_h^0(x) \ for \ x \in \Omega,$$

(3.4) $\qquad u_{\delta t,h}(x,t) = u_h^n(x) \ for \ x \in \Omega, t \in (t_{n-1}, t_n] \qquad n \in \{1, \cdots, N\},$

where $u_h^n = \sum_{D \in \mathcal{D}_h} u_D^n \varphi_D$;

(2) A function $\tilde{u}_{\delta t,h}$ such that

$$\tilde{u}_{\delta t,h}(x,0) = u_D^0 \ for \ x \in \mathring{D}, D \in \mathcal{D}_h,$$

(3.5) $\qquad \tilde{u}_{\delta t,h}(x,t) = u_D^n \ for \ x \in \mathring{D}, D \in \mathcal{D}_h, t \in (t_{n-1}, t_n] \qquad n \in \{1, \cdots, N\}.$

The function $u_{\delta t,h}$ is piecewise linear and continuous in the barycentres of the interior sides in space and piecewise constant in time; we will call it a *nonconforming finite element solution*. The function $\tilde{u}_{\delta t,h}$ is given by the values of u_D^n in side barycentres and is piecewise constant on the dual volumes in space and piecewise constant in time; we will call it a *finite volume solution*.

3.2. **The combined scheme.** For more clarity and for presentation simplicity, we present the combined scheme for a horizontal field and then we neglect the gravity effect. In remark 1, we indicate how to modify the scheme to include the gravity terms.

Definition 3. (Combined scheme) *The fully implicit combined finite volume-nonconforming finite element scheme for the problem* (2.1) *reads: find the values* $p_{\alpha,D}^n$, $D \in \mathcal{D}_h$, $n \in \{1, \cdots, N\}$, *such that*

(3.6) $\qquad p_{\alpha,D}^0 = \dfrac{1}{|D|} \displaystyle\int_D p_\alpha^0(x)\mathrm{d}x, \ s_{\alpha,D}^0 = \dfrac{1}{|D|} \int_D s_\alpha^0(x)\mathrm{d}x, \ for \ all \ D \in \mathcal{D}_h^{int},$

(3.7) $\quad |D|\phi_D \dfrac{\rho_l(p_{l,D}^n)s_{l,D}^n - \rho_l(p_{l,D}^{n-1})s_{l,D}^{n-1}}{\delta t} - \displaystyle\sum_{E \in \mathcal{N}(D)} \rho_{l,D,E}^n \, M_l(s_{l,D,E}^n) \, \Lambda_{D,E} \, \delta_{D|E}^n(p_l)$

$$+ |D| \, \rho_l(p_{l,D}^n)s_{l,D}^n f_{P,D}^n = |D| \, \rho_l(p_{l,D}^n)(s_{l,D}^I)^n f_{I,D}^n,$$

(3.8) $\quad |D|\phi_D \dfrac{\rho_g(p_{g,D}^n)s_{g,D}^n - \rho_g(p_{g,D}^{n-1})s_{g,D}^{n-1}}{\delta t} - \displaystyle\sum_{E \in \mathcal{N}(D)} \rho_{g,D,E}^n \, M_g(s_{g,D,E}^n) \, \Lambda_{D,E} \, \delta_{D|E}^n(p_g)$

$$+ |D| \, \rho_g(p_{g,D}^n)s_{g,D}^n f_{P,D}^n = |D| \, \rho_g(p_{g,D}^n)(s_{g,D}^I)^n f_{I,D}^n,$$

(3.9) $\qquad\qquad\qquad\qquad p_c(s_{l,D}^n) = p_{g,D}^n - p_{l,D}^n.$

We refer to the matrix $\mathbf{\Lambda}$ of the elements $\Lambda_{D,E}$, $D, E \in \mathcal{D}_h^{\text{int}}$, as to the diffusion matrix. This matrix, the stiffness matrix of the nonconforming finite element method writes in the form

(3.10) $\qquad\qquad \Lambda_{D,E} := - \displaystyle\sum_{K \in \mathcal{T}_h} (\mathbf{\Lambda}(x)\nabla\varphi_E, \nabla\varphi_D)_{0,K} \qquad D, E \in \mathcal{D}_h.$

Notice that the source terms are, for $n \in \{1, \ldots, N\}$

$$f_{P,D}^n := \frac{1}{\delta t \, |D|} \int_{t^{n-1}}^{t^n} \int_D f_P(t,x) \, dxdt, \quad f_{I,D}^n := \frac{1}{\delta t \, |D|} \int_{t^{n-1}}^{t^n} \int_D f_I(t,x) \, dxdt.$$

The mean value of the density of each phase on interfaces is not classical since it is given as

(3.11) $\qquad\qquad \dfrac{1}{\rho_{\alpha,D,E}^n} = \begin{cases} \frac{1}{p_{\alpha,E}^n - p_{\alpha,D}^n} \int_{p_{\alpha,D}^n}^{p_{\alpha,E}^n} \frac{1}{\rho_\alpha(\zeta)} \, d\zeta & \text{if } p_{\alpha,D}^n \neq p_{\alpha,E}^n, \\ \frac{1}{\rho_{\alpha,D}^n} & \text{otherwise.} \end{cases}$

This choice is crucial to obtain estimates on discrete pressures.

We denoted

(3.12) $\qquad\qquad G_\alpha(s_{\alpha,D}^n, s_{\alpha,E}^n; \delta_{D|E}^n(p_\alpha)) = -M_\alpha(s_{\alpha,D,E}^n)\delta_{D|E}^n(p_\alpha),$

the numerical fluxes, where $M_\alpha(s^n_{\alpha,D,E})$ denote the upwind discretization of $M_\alpha(s_\alpha)$ on the interface $\sigma_{D,E}$ and

$$(3.13) \qquad s^n_{\alpha,D,E} = \begin{cases} s^n_{\alpha,D} & \text{if } (D,E) \in \mathcal{E}^n_\alpha, \\ s^n_{\alpha,E} & \text{otherwise,} \end{cases}$$

with the set $\mathcal{E}^n_\alpha$ is subset of $\mathcal{E}_h$ such that

$$(3.14) \qquad \mathcal{E}^n_\alpha = \{(D,E) \in \mathcal{E}_h, \Lambda_{D,E}\,\delta^n_{D|E}(p_\alpha) = \Lambda_{D,E}\left(p^n_{\alpha,E} - p^n_{\alpha,D}\right) \leq 0\}.$$

Remark 1. *To take into account the gravity term, it is enough to modify, for example, the numerical fluxes to be*

$$G_\alpha(s^n_{\alpha,D}, s^n_{\alpha,E}; \delta^n_{D|E}(p_\alpha)) = -M_\alpha(s^n_{\alpha,D,E})\delta^n_{D|E}(p_\alpha) + \rho_\alpha(p^n_{\alpha,D,E})\Big(M_\alpha(s^n_{\alpha,D})g^+_{D|E} - M_\alpha(s^n_{\alpha,E})g^-_{D|E}\Big),$$

with $g^\pm_{D|E} = (g \cdot \eta_{D|E})^\pm$. This numerical flux satisfy is consistent, conservative and monotone, thus the convergence result remains valid.

In the sequel we shell consider apart the following special case:
All transmissibility's are non-negative, i.e.

$$(3.15) \qquad \Lambda_{D,E} \geq 0 \quad \forall D \in \mathcal{D}^{\text{int}}_h, E \in \mathcal{N}(D).$$

Since

$$\nabla\varphi_{D|K} = \frac{|\sigma_\mathcal{D}|}{|K|}\mathbf{n}_{\sigma_\mathcal{D}}, \quad K \in \mathcal{T}_h, \sigma_\mathcal{D} \in \mathcal{E}_K$$

with $\mathbf{n}_{\sigma_\mathcal{D}}$ the unit normal vector of the side $\sigma_\mathcal{D}$ outward to K, one can immediately see that Assumption (3.15) is satisfied when the diffusion tensor reduces to a scalar function and when the magnitude of the angles between $\mathbf{n}_{\sigma_\mathcal{D}}$, $\sigma_\mathcal{D} \in \mathcal{E}_K$, for all $K \in \mathcal{T}_h$ is greater or equal to $\pi/2$.

The main result of this paper is the following theorem.

Theorem 1. *There exists an approximate solutions $(p^n_{\alpha,D})_{n,D}$ corresponding to the system (3.7)-(3.8), which converges (up to a subsequence) to a weak solution p_α of (2.1) in the sense of the Definition 1.*

4. Preliminary fundamental lemmas

In this section, we will first present several technical lemmas that will be used in our latter analysis to obtain a priori estimate of the solution of the discrete problem.

Lemma 1. *For all $u_h = \sum_{D \in \mathcal{D}_h} u_D \varphi_D \in X_h$,*

$$\sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \Lambda_{D,E}(\delta_{D|E}u)^2 \geq c_\mathbf{\Lambda} \|u_h\|^2_{X_h}.$$

Proof. We have

$$\begin{aligned}
\sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \Lambda_{D,E}(\delta_{D|E}u)^2 &= \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \Lambda_{D,E}(u_E - u_D)^2 \\
&= -2 \sum_{D \in \mathcal{D}_h} u_D \sum_{E \in \mathcal{N}(D)} \Lambda_{D,E}(u_E - u_D) \\
&= -2 \sum_{D \in \mathcal{D}_h} u_D \sum_{E \in \mathcal{D}_h} \Lambda_{D,E} u_E \\
&= 2 \sum_{K \in \mathcal{T}_h} (\mathbf{\Lambda}\nabla u_h, \nabla u_h)_{0,K} \geq c_\mathbf{\Lambda} \|u_h\|^2_{X_h},
\end{aligned}$$

using (3.10) and assumption (A2). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

We state this lemma without a proof as well (cf. [24, Lemma 3.1]):

Lemma 2. *For all $u_h = \sum_{D \in \mathcal{D}_h} u_D \varphi_D \in X_h$, one has*

$$(4.1) \qquad \sum_{\sigma_{D,E} \in \mathcal{F}_h^{int}} \frac{|\sigma_{D,E}|}{d_{D,E}} (u_E - u_D)^2 \leq \frac{d+1}{2(d-1)\kappa_{\mathcal{T}}} \|u_h\|_{X_h}^2 .$$

In the continuous case, we have the following relationship between the global pressure, capillary pressure and the pressure of each phase

$$(4.2) \qquad M_l |\nabla p_l|^2 + M_g |\nabla p_g|^2 = M |\nabla p|^2 + \frac{M_l M_g}{M} |\nabla p_c|^2.$$

This relationship, means that, the control of the velocities ensures the control of the global pressure and the capillary terms $\mathcal{B}$ in the whole domain regardless of the presence or the disappearance of the phases.

In the discrete case, these relationship, are not obtained in a straightforward way. This equality is replaced by three discrete inequalities which we state in the following lemma.

Lemma 3. (*Total mobility, global pressure, Capillary term $\mathcal{B}$ and Dissipative terms*) *Under the assumptions $(A1) - (A7)$ and the notations (2.5). Then for all $(D, E) \in \mathcal{D}$ and for all $n \in [0, N]$ the following inequalities hold:*

$$(4.3) \qquad M_{l,D|E}^n + M_{g,D|E}^n \geq m_0,$$

$$(4.4) \qquad m_0 \left(\delta_{D|E}^n(p) \right)^2 \leq M_{l,D|E}^n \left(\delta_{D|E}^n(p_l) \right)^2 + M_{g,D|E}^n \left(\delta_{D|E}^n(p_g) \right)^2.$$

$$(4.5) \qquad \left(\delta_{D|E}^n(\mathcal{B}(s_l)) \right)^2 \leq M_{l,D|E}^n \left(\delta_{D|E}^n(p_l) \right)^2 + M_{g,D|E}^n \left(\delta_{D|E}^n(p_g) \right)^2,$$

$$(4.6) \qquad M_{l,D|E}^n (\delta_{D|E}^n(\bar{p}(s_l)))^2 \leq M_{l,D|E}^n \left(\delta_{D|E}^n(p_l) \right)^2 + M_{g,D|E}^n \left(\delta_{D|E}^n(p_g) \right)^2,$$

and

$$(4.7) \qquad M_{g,D|E}^n (\delta_{D|E}^n(\tilde{p}(s_l)))^2 \leq M_{l,D|E}^n \left(\delta_{D|E}^n(p_l) \right)^2 + M_{g,D|E}^n \left(\delta_{D|E}^n(p_g) \right)^2.$$

In [26], the authors prove this lemma on primal mesh satisfying the orthogonal condition. This proof use only two neighbors elements and it is based only on the definition of the global pressure. Thus, we state the above Lemma without proof since the proof made in [26] remains valid on the dual mesh.

5. A priori estimates and existence of the approximate solution

We derive new energy estimates on the discrete velocities $M_\alpha(s_{\alpha,D|E}^n)\delta_{D|E}^n(p_\alpha)$. Nevertheless, these estimates are degenerate in the sense that they do not permit the control of $\delta_{D|E}^n(p_\alpha)$, especially when a phase is missing. So, the global pressure has a major role in the analysis, we will show that the control of the discrete velocities $M_\alpha(s_{\alpha,D|E}^n)\delta_{D|E}^n(p_\alpha)$ ensures the control of the discrete gradient of the global pressure and the discrete gradient of the capillary term $\mathcal{B}$ in the whole domain regardless of the presence or the disappearance of the phases.

The following section gives us some necessary energy estimates to prove the theorem 1.

5.1. **The maximum principle.** Let us show in the following Lemma that the phase by phase upstream choice yields the L^∞ stability of the scheme which is a basis to the analysis that we are going to perform.

Lemma 4. (*Maximum principe*). *Under assumptions (A1)-(A7). Let $(s_{\alpha,D}^0)_{D \in \mathcal{D}_h} \in [0, 1]$ and assume that $(p_{\alpha,D}^n)_{D \in \mathcal{D}_h}$ is a solution of the finite volume (3.6)-(3.9). Then, the saturation $(s_{\alpha,D}^n)_{D \in \mathcal{D}_h}$, remains in $[0, 1]$ for all $D \in \mathcal{D}_h$, $n \in \{1, \dots, N\}$.*

Proof. Let us show by induction in n that for all $D \in \mathcal{D}_h$, $s_{\alpha,D}^n \geq 0$ where $\alpha = l, g$. For $\alpha = l$, the claim is true for $n = 0$ and for all $D \in \mathcal{D}_h$. We argue by induction that for all $D \in \mathcal{D}_h$, the claim is true up to order n. We consider the control volume D such that $s_{l,D}^n = \min\{s_{l,E}^n\}_{E \in \mathcal{D}_h}$ and we seek that $s_{l,D}^n \geq 0$.

For the above mentioned purpose, multiply the equation in (3.7) by $-(s_{l,D}^n)^-$, we obtain

$$
\begin{aligned}
(5.1) \quad & -|D|\,\phi_D \frac{\rho_l(p_{l,D}^n)s_{l,D}^n - \rho_l(p_{l,D}^{n-1})s_{l,D}^{n-1}}{\delta t}(s_{l,D}^n)^- \\
& - \sum_{E \in \mathcal{N}(D)} \tau_{D|E} \rho_{l,D,E}^n \, \Lambda_{D,E} \, G_l(s_{l,D}^n, s_{l,E}^n; \delta_{D|E}^n(p_l))(s_{l,D}^n)^- \\
& - |D|\,\rho_l(p_{l,D}^n)s_{l,D}^n f_{P,D}^n(s_{l,D}^n)^- = -|D|\,\rho_l(p_{l,D}^n)(s_{g,D}^I)^n f_{I,D}^n(s_{l,D}^n)^- \leq 0.
\end{aligned}
$$

The numerical flux G_l is nonincreasing with respect to $s_{l,E}^n$, and consistence, we get

$$
\begin{aligned}
(5.2) \quad G_l(s_{l,D}^n, s_{l,E}^n; \delta_{D|E}^n(p_l)) \, (s_{l,D}^n)^- & \leq G_l(s_{l,D}^n, s_{l,D}^n; \delta_{D|E}^n(p_l)) \, (s_{l,D}^n)^- \\
& = -\delta_{D|E}^n(p_l) \, M_l(s_{l,D}^n) \, (s_{l,D}^n)^- = 0.
\end{aligned}
$$

Using the identity $s_{l,D}^n = (s_{l,D}^n)^+ - (s_{l,D}^n)^-$, and the mobility M_l extended by zero on $]-\infty, 0]$, then $M_l(s_{l,D}^n)(s_{l,D}^n)^- = 0$ and

$$
(5.3) \quad -|D|\,\rho_l(p_{l,D}^n)s_{l,D}^n f_{P,K}^n(s_{l,D}^n)^- = |D|\,\rho_l(p_{l,D}^n)f_{P,K}^n((s_{l,D}^n)^-)^2 \geq 0.
$$

Then, we deduce from (5.1) that

$$
\rho_l(p_{l,D}^n)|(s_{l,D}^n)^-|^2 + \rho_l(p_{l,D}^{n-1})s_{l,D}^{n-1}(s_{l,D}^n)^- \leq 0,
$$

and from the nonnegativity of $s_{l,D}^{n-1}$, we obtain $(s_{l,D}^n)^- = 0$. This implies that $s_{l,D}^n \geq 0$ and

$$
0 \leq s_{l,D}^n \leq s_{l,E}^n \text{ for all } n \in [0, N-1] \text{ and } E \in \mathcal{D}_h.
$$

In the same way, we prove $s_{g,D}^n \geq 0$. $\square$

5.2. Estimations on the pressures.
We now give a priori estimates satisfied by the solution values p_D^n, $D \in \mathcal{D}_h$, $\{1, \ldots, N\}$.

Proposition 1. *Let $(p_{l,D}^n, p_{g,D}^n)$ be a solution of (3.6)-(3.9). Then, there exists a constant $C > 0$, which only depends on M_α, Ω, T, p_α^0, s_α^0, s_α^I, f_P, f_I and not on $\mathcal{D}_h$, such that the solution of the combined scheme satisfies*

$$
(5.4) \quad \sum_{n=1}^N \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \Lambda_{D,E} M_\alpha(s_{\alpha,D|E}^n) \left| p_{\alpha,E}^n - p_{\alpha,D}^n \right|^2 \leq C,
$$

and

$$
(5.5) \quad \sum_{n=1}^N \delta t \, \|p_h\|_{X_h}^2 \leq C.
$$

Proof. We define the function $\mathcal{A}_\alpha(p_\alpha) := \rho_\alpha(p_\alpha)g_\alpha(p_\alpha) - p_\alpha$, $\mathcal{P}_c(s_l) := \int_0^{s_l} p_c(z)\mathrm{d}z$ and $g_\alpha(p_\alpha) = \int_0^{p_\alpha} \frac{1}{\rho_\alpha(z)}\mathrm{d}z$. In the following proof, we denote by C_i various real values which independent on $\mathcal{D}$ and n. To prove the estimate (5.4), we multiply (3.7) and (3.8) respectively by $g_l(p_{l,D}^n)$, $g_g(p_{g,D}^n)$ and adding them, then summing the resulting equation over $D \in \mathcal{D}_h$ and $n \in \{1, \cdots, N\}$. We thus get:

$$
(5.6) \quad E_1 + E_2 + E_3 = 0,
$$

where

$$
\begin{aligned}
E_1 = \sum_{n=1}^N \sum_{D \in \mathcal{D}_h} |D|\,\phi_D \Big(& \left(\rho_l(p_{l,D}^n)s_{l,D}^n - \rho_l(p_{l,D}^{n-1})s_{l,D}^{n-1}\right) g_l(p_{l,D}^n) \\
& + \left(\rho_g(p_{g,D}^n)s_{g,D}^n - \rho_g(p_{g,D}^{n-1})s_{g,D}^{n-1}\right) g_g(p_{g,D}^n) \Big),
\end{aligned}
$$

$$E_2 = \sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(\mathcal{D})} \Lambda_{D,E} \Big(\rho_{l,D,E}^n G_l(s_{l,D}^n, s_{l,E}^n; \delta_{D|E}^n(p_l)) \, g_l(p_{l,D}^n)$$

$$+ G_g(s_{g,D}^n, s_{g,E}^n; \delta_{D|E}^n(p_g)) \, g_g(p_{g,D}^n) \Big),$$

$$E_3 = \sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} |D| \Big(\rho_l(p_{l,D}^n) s_{l,D}^n f_{P,D}^n g_l(p_{l,D}^n) - \rho_l(p_{l,D}^n)(s_{g,D}^I)^n f_{I,D}^n g_l(p_{l,D}^n)$$

$$+ \rho_g(p_{g,D}^n) s_{g,D}^n f_{P,D}^n g_g(p_{g,D}^n) - \rho_g(p_{g,D}^n)(s_{g,D}^I)^n f_{I,D}^n g_g(p_{g,D}^n) \Big).$$

To handle the first term of the equality (5.6), let us recall the following inequality :

$$(5.7) \quad \big(\rho_l(p_l^n)s_l^n - \rho_l(p_l^{n-1})s_l^{n-1}\big) g_l(p_l^n) + \big(\rho_g(p_g^n)s_g^n - \rho_g(p_g^{n-1})s_g^{n-1}\big) g_g(p_g^n)$$

$$\geq \mathcal{A}_l(p_l^n)s_l^n - \mathcal{A}_l(p_l^{n-1})s_l^{n-1} + \mathcal{A}_g(p_g^n)s_g^n - \mathcal{A}_g(p_g^{n-1})s_g^{n-1} - \mathcal{P}_c(s_l^n) + \mathcal{P}_c(s_l^{n-1}),$$

using the concavity property of g_α and $\mathcal{P}_c$, in [20] the authors prove the above inequality. So, this yields to

$$(5.8) \quad E_1 \geq \sum_{D \in \mathcal{D}_h} \phi_D \, |D| \left(s_{l,D}^N \mathcal{A}(p_{l,D}^N) - s_{l,D}^0 \mathcal{A}(p_{l,D}^0) + s_{g,D}^N \mathcal{A}(p_{g,D}^N) - s_{g,D}^0 \mathcal{A}(p_{g,D}^0) \right)$$

$$- \sum_{D \in \mathcal{D}_h} \phi_D \, |D| \, \mathcal{P}_c(s_{l,D}^N) + \sum_{D \in \mathcal{D}_h} \phi_D \, |D| \, \mathcal{P}_c(s_{l,D}^0).$$

Using the fact that the numerical fluxes G_l and G_g are conservative, we obtain by discrete integration by parts

$$E_2 = \frac{1}{2} \sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \Lambda_{D,E} \Big(\rho_{l,D,E}^n G_l(s_{l,D}^n, s_{l,E}^n; \delta_{D|E}^n(p_l))(g_l(p_{l,D}^n) - g_l(p_{l,E}^n))$$

$$+ \rho_{g,D,E}^n G_g(s_{g,D}^n, s_{g,E}^n; \delta_{D|E}^n(p_g))(g_g(p_{g,D}^n) - g_g(p_{g,E}^n)) \Big),$$

and due to the correct choice of the density of the phase α on each interface,

$$(5.9) \qquad \rho_{\alpha,D,E}^n(g_\alpha(p_{\alpha,D}^n) - g_\alpha(p_{\alpha,E}^n)) = p_{\alpha,D}^n - p_{\alpha,E}^n,$$

we obtain

$$E_2 = \frac{1}{2} \sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \Lambda_{D,E} \Big(G_l(s_{l,D}^n, s_{l,E}^n; \delta_{D|E}^n(p_l))(p_{l,D}^n - p_{l,E}^n)$$

$$+ G_g(s_{g,D}^n, s_{g,E}^n; \delta_{D|E}^n(p_g))(p_{g,D}^n - p_{g,E}^n) \Big).$$

The definition of the upwind fluxes in (3.12) implies

$$G_l(s_{l,D}^n, s_{l,E}^n; \delta_{D|E}^n(p_l))(p_{l,D}^n - p_{l,E}^n) + G_g(s_{g,D}^n, s_{g,E}^n; \delta_{D|E}^n(p_g))(p_{g,D}^n - p_{g,E}^n)$$

$$= M_l(s_{l,D,E}^n)(\delta_{D|E}^n(p_l))^2 + M_g(s_{g,D,E}^n)(\delta_{D|E}^n(p_g))^2.$$

Then, we obtain the following equality

(5.10)

$$E_2 = \frac{1}{2} \sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(\mathcal{D})} \Lambda_{D,E} \Big(M_l(s_{l,D,E}^n)(\delta_{D|E}^n(p_l))^2 + M_g(s_{g,D,E}^n)(\delta_{D|E}^n(p_g))^2 \Big).$$

In order to estimate E_3, using the fact that the densities are bounded and the map g_α is sublinear (a.e.$|g(p_\alpha)| \leq C|p_\alpha|$), we have

$$|E_3| \leq C_1 \sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} |D| \, (f_{P,D}^n + f_{I,D}^n)(|p_{l,D}^n| + |p_{g,D}^n|),$$

then

$$|E_3| \leq C_1 \sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} |D| \, (f_{P,D}^n + f_{I,D}^n)(2|p_D^n| + |\bar{p}_D^n| + |\tilde{p}_D^n|).$$

Hence, by the Hölder inequality, we get that

$$|E_4| \leq C_2 \|f_P + f_I\|_{L^2(Q_T)} \Big(\sum_{n=1}^{N} \delta t \|p_h^n\|_{L^2(\Omega)}^2 \Big)^{\frac{1}{2}},$$

and, from the discrete Poincaré–Friedrichs inequality [27], we get

$$(5.11) \qquad |E_4| \leq C_3 \Big(\sum_{n=1}^{N} \delta t \|p_h^n\|_{X_h}^2 \Big)^{\frac{1}{2}} + C_4.$$

The equality (5.6) with the inequalities (5.8), (5.10), (5.11) give (5.4). Then we deduce (5.5) from (4.4) and Lemma 1. □

We now state the following corollary, which is essential for the compactness and limit study.

Corollary 1. *From the previous Proposition, we deduce the following estimations:*

$$(5.12) \qquad \sum_{n=1}^{N} \delta t \left\|\mathcal{B}(s_{l,h}^n)\right\|_{X_h}^2 \leq C,$$

$$(5.13) \qquad \sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \Lambda_{D,E} M_{l,D|E}^n (\delta_{D|E}^n(\bar{p}(s_l)))^2 \leq C,$$

and

$$(5.14) \qquad \sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \Lambda_{D,E} M_{g,D|E}^n (\delta_{D|E}^n(\tilde{p}(s_l)))^2 \leq C.$$

Proof. The prove of the estimates (5.12), (5.13) and (5.14) are a direct consequence of the inequality (4.5), (4.6), (4.7), the Lemma 1 and the Proposition 1. □

5.3. **Existence of the finite volume scheme.**

Proposition 2. *The problem* (3.7)-(3.8) *admits at least one solution* $(p_{l,D}^n, p_{g,D}^n)_{(D,n) \in \mathcal{D}_h \times \{1, \cdots, N\}}$.

The proof is based on a technical assertion to characterize the zeros of a vector field which stated and proved in [13]. This method is used in [3] and [26], so it is easy to adopt their proof in our case, thus we omit it.

6. COMPACTNESS PROPERTIES

In this section we derive estimates on differences of space and time translates of the function $\mathbf{U}_{\alpha,\delta t,h} = \phi \rho_\alpha(p_{\alpha,\delta t,h}) s_{\alpha,\delta t,h}$ which imply that the sequence $\phi \rho_\alpha(p_{\alpha,\delta t,h}) s_{\alpha,\delta t,h}$ is relatively compact in $L^1(Q_T)$.

The following important relation between $\mathbf{U}_{\delta t,h}$ and $\tilde{\mathbf{U}}_{\delta t,h}$ (see definition 2) is valid:

Lemma 5. $\Big($*Relation between* $\boldsymbol{U}_{\alpha,\delta t,h}$ *and* $\tilde{\boldsymbol{U}}_{\alpha,\delta t,h}\Big)$. *There holds*

$$\left\|\boldsymbol{U}_{\alpha,\delta t,h} - \tilde{\boldsymbol{U}}_{\alpha,\delta t,h}\right\|_{L^1(Q_T)} \longrightarrow 0 \ as \ h \to 0.$$

Proof.

$$\begin{aligned}
\left\|\mathbf{U}_{l,\delta t,h} - \tilde{\mathbf{U}}_{l,\delta t,h}\right\|_{L^1(Q_T)} &= \int_{Q_T} \left|U_{l,\delta t,h}(t,x) - \tilde{U}_{l,\delta t,h}(t,x)\right| \mathrm{d}x \, \mathrm{d}t \\
&\leq \int_{Q_T} \left|s_{l,\delta t,h}(t,x)\Big(\rho_l(p_{l,\delta t,h}(t,x)) - \rho_l(\tilde{p}_{l,\delta t,h}(t,x))\Big)\right| \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_{Q_T} \left|\rho_l(\tilde{p}_{l,\delta t,h}(t,x))\Big(s_{l,\delta t,h}(t,x) - \tilde{s}_{l,\delta t,h}(t,x)\Big)\right| \mathrm{d}x \, \mathrm{d}t \\
&\leq \mathrm{T}_1 + \mathrm{T}_2,
\end{aligned}$$

where T_1 and T_2 defined as follows

$$(6.1) \qquad \mathrm{T}_1 = \rho_M \int_{Q_T} \left|s_{l,\delta t,h}(t,x) - \tilde{s}_{l,\delta t,h}(t,x)\right| \mathrm{d}x \, \mathrm{d}t,$$

(6.2) $$\mathsf{T}_2 = \int_{Q_T} |\rho_l(p_{l,\delta t,h}(t,x)) - \rho_l(\tilde{p}_{l,\delta t,h}(t,x))|\, \mathrm{d}x\, \mathrm{d}t.$$

To handle the term on saturation T_1, we use the fact that $\mathcal{B}^{-1}$ is a Hölder function, then

$$\mathsf{T}_1 \le \rho_M\, C \int_{Q_T} |\mathcal{B}(s_{l,\delta t,h}(t,x)) - \mathcal{B}(\tilde{s}_{l,\delta t,h}(t,x))|^{\theta}\, \mathrm{d}x\, \mathrm{d}t,$$

and by application of the Cauchy-Schwarz inequality, we deduce

$$\mathsf{T}_1 \le C \Big(\int_{Q_T} |\mathcal{B}(s_{l,\delta t,h}(t,x)) - \mathcal{B}(\tilde{s}_{l,\delta t,h}(t,x))|\, \mathrm{d}x\, \mathrm{d}t \Big)^{\theta} \le C\, (\mathsf{T}_1')^{\theta},$$

where T_1' defined as follows

$$T_1' = \int_{Q_T} |\mathcal{B}(s_{l,\delta t,h}(t,x)) - \mathcal{B}(\tilde{s}_{l,\delta t,h}(t,x))|\, \mathrm{d}x\, \mathrm{d}t.$$

We have

(6.3)
$$
\begin{aligned}
T_1' &= \sum_{n=1}^{N} \delta t \sum_{K \in \mathcal{T}_h} \sum_{\sigma_D \in \mathcal{E}_K} \int_{K \cap D} |\mathcal{B}(s_{l,\delta t,h}(t,x)) - \mathcal{B}(\tilde{s}_{l,\delta t,h}(t,x))|\, \mathrm{d}x \\
&= \sum_{n=1}^{N} \delta t \sum_{K \in \mathcal{T}_h} \sum_{\sigma_D \in \mathcal{E}_K} \int_{K \cap D} |\mathcal{B}(s_{l,\delta t,h}(t,x)) - \mathcal{B}(s_{l,\delta t,h}(t,Q_D))|\, \mathrm{d}x \\
&= \sum_{n=1}^{N} \delta t \sum_{K \in \mathcal{T}_h} \sum_{\sigma_D \in \mathcal{E}_K} \int_{K \cap D} |\nabla \mathcal{B}(s_{l,\delta t,h}(t,x)) \cdot (x - Q_D)|\, \mathrm{d}x \\
&\le \sum_{n=1}^{N} \delta t \sum_{K \in \mathcal{T}_h} \sum_{\sigma_D \in \mathcal{E}_K} \left| \nabla \mathcal{B}(s_{l,\delta t,h})|_K \right| \mathrm{diam}(D)\, |K \cap D| \\
&\le h \sum_{n=1}^{N} \delta t \sum_{K \in \mathcal{T}_h} \left| \nabla \mathcal{B}(s_{l,\delta t,h})|_K \right| |K| \\
&\le h \left(\sum_{n=1}^{N} \delta t\, \|\mathcal{B}(s_{l,\delta t,h})\|_{X_h}^2 + C \right) \le Ch,
\end{aligned}
$$

where we have used the definitions of $s_{l,\delta t,h}$ and $\tilde{s}_{l,\delta t,h}$, the Cauchy-Schwarz inequality and the estimate (5.12), thus

(6.4) $$\mathsf{T}_1 \le Ch^{\theta}.$$

To treat T_2, we use the fact that the map ρ_l' is bounded and the relationship between the gas pressure and the global pressure, namely : $p_l = p - \bar{p}$ defined in (2.5), then we have

(6.5)
$$
\begin{aligned}
\mathsf{T}_2 &\le \max_{\mathbb{R}} |\rho_l'| \int_{Q_T} |p_{l,\delta t,h}(t,x) - \tilde{p}_{l,\delta t,h}(t,x)|\, \mathrm{d}x\mathrm{d}t \\
&\le \max_{\mathbb{R}} |\rho_l'| \int_{Q_T} |p_{\delta t,h}(t,x) - \tilde{p}_{\delta t,h}(t,x)|\, \mathrm{d}x\mathrm{d}t \\
&\quad + \max_{\mathbb{R}} |\rho_l'| \int_{Q_T} |\bar{p}(s_{l,\delta t,h}(t,x)) - \bar{p}(\tilde{s}_{l,\delta t,h}(t,x))|\, \mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

furthermore one can easily show that $\bar{p}$ is a $C^1([0,1];\mathbb{R})$, it follows, there exists a positive constant $C > 0$ such that

$$\mathsf{T}_2 \le C \int_{Q_T} |p_{\delta t,h}(t,x) - \tilde{p}_{\delta t,h}(t,x)|\mathrm{d}x\mathrm{d}t + C \int_{Q_T} |s_{l,\delta t,h}(t,x) - \tilde{s}_{l,\delta t,h}(t,x)|\mathrm{d}x\mathrm{d}t.$$

The last term in the previous inequality is proportional to T_1, and consequently it remains to show that the term on the global pressure is small with h. In fact, follows (6.3), one

gets

$$
\begin{aligned}
\int_{Q_T} |p_{\delta t,h}(t,x) - \tilde{p}_{\delta t,h}(t,x)| \mathrm{d}x \mathrm{d}t &= \sum_{n=1}^{N} \delta t \sum_{K \in \mathcal{T}_h} \sum_{\sigma_D \in \mathcal{E}_K} \int_{K \cap D} |p_{\delta t,h}(t,x) - \tilde{p}_{\delta t,h}(t,x)| \, \mathrm{d}x \\
&= \sum_{n=1}^{N} \delta t \sum_{K \in \mathcal{T}_h} \sum_{\sigma_D \in \mathcal{E}_K} \int_{K \cap D} |p_{\delta t,h}(t,x) - p_{\delta t,h}(t,Q_D)| \, \mathrm{d}x \\
&= \sum_{n=1}^{N} \delta t \sum_{K \in \mathcal{T}_h} \sum_{\sigma_D \in \mathcal{E}_K} \int_{K \cap D} |\nabla p_{\delta t,h}(t,x) \cdot (x - Q_D)| \, \mathrm{d}x \\
&\leq h \Big(\sum_{n=1}^{N} \delta t \, \|p_{\delta t,h}\|_{X_h}^2 + C \Big) \leq Ch.
\end{aligned}
$$

Finally, we consider the case where $\alpha = g$ in the same manner.   □

We now give the space translate estimate for $\tilde{\mathbf{U}}_{\alpha,\delta t,h}$ given by (3.5).

Lemma 6. $\Big(Space\ translate\ of\ \tilde{\boldsymbol{U}}_{\alpha,\delta t,h}\Big)$. *Under the assumptions* $(A1)-(A7)$. *Let* $p_{\alpha,\delta t,h}$ *be a solution of* (3.6)–(3.9). *Then, the following inequality hold:*

$$
(6.6) \qquad \int_{\Omega' \times (0,T)} \left| \tilde{\boldsymbol{U}}_{\alpha,\delta t,h}(t,x+y) - \tilde{\boldsymbol{U}}_{\alpha,\delta t,h}(t,x) \right| \mathrm{d}x \mathrm{d}t \leq \omega(|y|),
$$

for all $y \in \mathbb{R}^d$ *with* $\Omega' = \{x \in \Omega,\ [x, x+y] \subset \Omega\}$ *and* $\omega(|y|) \to 0$ *when* $|y| \to 0$.

Proof. For $\alpha = l$ and from the definition of $\mathbf{U}_{l,\delta t,h}$, one gets

$$
\begin{aligned}
&\int_{(0,T) \times \Omega'} \left| \tilde{\mathbf{U}}_{l,\delta t,h}(t,x+y) - \tilde{\mathbf{U}}_{l,\delta t,h}(t,x) \right| \mathrm{d}x \mathrm{d}t \\
&= \int_{(0,T) \times \Omega'} \left| \Big(\rho_l(\tilde{p}_{l,\delta t,h}) \tilde{s}_{l,\delta t,h} \Big)(t,x+y) - \Big(\rho_l(\tilde{p}_{l,\delta t,h}) \tilde{s}_{l,\delta t,h} \Big)(t,x) \right| \mathrm{d}x \mathrm{d}t \\
&\leq E_1 + E_2,
\end{aligned}
$$

where E_1 and E_2 defined as follows

$$
(6.7) \qquad E_1 = \rho_M \int_{(0,T) \times \Omega'} |\tilde{s}_{l,\delta t,h}(t,x+y) - \tilde{s}_{l,\delta t,h}(t,x)| \, \mathrm{d}x \mathrm{d}t,
$$

$$
(6.8) \qquad E_2 = \int_{(0,T) \times \Omega'} |\rho_l(\tilde{p}_{l,\delta t,h}(t,x+y)) - \rho_l(\tilde{p}_{l,\delta t,h}(t,x))| \, \mathrm{d}x \mathrm{d}t.
$$

To handle with the space translation on saturation, we use again the fact that $\mathcal{B}^{-1}$ is a Hölder function, then

$$
E_1 \leq \rho_M C \int_{(0,T) \times \Omega'} |\mathcal{B}(\tilde{s}_{l,\delta t,h}(t,x+y)) - \mathcal{B}(\tilde{s}_{l,\delta t,h}(t,x))|^{\theta} \, \mathrm{d}x \mathrm{d}t
$$

and by application of the Cauchy-Schwarz inequality, we deduce

$$
E_1 \leq C \Big(\int_{(0,T) \times \Omega'} |\mathcal{B}(\tilde{s}_{l,\delta t,h}(t,x+y)) - \mathcal{B}(\tilde{s}_{l,\delta t,h}(t,x))| \, \mathrm{d}x \mathrm{d}t \Big)^{\theta}.
$$

According to [14]), let $y \in \mathbb{R}^d$, $x \in \Omega'$, and $L \in N(K)$. We define a function $\beta_\sigma(x)$ for each $\sigma \in \mathcal{F}_h^{\mathrm{int}}$ by

$$
\beta_\sigma = \begin{cases} 1, & \text{if the line segment } [x, x+y] \text{ intersects } \sigma, \\ 0, & \text{otherwise.} \end{cases}
$$

We observe that (see [14] for more details) $\int_{\Omega'} \beta_{\sigma_{D,E}}(x)\,dx \leq |\sigma_{D,E}|$. Now, denote that

$$E_1 \leq C\Big(\sum_{n=1}^{N} \delta t \sum_{\sigma_{D,E} \in \mathcal{F}_h^{\text{int}}} \Big|\mathcal{B}(s_{l,E}) - \mathcal{B}(s_{l,D})\Big| \int_{\Omega'} \beta_{\sigma_{D,E}}(x)\mathrm{d}x\Big)^{\theta}$$

$$\leq C\Big(|y|\sum_{n=1}^{N}\delta t \sum_{\sigma_{D,E}\in\mathcal{F}_h^{\text{int}}} |\sigma_{D,E}|\,\Big|\mathcal{B}(s_{l,E}) - \mathcal{B}(s_{l,D})\Big|\Big)^{\theta}.$$

Let us write $|\sigma_{D,E}| = (d_{D,E}|\sigma_{D,E}|)^{\frac{1}{2}}(\frac{|\sigma_{D,E}|}{d_{D,E}})^{\frac{1}{2}}$. Obviously $d_{D,E} \leq \frac{\text{diam}(K_{D,E})}{d}$, and $|\sigma_{D,E}| \leq \frac{\text{diam}(K_{D,E})^{d-1}}{d-1}$, thus by the regularity shape assumption (3.1), we have

$$(6.9) \qquad \exists\, C_{te} > 0, \quad \forall h,\ \forall D \in \mathcal{D}_h\ \forall E \in \mathcal{N}(D) \qquad |\sigma_{D,E}|\,d_{D,E} \leq C_{te}\,|K|\,.$$

Applying again the Cauchy-Schwarz inequality, using (6.9), (4.1) and the fact that the discrete gradient of the function $\mathcal{B}$ is bounded (5.12) to obtain

$$(6.10) \qquad\qquad\qquad\qquad E_1 \leq C\,|y|^{\theta}\,.$$

To treat the space translate of E_2, we use the fact that the map ρ_l' is bounded and the relationship between the gas pressure and the global pressure, namely : $p_l = p - \bar{p}$ defined in (2.5), then we have

$$E_2 \leq \max_{\mathbb{R}} |\rho_l'| \int_{(0,T)\times\Omega'} |\tilde{p}_{l,\delta t,h}(t, x+y) - \tilde{p}_{l,\delta t,h}(t,x)|\,\mathrm{d}x\mathrm{d}t$$

$$\leq \max_{\mathbb{R}} |\rho_l'| \int_{(0,T)\times\Omega'} |\tilde{p}_{\delta t,h}(t, x+y) - \tilde{p}_{\delta t,h}(t,x)|\,\mathrm{d}x\mathrm{d}t$$

$$+ \max_{\mathbb{R}} |\rho_l'| \int_{(0,T)\times\Omega'} |\bar{p}(\tilde{s}_{l,\delta t,h}(t, x+y)) - \bar{p}(\tilde{s}_{l,\delta t,h}(t,x))|\,\mathrm{d}x\mathrm{d}t,$$

furthermore one can easily show that $\bar{p}$ is a $C^1([0,1];\mathbb{R})$, it follows, there exists a positive constant $C > 0$ such that

$$E_2 \leq C \int_{(0,T)\times\Omega'} |\tilde{p}_{\delta t,h}(t, x+y) - \tilde{p}_{\delta t,h}(t,x)|\mathrm{d}x\mathrm{d}t$$

$$+ C \int_{(0,T)\times\Omega'} |\tilde{s}_{l,\delta t,h}(t, x+y) - \tilde{s}_{l,\delta t,h}(t,x)|\mathrm{d}x\mathrm{d}t.$$

The last term in the previous inequality is proportional to E_1, and consequently it remains to show that the space translate on the global pressure is small with y. In fact

$$\int_{(0,T)\times\Omega'} |\tilde{p}_{\delta t,h}(t, x+y) - \tilde{p}_{\delta t,h}(t,x)|\mathrm{d}x\mathrm{d}t \leq \sum_{n=1}^{N}\delta t \sum_{\sigma_{D,E}} |p_E^n - p_D^n| \int_{\Omega'} \beta_{\sigma_{D,E}}(x)\mathrm{d}x$$

$$\leq |y|\sum_{n=1}^{N}\delta t \sum_{\sigma_{D,E}} |\sigma_{D,E}|\,|p_E^n - p_D^n|.$$

Finally, using (4.1) and the fact that the discrete gradient of global pressure is bounded (5.5), we deduce that

$$(6.11) \qquad\qquad \int_{(0,T)\times\Omega'} \Big|\tilde{\mathbf{U}}_{l,D}(t, x+y) - \tilde{\mathbf{U}}_{l,D}(t,x)\Big|\,\mathrm{d}x \leq C(|y| + |y|^{\theta}),$$

for some constant $C > 0$.

In the same way, we prove the space translate for $\alpha = g$. $\qquad\qquad\qquad\square$

We give below a time translate estimate for $\tilde{\mathbf{U}}_{\alpha,\delta t,h}$ given by (3.5).

Lemma 7. $\Big($*Time translate of* $\tilde{\boldsymbol{U}}_{\alpha,\delta t,h}\Big)$. *Under the assumptions* $(A1) - (A7)$. *Let* $p_{\alpha,\delta t,h}$ *be a solution of* (3.6)–(3.9). *Then, there exists a positive constant* $C > 0$ *depending on* Ω, T *such that the following inequality hold:*

$$(6.12) \qquad\qquad \int_{\Omega\times(0,T-\tau)} \Big|\tilde{\boldsymbol{U}}_{\alpha,\delta t,h}(t+\tau, x) - \tilde{\boldsymbol{U}}_{\alpha,\delta t,h}(t,x)\Big|^2\,dx\,dt \leq \tilde{\omega}(\tau),$$

for all $\tau \in (0, T)$. Here $\tilde{\omega} : \mathbb{R}^+ \to \mathbb{R}^+$ is a modulus of continuity, i.e. $\lim_{\tau \to 0} \tilde{\omega}(\tau) = 0$.

We state without proof the following lemma on time translate of $\mathbf{U}_{\alpha, \delta t, h}$. Following [14] and [24], the proof is a direct consequence of (4.1) and the estimations (5.5) and (5.12), then we omit it.

7. CONVERGENCE AND STUDY OF THE LIMIT

Using the a priori estimates of the previous section and the Kolmogorov relative compactness theorem, we show in this section that the approximate solutions $p_{\alpha, h, \delta t}$ converge strongly in $L^1(Q_T)$ to a function p_α and we prove that p_α is a weak solution of the continuous problem.

7.1. **Strong convergence in $L^1(Q_T)$ and convergence almost everywhere in Q_T.**

Theorem 2. *(Strong convergence in $L^1(Q_T)$) There exist subsequences of $s_{\alpha, \delta t, h}$, $p_{\alpha, \delta t, h}$, $\tilde{s}_{\alpha, \delta t, h}$ and $\tilde{p}_{\alpha, \delta t, h}$ verify the following convergence*

$$\tag{7.1} \tilde{\mathbf{U}}_{\alpha, \delta t, h} \text{ and } \mathbf{U}_{\alpha, \delta t, h} \longrightarrow \mathbf{U}_\alpha \qquad \textit{strongly in } L^1(Q_T) \textit{ and a.e. in } Q_T,$$

$$\tag{7.2} \tilde{s}_{\alpha, \delta t, h} \text{ and } s_{\alpha, \delta t, h} \longrightarrow s_\alpha \qquad \textit{almost everywhere in } Q_T,$$

$$\tag{7.3} \tilde{p}_{\alpha, \delta t, h} \text{ and } p_{\alpha, \delta t, h} \longrightarrow p_\alpha \qquad \textit{almost everywhere in } Q_T.$$

Furthermore, $\mathcal{B}(s_\alpha)$ and p_α belongs in $L^1(0, T; H^1_{\Gamma_l}(\Omega))$ and

$$\tag{7.4} 0 \le s_\alpha \le 1 \textit{ a.e. in } Q_T,$$

$$\tag{7.5} \mathbf{U}_\alpha = \phi \rho_\alpha(p_\alpha) s_\alpha \textit{ a.e. in } Q_T.$$

Proof. Observe that from Lemma 6 and 7 and Kolmogorov's compactness criterion ([5, Theorem IV.25], [14, Theorem 14.1]), we deduce that $\tilde{\mathbf{U}}_{\alpha, \delta t, h}$ is relatively compact in $L^1(Q_T)$. This ensures the following strong convergences of a subsequence of $\tilde{\mathbf{U}}_{\alpha, \delta t, h}$

$$\rho_\alpha(\tilde{p}_{\alpha, \delta t, h}) \tilde{s}_{\alpha, \delta t, h} \longrightarrow l_\alpha \quad \text{in } L^1(Q_T) \text{ and a.e. in } Q_T \,,$$

and due to the Lemma 5 , we deduce that $\mathbf{U}_{\alpha, \delta t, h}$ converges to the same l_α.

Denote by $u_\alpha = \rho_\alpha(p_\alpha) s_\alpha$. Define the map $\mathbb{A} : \mathbb{R}^+ \times \mathbb{R}^+ \mapsto \mathbb{R}^+ \times [0, \mathcal{B}(1)]$ defined by

$$\tag{7.6} \mathbb{A}(u_l, u_g) = (p, \mathcal{B}(s_l))$$

where u_α are solutions of the system

$$u_l(p, \mathcal{B}(s_l)) = \rho_l(p - \bar{p}(\mathcal{B}^{-1}(\mathcal{B}(s_l)))) \mathcal{B}^{-1}(\mathcal{B}(s_l))$$
$$u_g(p, \mathcal{B}(s_l)) = \rho_g(p - \tilde{p}(\mathcal{B}^{-1}(\mathcal{B}(s_l))))(1 - \mathcal{B}^{-1}(\mathcal{B}(s_l))).$$

Note that $\mathbb{A}$ is well defined as a diffeomorphism [19], [20] and [6]. As the map $\mathbb{A}$ defined in (7.6) is continuous, we deduce

$$p_{\delta t, h} \longrightarrow p \quad \text{a.e. in } Q_T,$$
$$\mathcal{B}(s_{l, \delta t, h}) \longrightarrow \mathcal{B}^* \quad \text{a.e. in } Q_T.$$

Then, as $\mathcal{B}^{-1}$ is continuous, this leads to the desired estimate (7.2)

$$s_{l, \delta t, h} \longrightarrow s_l = \mathcal{B}^{-1}(\mathcal{B}^*) \quad \text{a.e. in } Q_T.$$

Consequently and due to the relationship between the pressure of each phase and the global pressure defined in (2.5), then the convergences (7.3) hold

$$p_{\alpha, \delta t, h} \longrightarrow p_\alpha \quad \text{a.e. in } Q_T.$$

Moreover, due to the space translate estimate on the saturation and the global pressure (6.7)-(6.8), [14, Theorem 3.10] gives that $\mathcal{B}(s_\alpha)$ and $p \in L^1(0, T; H^1_{\Gamma_l}(\Omega))$. The identification of the limit in (7.5) follows from the previous convergence. $\qquad \square$

7.2. **Proof of theorem 1.** In order to achieve the proof of Theorem 1 and show that p_α is a weak solution of the continuous problem, it remains to pass to the limit as $(\delta t, h)$ goes to zero in the formulations (3.7)–(3.8). For this purpose, we introduce

$$(7.7) \qquad \mathcal{G} := \{\psi \in \mathcal{C}^{2,1}(\Omega \times [0,T]), \psi = 0 \text{ on } \partial\Omega \times [0,T], \psi(.,T) = 0\}.$$

Let T be a fixed positive constant and $\psi \in \mathcal{G}$. Set $\psi_D^n := \psi(t^n, Q_D)$ for all $D \in \mathcal{D}_h$ and $n \in [0,N]$.

For the discrete liquid equation, we multiply the equation (3.7) by $\delta t \psi_D^n$ and sum the result over $D \in \mathcal{D}_h^{\text{int}}$ and $n \in \{1, \cdots, N\}$. This yields

$$\mathfrak{C}_1 + \mathfrak{C}_2 + \mathfrak{C}_3 = 0,$$

where

$$\mathfrak{C}_1 = \sum_{n=1}^{N} \sum_{D \in \mathcal{D}_h} |D| \, \phi_D \left(\rho_l(p_{l,D}^n) s_{l,D}^n - \rho_l(p_{l,D}^{n-1}) s_{l,D}^{n-1}\right) \psi_D^n,$$

$$\mathfrak{C}_2 = \sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \rho_{l,D,E}^n \Lambda_{D,E} G_l(s_{l,D}^n, s_{l,E}^n; \delta_{D|E}^n(p_l)) \psi_D^n,$$

$$\mathfrak{C}_3 = \sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} |D| \left(\rho_l(p_{l,D}^n) s_{l,D}^n f_{P,D}^n \psi_D^n - \rho_l(p_{l,D}^n)(s_{l,D}^I)^n f_{I,D}^n \psi_D^n\right).$$

using $\psi_D^n = 0$ for all $D \in \mathcal{D}_h^{\text{ext}}$ and $n \in \{0, \cdots, N\}$. We now show that each of the above terms converges to its continuous version as h and δt tend to zero.

Firstly, for the evolution term. Making summation by parts in time and keeping in mind that $\psi(T = t^N, Q_D) = \psi_D^N = 0$. For all $D \in \mathcal{D}_h$, we get

$$\mathfrak{C}_1 = -\sum_{n=1}^{N} \sum_{D \in \mathcal{D}_h} |D| \, \phi_D \rho_l(p_{l,D}^n) s_{l,D}^n \left(\psi_D^n - \psi_D^{n-1}\right) - \sum_{D \in \mathcal{D}_h} |D| \, \phi_D \rho_l(p_{l,D}^0) s_{l,D}^0 \psi_D^0$$

$$= -\sum_{n=1}^{N} \sum_{D \in \mathcal{D}_h} \int_{t^{n-1}}^{t^n} \int_D \phi_D \rho_l(p_{l,D}^n) s_{l,D}^n \partial_t \psi(t, Q_D) \mathrm{d}x \mathrm{d}t - \sum_{D \in \mathcal{D}_h} \int_D \phi_D \rho_l(p_{l,D}^0) s_{l,D}^0 \psi(0, Q_D) \mathrm{d}x.$$

Since $\phi_h \rho_l(p_{l,\delta t,h}) s_{l,\delta t,h}$ and $\phi_h \rho_l(p_{l,\delta t,h}^0) s_{l,\delta t,h}^0$ converge almost everywhere respectively to $\phi \rho_l(p_l) s_l$ and $\phi \rho_l(p_l^0) s_l^0$, and as a consequence of Lebesgue dominated convergence theorem, we get

$$\mathfrak{C}_1 \longrightarrow -\int_{Q_T} \phi \rho_l(p_l) s_l \partial_t \psi(t,x) \mathrm{d}x \mathrm{d}t - \int_\Omega \phi \rho_l(p_l^0) s_l^0 \psi(0,x) \mathrm{d}x, \text{ as } h, \delta t \to 0.$$

Now, let us focus on convergence of the degenerate diffusive term to show

$$(7.8) \qquad \mathfrak{C}_2 \longrightarrow -\int_{Q_T} \rho_l(p_l) M_l(s_l) \nabla p_l \cdot \nabla \psi \mathrm{d}x \mathrm{d}t, \text{ as } h, \delta t \to 0.$$

Since the discrete gradient of each phase is not bounded, it is not possible to justify the pass to the limit in a straightforward way. To do this, we use the feature of global pressure and the auxiliary pressures defined in (2.5) and the discrete energy estimates in proposition 1 and corollary 1.

We rewrite $\mathfrak{C}_2$ as

$$\mathfrak{C}_2 = \mathfrak{C}_{2,1} + \mathfrak{C}_{2,2}$$

with, by using the definition (2.5),

$$\mathfrak{C}_{2,1} = -\sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \Lambda_{D,E} \rho_{l,D,E}^n M_l(s_{l,D,E}^n) \delta_{D|E}^n(p) \psi_D^n,$$

$$\mathfrak{C}_{2,2} = \sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \Lambda_{D,E} \rho_{l,D,E}^n M_l(s_{l,D,E}^n) \delta_{D|E}^n(\bar{p}(s_l)) \psi_D^n.$$

Let us show that

$$(7.9) \qquad \mathfrak{C}_{2,1} \longrightarrow - \int_{Q_T} \mathbf{\Lambda}(x)\, \rho_l(p_l) M_l(s_l) \nabla p \cdot \nabla \psi \, \mathrm{d}x \mathrm{d}t \text{ as } \delta t, h \to 0.$$

For each couple of neighbours D and E we denote $s^n_{l,min}$ the minimum of $s^n_{l,D}$ and $s^n_{l,E}$ and we introduce

$$(7.10) \qquad \mathfrak{C}^*_{2,1} = - \sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \Lambda_{D,E} \, \rho^n_{l,D,E} M_l(s^n_{l,min}) \delta^n_{D|E}(p) \psi^n_D$$

We now show

$$(7.11) \qquad \mathfrak{C}^*_{2,1} \longrightarrow - \int_{Q_T} \mathbf{\Lambda}(x)\, \rho_l(p_l) M_l(s_l) \nabla p \cdot \nabla \psi \, \mathrm{d}x \mathrm{d}t \text{ as } \delta t, h \to 0$$

as $\delta t, h \to 0$. Define $\overline{s}_{\alpha,\delta t,h}$ and $\underline{s}_{\alpha,\delta t,h}$ by

$$\overline{s}_{\alpha,\delta t,h}|_{(t^n,t^n] \times K_{D,E}} := \max\{s^n_{\alpha,D}, s^n_{\alpha,E}\}, \quad \underline{s}_{\alpha,\delta t,h}|_{(t^n,t^n] \times K_{D,E}} := \min\{s^n_{\alpha,D}, s^n_{\alpha,E}\}$$

Remark that

$$\mathfrak{C}^*_{2,1} = - \sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \Lambda_{D,E} \, \rho^n_{l,D,E} M_l(s^n_{l,min}) \delta^n_{D|E}(p) \psi^n_D$$

$$= - \sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{D}_h} \Lambda_{D,E} \, \rho^n_{l,D,E} M_l(s^n_{l,min}) p^n_E \psi^n_D$$

$$= - \sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{D}_h} \rho^n_{l,D,E} M_l(s^n_{l,min}) \sum_{K \in \mathcal{T}_h} (\mathbf{\Lambda}(x) \nabla \varphi_E, \nabla \varphi_D)_{0,K} \, p^n_E \psi^n_D$$

$$= \sum_{n=1}^{N} \delta t \sum_{K \in \mathcal{T}_h} \int_K \mathbf{\Lambda}(x) \rho_l(p^n_{l,h}) M_l(\underline{s}^n_{l,h}) \nabla p^n_h \cdot \nabla \left(\sum_{D \in \mathcal{D}_h} \psi(t^n, Q_D) \varphi_D(x) \right) \mathrm{d}x$$

We will show the validity of two passages to the limit. We begin by defining :

$$\mathfrak{D}_1 = \mathfrak{C}^*_{2,1} - \sum_{n=1}^{N} \delta t \sum_{K \in \mathcal{T}_h} \int_K \mathbf{\Lambda}(x) \rho_l(p^n_{l,h}) M_l(\underline{s}^n_{l,h}) \nabla p^n_h \cdot \nabla \psi(t^n, x) \mathrm{d}x.$$

We then estimate

$$|\mathfrak{D}_1| \leq Ch,$$

using the estimate (5.5), for more details see [9, Theorem 15.3] and [24, section 6.2]. Then,

$$\mathfrak{D}_1 \longrightarrow 0 \text{ as } h \to 0.$$

We next show that

$$(7.12) \quad \sum_{n=1}^{N} \delta t \sum_{K \in \mathcal{T}_h} \int_K \mathbf{\Lambda}(x) \rho_l(p^n_{l,h}) M_l(\underline{s}^n_{l,h}) \nabla p^n_h \cdot \nabla \psi(t^n, x) \mathrm{d}x \longrightarrow$$

$$\int_0^T \int_\Omega \mathbf{\Lambda}(x) \rho_l(p_l) M_l(s_l) \nabla p(t,x) \cdot \nabla \psi(t,x) \, \mathrm{d}x \mathrm{d}t$$

as $\delta t, h \to 0$. We see that both $p^n_h(x)$ and $\psi(t^n, x)$ are constant in time, so that we can easily introduce an integral with respect to time into the first term of (7.12). We further add and subtract

$$\sum_{n=1}^{N} \int_{t^{n-1}}^{t^n} \int_\Omega \mathbf{\Lambda}(x) \rho_l(p^n_{l,h}) M_l(\underline{s}^n_{l,h}) \nabla p^n_h \cdot \nabla \psi(t,x) \, \mathrm{d}x \mathrm{d}t$$

and introduce

$$\mathfrak{D}_2 := \sum_{n=1}^{N} \int_{t^{n-1}}^{t^n} \sum_{K \in \mathcal{T}_h} \int_K \mathbf{\Lambda}(x) \rho_l(p^n_{l,h}) M_l(\underline{s}^n_{l,h}) \nabla p^n_h \cdot (\nabla \psi(t^n, x) - \nabla \psi(t,x)) \, \mathrm{d}x \mathrm{d}t,$$

$$\mathfrak{D}_3 := \int_0^T \sum_{K \in \mathcal{T}_h} \int_K \mathbf{\Lambda}(x) \rho_l(p_{l,\delta t,h}) M_l(\underline{s}_{l,\delta t,h}) \nabla p_{\delta t,h}(t,x) \cdot \nabla \psi(t,x)$$

$$- \int_0^T \int_\Omega \mathbf{\Lambda}(x)\, \rho_l(p_l) M_l(s_l) \nabla p(t,x) \cdot \nabla \psi(t,x)\, \mathrm{d}x \mathrm{d}t$$

where $p_{\delta t,h}$ is given by (3.4). Clearly, (7.12) is valid when $\mathfrak{D}_2$ and $\mathfrak{D}_3$ tend to zero as $\delta t, h \to 0$. We first estimate $\mathfrak{D}_2$. We have, for $t \in (t^{n-1}, t^n]$,

$$|\nabla \psi(t^n, x) - \nabla \psi(t, x)| \le g(\delta t),$$

where g satisfies $g(\delta t) > 0$ and $g(\delta t) \longrightarrow 0$ as $\delta t \to 0$. Thus

$$|\mathfrak{D}_2| \le Cg(\delta t) \sum_{n=1}^N \delta t \sum_{K \in \mathcal{T}_h} |\nabla p_h^n|_K| \, |K| \le Cg(\delta t) T^{\frac{1}{2}} \, |\Omega|^{\frac{1}{2}}$$

using the Cauchy-Schwarz inequality and the estimate (5.5).

We now turn to $\mathfrak{D}_3$. We easily notice that we cannot use the Green theorem for p_h^n on Ω, since $p_h^n \notin H^1(\Omega)$. So, we are thus forced to apply it on each $K \in \mathcal{T}_h$.

To show that $\mathfrak{D}_3 \longrightarrow 0$ as $\delta t, h \to 0$, we begin by showing that

$$(7.13) \qquad \int_0^T \sum_{K \in \mathcal{T}_h} \int_K \left(\nabla p_{\delta t,h}(t,x) - \nabla p(t,x) \right) \cdot \mathbf{w}(t,x)\, \mathrm{d}x \mathrm{d}t \longrightarrow 0$$

as $\delta t, h \to 0$ for all $\mathbf{w} \in (\mathcal{C}^1(\overline{Q_T}))^d$. To this purpose, we use the a priori estimate (5.5) and [24, Section 6.2]. Using the density of the set $[\mathcal{C}^1(\overline{Q_T})]^d$ in $[L^2(Q_T)]^d$ and (7.13), we will conclude a weak convergence of $\nabla p_{\delta t,h}$ (piecewise constant function is space and time) to ∇p.

We now finally conclude that $\mathfrak{D}_3 \longrightarrow 0$ as $\delta t, h \to 0$. To do that, we begin by showing that $\underline{s}_{l,\delta t,h} \to s_l$ a.e on Q_T. Define $\overline{s}_{\alpha,\delta t,h}$ and $\underline{s}_{\alpha,\delta t,h}$ by

$$\overline{s}_{\alpha,\delta t,h}|_{(t^n,t^n] \times K_{D,E}} := \max\{s_{\alpha,D}^n, s_{\alpha,E}^n\}, \quad \underline{s}_{\alpha,\delta t,h}|_{(t^n,t^n] \times K_{D,E}} := \min\{s_{\alpha,D}^n, s_{\alpha,E}^n\}$$

By the monotonicity of $\mathcal{B}$, we have

$$\int_0^T \int_\Omega \left| \mathcal{B}(\overline{s}_{l,\delta t,h}) - \mathcal{B}(\underline{s}_{l,\delta t,h}) \right|^2 \mathrm{d}x \mathrm{d}t \le \sum_{n=1}^N \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \int_{K_{D,E}} \left(\mathcal{B}(s_{l,E}^n) - \mathcal{B}(s_{l,D}^n) \right)^2 \mathrm{d}x$$

$$\le \sum_{n=1}^N \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \int_{K_{D,E}} \left| \nabla \mathcal{B}(s_{l,h}^n)|_{K_{D,E}} \right|^2 d_{D,E}^2 \, \mathrm{d}x$$

$$\le \sum_{n=1}^N \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} \left| \nabla \mathcal{B}(s_{l,h}^n)|_{K_{D,E}} \right|^2 d_{D,E}^2 \, |K_{D,E}|$$

$$\le h^2 \sum_{n=1}^N \delta t \sum_{\sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}}} \left| \nabla \mathcal{B}(s_{l,h}^n)|_{K_{D,E}} \right|^2 |K_{D,E}|$$

$$\le h^2 \sum_{n=1}^N \delta t \left\| \mathcal{B}(s_{l,h}^n) \right\|_{X_h}^2 \le Ch^2$$

where we have used the estimate (5.12).

Since $\mathcal{B}^{-1}$ is continuous, we deduce up to a subsequence

$$(7.14) \qquad \left| \underline{s}_{\alpha,\mathcal{D}_m} - \overline{s}_{\alpha,\mathcal{D}_m} \right| \to 0 \text{ a.e. on } Q_T.$$

Moreover, we have $\underline{s}_{\alpha,\delta t,h} \le s_{\alpha,\delta t,h} \le \overline{s}_{\alpha,\delta t,h}$ and $s_{\alpha,\delta t,h} \to s_\alpha$ a.e. on Q_T. Consequently, and due to the continuity of the mobility function M_l we have

$$(7.15) \qquad M_l(\underline{s}_{l,\delta t,h}) \to M_l(s_l)$$

a.e on Q_T and in $L^p(Q_T)$ for $p < +\infty$.

Finally, we further add and subtract $\int_0^T \int_\Omega \mathbf{\Lambda}(x)\, \rho_l(p_l) M_l(s_l) \nabla p_{\delta t,h}(t,x) \cdot \nabla \psi(t,x)\, \mathrm{d}x \mathrm{d}t$ to $\mathfrak{D}_3$ and using (7.3), (7.15), the a priori estimate (5.5), the weak convergence of $\nabla p_{\delta t,h}$

to ∇p (7.13), to conclude that $\mathfrak{D}_3 \longrightarrow 0$ as $\delta t, h \to 0$. Altogether, combining (7.10) and (7.12) gives

$$\mathfrak{C}_{2,1}^* \longrightarrow -\int_{Q_T} \mathbf{\Lambda}(x)\, \rho_l(p_l) M_l(s_l) \nabla p \cdot \nabla \psi \, \mathrm{d}x \mathrm{d}t \text{ as } \delta t, h \to 0.$$

It remains to show that

(7.16) $$\left| \mathfrak{C}_{2,1} - \mathfrak{C}_{2,1}^* \right| \longrightarrow 0 \text{ as } \delta t, h \to 0.$$

Remark that

$$\left| M_l(s_{l,D,E}^n) \delta_{D|E}^n(p) - M_l(s_{l,min}^n) \delta_{D|E}^n(p) \right| \leq C \left| s_{l,E}^n - s_{l,D}^n \right| \left| \delta_{D|E}^n(p) \right|.$$

Consequently

$$\left| \mathfrak{C}_{2,1} - \mathfrak{C}_{2,1}^* \right| \leq C \sum_{n=1}^{N} \delta t \sum_{K \in \mathcal{T}_h} \int_K \left| s_{l,E}^n - s_{l,D}^n \right| \nabla p_h^n \cdot \nabla \left(\sum_{D \in \mathcal{D}_h} \psi(t^n, Q_D) \varphi_D(x) \right) \mathrm{d}x$$

Applying the Cauchy-Schwarz inequality, and thanks to the uniform bound (5.5) and the convergence (7.14), we establish (7.16).

To prove the pass to limit of $\mathfrak{C}_{2,2}$, we need to prove firstly that

$$\|\delta_{D|E}^n(\Gamma(s_l)) - \sqrt{M_l(s_{l,D,E}^n)} \delta_{D|E}^n(\bar{p}(s_l))\|_{L^2(Q_T)} \to 0 \text{ as } \delta t, h \to 0,$$

where $\Gamma(s_l) = \int_0^{s_l} \sqrt{M_l(z)} \frac{\mathrm{d}\bar{p}}{\mathrm{d}s_l}(z) \mathrm{d}z$.
In fact, remark that there exist $a \in [s_{l,D}, s_{l,E}]$ such as:

$$\begin{aligned}
|\delta_{D|E}^n(\Gamma(s_l)) - \sqrt{M_l(s_{l,D,E}^n)} \delta_{D|E}^n(\bar{p}(s_l))| &= |\sqrt{M_l(a)} - \sqrt{M_l(s_{l,D,E}^n)}||\delta_{D|E}^n(\bar{p}(s_l))| \\
&\leq C |\delta_{D|E}^n(\bar{p}(s_l))| \leq C \left| s_{l,E}^n - s_{l,D}^n \right| \\
&\leq C \left| \mathcal{B}(s_{l,E}^n) - \mathcal{B}(s_{l,D}^n) \right|^\theta,
\end{aligned}$$

since $\mathcal{B}^{-1}$ is an Hölder function. Thus we get,

$$\begin{aligned}
&\|\delta_{D|E}^n(\Gamma(s_l)) - \sqrt{M_l(s_{l,D,E}^n)} \delta_{D|E}^n(\bar{p}(s_l))\|_{L^2(Q_T)}^2 \\
&= \sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} |K_{D,E}| |\delta_{D|E}^n(\Gamma(s_l)) - \sqrt{M_l(s_{l,D,E}^n)} \delta_{D|E}^n(\bar{p}(s_l))|^2 \\
&\leq \sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} |K_{D,E}|^{1-\theta} |K_{D,E}|^\theta \left| \mathcal{B}(s_{l,E}^n) - \mathcal{B}(s_{l,D}^n) \right|^{2\theta},
\end{aligned}$$

and using the Cauchy-Schwarz inequality and the estimate , we deduce

$$\begin{aligned}
&\|\delta_{D|E}^n(\Gamma(s_l)) - \sqrt{M_l(s_{l,D,L}^n)} \delta_{D|E}^n(\bar{p}(s_l))\|_{L^2(Q_T)}^2 \\
&\leq \left(\sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} |K_{D,E}| \right)^{1-\theta} \left(\sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} |K_{D,E}| \left| \mathcal{B}(s_{l,E}^n) - \mathcal{B}(s_{l,D}^n) \right|^2 \right)^\theta \\
&\leq \left(\sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} |K_{D,E}| \right)^{1-\theta} \left(\sum_{n=1}^{N} \delta t \sum_{D \in \mathcal{D}_h} \sum_{E \in \mathcal{N}(D)} |K_{D,E}| \left| \nabla \mathcal{B}(s_{l,h}^n)|_{K_{D,E}} \right|^2 d_{D,E}^2 \right)^\theta \\
&\leq C h^{2\theta} \left(\sum_{n=1}^{N} \delta t \sum_{\sigma_{D,E} \in \mathcal{F}_h^{\mathrm{int}}} |K_{D,E}| \left| \nabla \mathcal{B}(s_{l,h}^n)|_{K_{D,E}} \right|^2 \right)^\theta \\
&\leq C h^{2\theta} \left(\sum_{n=1}^{N} \delta t \left\| \nabla \mathcal{B}(s_{l,h}^n) \right\|_{X_h}^2 \right)^\theta \leq C h^{2\theta}
\end{aligned}$$

which shows that $\|\delta_{D|E}^n(\Gamma(s_l)) - \sqrt{M_l(s_{l,D,E}^n)}\delta_{D|E}^n(\bar{p}(s_l))\|_{L^2(Q_T)}^2 \to 0$ as $h \to 0$. And from (5.13) in corollary 1, we deduce that there exists a constant $C > 0$ where the following inequalities hold:

$$(7.17) \qquad \sum_{n=1}^N \delta t \left\|\Gamma(s_{l,h}^n)\right\|_{X_h}^2 \le C.$$

Furthermore, by (7.2), we have

$$\Gamma(s_{l,\delta t,h}) \longrightarrow \Gamma(s_l) \text{ in } L^2(Q_T).$$

In the same manner of (7.13), we prove a weak convergence of $\nabla\Gamma(s_{l,\delta t,h})$ (piecewise constant function is space and time) to $\nabla\Gamma(s_l)$. As consequence $\sqrt{M_l(s_{l,\delta t,h})}\nabla\bar{p}(s_{l,\delta t,h})$ converges to $\nabla\Gamma(s_l)$ in $L^2(Q_T)$, and

$$(7.18) \qquad \mathfrak{C}_{2,2} \longrightarrow -\int_0^T \int_\Omega \rho_l(p_l)\sqrt{M_l(s_l)}\nabla\Gamma(s_l) \cdot \nabla\psi \mathrm{d}x\mathrm{d}t$$

$$(7.19) \qquad = -\int_0^T \int_\Omega \rho_l(p_l)M_l(s_l)\nabla\bar{p}(s_l) \cdot \nabla\psi \mathrm{d}x\mathrm{d}t.$$

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